Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN 28 (2011) 1–26

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Continuity of the blow-up profile with respect to initial data and to the blow-up point for a semilinear heat equation

S. Khenissy [a,b,1], Y. Rébaï [c,b,2], H. Zaag [d,*,3]

[a] *Université de Tunis El Manar, Institut Supérieur d'Informatique, Département de Mathématiques Appliquées, 2037 Ariana, Tunisia*
[b] *UR "Analyse non linéaire et géométrie" (03/UR/15-01), Université de Tunis El Manar (FST), Tunisia*
[c] *Université du 7 novembre à Carthage, Faculté des Sciences de Bizerte, Département de Mathématiques, 7021 Jarzouna, Bizerte, Tunisia*
[d] *LAGA, UMR 7539, Université Paris 13, Institut Galilée, 99 Avenue Jean-Baptiste Clément, 93430 Villetaneuse, France*

Received 23 February 2009; received in revised form 4 September 2010; accepted 15 September 2010

Available online 8 October 2010

Abstract

We consider blow-up solutions for semilinear heat equations with Sobolev subcritical power nonlinearity. Given a blow-up point $\hat{a}$, we have from earlier literature, the asymptotic behavior in similarity variables. Our aim is to discuss the stability of that behavior, with respect to perturbations in the blow-up point and in initial data. Introducing the notion of "profile order", we show that it is upper semicontinuous, and continuous only at points where it is a local minimum.

Résumé

Nous considérons des solutions explosives de l'équation semilinéaire de la chaleur avec une nonlinéarité sous-critique au sens de Sobolev. Etant donné un point d'explosion $\hat{a}$, grâce à des travaux antérieurs, on connaît le comportement asymptotique des solutions en variables auto-similaires. Notre objectif est de discuter la stabilité de ce comportement, par rapport à des perturbations du point d'explosion et de la donnée initiale. Introduisant la notion de « l'ordre du profil », nous montrons qu'il est semi-continu supérieurement, et continu uniquement aux points où il est un minimum local.

* Corresponding author.
E-mail addresses: saima.khenissy@insat.rnu.tn (S. Khenissy), Yomna.Rebai@fsb.rnu.tn, yomna.rebai@gmail.com (Y. Rébaï), Hatem.Zaag@univ-paris13.fr (H. Zaag).

1 This author is supported by a grant SSHN from the French Embassy in Tunis and the Tunisian Ministry of Higher Education, Scientific Research and Technology, Numéro d'affaire d'Egide 626836F.

2 This author is supported by a grant SSHN from French Embassy in Tunis and the Tunisian Ministry of Higher Education, Scientific Research and Technology, Numéro d'affaire d'Egide 531119E.

3 This author is supported by a grant from the French Agence Nationale de la Recherche, project ONDENONLIN, reference ANR-06-BLAN-0185.

0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2010.09.006

1. Introduction

We consider the parabolic problem

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u, \\ u(x,0) = u_0(x) \end{cases} \tag{1.1}$$

where $u(t) \in L^\infty(\mathbb{R}^N)$, $u : \mathbb{R}^N \times [0,T) \to \mathbb{R}$, the exponent $p > 1$ is subcritical (that means that $p < \frac{N+2}{N-2}$ if $N \geqslant 3$) and Δ stands for the Laplacian in $\mathbb{R}^N$.

Given $u_0 \in L^\infty(\mathbb{R}^N)$, by standard results, the parabolic problem (1.1) has a unique classical solution $u(x,t)$, which exists at least for small times. The solution $u(x,t)$ may develop singularities in some finite time, no matter how smooth $u_0(x)$ is. We say that $u(x,t)$ blows up in a finite time T if $u(x,t)$ satisfies (1.1) in $\mathbb{R}^N \times [0,T)$ and

$$\lim_{t\to T} \|u(t)\|_{L^\infty} = +\infty.$$

T is called the blow-up time of $u(x,t)$.

A point $a \in \mathbb{R}^N$ is a blow-up point if and only if there exist $(a_n, t_n) \to (a,T)$ such that $|u(a_n,t_n)| \to +\infty$ as $n \to +\infty$. We know from [25] that an equivalent definition could be a point $a \in \mathbb{R}^N$ such that $|u(x,t)| \to +\infty$ as $(x,t) \to (a,T)$. The blow-up set $S_u \subset \mathbb{R}^N$ at time T is the set of all blow-up points.

Problem (1.1) has been addressed in different ways in the literature. A major direction was developed by authors looking for sufficient blow-up conditions on initial data (cf. Levine [20], Ball [3]) or on the exponent (cf. Fujita [9]). The second main direction is about the description of the asymptotic blow-up behavior, locally near a given blow-up point (cf. Giga and Kohn [12], Bricmont and Kupiainen [4], Herrero and Velázquez [17], Velázquez [28], Merle and Zaag [25]). It happens however that most contributions concern the case of isolated blow-up, which is better understood (see Weissler [32], Bricmont and Kupiainen [4], Fermanian Kammerer, Merle and Zaag [6,5]), and much less the case of non-isolated points. In this paper, we make contributions to the asymptotic behavior question, in particular in the much less studied case of non-isolated blow-up points.

Consider $u(x,t)$ a solution of (1.1) which blows up at a time T on some blow-up set S_u. The very first question to be answered is the blow-up rate. According to Giga and Kohn [11] and Giga, Matsui and Sasayama [13], we know that

$$\forall t \in [0,T), \quad \|u(t)\|_{L^\infty} \leqslant C(T-t)^{-\frac{1}{p-1}}. \tag{1.2}$$

This fundamental step opens the door to the notion of blow-up profile which has been initiated by Herrero and Velázquez in [14,17], Velázquez in [28,29], Filippas and Kohn in [7] and Filippas and Liu in [8]. The following selfsimilar change of variables is particularly well adapted to the study of the blow-up profile.

Given a be a blow-up point of $u(x,t)$ (a solution to (1.1)) at time T, we set

$$u(x,t) = (T-t)^{-\frac{1}{p-1}} w_{a,T}(y,s) \quad \text{where } x - a = y(T-t)^{\frac{1}{2}},\ s = -\log(T-t) \tag{1.3}$$

so that the selfsimilar solution $w_{a,T}(y,s)$ satisfies for all $s \in [-\log T, +\infty)$ and for all $y \in \mathbb{R}^N$,

$$\partial_s w_{a,T} = \Delta w_{a,T} - \frac{1}{2} y \cdot \nabla w_{a,T} - \frac{1}{p-1} w_{a,T} + |w_{a,T}|^{p-1} w_{a,T}. \tag{1.4}$$

The study of u in the neighborhood of (a,T) is equivalent to the study of $w_{a,T}$ for large values of the time s. We note that, considering $-w_{a,T}$ if necessary, we have by [12],

$$w_{a,T}(y,s) \underset{s\to+\infty}{\longrightarrow} \kappa = (p-1)^{-\frac{1}{p-1}},$$

uniformly on compact sets. Moreover, we know that the speed of convergence is either $|\log(T-t)|^{-1}$ (slow) or $(T-t)^\mu$ (fast) for some $\mu > 0$ (see Velázquez [29] for example).

To learn more about the way $w_{a,T}$ approaches κ, it is natural to linearize Eq. (1.4) about κ. If we set

$$v_{a,T}(y,s) = w_{a,T}(y,s) - \kappa, \tag{1.5}$$

then $v_{a,T}$ (or v for simplicity) satisfies the following equation

$$\partial_s v = \Delta v - \frac{1}{2} y \cdot \nabla v + v + f(v) \equiv \mathcal{L}v + f(v), \tag{1.6}$$

where $f(v) = |v+\kappa|^{p-1}(v+\kappa) - \frac{\kappa}{p-1} - \frac{p}{p-1}v$. We easily see from (1.2) that $|f(v)| \leqslant C|v|^2$ for some positive constant C.

It is natural to consider (1.6) as a dynamical system in the weighted Hilbert space

$$L^2_\rho(\mathbb{R}^N) = \left\{ g(y) \in L^2_{\mathrm{loc}}(\mathbb{R}^N) \colon \int_{\mathbb{R}^N} g^2(y)\rho(y)\,dy < +\infty \right\}, \quad \text{with } \rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{(4\pi)^{\frac{N}{2}}},$$

endowed with the norm defined by

$$\|g\|^2_{L^2_\rho} = \langle g, g\rangle_{L^2_\rho} = \int_{\mathbb{R}^N} \big(g(y)\big)^2 \rho(y)\,dy,$$

since the operator $\mathcal{L}$ is self-adjoint on $L^2_\rho(\mathbb{R}^N)$ and has eigenvalues

$$\lambda_m = 1 - \frac{m}{2}, \quad m = 0, 1, 2, \ldots. \tag{1.7}$$

If $N = 1$, then all the eigenvalues of $\mathcal{L}$ are simple and to $1 - \frac{m}{2}$ corresponds the eigenfunction

$$h_m(y) = \sum_{n=0}^{[\frac{m}{2}]} \frac{m!}{n!(m-2n)!}(-1)^n y^{m-2n}. \tag{1.8}$$

If $N \geqslant 2$, the eigenfunctions corresponding to $1 - \frac{m}{2}$ are

$$H_\alpha(y) = h_{\alpha_1}(y_1)\cdots h_{\alpha_N}(y_N), \quad \text{with } \alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N \text{ and } |\alpha| = m. \tag{1.9}$$

Since the eigenfunctions of $\mathcal{L}$ span $L^2_\rho(\mathbb{R}^N)$, we expand v as follows

$$v(y,s) = \sum_{k=0}^{\infty} v_k(y,s), \quad \text{where } v_k(y,s) = P_k(v)(y,s) \tag{1.10}$$

is the orthogonal projection of v on the eigenspace associated to $\lambda_k = 1 - \frac{k}{2}$.

With these new notations, we know from Velázquez [29] that if $v(\cdot,s)$ is not equal to the null function for some $s > 0$, then, it holds that

$$\big\|v(s) - P_m(v)(s)\big\|_{L^2_\rho} = o\big(\big\|v(s)\big\|_{L^2_\rho}\big) \quad \text{as } s \to +\infty, \tag{1.11}$$

for some even number $m = m(u_0, a) \geqslant 2$.

Moreover, the following possibilities arise according to the value of $m(u_0, a)$:

- If $m(u_0, a) = 2$, then there exists an orthogonal transformation of coordinate axes such that, denoting still by y the new coordinates

$$v(y,s) = -\frac{\kappa}{4ps}\sum_{k=1}^{l_a}\big(y_k^2 - 2\big) + o\left(\frac{1}{s}\right) \quad \text{as } s \to +\infty, \tag{1.12}$$

  and then for all $K_0 > 0$,

$$\sup_{|z| \leqslant K_0} \Big|(T-t)^{\frac{1}{p-1}} u\big(a + z\sqrt{(T-t)\big|\log(T-t)\big|}, t\big) - f_{l_a}(z)\Big| \to 0 \quad \text{as } t \to T, \tag{1.13}$$

  where $l_a = 1, \ldots, N$ and $f_{l_a}(z) = (p - 1 + \frac{(p-1)^2}{4p}\sum_{i=1}^{l_a} z_i^2)^{-\frac{1}{p-1}}$.
- If $m(u_0, a) \geqslant 4$ and even, there exist constants c_α not all zero such that

$$v(y,s) = -e^{(1-\frac{m(u_0,a)}{2})s} \sum_{|\alpha| = m(u_0,a)} c_\alpha H_\alpha(y) + o\big(e^{(1-\frac{m(u_0,a)}{2})s}\big) \quad \text{as } s \to +\infty, \tag{1.14}$$

and then for all $K_0 > 0$,

$$\sup_{|z|\leqslant K_0}\left|(T-t)^{\frac{1}{p-1}}u\big(a+z(T-t)^{\frac{1}{m}},t\big)-\Big(p-1+\sum_{|\alpha|=m(u_0,a)}C_\alpha z^\alpha\Big)^{-\frac{1}{p-1}}\right|\to 0 \quad \text{as } t\to T, \tag{1.15}$$

where $C_\alpha = -\frac{\kappa}{(p-1)^2}c_\alpha$, $x^\alpha = x_1^{\alpha_1}\cdots x_N^{\alpha_N}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_N$ if $\alpha = (\alpha_1,\dots,\alpha_N)$ and $B_a(x) = \sum_{|\alpha|=m(u_0,a)} C_\alpha x^\alpha \geqslant 0$ for all $x \in \mathbb{R}^N$.

We recall that in (1.12) and (1.14), convergence takes place in $L^2_\rho(\mathbb{R}^N)$ as well as in $C^{k,\gamma}_{\mathrm{loc}}(\mathbb{R}^N)$ for any $k \in \mathbb{N}$ and $\gamma \in (0,1)$.

In our paper, we call the even number $m(u_0,a)$ *the profile order* at the blow-up point a. One may think that this description of Velázquez is exhaustive, since it gives a "profile" near any blow-up point $a \in S_u$. In our opinion, this description lets two fundamental questions unanswered:

- **Question 1:** Are the descriptions (1.11)–(1.15) uniform with respect to the blow-up point and initial data?
- **Question 2:** What about the geometry of the blow-up set? In other words, is it possible to sum up the *local* information given in (1.12)–(1.15) for all $a \in S_u \cap B(\hat{a},\hat{\delta})$ for some $\hat{a} \in S_u$ and $\hat{\delta} > 0$, in order to derive *global* information about the geometry of $S_u \cap B(\hat{a},\hat{\delta})$?

In this paper, we address the first question. The second question was the very first motivation of our work. Indeed, we initially wanted to extend the work done in [33] in the case $m(\hat{u}_0,\hat{a}) = 2$ to the case $m(\hat{u}_0,\hat{a}) \geqslant 4$. In fact, in [33], the author could successfully use local information to show a global information. Namely, he proved that the blow-up set is a smooth manifold, assuming only continuity of the blow-up set. Unfortunately, we feel far from obtaining an analogous result when $m(\hat{u}_0,\hat{a}) \geqslant 4$, which is a much more complicated case. Thus, we leave the second question open.

In the following, we give various answers for Question 1 in Section 1.1. In Section 1.2, we discuss the difficulty of answering Question 2.

1.1. Uniform convergence to the blow-up profile in selfsimilar variables

We address Question 1 in this subsection. Up to our knowledge, Question 1 was first addressed by Zaag in [33] in the case $m(u_0,a) = 2$, under the assumption that S_u locally contains a continuum. In [33], the author proves that *the profile remains unchanged and that the convergence is uniform with respect to the blow-up point*. This uniform estimate allowed to derive local geometrical information on the blow-up set, namely that it is a C^1 manifold, and if its codimension is 1, then, it is of class C^2 (see [36]).

The result of [33,34] and [36] relies on a dynamical system formulation of Eq. (1.4) and on the following Liouville theorem by Merle and Zaag [24,25].

Liouville theorem for Eq. (1.4). *Assume that w is a solution of* (1.4) *defined on $\mathbb{R}^N \times \mathbb{R}$ such that $w \in L^\infty(\mathbb{R}^N)$. Then $w \equiv 0$ or $w \equiv \pm\kappa$ or $w(y,s) \equiv \pm\theta(s+s_0)$ for some $s_0 \in \mathbb{R}$, where $\theta(s) = \kappa(1+e^s)^{-\frac{1}{p-1}}$.*

In this paper, we want to see if the uniform convergence to the blow-up profile proved in [33] can be extended in a double way:

- to the case where $m(u_0,a) \geqslant 4$;
- by allowing perturbations, not only with respect to the blow-up point, but also with respect to initial data.

Our first result states that the profile order $m(u_0,a)$ is upper semicontinuous with respect to perturbations in the initial data and the blow-up point. More precisely, we prove the following:

Theorem 1 *(Upper semicontinuity of the profile order). Let $\hat{u}$ be a solution of* (1.1) *associated to the initial data $\hat{u}_0$ and blowing up at a point $\hat{a}$ and at the time $\hat{T}$ such that $\hat{u}(x,t) \not\equiv \pm\kappa(\hat{T}-t)^{-\frac{1}{p-1}}$. Then, there exist $\mathcal{V}_{\hat{u}_0}$ a neighborhood* *of $\hat{u}_0$ in $L^\infty(\mathbb{R}^N)$ and $\hat{\delta} > 0$ such that for all $u_0 \in \mathcal{V}_{\hat{u}_0}$, u, the solution of* (1.1) *with initial data u_0, blows up at T and we have this alternative*:

(1) *either $S_u \cap B(\hat{a}, \hat{\delta}) = \emptyset$, or*
(2) *for all $a \in S_u \cap B(\hat{a}, \hat{\delta})$, $m(u_0, a) \leqslant \hat{m} \equiv m(\hat{u}_0, \hat{a})$.*

Moreover, we have

$$\sup_{u_0 \in \mathcal{V}_{\hat{u}_0}} \sup_{a \in S_u \cap B(\hat{a}, \hat{\delta})} \frac{\| v_{a,T}(s) - \sum_{i=2}^{\hat{m}} P_i(v_{a,T})(s) \|_{L^2_\rho}}{\| \sum_{i=2}^{\hat{m}} P_i(v_{a,T})(s) \|_{L^2_\rho}} \to 0 \quad \text{as } s \to +\infty. \tag{1.16}$$

Remark. Case (1) may occur as one can see from the example constructed by Merle in [23]. Indeed, given $\hat{a}$ and $\hat{b}$ in $\mathbb{R}$, Merle gives a family of blow-up solutions $u_\lambda(x, t)$ to (1.1), where $\lambda > 0$, with initial data $u_{0,\lambda}$ (continuous in λ) such that for a critical value $\lambda^* > 0$, the following occurs:

- If $\lambda = \lambda^*$, then u_{λ^*} blows up exactly at two points, $\hat{a}$ and $\hat{b}$ with $m(u_{0,\lambda^*}, \hat{a}) = m(u_{0,\lambda^*}, \hat{b}) = 2$.
- If $\lambda < \lambda^*$, then u_λ blows up only at a point a_λ with $m(u_{0,\lambda}, a_\lambda) = 2$ and $a_\lambda \to \hat{a}$ as $\lambda \to \lambda^{*-}$.
- If $\lambda > \lambda^*$, then u_λ blows up only at a point b_λ with $m(u_{0,\lambda}, b_\lambda) = 2$ and $b_\lambda \to \hat{b}$ as $\lambda \to \lambda^{*+}$.

Since $u_{0,\lambda} \to u_{0,\lambda^*}$ as $\lambda \to \lambda^*$, we see that for some $\varepsilon_0 > 0$, $\hat{\delta} > 0$, we have the following:

- If $\lambda^* < \lambda < \lambda^* + \varepsilon_0$, then $S_{u_\lambda} \cap B(\hat{a}, \hat{\delta}) = \emptyset$.
- If $\lambda^* - \varepsilon_0 < \lambda < \lambda^*$, then $S_{u_\lambda} \cap B(\hat{a}, \hat{\delta}) = \{a_\lambda\}$.

Thus, this example illustrates the alternative in Theorem 1.

Remark. The existence of the blow-up profile order for (u_0, a) means that $u(x, t)$ is different from the trivial solution $\pm\kappa(T - t)^{-\frac{1}{p-1}}$ (see the line before (1.11)). Since the profile order is by definition greater than or equal to 2, when $m(\hat{u}_0, \hat{a}) = 2$, we get $m(u_0, a) = 2$ for all $u_0 \in \mathcal{V}_{\hat{u}_0}$ and $a \in S_u \cap B(\hat{a}, \hat{\delta})$. In other words, the profile order is continuous near its minimal value 2. Theorem 1 was already obtained when $m(\hat{u}_0, \hat{a}) = 2$ by Fermanian Kammerer, Merle and Zaag [6] (for $l_{\hat{a}} = N$) and Zaag [33] (for $l_{\hat{a}} \leqslant N - 1$).

Remark. Unlike Zaag [34,33,36], there is no need to assume that $S_u \cap B(\hat{a}, \hat{\delta})$ contains a continuum.

Theorem 1 gives the uniform predominance of $\| \sum_{i=2}^{\hat{m}} P_i(v_{a,T})(s) \|_{L^2_\rho}$ with respect to the initial data u_0 in a neighborhood of $\hat{u}_0$ and with respect to the singular point a in a neighborhood of $\hat{a}$. It also provides the upper semicontinuity of the profile order $m(u_0, a)$. In order to get the continuity (in fact, the property of being locally constant, since $m(u_0, a) \in \mathbb{N}$), we give in the following theorem a necessary and sufficient condition:

Theorem 2 *(Necessary and sufficient conditions for the continuity of the profile order). Under the hypotheses of Theorem* 1*, the following statements are equivalent*:

(i) *For some $\hat{\delta}' > 0$ and some neighborhood $\mathcal{V}'_{\hat{u}_0}$ of $\hat{u}_0$, for all $u_0 \in \mathcal{V}'_{\hat{u}_0}$ and $a \in S_u \cap B(\hat{a}, \hat{\delta}')$, $m(u_0, a) = m(\hat{u}_0, \hat{a})$.*
(ii) *For some $\delta_0 > 0$ and some neighborhood $\mathcal{V}_0$ of $\hat{u}_0$,*

$$m(\hat{u}_0, \hat{a}) = \min_{u_0 \in \mathcal{V}_0,\, a \in S_u \cap B(\hat{a}, \delta_0)} m(u_0, a).$$

(iii) *For some $\hat{\delta}' > 0$ and some neighborhood $\mathcal{V}'_{\hat{u}_0}$ of $\hat{u}_0$,*

$$\sup_{u_0 \in \mathcal{V}'_{\hat{u}_0}} \sup_{a \in S_u \cap B(\hat{a}, \hat{\delta}')} \frac{\| v_{a,T}(s) - P_{\hat{m}}(v_{a,T})(s) \|_{L^2_\rho}}{\| P_{\hat{m}}(v_{a,T})(s) \|_{L^2_\rho}} \to 0 \quad \text{as } s \to +\infty.$$

Remark. Since $m(u_0, a) \in \mathbb{N}$, the set $\{(u_0, a) \mid u_0 \in \mathcal{V}_0$ and $a \in S_u \cap B(\hat{a}, \delta_0)\} \neq \emptyset$ (it contains $(\hat{u}_0, \hat{a})$), the minimum in (ii) is realized for some $(\hat{u}_1, \hat{a}_1)$. Up to replacing $(\hat{u}_0, \hat{a})$ by $(\hat{u}_1, \hat{a}_1)$ and shrinking the neighborhoods, (ii) is satisfied. Thus, Theorem 2 is not an empty statement.

Remark. Taking $u_0 = \hat{u}_0$ in Theorem 2, we obtain a new version of Theorem 2 given in the next subsection (see Theorem 2′).

Following Velázquez [29] and Filippas and Kohn [7], we find the asymptotic behavior (uniformly in u_0 and a) in the following

Proposition 3 *(Asymptotic behavior and blow-up profiles uniform in u_0 and a). The assertions of Theorem 2 are equivalent to the following: For some $\hat{\delta}' > 0$ and $\mathcal{V}'_{\hat{u}_0}$ a neighborhood of $\hat{u}_0$ in $L^\infty(\mathbb{R}^N)$:*

- *If $\hat{m} = 2$, then for some $C > 0$ and $s' \in \mathbb{R}$, we have for all $u_0 \in \mathcal{V}'_{\hat{u}_0}$, $a \in S_u \cap B(\hat{a}, \hat{\delta}')$ and $s \geqslant s'$*
$$\frac{1}{Cs} \leqslant \left\| v_{a,T}(s) \right\|_{L^2_\rho} \leqslant \frac{C}{s}. \tag{1.17}$$
- *If $\hat{m} \geqslant 4$, then for each $\alpha \in \mathbb{N}^N$ with $|\alpha| = \hat{m}$, there exists $c_\alpha(u_0, a) \in \mathbb{R}$ such that*
$$\sup_{u_0 \in \mathcal{V}'_{\hat{u}_0}} \sup_{a \in S_u \cap B(\hat{a}, \hat{\delta}')} \frac{\| v_{a,T}(s) - e^{(1-\frac{\hat{m}}{2})s} \sum_{|\alpha| = \hat{m}} c_\alpha(u_0, a) H_\alpha \|_{L^2_\rho}}{e^{(1-\frac{\hat{m}}{2})s}} \to 0 \quad \text{as } s \to +\infty,$$
and for all $K_0 > 0$,
$$\sup_{u_0 \in \mathcal{V}'_{\hat{u}_0}} \sup_{a \in S_u \cap B(\hat{a}, \hat{\delta}')} \sup_{|z| \leqslant K_0} \left| (T-t)^{\frac{1}{p-1}} u\big(a + z(T-t)^{\frac{1}{\hat{m}}}, t\big) - \left(p - 1 + \sum_{|\alpha| = \hat{m}} C_\alpha z^\alpha \right)^{-\frac{1}{p-1}} \right| \to 0 \quad \text{as } t \to T,$$
where $C_\alpha = -\frac{\kappa}{(p-1)^2} c_\alpha$ and the multilinear form $\sum_{|\alpha| = \hat{m}} C_\alpha x^\alpha \geqslant 0$ for all $x \in \mathbb{R}^N$.
Moreover $c_\alpha(u_0, a)$ is continuous with respect to u_0 and a.

Remark. Proposition 3 has already been obtained by Herrero and Velázquez [31,17] and Filippas and Liu in [8] (when $\hat{m} \geqslant 2$), with no uniform character in (u_0, a). Our contribution is exactly to prove this uniform character. In fact, when $\hat{m} \geqslant 2$, one has to slightly adapt the argument of [17] and [8] to get the uniform character. See the proof of Proposition 3 in Section 3.3.

Remark. Unlike the case $\hat{m} \geqslant 4$, we don't have a uniform convergence to some profile when $\hat{m} = 2$ systematically. The situation is indeed more complicated.

Indeed, if $l_{\hat{a}} = N$, then we know from [6, Theorem 2, p. 350] that for all $u_0 \in \mathcal{V}'_{\hat{u}_0}$, u_0 has a single blow-up point $a(u_0)$ in $B(\hat{a}, \hat{\delta}')$. Moreover, we have the uniform convergence to the profile, in the sense that
$$\sup_{u_0 \in \mathcal{V}'_{\hat{u}_0}} s \left\| v_{a(u_0),T} - \frac{\kappa}{2ps} \left(N - \frac{|y|^2}{2} \right) \right\|_{L^2_\rho} \to 0 \quad \text{as } s \to +\infty. \tag{1.18}$$
If $l_{\hat{a}} \leqslant N - 1$, then the uniform convergence to some profile is known only under the additional hypothesis that the blow-up set of $\hat{u}$ contains a continuum going through $\hat{a}$ of codimension $l_{\hat{a}}$. The question remains open without this hypothesis.

Remark. Since we expect from the announced result of Herrero and Velázquez [18] that $m(u_0, a) = 2$ is the generic behavior, the minimum in (ii) of Theorem 2 should be 2, hence the case $\hat{m} \geqslant 4$ in Proposition 3 is an empty case.

If $N = 1$, we know from Herrero and Velázquez [16,15] that the situation $m(u_0, a) = 2$ is generic, in the sense that: given initial data $\hat{u}_0 \in L^\infty(\mathbb{R}^N)$ such that the corresponding solution of Eq. (1.1) blows up at some time $\hat{T}$ at some point $\hat{a}$ with $m(\hat{u}_0, \hat{a}) \geqslant 4$, then any neighborhood of $\hat{u}_0$ contains initial data u_0 such that the corresponding solution of Eq. (1.1) blows up at some time T at only one point a with $m(u_0, a) = 2$.

Therefore, the minimum in (ii) of Theorem 2 is equal to 2 and our Theorem 2 reads as follows:

Corollary 4. *If* $N = 1$ *and under the hypotheses of Theorem* 1*, the following statements are equivalent*:

(i) *For some* $\hat{\delta}' > 0$ *and some neighborhood* $\mathcal{V}'_{\hat{u}_0}$ *of* $\hat{u}_0$*, for all* $u_0 \in \mathcal{V}'_{\hat{u}_0}$ *and* $a \in S_u \cap B(\hat{a}, \hat{\delta}')$*,* $m(u_0, a) = m(\hat{u}_0, \hat{a})$*.*
(iv) $m(\hat{u}_0, \hat{a}) = 2$.

Moreover, they are both equivalent to (ii) *and* (iii) *in Theorem* 2*.*

Remark. Following the third remark after Proposition 3, if the result of [18] is confirmed, then Corollary 4 becomes true for $N \geqslant 2$ too. From [16], one would derive that all the behaviors where $\hat{m} \geqslant 4$ or $\hat{m} = 2$ with $l_{\hat{a}} \leqslant N - 1$ are unstable.

1.2. Discussion of the geometry of the blow-up set

Regarding the blow-up set, two questions arise:

- **The description:** Given a blow-up solution $u(x, t)$ of (1.1), what can we say about its blow-up set S_u? The only general answer available with no restriction on initial data is due to Velázquez who proved in [30] that S_u is closed and that its Hausdorff dimension is at most equal to $N - 1$. Our Question 2 stated before Section 1.1 is a description question, to which we devote the following subsection.
- **The construction:** Given a closed set S whose Hausdorff dimension is at most equal to $N - 1$, is there a blow-up solution $u(x, t)$ of (1.1) such that $S_u = S$? The answer is yes when S is one of the following cases
  - a finite number of points from Merle [23];
  - a sphere thanks to Giga and Kohn in [12] (see (1.15), p. 848 and Corollary 5.7, p. 877);
  - a finite number of concentric spheres, as suggested by Matano and Merle in [22, Theorem 1.11, p. 1499]. To prove the existence of such a solution, one has to adapt the method used by Merle in [23].

  Note that the solution is radial in the two last cases. No other geometries for the blow-up sets are known (except those artificially generated from the above cases by adding irrelevant space variables to the domain of definition of the solution, giving rise to affine subspaces, cylinders, etc.). The question remains open in the other cases, in particular when S is an ellipse in 2 dimensions.

As we said above, this subsection is devoted to Question 2. Unfortunately, we don't give any answer, apart from recalling the results of [33,34,36] proved in the case where $m(\hat{u}_0, \hat{a}) = 2$. Indeed, the case $m(\hat{u}_0, \hat{a}) \geqslant 4$ is much more complicated. Our goal is to give the reader a flavor of the complexity of Question 2.

In the following, we fix initial data $u_0 = \hat{u}_0$ and allow a to move in $S_{\hat{u}}$ near some $\hat{a} \in S_{\hat{u}}$, a non-isolated blow-up point.

Question 2 asks whether one can derive any information on the geometry of the blow-up set near $\hat{a}$, from local information in (1.12)–(1.15) on the blow-up profile near $a \in S_{\hat{u}}$, where a is close to $\hat{a}$.

Knowing that $\hat{a}$ is a non-isolated blow-up point, we remark that two cases in (1.12)–(1.15) cannot hold since they lead to an isolated point:

- when $m(\hat{u}_0, a) = 2$ with $l_a = N$: we locally have a bump, see (1.18), or
- when case (1.14) occurs with a definite positive $B_a(x)$: in that case, we know from Velázquez [29] that a is an isolated blow-up point, i.e. $S_{\hat{u}} \cap B(a, \delta) = \{a\}$ for some $\delta > 0$.

Therefore, we either have ($m(\hat{u}_0, \hat{a}) = 2$ with $l_{\hat{a}} \leqslant N - 1$ or $m(\hat{u}_0, \hat{a}) \geqslant 4$ with a non-definite positive $B_{\hat{a}}(x)$).

– When $m(\hat{u}_0, \hat{a}) = 2$ and $l_{\hat{a}} \leqslant N - 1$, Zaag assumed in [33] (see also the note [35]) that $S_{\hat{u}}$ contains a continuum going through $\hat{a}$. He shows that $S_{\hat{u}}$ is locally a $\mathcal{C}^1$ manifold. In [36], he shows that when $l_{\hat{a}} = N - 1$, $S_{\hat{u}}$ is locally a $\mathcal{C}^2$ manifold. The proof relies on two steps:

- **Step 1:** *Stability of the blow-up profile with respect to perturbations in the blow-up point a and uniform convergence to the profile*
  The author proves the stability of the blow-up profile and the uniform convergence to the profile with respect to the blow-up point a near $\hat{a}$. The Liouville theorem in [24] and [25], stated in Section 1.1 of our paper, is the key tool in this step.
- **Step 2:** *A covering geometrical argument*
  From Step 1, the author derives an asymptotic profile for $u(x,t)$ in every ball $B(a, K_0\sqrt{T-t})$ for some $K_0 > 0$ and a a blow-up point close to $\hat{a}$. Most importantly, these profiles are continuous in a and the speed of convergence of u to each one in the ball $B(a, K_0\sqrt{T-t})$ is uniform with respect to a. Now, if a and b are in $S_{\hat{u}}$ and $0 < |a-b| \leqslant K_0\sqrt{T-t}$, then the balls $B(a, K_0\sqrt{T-t})$ and $B(b, K_0\sqrt{T-t})$ intersect each other, leading to two different profiles for $u(x,t)$ in the intersection. Of course, these profiles have to coincide, up to the error terms. This makes a geometric constraint which gives more regularity for the blow-up set near $\hat{a}$.
  The fact that the rate of convergence of the expansion of $u(x,t)$ in $B(a, K_0\sqrt{T-t})$ is uniform in a is "essential". By the way, Velázquez, Filippas and Liu obtain those profiles, with no uniform character with respect to a (see [8,28,29]).

This two-step technique was successfully used by Nouaili in [27] for the case of the semilinear wave equation

$$u_{tt} = u_{xx} + |u|^{p-1}u \tag{1.19}$$

where $u = u(x,t)$, $x \in \mathbb{R}$, $0 \leqslant t \leqslant T(x)$ and $p > 1$. More precisely, in [27], the author started from the $\mathcal{C}^1$ regularity of the blow-up set proved by Merle and Zaag in [26] and could prove the $\mathcal{C}^{1,\alpha}$ regularity using this two-step technique. Note that for Eq. (1.19), non-global solutions blow up on a graph $\Gamma = \{(x, T(x)) \mid x \in \mathbb{R}\}$, where $x \mapsto T(x)$ is 1-Lipschitz (see Alinhac [1,2] or Lindblad and Sogge [21]).

– When $m(\hat{u}_0, \hat{a}) \geqslant 4$ and $B_{\hat{a}}(x)$ is not positive definite, our ambition was to adapt the two-step technique of [33] here. We could obtain the first step provided that $m(\hat{u}_0, \hat{a}) = \min_{a \in S_{\hat{u}} \cap B(\hat{a}, \delta_0)} m(\hat{u}_0, a)$. More precisely, let us write the following two versions of Theorem 2 and Proposition 3 that we obtain taking $u_0 = \hat{u}_0$.

Theorem 2′ *(Stability of the profile order near a local minimum). Under the hypotheses of Theorem* 1*, the following statements are equivalent*:

(i)′ *For some $\hat{\delta}' > 0$, for all $a \in S_{\hat{u}} \cap B(\hat{a}, \hat{\delta}')$, $m(\hat{u}_0, a) = m(\hat{u}_0, \hat{a})$.*
(ii)′ *For some $\delta_0 > 0$, $m(\hat{u}_0, \hat{a}) = \min_{a \in S_{\hat{u}} \cap B(\hat{a}, \delta_0)} m(\hat{u}_0, a)$.*
(iii)′ *For some $\hat{\delta}' > 0$,*

$$\sup_{a \in S_{\hat{u}} \cap B(\hat{a}, \hat{\delta}')} \frac{\|v_{a,T}(s) - P_{\hat{m}}(v_{a,T})(s)\|_{L^2_\rho}}{\|P_{\hat{m}}(v_{a,T})(s)\|_{L^2_\rho}} \to 0 \quad \text{as } s \to +\infty.$$

We also have the following equivalent statements to those of Theorem 2′:

Proposition 3′ *(Asymptotic behavior and blow-up profiles uniform in a). The assertions of Theorem* 2′ *are equivalent to the following*: *For some $\hat{\delta}' > 0$*:

- *If $\hat{m} = 2$, then for some $C > 0$ and $s' \in \mathbb{R}$, we have for all $a \in S_{\hat{u}} \cap B(\hat{a}, \hat{\delta}')$ and $s \geqslant s'$*
  $$\frac{1}{Cs} \leqslant \|v_{a,T}(s)\|_{L^2_\rho} \leqslant \frac{C}{s}.$$
- *If $\hat{m} \geqslant 4$, then for each $\alpha \in \mathbb{N}^N$ with $|\alpha| = \hat{m}$, there exists $c_\alpha(a) \in \mathbb{R}$ such that*
  $$\sup_{a \in S_{\hat{u}} \cap B(\hat{a}, \hat{\delta}')} \frac{\|v_{a,T}(s) - e^{(1-\frac{\hat{m}}{2})s} \sum_{|\alpha|=\hat{m}} c_\alpha(a) H_\alpha\|_{L^2_\rho}}{e^{(1-\frac{\hat{m}}{2})s}} \to 0 \quad \text{as } s \to +\infty,$$

and for all $K_0 > 0$,

$$\sup_{a\in S_{\hat u}\cap B(\hat a,\hat\delta')}\ \sup_{|z|\leqslant K_0}\left|(T-t)^{\frac{1}{p-1}}u\big(a+z(T-t)^{\frac{1}{\hat m}},t\big)-\Big(p-1+\sum_{|\alpha|=\hat m}C_\alpha z^\alpha\Big)^{-\frac{1}{p-1}}\right|\to 0\quad \textit{as } t\to T,$$

where $C_\alpha = -\frac{\kappa}{(p-1)^2}c_\alpha$ *and the multilinear form* $\sum_{|\alpha|=\hat m} C_\alpha x^\alpha \geqslant 0$ *for all* $x\in\mathbb{R}^N$.
Moreover, $c_\alpha(a)$ *is continuous with respect to* a.

Thanks to Theorem 2′, it is enough to choose $\hat a$ such that $m(\hat u_0,\hat a)=\min_{a\in S_{\hat u}\cap B(\hat a,\delta_0)} m(\hat u_0,a)$ in order to get the stability of the blow-up profile and the uniform convergence to those profiles. This achieves Step 1 in the technique of [33].

As for the geometrical covering argument of Step 2 of [33], we could not do the same, since the profiles for $m\geqslant 4$ are much more complicated to describe than for $m=2$.

Step 1 revealed to be a fundamental step towards the regularity of the blow-up set in the case $m=2$ treated in [33] and for the semilinear wave equation treated by Nouaili [27]. Similarly, we believe that in the case $m\geqslant 4$ for the heat equation (1.1), we made a step towards further geometrical results for the blow-up set.

Remark. Unlike in Theorem 2 (see the third remark following Proposition 3), we may have here $m(\hat u_0,\hat a)\geqslant 4$ and the assertion in Proposition 3′ is totally meaningful. More precisely for any even integer $m\in\mathbb{N}^*$, there exists a blow-up solution u such that for all $a\in S_u$, $m(u_0,a)=m$. Indeed, one has just to adapt the method of Bricmont and Kupiainen [4] to the radial version of (1.1):

$$\partial_t U=\partial_r^2 U+\frac{N-1}{r}\partial_r U+|U|^{p-1}U$$

to find a solution blowing up for $r=1$ with:

- If $m=2$, $\forall K_0>0$,

$$\sup_{|z|<K_0}\left|(T-t)^{\frac{1}{p-1}}U(1+z\sqrt{T-t},t)-\Big(p-1+\frac{(p-1)^2}{4p}z^2\Big)^{-\frac{1}{p-1}}\right|\to 0\quad \text{as } t\to T.$$

- If $m\geqslant 4$ and even, $\forall K_0>0$,

$$\sup_{|z|<K_0}\left|(T-t)^{\frac{1}{p-1}}U\big(1+z(T-t)^{\frac{1}{m_0}},t\big)-\big(p-1+cz^{m_0}\big)^{-\frac{1}{p-1}}\right|\to 0\quad \text{as } t\to T.$$

In fact, Bricmont and Kupiainen [4] did the work in one dimension and in higher dimensions, the term $\frac{N-1}{r}\partial_r U$ can be controlled as a lower order term in selfsimilar variables.

Remark. As we said in the first remark after Proposition 3, the estimate in the case $\hat m=2$ has already been proved in [8] with no uniform character.

Since $m(\hat u_0,a)\in\mathbb{N}$, the mapping $a\mapsto m(\hat u_0,a)$ has local minima. In particular, it realizes its global minimum at some $\hat a\in S_{\hat u}$ and we have the following:

Corollary 5. *Let* $\hat u$ *be a solution of* (1.1) *associated to the initial data* $\hat u_0$ *and blowing up at some time* $\hat T$. *Then, there exists* $\hat a\in S_{\hat u}$ *such that* (i)′, (ii)′ *and* (iii)′ *of Theorem* 2′ *are satisfied.*

Remark. Following this corollary, we conjecture that the profile order (for fixed initial data $\hat u_0$) is constant on the connected components of $S_{\hat u}$, and that the convergence in (iii)′ is uniform on the connected component.

Remark. This corollary is meaningful when $\hat a$ is a non-isolated blow-up point. Note also that we don't prove the stability of the blow-up profile with respect to the blow-up point and that we only prove that the order of the multilinear form $B_a(x)$ is locally constant (hence, is stable).

Remark. If $m(\hat{u}_0, a) = 2$, then it is automatically a local minimum and (ii)$'$ of Theorem 2$'$ is satisfied. Moreover:

- If $l_{\hat{a}} = N$, then $\hat{a}$ is an isolated blow-up point as written earlier.
- If $l_{\hat{a}} \leqslant N-1$, then with the additional hypothesis that $S_{\hat{u}}$ contains a continuum of dimension $N - l_{\hat{a}}$ going through $\hat{a}$, we know from [33] that the profile is stable with respect to the blow-up point.

The proof of our results relies on the Liouville theorem of [24] and [25], and on a dynamical system formulation in selfsimilar variables. Note that we don't prove Corollaries 4 and 5 since they are immediate consequences of Theorems 2, 2$'$ and the results of Herrero and Velázquez [15] and [16].

This paper is organized as follows: In Section 2, we prove uniform estimates in the study of Eq. (1.6) satisfied by v. In Section 3, we give the proof of Theorems 1, 2 and 2$'$ as well as Propositions 3 and 3$'$.

We note that in the remaining of this paper, we will denote by C all positive constants.

2. Uniform estimates and dynamical study in selfsimilar coordinates

Let $\hat{u}(x,t)$ be a solution of (1.1) with initial data $\hat{u}_0(x)$ and blowing up at some point $\hat{a}$ and at time $\hat{T}$ and

$$w \not\equiv \pm\kappa(\hat{T} - t)^{-\frac{1}{p-1}}. \tag{2.1}$$

From Giga and Kohn [12], and up to replacing $\hat{u}$ by $-\hat{u}$, we assume that

$$\hat{w}_{\hat{a},\hat{T}}(y,s) \to \kappa \quad \text{as } s \to +\infty \text{ in } L^2_\rho(\mathbb{R}^N) \text{ and in } \mathcal{C}^{k,\gamma}_{\text{loc}}(\mathbb{R}^N) \tag{2.2}$$

for any $k \in \mathbb{N}$ and $\gamma \in (0,1)$. From (2.1), as mentioned in the introduction, the blow-up profile of $\hat{u}$ near $(\hat{a}, \hat{T})$ is given according to the value of some even parameter $\hat{m} \equiv m(\hat{u}_0, \hat{a}) \geqslant 2$ defined in (1.11).

From now on, given initial data u_0, we denote by u the solution to (1.1) corresponding to u_0 and blowing up at some time T. If $a \in S_u$, we denote by $w_{a,T}$ the corresponding selfsimilar variables solution given by (1.3) and by $v_{a,T}$ the function given by (1.5).

We first derive the following uniform L^∞ bound in a neighborhood of $\hat{u}_0$ and a constant sign property of $u(x,t)$ for x close to the blow-up point $\hat{a}$:

Proposition 2.1 *(Uniform L^∞ bound and ODE localization). (See Fermanian Kammerer, Merle and Zaag [6].) There exist $\mathcal{V}_1$ a neighborhood of $\hat{u}_0$ in $L^\infty(\mathbb{R}^N)$, $C > 0$ and $\{C_\varepsilon\}_\varepsilon$ such that for all initial data u_0 in $\mathcal{V}_1$:*

(i) *$u(t)$ blows up in T and $T \to \hat{T}$ as $u_0 \to \hat{u}_0$ in $L^\infty(\mathbb{R}^N)$.*
(ii) $\forall t \in [0,T)$, $\|u(t)\|_{L^\infty} \leqslant C(T-t)^{-\frac{1}{p-1}}$.
(iii) $\forall \varepsilon > 0$, $\forall t \in [\frac{T}{2}, T)$, $|\partial_t u - |u|^{p-1}u| \leqslant \varepsilon |u|^p + C_\varepsilon$.
(iv) *There exists $\delta_1 > 0$ such that*

$$\forall t \in [T - \delta_1, T),\ \forall |x - \hat{a}| \leqslant 2\delta_1, \quad u(x,t) \geqslant 0.$$

(v) *For all $a \in S_u \cap B(\hat{a}, \delta_1)$,*

$$v_{a,T}(y,s) \to 0 \quad \text{as } s \to +\infty \text{ in } \mathcal{C}^{k,\gamma}_{\text{loc}}(\mathbb{R}^N) \quad \text{for any } k \in \mathbb{N} \text{ and } \gamma \in (0,1).$$

Proof. • For (i) to (iv), see [6, Lemma 2.2, Proposition 1.7 and Corollary 1.8, p. 358 and p. 355]. Note that those results of [6] are valid without the assumption made in [6] about the blow-up profile.

For the reader's convenience, we show how to derive (iv) from (iii). Let us consider $\mathcal{V}_1'$ a neighborhood of $\hat{u}_0$ in $L^\infty(\mathbb{R}^N)$ such that for any $u_0 \in \mathcal{V}_1'$, points (i), (ii) and (iii) hold.

Applying (iii) for $\varepsilon = \frac{1}{2}$, we get the existence of a positive constant $C_{\frac{1}{2}}$ such that

$$\forall u_0 \in \mathcal{V}_1',\ \forall x \in \mathbb{R}^N,\ \forall t \in \left[\frac{T}{2}, T\right), \quad \partial_t u \geqslant |u|^{p-1}u - \frac{1}{2}|u|^p - C_{\frac{1}{2}}. \tag{2.3}$$

We now choose $A > 0$ such that

$$\frac{1}{2}A^p - C_{\frac{1}{2}} > 0. \tag{2.4}$$

Using (2.2), we deduce the existence of $\delta > 0$ and $\delta' > 0$ such that for all $|x - \hat{a}| \leqslant \delta$, $\hat{u}(x, \hat{T} - \delta') > 2A$.

Then, from continuity arguments applied to Eq. (1.1) and the continuity of the blow-up time (cf. (i) of this proposition), there exists $\mathcal{V}_1$ a neighborhood of $\hat{u}_0$ such that $\mathcal{V}_1 \subset \mathcal{V}_1'$ and

$$\forall u_0 \in \mathcal{V}_1,\ \forall |x - \hat{a}| \leqslant \delta, \quad u\big(x, T - \delta'\big) > A. \tag{2.5}$$

Therefore, thanks to (2.4), we can prove from (2.3) and (2.5), by a priori estimates, that $u(x,t) > A > 0$ for all $u_0 \in \mathcal{V}_1$, $t \in [T - \delta', T)$ and $|x - \hat{a}| \leqslant \delta$. Taking $\delta_1 = \frac{1}{2}\min(\delta, \delta')$ concludes the proof of (iv).

• For (v), we just remark that thanks to Giga and Kohn [10], we know that we have the convergence of $w_{a,T}$ to $\pm\kappa$ and that since we have the positivity of the solution locally near $(\hat{a}, T)$ (see (iv)), we deduce that $w_{a,T}$ converges to κ.

This ends the proof of Proposition 2.1. □

Note that at this stage, we don't know if the convergence in (v) is uniform with respect to u_0 and a or not. Using the Liouville theorem of Merle and Zaag [24,25], we can show that uniform character.

We then have:

Proposition 2.2 *(Uniform smallness of $v_{a,T}$). There exist a neighborhood $\mathcal{V}_2$ of $\hat{u}_0$ in $L^\infty(\mathbb{R}^N)$ and a positive constant δ_2 such that as $s \to +\infty$,*

(i) $\displaystyle \sup_{u_0 \in \mathcal{V}_2} \sup_{a \in S_u \cap B(\hat{a}, \delta_2)} \big\| v_{a,T}(s) \big\|_{L^2_\rho} \to 0,$

(ii) $\displaystyle \forall R > 0, \quad \sup_{u_0 \in \mathcal{V}_2} \sup_{a \in S_u \cap B(\hat{a}, \delta_2)} \Big(\sup_{|y| \leqslant R} \big| v_{a,T}(y,s) \big| \Big) \to 0.$

Proof. We only prove (i), since (ii) follows from (i) by standard parabolic regularity arguments. Let us assume that we cannot find a neighborhood of $\hat{u}_0$ and a constant $\delta_2 > 0$ such that (i) holds. Then there exist $\eta_0 > 0$, $s_n \to +\infty$, $u_{0,n} \to \hat{u}_0$ and $a_n \to \hat{a}$, $a_n \in S_{u_{0,n}}$ when $n \to +\infty$ such that

$$\forall n \in \mathbb{N}, \quad \big\| w_{n,a_n,T_n}(s_n) - \kappa \big\|_{L^2_\rho} > \eta_0. \tag{2.6}$$

By Proposition 2.1, we know that $w_{n,a_n,T_n}(y,s) \to \kappa$ as $s \to +\infty$ in $C^{k,\gamma}_{\mathrm{loc}}(\mathbb{R}^N)$ for any $k \in \mathbb{N}$ and $\gamma \in (0,1)$. Then $E(w_{n,a_n,T_n}(s)) \to E(\kappa)$ as $s \to +\infty$, where

$$E(w)(s) = \int_{\mathbb{R}^N} \left(\frac{1}{2}\big|\nabla w(y,s)\big|^2 + \frac{1}{2(p-1)}\big|w(y,s)\big|^2 - \frac{1}{p+1}\big|w(y,s)\big|^{p+1} \right) \rho(y)\,dy \tag{2.7}$$

is a decreasing function in time. Therefore we have

$$E\big(w_{n,a_n,T_n}(s)\big) \geqslant E(\kappa). \tag{2.8}$$

Since $s_n \to +\infty$, the point (iv) of Proposition 2.1 implies for n large,

$$w_{n,a_n,T_n}(0, s_n) = e^{-\frac{s_n}{p-1}} u_n\big(a_n, T_n - e^{-s_n}\big) \geqslant 0. \tag{2.9}$$

We introduce

$$W_n(y,s) = w_{n,a_n,T_n}(y, s + s_n). \tag{2.10}$$

Then W_n satisfies Eq. (1.4), and estimates (2.8), (2.9) and (2.6) yield for n large

$$E\big(W_n(0)\big) \geqslant E(\kappa), \qquad W_n(0,0) \geqslant 0 \quad \text{and} \quad \big\| W_n(\cdot, 0) - \kappa \big\|_{L^2_\rho} > \eta_0. \tag{2.11}$$

By (ii) in Proposition 2.1, (2.9) and (2.10), there exists $C > 0$ such that

$$\forall s \in [-\log T_n - s_n, +\infty), \quad \big\| W_n(s) \big\|_{L^\infty} \leqslant C. \tag{2.12}$$

By the parabolic regularity and a compactness procedure, and since $s_n \to +\infty$, there exists $W(y,s)$ such that up to a subsequence

$$W_n \to W \quad \text{as } n \to +\infty \text{ in } \mathcal{C}^{2,1}_{\text{loc}}\big(\mathbb{R}^N \times \mathbb{R}\big). \tag{2.13}$$

Moreover, W satisfies (1.4), and we have from (2.11) and (2.12),

$$\|W\|_{L^\infty} \leqslant C, \qquad E\big(W(0)\big) \geqslant E(\kappa), \qquad W(0,0) \geqslant 0 \quad \text{and} \quad \big\|W(0)-\kappa\big\|_{L^2_\rho} > \eta_0. \tag{2.14}$$

Therefore, by the Liouville theorem, we get

$$W \equiv \pm\kappa, \qquad W \equiv 0 \quad \text{or} \quad W(y,s) = \pm\theta(s+s_0), \quad \text{for some } s_0 \in \mathbb{R}. \tag{2.15}$$

This is in contradiction with (2.14). Indeed, $W \equiv -\kappa$ contradicts $W(0,0) \geqslant 0$, $W = \kappa$ contradicts $\|W(0)-\kappa\|_{L^2_\rho} > \eta_0$ and $W \equiv 0$ or $W(y,s) = \pm\theta(s+s_0)$ contradict $E(W(0)) \geqslant E(\kappa)$ (for $E(0) = 0 < E(\kappa)$ and $\forall s \in \mathbb{R}$, $E(\pm\theta(s)) < E(\kappa)$). This concludes the proof of (i) of Proposition 2.2. □

Note that Proposition 2.2 gives the uniform smallness in time and space of $v_{a,T}(y,s)$ with respect to the initial data $u_0(x)$ in a neighborhood $\mathcal{V}_2$ of $\hat{u}_0$ in $L^\infty(\mathbb{R}^N)$ and $a \in S_u \cap B(\hat{a},\delta_2)$. From the result of Velázquez [29] stated in (1.11), we know that $v_{a,T}(s) \sim P_m(v_{a,T}(s))$ in L^2_ρ as $s \to +\infty$, for some even $m(u_0,a) \geqslant 4$, with no uniform information with respect to u_0 and a. If $\hat{m} = 2$, we have already the uniform character from [6] and [33]. When $\hat{m} \geqslant 4$, we believe that we can get the uniform character if we consider the block $\sum_{i=2}^{\hat{m}} P_i(v_{a,T})(s)$. Accordingly, we decompose $v_{a,T}$ with respect to the spectrum of $\mathcal{L}$ as follows:

$$\begin{cases} \text{(expanding modes block)} & x_+(s) = \Big\|\displaystyle\sum_{k=0}^{1} P_k(v_{a,T})(s)\Big\|_{L^2_\rho}, \\ \text{(low frequency block)} & y_m(s) = \Big\|\displaystyle\sum_{k=2}^{m} P_k(v_{a,T})(s)\Big\|_{L^2_\rho}, \\ \text{(high frequency block)} & \tilde{z}_m(s) = \Big\|v_{a,T}(s) - \displaystyle\sum_{k=0}^{m} P_k(v_{a,T})(s)\Big\|_{L^2_\rho}, \end{cases} \tag{2.16}$$

where the projection P_k is defined in (1.10). Since the nonlinear term in (1.6) is not quadratic in L^2_ρ, we need to estimate an additional variable

$$J_m(s) = \bigg(\int_{\mathbb{R}^N} \big|v_{a,T}(y,s)\big|^4 |y|^k \rho(y)\,dy\bigg)^{\frac{1}{2}} \quad \big(k = k(m)\big) \tag{2.17}$$

where $k(m) > 0$ will be fixed in Lemma 2.5 below as an increasing sequence. We need also to introduce

$$z_m(s) = \tilde{z}_m(s) + J_m(s). \tag{2.18}$$

When $(u_0,a) = (\hat{u}_0,\hat{a})$, we add a "ˆ" to the notation ($\hat{x}_+$, $\hat{y}_m$ and $\hat{z}_m$).

Using the notation (2.16), we claim that estimate (1.11) yields $x_+(s) + z_m(s) = o(y_m(s))$ as $s \to +\infty$ which we write more precisely in the following:

Lemma 2.3. *If u is a solution of* (1.1) *blowing up at time T and some point a with the profile given in* (1.12) *or* (1.14) *according to the value of $m = m(u_0,a)$, then*

$$\forall \varepsilon > 0, \quad \exists s_0(\varepsilon,u_0,a), \quad \forall s \geqslant s_0(\varepsilon,u_0,a), \quad \varepsilon y_m(s) \geqslant x_+(s) + z_m(s). \tag{2.19}$$

Proof. Using (1.11), (1.12) and (1.14), we see that

$$\big\|v(s) - P_m(v)(s)\big\|_{L^2_\rho} = o\big(\big\|v(s)\big\|_{L^2_\rho}\big) \quad \text{as } s \to +\infty \tag{2.20}$$

and

$$\text{either } m(u_0,a)=2 \quad\text{and}\quad \|v(s)\|_{L^2_\rho}\sim\frac{C_0}{s} \quad\text{as } s\to+\infty, \quad\text{or}$$
$$m(u_0,a)\geqslant 4 \quad\text{and}\quad \|v(s)\|_{L^2_\rho}\sim C_0 e^{(1-\frac{m}{2})s} \quad\text{as } s\to+\infty. \tag{2.21}$$

Since we have from (2.18), $x_+ + z_m = x_+ + \tilde{z}_m + J_m$, we first show that $x_+ + \tilde{z}_m = o(y_m)$, then we show that $J_m = o(y_m)$.

Since we have from (2.16), $\|v(s)\|_{L^2_\rho} \leqslant C(x_+ + y_m + \tilde{z}_m)(s)$ and $x_+(s)+\tilde{z}_m(s) \leqslant C\|v(s) - P_m(v)(s)\|_{L^2_\rho}$ as $s\to+\infty$, we use (2.20) to get

$$x_+ + \tilde{z}_m = o(y_m). \tag{2.22}$$

Now, we recall from Herrero and Velázquez [17], the following regularizing effect for the operator $\mathcal{L}$:

Claim 2.4. *(See Herrero and Velázquez [17].) There exist positive s_* and C^* such that for s large enough, we have*:

$$\left(\int_{\mathbb{R}^N} |v(y,s)|^8\rho(y)\,dy\right)^{\frac{1}{8}} \leqslant C^*\left(\int_{\mathbb{R}^N} |v(y,s-s^*)|^2\rho(y)\,dy\right)^{\frac{1}{2}}.$$

Proof. See [17, Lemma 2.3, p. 142]. Note that the result holds for sign-changing solutions with the same proof. □

Using (2.17), the Cauchy–Schwarz inequality and Claim 2.4, we write

$$J_m(s) \leqslant \left(\int_{\mathbb{R}^N} |v(y,s)|^8\rho(y)\,dy\right)^{\frac{1}{4}}\left(\int_{\mathbb{R}^N} |y|^{2k}\rho(y)\,dy\right)^{\frac{1}{4}} \leqslant C\left(\int_{\mathbb{R}^N} v(y,s-s^*)^2\rho(y)\,dy\right). \tag{2.23}$$

We claim that for s large enough, we have

$$\int_{\mathbb{R}^N} v(y,s-s^*)^2\rho(y)\,dy \leqslant C\int_{\mathbb{R}^N} v(y,s)^2\rho(y)\,dy. \tag{2.24}$$

Indeed, if $m(u_0,a)=2$, then we write from (2.21), for s large enough

$$\int_{\mathbb{R}^N} v(y,s-s^*)^2\rho(y)\,dy \leqslant \frac{2C_0}{s-s^*} \leqslant \frac{3C_0}{s} \leqslant 4\|v(s)\|_{L^2_\rho}$$

and the same proof holds when $m(u_0,a)\geqslant 4$.

Using (2.23), (2.24), (2.16) and (2.22), we write

$$J_m(s) \leqslant C\left(\int_{\mathbb{R}^N} v(y,s-s^*)^2\rho(y)\,dy\right) \leqslant C\int_{\mathbb{R}^N} v(y,s)^2\rho(y)\,dy \leqslant C\left(x_+^2+y_m^2+\tilde{z}_m^2\right)(s) \leqslant Cy_m^2(s).$$

Hence $J_m(s) = o(y_m(s))$ as $s\to+\infty$. Using (2.22), we conclude the proof of Lemma 2.3. □

In order to prove the stability of the block $\sum_{i=2}^{\hat{m}} P_i(v_{a,T})(s)$, we use the decomposition (2.16) and (2.18) to project Eq. (1.6) in the following:

Lemma 2.5 *(Differential inequalities on the components of $v_{a,T}(s)$). For all $i\geqslant 2$, there exist $k=k(i)>0$ an increasing sequence, a neighborhood $\mathcal{V}_3(i)\subset\mathcal{V}_2$ of $\hat{u}_0$ in $L^\infty(\mathbb{R}^N)$ and $\delta_3(i)>0$ such that for all $\varepsilon>0$, there exists $s_3(\varepsilon,i)\in\mathbb{R}$ such that for all $s\geqslant s_3(\varepsilon,i)$, for all $u_0\in\mathcal{V}_3(i)$ and $a\in S_u\cap B(\hat{a},\delta_3(i))$, we have*

$$x_+'(s) \geqslant \frac{1}{2}x_+(s) - \varepsilon\big(x_+(s)+y_i(s)+z_i(s)\big), \tag{2.25}$$

$$\varepsilon\big(x_+(s)+y_i(s)+z_i(s)\big) \geqslant y_i'(s) \geqslant \left(1-\frac{i}{2}\right)y_i(s) - \varepsilon\big(x_+(s)+y_i(s)+z_i(s)\big), \tag{2.26}$$

$$z_i'(s) \leqslant \left(1-\frac{i+1}{2}\right)z_i(s) + \varepsilon\big(x_+(s)+y_i(s)+z_i(s)\big). \tag{2.27}$$

Proof. See Appendix A. □

With these inequalities, we are in a position to prove that for $i = \hat{m}$, $y_{\hat{m}}$ dominates x_+ and $z_{\hat{m}}$ as $s \to +\infty$, uniformly with respect to u_0 and a. In the following, we start by neglecting x_+ with respect to $y_i + z_i$.

Lemma 2.6 *(Uniform smallness of the expanding modes block). For all $i \geqslant 2$, $u_0 \in \mathcal{V}_3(i)$, $a \in S_u \cap B(\hat{a}, \delta_3(i))$, $\varepsilon > 0$ and $s \geqslant s_3(\varepsilon, i)$, we have*

$$x_+(s) \leqslant \varepsilon\big(y_i(s) + z_i(s)\big). \tag{2.28}$$

This lemma is an immediate consequence of the following:

Lemma 2.7. *Consider $s_* \in \mathbb{R}$ and $Y, Z \in C^1([s_*, +\infty), \mathbb{R}^+)$ such that*

(1) *for all $\varepsilon > 0$, there exists $s_5(\varepsilon)$ such that*

$$\text{for all } s \geqslant s_5(\varepsilon) \quad \begin{cases} Y' \geqslant -\varepsilon(Y + Z), \\ Z' \leqslant -\frac{1}{3}Z + \varepsilon Y. \end{cases}$$

(2) *If for some $\hat{s} \geqslant s_*$, we have $Y(\hat{s}) + Z(\hat{s}) = 0$, then for all $s \geqslant \hat{s}$, $Y(s) + Z(s) = 0$.*

Then, either $Z = o(Y)$ or $Y = o(Z)$ as $s \to +\infty$. Moreover, in this latter case, we have

$$\forall \varepsilon > 0 \text{ and } s \geqslant s_5(\varepsilon), \quad Y(s) \leqslant C\varepsilon Z(s).$$

Indeed, let us first derive Lemma 2.6 from Lemma 2.7 and then prove this latter.

Proof of Lemma 2.6. Let $Y(s) = e^{\frac{s}{2}} x_+(s)$ and $Z(s) = e^{\frac{s}{2}}(y_i(s) + z_i(s))$. Using Lemma 2.5, we see that Y and Z satisfy condition (1) of Lemma 2.7. It also satisfies condition (2). Indeed, if we assume that $Y(\hat{s}) + Z(\hat{s}) = 0$ for some $\hat{s} \geqslant -\log T$, then from the definitions (2.16) and (2.18) of x_+, y_m and z_m, we have $\|v_{a,T}(\hat{s})\|_{L^2_\rho} = 0$, hence $v_{a,T}(\cdot, \hat{s}) \equiv 0$. From the uniqueness of the solution to the Cauchy problem of Eq. (1.6), we get $v_{a,T}(\cdot, s) \equiv 0$, hence $Y(s) + Z(s) = 0$ for all $s \geqslant \hat{s}$. Therefore, the conclusion of Lemma 2.6 directly follows from Lemma 2.7. □

Let us now prove Lemma 2.7.

Proof of Lemma 2.7. *Part* 1: Let $\varepsilon > 0$, we prove in this part that

$$\text{either } \exists s_5' = s_5'(\varepsilon) \quad \text{such that} \quad \forall s \geqslant s_5', \quad Z(s) \leqslant C\varepsilon Y(s), \quad \text{or} \tag{2.29}$$

$$\forall s \geqslant s_5(\varepsilon), \quad Y(s) \leqslant C\varepsilon Z(s). \tag{2.30}$$

We set $\gamma(s) = 6\varepsilon Y(s) - Z(s)$. Two cases arise:

Case 1: $\exists s_5' \geqslant s_5(\varepsilon)$ such that $\gamma(s_5') > 0$.

If for all $s \geqslant s_5'$, $\gamma(s) \geqslant 0$, then (2.29) holds.

If not, then we have $\gamma(s_*) = 0$ for some $s_* \geqslant s_5'$ where s_* is the smallest s satisfying $\gamma(s) = 0$. Therefore $\gamma'(s_*) \leqslant 0$. We then compute $\gamma'(s_*)$. We have from hypothesis (1)

$$\gamma'(s_*) = 6\varepsilon Y'(s_*) - Z'(s_*) \geqslant Z(s_*)\left(-\varepsilon - 6\varepsilon^2 + \frac{1}{6}\right) \geqslant \frac{1}{7}Z(s_*) \quad \text{for } \varepsilon \text{ small enough.}$$

Knowing that $\gamma'(s_*) \leqslant 0$, we deduce that $Z(s_*) = Y(s_*) = 0$. By hypothesis (2), we have $Z(s) = Y(s) = 0$ for all $s \geqslant s_*$ and (2.29) follows with $s_5' = s_*$.

Case 2: $\forall s \geqslant s_5(\varepsilon)$, $\gamma(s) \leqslant 0$, that is

$$6\varepsilon Y(s) \leqslant Z(s). \tag{2.31}$$

Then, we have from hypothesis (1)

$$\forall s \geqslant s_5(\varepsilon), \quad Z'(s) \leqslant -\frac{1}{6}Z(s) \quad \text{and} \quad Z(s) + Y(s) \to 0 \quad \text{as } s \to +\infty. \tag{2.32}$$

Using (2.31) and hypothesis (1), we then deduce that $Y' \geqslant -(\frac{1}{6}+\varepsilon)Z \geqslant (1+6\varepsilon)Z'$. Integrating between s and $+\infty$, we get $Y \leqslant (1+6\varepsilon)Z$. Writing again hypothesis (1) using this last inequality and (2.32), we get

$$Y' \geqslant \big(-\varepsilon(1+6\varepsilon)-\varepsilon\big)Z \geqslant 6\varepsilon(2+6\varepsilon)Z'.$$

Integrating again between s and $+\infty$, we get

$$Y(s) \leqslant 6\varepsilon(2+6\varepsilon)Z(s),$$

which gives the second case.

Part 2: Let $\varepsilon < \frac{1}{C}$, then either (2.29) or (2.30) occurs.

Assuming that (2.29) occurs, it is clear from Part 1, that for any $\varepsilon' < \varepsilon$, (2.29) occurs as well. Hence $Z = o(Y)$ as $s \to +\infty$. If (2.30) occurs for ε, then we are lead to $Y = o(Z)$ as $s \to +\infty$. This ends the proof of Lemma 2.7. □

Using Lemmas 2.5 and 2.7, we get the following:

Corollary 2.8 *(Either the high or the low frequency block of $v_{a,T}$ dominates). For all $i \geqslant 2$, $u_0 \in \mathcal{V}_3(i)$ and $a \in S_u \cap B(\hat{a}, \delta_3(i))$, we have either $z_i(s) = o(y_i(s))$ or $y_i(s) = o(z_i(s))$ as $s \to +\infty$. Moreover, in this latter case, we have*

$$\text{for all } \varepsilon > 0 \text{ and } s \geqslant s_3(\varepsilon, i), \quad y_i(s) \leqslant C\varepsilon z_i(s). \tag{2.33}$$

Proof. Just apply Lemmas 2.5 and 2.7 with $Y(s) = e^{(1-\frac{i}{2})s} y_i(s)$ and $Z(s) = e^{(1-\frac{i}{2})s} z_i(s)$. □

Remark. Unlike the case where $y_i = o(z_i)$, when $z_i = o(y_i)$, the inequalities (2.26) and (2.27) alone do not yield an estimate like (2.33), uniform with respect to u_0 and a. As a matter of fact, when $i = \hat{m}$, we will use other ideas to derive such a uniform estimate. That will be the heart of our argument.

We now establish the following result giving the uniform stability of the dynamics where y_m is predominant.

Lemma 2.9 *(Uniform stability of the dynamics where the low frequency block is predominant). For all $i \geqslant 2$ and $C^* > 0$, there exists $s^*(i, C^*) \in \mathbb{R}$, such that for all initial data u_0 in $\mathcal{V}_3(i)$, $a \in S_u \cap B(\hat{a}, \delta_3(i))$ and $s_0 \geqslant s^*$,*

$$\text{if } y_i(s_0) \geqslant C^* z_i(s_0), \quad \text{then } \forall s \geqslant s_0, \quad y_i(s) \geqslant \frac{C^*}{2} z_i(s). \tag{2.34}$$

Proof. Consider $i \geqslant 2$. Following closely the proof in [6, Lemma 3.3, p. 375] and considering Lemmas 2.5 and 2.6, we have for all $\varepsilon \in (0, \frac{1}{2})$, $u_0 \in \mathcal{V}_3(i)$, $a \in S_u \cap B(\hat{a}, \delta_3(i))$ and $s \geqslant s_3(\varepsilon, i)$

$$\begin{cases} y_i'(s) \geqslant \left(1 - \dfrac{i}{2}\right) y_i(s) - \dfrac{3}{2}\varepsilon\big(y_i(s) + z_i(s)\big), \\ z_i'(s) \leqslant \left(1 - \dfrac{i+1}{2}\right) z_i(s) + \dfrac{3}{2}\varepsilon\big(y_i(s) + z_i(s)\big). \end{cases} \tag{2.35}$$

We argue by contradiction. Suppose that there exist $C > 0$, $s_0 > s_3(\varepsilon, i)$ where $\varepsilon = \frac{C}{4(2+C)^2}$, $u_0 \in \mathcal{V}_3(i)$ and $a \in S_u \cap B(\hat{a}, \delta_3(i))$ such that

$$y_i(s_0) \geqslant C z_i(s_0) \quad \text{and} \quad \exists s_0^* > s_0, \quad y_i\big(s_0^*\big) < \frac{C}{2} z_i\big(s_0^*\big).$$

Let $\gamma(s) = y_i(s) - \frac{C}{2} z_i(s)$, then $\gamma(s_0) \geqslant 0$ and $\gamma(s_0^*) < 0$. Therefore, there exists $s_2 \in [s_0, s_0^*[$ such that

$$\gamma(s_2) = 0, \qquad \gamma(s) < 0 \quad \text{for all } s \in \big[s_2, s_0^*\big), \quad \text{hence } \gamma'(s_2) \leqslant 0 \tag{2.36}$$

on the one hand. On the other hand, we have from (2.35)

$$\gamma'(s_2) = y_i'(s_2) - \frac{C}{2} z_i'(s_2) \geqslant \frac{C}{4} z_i(s_2) + \left(1 - \frac{i}{2}\right)\left(y_i(s) - \frac{C}{2} z_i(s_2)\right) - \frac{3}{2}\varepsilon\left(1 + \frac{C}{2}\right)\big(y_i(s_2) + z_i(s_2)\big). \tag{2.37}$$

Using (2.36) and (2.37), we obtain

$$\gamma'(s_2) \geqslant \left[\frac{C}{4} - \frac{3}{2}\varepsilon\left(1+\frac{C}{2}\right)^2\right] z_i(s_2) > y_i(s_2), \tag{2.38}$$

since $\varepsilon = \frac{C}{4(2+C)^2}$ and $z_i(s_2) > 0$ (in case $z_i(s_2) = 0$ it follows that for all $s \geqslant s_2$, $v_a \equiv 0$ and $\gamma(s) = 0$ which contradicts $\gamma(s_0^*) < 0$). This contradicts (2.36) and concludes the proof of Lemma 2.9. □

3. Proof of the main results

Our aim in this section consists in proving Theorems 1, 2 and 2′ as well as Propositions 3 and 3′.

3.1. Proof of Theorem 1

We claim that it is enough to prove (1.16). Indeed, assuming (1.16) true and taking $a \in S_u \cap B(\hat{a}, \hat{\delta})$, we see from the definition of $m(u_0, a)$ that $v_{a,T}(s) \sim P_{m(u_0,a)}(v_{a,T}(s))$ as $s \to +\infty$ in L^2_ρ on the one hand. On the other hand, from (1.16), $v_{a,T}(s) \sim \sum_{i=2}^{\hat{m}} P_i(v_{a,T}(s))$ as $s \to +\infty$ in L^2_ρ. Thus, $m(u_0, a) \leqslant \hat{m}$ and the alternative (1), (2) in Theorem 1 holds. Therefore, we only prove (1.16).

We proceed in three parts. In Part 1, we prove that $z_{\hat{m}}(s) \leqslant y_{\hat{m}}(s)$ uniformly in u_0 and a. In Part 2, we prove that $z_{\hat{m}}(s) = o(y_{\hat{m}}(s))$ as $s \to +\infty$, with no uniform character (with respect to a and u_0). Finally, in Part 3, we prove the uniform character of $z_{\hat{m}}(s) = o(y_{\hat{m}}(s))$ as $s \to +\infty$.

Part 1: We claim the following:

Lemma 3.1. *There exist $\hat{s}_0 \in \mathbb{R}$, a neighborhood $\mathcal{V}_4 \subset \mathcal{V}_3(\hat{m})$ of $\hat{u}_0$ in $L^\infty(\mathbb{R}^N)$ and $\delta_4 \in (0, \delta_3(\hat{m}))$, such that for all $u_0 \in \mathcal{V}_4$ and for all $a \in S_u \cap B(\hat{a}, \delta_4)$,*

$$\forall s \geqslant \hat{s}_0, \quad y_{\hat{m}}(s) \geqslant z_{\hat{m}}(s). \tag{3.1}$$

Proof. Rewriting Lemma 2.9 with $C^* = 2$, we have the existence of some s^* such that

$$\begin{aligned} &\forall u_0 \in \mathcal{V}_3,\ \forall a \in S_u \cap B(\hat{a}, \delta_3),\\ &\text{if } \exists s_0 \geqslant s^* \quad \text{such that} \quad y_{\hat{m}}(s_0) \geqslant 2 z_{\hat{m}}(s_0), \quad \text{then } \forall s \geqslant s_0, \quad y_{\hat{m}}(s) \geqslant z_{\hat{m}}(s). \end{aligned} \tag{3.2}$$

By Lemma 2.3 applied to $(\hat{u}_0, \hat{a})$ with $\varepsilon = \frac{1}{3}$,

$$\exists s_0\left(\frac{1}{3}, \hat{u}_0, \hat{a}\right): \quad \forall s \geqslant s_0\left(\frac{1}{3}, \hat{u}_0, \hat{a}\right), \quad \hat{y}_{\hat{m}}(s) \geqslant 3\hat{z}_{\hat{m}}(s).$$

We set

$$\hat{s}_0 := \max\left(s_0\left(\frac{1}{3}, \hat{u}_0, \hat{a}\right), s^*\right).$$

Then, using continuity arguments at $s = \hat{s}_0$, applied to Eq. (1.1), we obtain the existence of a neighborhood $\mathcal{V}_3''$ of $\hat{u}_0$ in $L^\infty(\mathbb{R}^N)$ and $\delta_3'' > 0$ such that

$$\forall u_0 \in \mathcal{V}_3'',\ \forall a \in S_u \cap B\big(\hat{a}, \delta_3''\big), \quad y_{\hat{m}}(\hat{s}_0) \geqslant 2 z_{\hat{m}}(\hat{s}_0).$$

Finally, by (3.2), we obtain for all $u_0 \in \mathcal{V}_4 = \mathcal{V}_3 \cap \mathcal{V}_3''$ and for all $a \in S_u \cap B(\hat{a}, \delta_4)$ (where $\delta_4 = \inf(\delta_3, \delta_3'')$),

$$\forall s \geqslant \hat{s}_0, \quad y_{\hat{m}}(s) \geqslant z_{\hat{m}}(s).$$

This ends the proof of Lemma 3.1. □

Part 2: We claim the following:

Lemma 3.2. *For all $\varepsilon > 0$, $u_0 \in \mathcal{V}_4$ and $a \in S_u \cap B(\hat{a}, \delta_4)$, there exists $s_5'(\varepsilon, u_0) \in \mathbb{R}$ such that*

$$\forall s \geqslant s_5'(\varepsilon, u_0, a), \quad 4\varepsilon y_{\hat{m}}(s) \geqslant z_{\hat{m}}(s). \tag{3.3}$$

Proof. Let $\varepsilon > 0$, $u_0 \in \mathcal{V}_4$ and $a \in S_u \cap B(\hat{a}, \delta_4)$. We shall restrict ε to small ones in the following. Using Lemmas 2.5, 2.6 and 3.1, setting

$$s_4(\varepsilon) = \sup\big(\hat{s}_0, s_3(\varepsilon, \hat{m})\big), \tag{3.4}$$

we have for all $s \geqslant s_4(\varepsilon)$, the inequalities (2.35) and (3.1) hold. Using Corollary 2.8, we see that: either $y_{\hat{m}}(s) = o(z_{\hat{m}}(s))$ or $z_{\hat{m}}(s) = o(y_{\hat{m}}(s))$ as $s \to +\infty$ and in view of (3.1), we must have $z_{\hat{m}}(s) = o(y_{\hat{m}}(s))$ as $s \to +\infty$ and Lemma 3.2 follows. □

Part 3: From Lemma 3.2, we can introduce for all $\varepsilon > 0$, $u_0 \in \mathcal{V}_4$ and $a \in S_u \cap B(\hat{a}, \delta_4)$

$$s_5(\varepsilon, u_0, a) := \inf\big\{s \geqslant s_4(\varepsilon)\colon \forall \sigma \geqslant s,\ 4\varepsilon y_{\hat{m}}(\sigma) \geqslant z_{\hat{m}}(\sigma)\big\}. \tag{3.5}$$

We claim the following:

Lemma 3.3. *$s_5(\varepsilon, u_0, a) - s_4(\varepsilon)$ is bounded only in terms of ε independently from u_0 and a.*

Proof. If $s_5(\varepsilon, u_0, a) = s_4(\varepsilon)$, then the answer is trivial. Hence, we assume in the following that $s_4(\varepsilon) < s_5(\varepsilon, u_0, a)$. We note that in this case, by minimality, there exists a sequence (s_n) such that

$$s_n \underset{n\to+\infty}{\longrightarrow} s_5(\varepsilon, u_0, a), \quad \text{with } s_n \in \big[s_4(\varepsilon), s_5(\varepsilon, u_0, a)\big] \quad \text{and} \quad 4\varepsilon y_{\hat{m}}(s_n) < z_{\hat{m}}(s_n). \tag{3.6}$$

Step 1: We prove that

$$\exists \varepsilon_0 > 0\colon \quad \forall \varepsilon \in (0, \varepsilon_0)\colon \quad \forall s \in \big[s_4(\varepsilon), s_5(\varepsilon, u_0, a)\big], \quad 4\varepsilon y_{\hat{m}}(s) \leqslant z_{\hat{m}}(s). \tag{3.7}$$

We argue by contradiction. If (3.7) does not hold, then we can construct from (3.6) $\sigma^*(= \sigma^*(n)) \in [s_4(\varepsilon), s_n)$ such that

$$4\varepsilon y_{\hat{m}}\big(\sigma^*\big) = z_{\hat{m}}\big(\sigma^*\big) \quad \text{and} \quad \forall \sigma \in \big(\sigma^*, s_n\big], \quad 4\varepsilon y_{\hat{m}}(\sigma) < z_{\hat{m}}(\sigma). \tag{3.8}$$

By minimality, this yields

$$z_{\hat{m}}'\big(\sigma^*\big) - 4\varepsilon y_{\hat{m}}'\big(\sigma^*\big) \geqslant 0, \tag{3.9}$$

on the one hand. On the other hand, using (2.35), there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, we have

$$\begin{aligned} z_{\hat{m}}'\big(\sigma^*\big) - 4\varepsilon y_{\hat{m}}'\big(\sigma^*\big) &\leqslant \left(1 - \frac{\hat{m}+1}{2}\right) z_{\hat{m}}\big(\sigma^*\big) + \frac{3}{2}\varepsilon\big(y_{\hat{m}}\big(\sigma^*\big) + z_{\hat{m}}\big(\sigma^*\big)\big) \\ &\quad - 4\varepsilon\left[\left(1 - \frac{\hat{m}}{2}\right) y_{\hat{m}}\big(\sigma^*\big) - \frac{3}{2}\varepsilon\big(y_{\hat{m}}\big(\sigma^*\big) + z_{\hat{m}}\big(\sigma^*\big)\big)\right] \\ &\leqslant z_{\hat{m}}\big(\sigma^*\big)\left[-\frac{1}{8} + 3\varepsilon + 6\varepsilon^2\right] \leqslant 0. \end{aligned} \tag{3.10}$$

Using (3.9) and (3.10), we see that $z_{\hat{m}}(\sigma^*) = 0$. Therefore, $v_{a,T}(y, s) \equiv 0$ for all $s \geqslant \sigma^*$, by uniqueness in the Cauchy problem of (1.6). By definitions (2.16) and (2.18) of $y_{\hat{m}}$ and $z_{\hat{m}}$, we see that for all $s \geqslant \sigma^*$, $z_{\hat{m}}(s) = y_{\hat{m}}(s) = 0$, hence $s_5(\varepsilon, u_0, a) \leqslant \sigma^*$ from (3.5). This contradicts the fact that $\sigma^* < s_n \leqslant s_5(\varepsilon, u_0, a)$. Thus, (3.7) holds. Finally, using (3.5) and (3.7), we are led to

$$z_{\hat{m}}\big(s_5(\varepsilon, u_0, a)\big) = 4\varepsilon y_{\hat{m}}\big(s_5(\varepsilon, u_0, a)\big). \tag{3.11}$$

Step 2: We prove that

$$s_5(\varepsilon, u_0, a) - s_4(\varepsilon) \leqslant 10|\log\varepsilon|. \tag{3.12}$$

In fact, using (3.1), (3.4) and (3.7), we have

$$\forall s \in \left[s_4(\varepsilon), s_5(\varepsilon, u_0, a)\right], \quad 4\varepsilon y_{\hat m}(s) \leqslant z_{\hat m}(s) \leqslant y_{\hat m}(s). \tag{3.13}$$

If for some $\bar s \in [s_4, s_5]$, $y_{\hat m}(\bar s) = 0$, then, $y_{\hat m}(\bar s) = z_{\hat m}(\bar s)$ and $v_{a,T}(y,s) = 0$ for all $s \geqslant \bar s$ by uniqueness in the Cauchy problem of (1.6). This implies $s_5(\varepsilon, u_0, a) = s_4(\varepsilon)$ and (3.12) follows.

If for all $s \in [s_4(\varepsilon), s_5(\varepsilon, u_0, a)]$, $y_{\hat m}(s) > 0$, using (3.13), (2.35) becomes: $\forall s \in [s_4(\varepsilon), s_5(\varepsilon, u_0, a)]$,

$$y'_{\hat m}(s) \geqslant \left(1 - \frac{\hat m}{2} - 3\varepsilon\right) y_{\hat m}(s) \quad \text{and} \quad z'_{\hat m}(s) \leqslant \left(\frac{11}{8} - \frac{\hat m + 1}{2} + \frac{3}{2}\varepsilon\right) z_{\hat m}(s).$$

Therefore,

$$\left[\log\left(\frac{z_{\hat m}}{y_{\hat m}}\right)\right]' \leqslant -\frac{1}{8} + 5\varepsilon \leqslant -\frac{1}{9} \quad \text{if } \varepsilon \text{ is small enough}$$

and by (3.13), we have

$$4\varepsilon \leqslant \frac{z_{\hat m}(s_5)}{y_{\hat m}(s_5)} \leqslant \frac{z_{\hat m}(s_4)}{y_{\hat m}(s_4)} \exp\left(-\frac{(s_5 - s_4)}{9}\right),$$

which yields

$$s_5(\varepsilon, u_0, a) - s_4(\varepsilon) \leqslant 9|\log 4\varepsilon|,$$

and Lemma 3.3 is proved. □

As a conclusion: for all $\varepsilon < \varepsilon'_0$, $u_0 \in \mathcal{V}' = \mathcal{V}_4$, $a \in S_u \cap B(0, \delta_4)$, we have from (3.5) and (3.12):

$$\forall s \geqslant s'_0(\varepsilon) = s_4(\varepsilon) + 10|\log \varepsilon| \geqslant s_5(\varepsilon, u_0, a), \quad 4\varepsilon y_{\hat m}(s) \geqslant z_{\hat m}(s). \tag{3.14}$$

Therefore, from (2.18) and (2.28), we have

$$\text{for all } s \geqslant \max\left(s'_0(\varepsilon), s_3(\varepsilon, \hat m)\right), \quad \tilde z_{\hat m} + x_+ \leqslant C\varepsilon y_{\hat m}.$$

Using (2.16), we get

$$\left\| v_{a,T}(s) - \sum_{k=2}^{\hat m} P_k\left(v_{a,T}(s)\right) \right\|_{L^2_\rho} \leqslant C\varepsilon \left\| v_{a,T}(s) \right\|_{L^2_\rho}$$

which concludes the proof of (1.16) and Theorem 1.

3.2. Proof of Theorems 2 and 2′

We only prove Theorem 2, since the proof of Theorem 2′ is quite similar. Indeed, in order to get the proof of Theorem 2′, just follow the proof of Theorem 2 and take $\mathcal{V}_0 = \mathcal{V}_{\hat u_0} = \{\hat u_0\}$.

Proof of Theorem 2. (iii) ⇒ (i): This follows by the definition (1.11) of the profile order.

(i) ⇒ (ii): Trivial.

(ii) ⇒ (iii): Let $\mathcal{V}_6 = \mathcal{V}_4 \cap \mathcal{V}_3(\hat m) \cap \mathcal{V}_1 \cap \mathcal{V}_3(\hat m - 1) \cap \mathcal{V}_0$ (hence, for Theorem 2′, $\mathcal{V}_6 = \{\hat u_0\}$) and $\delta_6 = \min(\delta_3(\hat m), \delta_4, \delta_3(\hat m - 1), \delta_0)$ and introduce

$$p_{\hat m}(s) = \left\| P_{\hat m}(v)(s) \right\|_{L^2_\rho}. \tag{3.15}$$

We claim that (iii) follows from the following:

Lemma 3.4. *For all $\varepsilon > 0$, there exists $s_6(\varepsilon)$ such that for all $s \geqslant s_6(\varepsilon)$, $u_0 \in \mathcal{V}_6$ and $a \in S_u \cap B(\hat a, \delta_6)$, we have*

(a) $x_+(s) \leqslant \varepsilon(y_{\hat m - 1} + p_{\hat m} + z_{\hat m})(s)$,

(b) $z_{\hat{m}}(s) \leqslant \varepsilon(y_{\hat{m}-1} + p_{\hat{m}})(s)$,
(c) $y_{\hat{m}-1}(s) \leqslant \varepsilon(x_+ + y_{\hat{m}-1} + p_{\hat{m}} + z_{\hat{m}})(s)$.

Indeed, considering $\varepsilon \in (0, \frac{1}{4})$, $s \geqslant s_6(\varepsilon)$, $u_0 \in \mathcal{V}_6$ and $a \in S_u \cap B(\hat{a}, \delta_6)$ and summing the three inequalities in this lemma, we get

$$x_+(s) + z_{\hat{m}}(s) + y_{\hat{m}-1}(s) \leqslant 3\varepsilon p_{\hat{m}}(s) + 2\varepsilon(x_+ + z_{\hat{m}} + y_{\hat{m}-1})(s).$$

Hence, using the definition (2.18) of $\tilde{z}_{\hat{m}}$, we get

$$x_+(s) + \tilde{z}_{\hat{m}}(s) + y_{\hat{m}-1}(s) \leqslant x_+(s) + z_{\hat{m}}(s) + y_{\hat{m}-1}(s) \leqslant 6\varepsilon p_{\hat{m}}(s).$$

Using the definitions (2.16), (3.15) of x_+, $\tilde{z}_{\hat{m}}$, $y_{\hat{m}-1}$ and $p_{\hat{m}}$, we get

$$\big\| v(s) - P_{\hat{m}}(v)(s) \big\|_{L^2_\rho} \leqslant 6\varepsilon \big\| P_{\hat{m}}(v)(s) \big\|_{L^2_\rho},$$

which is the desired conclusion in (iii).

It remains to prove Lemma 3.4 to conclude the proof of Theorems 2 and 2′.

Proof of Lemma 3.4. Consider $\varepsilon > 0$, $u_0 \in \mathcal{V}_6$ and $a \in S_u \cap B(\hat{a}, \delta_6)$.

(a) From Lemma 2.6, we have for all $s \geqslant s_3(\varepsilon, \hat{m})$,

$$x_+(s) \leqslant \varepsilon(y_{\hat{m}} + z_{\hat{m}})(s).$$

Since we have the definition (2.16) of $y_{\hat{m}}$,

$$y_{\hat{m}}(s) \leqslant p_{\hat{m}}(s) + y_{\hat{m}-1}(s) \tag{3.16}$$

and (a) follows.

(b) This is a direct consequence of (3.14) and (3.16).

(c) Since by (ii) of Theorem 2, we have $m(u_0, a) \geqslant \hat{m}$, we have from the definitions (1.11), (2.16), (2.18) of $m(u_0, a)$, $y_{\hat{m}-1}$, $\tilde{z}_{\hat{m}-1}$ and $z_{\hat{m}-1}$

$$y_{\hat{m}-1}(s) = o\big(\tilde{z}_{\hat{m}-1}(s)\big), \quad \text{hence} \quad y_{\hat{m}-1}(s) = o\big(z_{\hat{m}-1}(s)\big) \quad \text{as } s \to +\infty \tag{3.17}$$

(with no uniform character with respect to u_0 and a).

Applying Corollary 2.8 with $i = \hat{m} - 1$, we see then that the second estimate in (3.17) holds uniformly in the sense that

$$\text{for all } \varepsilon > 0 \text{ and } s \geqslant s_3(\varepsilon, \hat{m} - 1), \quad y_{\hat{m}-1}(s) \leqslant C\varepsilon z_{\hat{m}-1}(s).$$

Using the definitions (2.18), (2.16) and (3.15) of $z_{\hat{m}-1}$, $\tilde{z}_{\hat{m}-1}$ and $p_{\hat{m}}$, we write

$$z_{\hat{m}-1}(s) = \tilde{z}_{\hat{m}-1}(s) + J_{\hat{m}-1}(s), \tag{3.18}$$

$$\tilde{z}_{\hat{m}-1}(s) \leqslant p_{\hat{m}}(s) + \tilde{z}_{\hat{m}-1}(s) \leqslant p_{\hat{m}}(s) + z_{\hat{m}}(s). \tag{3.19}$$

Since $\|v\|_{L^\infty} \leqslant M + \kappa$ from (ii) of Proposition 2.1, and knowing that the sequence $k(i)$ is increasing, we write from the definitions (2.17) and (2.18) of $J_{\hat{m}}$ and $z_{\hat{m}}$:

$$\begin{aligned} J_{\hat{m}-1}(s) &\leqslant \bigg(\int\limits_{|y|<1} \big|v(y,s)\big|^4 |y|^{k(\hat{m}-1)} \rho(y)\,dy \bigg)^{\frac{1}{2}} + \bigg(\int\limits_{|y|>1} \big|v(y,s)\big|^4 |y|^{k(\hat{m}-1)} \rho(y)\,dy \bigg)^{\frac{1}{2}} \\ &\leqslant M \bigg(\int\limits_{|y|<1} \big|v(y,s)\big|^2 \rho(y)\,dy \bigg)^{\frac{1}{2}} + \bigg(\int\limits_{|y|>1} \big|v(y,s)\big|^4 |y|^{k(\hat{m})} \rho(y)\,dy \bigg)^{\frac{1}{2}} \\ &\leqslant M \big\| v(s) \big\|_{L^2_\rho} + J_{\hat{m}}(s) \end{aligned}$$

which gives

$$J_{\hat{m}-1}(s) \leqslant M \big\| v(s) \big\|_{L^2_\rho} + z_{\hat{m}}(s). \tag{3.20}$$

Since we have from the definitions (2.16), (3.15) and (2.18) of x_+, $y_{\hat m-1}$, $p_{\hat m}$, $\tilde z_{\hat m}$ and $z_{\hat m}$,

$$\|v(s)\|_{L^2_\rho} \leqslant (x_+ + y_{\hat m-1} + p_{\hat m} + \tilde z_{\hat m})(s) \leqslant (x_+ + y_{\hat m-1} + p_{\hat m} + z_{\hat m})(s),$$

(c) follows from (3.18), (3.19) and (3.20).

This concludes the proof of Lemma 3.4 as well as (ii) ⇒ (iii). This concludes also the proof of Theorems 2 and 2′. □

3.3. Proof of Propositions 3 and 3′

We only prove Proposition 3 since Proposition 3′ follows by the same argument.

Proof of Proposition 3. We will prove that the assertion in Proposition 3 is equivalent to assertion (i) in Theorem 2.

Assertion of Proposition 3 ⇒ (i) of Theorem 2: If the assertion of Proposition 3 is true, then by definition of $m(u_0, a)$, for all $u_0 \in \mathcal{V}'_{\hat u_0}$, $a \in S_u \cap B(\hat a, \hat\delta')$, $m(u_0, a) = \hat m$ and (i) of Theorem 2 follows.

(i) of Theorem 2 ⇒ assertion of Proposition 3: Here, we redo the analysis of Herrero and Velázquez [17] and Filippas and Liu [8], paying attention to getting the uniform character with respect to u_0 and a. We will distinguish two cases: $\hat m \geqslant 4$ and even and $\hat m = 2$.

If $\hat m \geqslant 4$ and even: For a multi-index α in $\mathbb{N}^N$, we introduce v_α the projection of v over H_α. It is defined by

$$v_\alpha(s) = \int_{\mathbb{R}^N} v(y,s) \frac{H_\alpha(y)}{\|H_\alpha\|^2_{L^2_\rho}} \rho(y)\,dy. \tag{3.21}$$

Note that for any $m \in \mathbb{N}$, $P_m(v)$ defined in (1.10) satisfies

$$P_m(v)(y,s) = \sum_{|\alpha|=m} v_\alpha(s) H_\alpha(y).$$

Taking $|\alpha| = \hat m$ and projecting Eq. (1.6) on the eigenfunction H_α, we write:

$$v'_\alpha(s) = \left(1 - \frac{\hat m}{2}\right) v_\alpha(s) + \int_{\mathbb{R}^N} f\big(v(y,s)\big) \frac{H_\alpha(y)}{\|H_\alpha\|^2_{L^2_\rho}} \rho(y)\,dy.$$

Since $|v| \leqslant C$ and $|f(v)| \leqslant C|v|^2 \leqslant C|v|^{\frac32}$, we use the Hölder inequality to write

$$\left| \int_{\mathbb{R}^N} f\big(v(y,s)\big) H_\alpha(y) \rho(y)\,dy \right| \leqslant C \int_{\mathbb{R}^N} \big|v(y,s)\big|^{\frac32} \big(1 + |y|^{\hat m}\big) \rho(y)\,dy \leqslant C \left(\int_{\mathbb{R}^N} \big|v(y,s)\big|^2 \rho(y)\,dy \right)^{\frac34}.$$

Therefore, from (iii) of Theorem 2, we know that for all $|\alpha| = \hat m$ and $s \geqslant s_9$ for some $s_9 \in \mathbb{R}$,

$$\left| v'_\alpha(s) - \left(1 - \frac{\hat m}{2}\right) v_\alpha(s) \right| \leqslant C \left(\int_{\mathbb{R}^N} \big|v(y,s)\big|^2 \rho(y)\,dy \right)^{\frac34} \leqslant C p^{\frac32}_{\hat m}(s), \tag{3.22}$$

with $p^2_{\hat m}(s) = \|P_{\hat m}(v)(s)\|^2_{L^2_\rho} = \sum_{|\beta|=\hat m} (v_\beta(s))^2 \|H_\beta\|^2_{L^2_\rho}$ and

$$p'_{\hat m}(s) \leqslant \left(1 - \frac{\hat m}{2}\right) p_{\hat m}(s) + C p^{\frac32}_{\hat m}(s),$$

since $\|v(s)\|_{L^2_\rho} \to 0$, hence $p_{\hat m}(s) \to 0$ as $s \to +\infty$, uniformly in u_0 and a (see Proposition 2.2 and (i) of Theorem 2). Therefore, this yields $p_{\hat m}(s) \leqslant C e^{(1-\frac{\hat m}{2})s}$ for $s \geqslant s_{10}$, for some s_{10}, and for all $u_0 \in \mathcal{V}_{10}$ and $a \in S_u \cap B(\hat a, \delta_{10})$. Injecting this in (3.22), we get the existence of $C_\alpha(u_0, a)$ such that $v_\alpha \sim C_\alpha e^{(1-\frac{\hat m}{2})s}$. More precisely, for all $s \geqslant s_{10}$, $u_0 \in \mathcal{V}_{10}$ and $a \in S_u \cap B(\hat a, \delta_{10})$,

$$\big| v_\alpha(s) - C_\alpha e^{(1-\frac{\hat m}{2})s} \big| \leqslant C e^{\frac32(1-\frac{\hat m}{2})s}$$

and C_α is continuous with respect to u_0 and a.

If $\hat{m} = 2$: This case has been treated by Filippas and Liu in [8] and Velázquez in [29], with no uniform character. Our contribution is to prove this uniform character.

From (1.8) and (1.9) (with $m = 2$), we know that the eigenvalue $\lambda_2 = 0$ is of multiplicity $\frac{N(N+1)}{2}$ and that its eigenspace is generated by the orthogonal basis

$$\{y_i y_j \mid i < j\} \cup \left\{y_i^2 - 2 \,\middle|\, i = 1, \dots, N\right\}. \tag{3.23}$$

Therefore, defining the $N \times N$ symmetric matrix $A(u_0, a, s)$ (or $A(s)$ for simplicity) by

$$A(u_0, a, s) \equiv A(s) = \int_{\mathbb{R}^N} v_{a,T}(y, s) M(y) \rho(y)\, dy \quad \text{where } M_{i,j}(y) = \frac{1}{4} y_i y_j - \frac{1}{2}\delta_{ij}, \tag{3.24}$$

we see that the coefficients of $A(s)$ are (up to a multiplicity factor) the projections of $v_{a,T}(y, s)$ on the eigenspace generated by (3.23). Moreover, we have the following nice expression for $P_2(v)$

$$P_2(v)(y, s) = \frac{1}{2} y^T A(s) y - \operatorname{tr} A(s) \quad \text{with } \frac{1}{C_0} \leqslant \frac{\|P_2(v)(s)\|_{L^2_\rho}}{\|A(s)\|} \leqslant C_0. \tag{3.25}$$

We have the following result:

Lemma 3.5 *(An ODE satisfied by the matrix $A(s)$). There exist $\mathcal{V}_{10}$ a neighborhood of u_0 in $L^\infty(\mathbb{R}^N \times \mathbb{R})$ and a constant $\delta_{10} > 0$ such that for all $\varepsilon > 0$, there exists $s_{10}(\varepsilon)$ satisfying for all $s \geqslant s_{10}(\varepsilon)$, $u_0 \in \mathcal{V}_{10}$, and $a \in S_u \cap B(\hat{a}, \delta_{10})$,*

$$\left\| A'(s) - \frac{1}{\beta} A(s)^2 \right\| \leqslant \varepsilon \left\| A(s) \right\|^2 \quad \textit{with } \beta = \frac{\kappa}{2p}. \tag{3.26}$$

Proof. As for Proposition 3.2 in [33, p. 514], there is no difficulty in adapting the proof of Filippas and Liu [8] to this uniform context. □

In the following lemma, we define eigenvalues for $A(s)$ and project (3.26) on the eigenvectors:

Lemma 3.6 *(Eigenvalues for the matrix $A(s)$).*

(i) *There exist N real functions $l_i(u_0, a, s) = l_i(s)$, eigenvalues of $A(s)$ $\mathcal{C}^1$ in terms of s.*
For any $(\bar{u}_0, \bar{a}, \bar{s}) \in \mathcal{V}_{10} \times S_{\bar{u}} \cap B(\hat{a}, \delta_{10}) \times [-\log T, +\infty)$ and $\epsilon > 0$, there exists $\eta > 0$ such that if $(u_0, a, s) \in \mathcal{V}_{10} \times S_u \cap B(\hat{a}, \delta_{10}) \times [-\log T, +\infty)$ and $\|u_0 - \bar{u}_0\|_{L^\infty(\mathbb{R}^N)} + |a - \bar{a}| + |s - \bar{s}| \leqslant \eta$, then for all $i \in \{1, \dots, N\}$, $|l_{\sigma(i)}(u_0, a, s) - l_i(\bar{u}_0, \bar{a}, \bar{s})| \leqslant \epsilon$ for some permutation σ of $\{1, \dots, N\}$.

(ii) *The eigenvalues of $A(s)$ satisfy for all $s \geqslant s_{10}(\varepsilon)$, $u_0 \in \mathcal{V}_{10}$, and $a \in S_u \cap B(\hat{a}, \delta_{10})$ (where $s_{10}(\varepsilon)$, $\mathcal{V}_{10}$ and δ_{10} are defined in Lemma 3.5),*

$$\left| {l_i}'(s) - \frac{1}{\beta} \left| l_i(s) \right|^2 \right| \leqslant \varepsilon \left(\sum_{i=1}^N \left| l_i(s) \right|^2 \right), \quad i = 1, \dots, N. \tag{3.27}$$

Remark. We take $\sum_{i=1}^N |l_i(s)|$ or $\sqrt{\sum_{i=1}^N |l_i(s)|^2}$ as a norm of $A(s)$ depending on the convenience.

We note that $\sum_{i=1}^N |l_i(s)|$ is always different from zero. Indeed, if we assume that there exist a time s_0, u_0 and a such that $\sum_{i=1}^N |l_i(s_0)| = 0$, then $A(s_0) = 0$. Using (3.25) and (iii) of Theorem 2, we see that $v(y, s_0) \equiv 0$. This yields $v(y, s) \equiv 0$ and $A(s) = 0$ for all the times thanks to the uniqueness of the initial value problem for Eq. (1.1). Consequently, we get

$$u \equiv \kappa (T - t)^{-\frac{1}{p-1}}. \tag{3.28}$$

Since Proposition 3 holds under the hypotheses of Theorem 1 and (3.28) is excluded by the hypotheses of Theorem 1, we get a contradiction. Thus, $\sum_{i=1}^N |l_i(s)|$ is always different from zero.

Proof of Lemma 3.6. (i) As for [33, Lemma 3.1, p. 514], from the regularity of $w_{a,T}$, it is clear that for each $a \in \mathbb{R}^N$, the symmetric matrix $A(s)$ is a $\mathcal{C}^1$ function of s. Therefore, according to Kato [19], we can define N $\mathcal{C}^1$ functions of s that we denote by $l_i(u_0, a, s) \equiv l_i(s)$, $1 \leqslant i \leqslant N$, eigenvalues of $A(s)$.

Since $A(s)$ is a continuous function of (u_0, a, s) and the eigenvalues of a matrix vary continuously with respect to the coefficients, the eigenvalues $l_i(s)$ are continuous in terms of (u_0, a, s), after appropriate renaming.

(ii) The ODE (3.27) follows from (3.26) by projection on the eigenvectors. We refer to [8] and [29] for more details. □

Since we have from (iii) in Theorem 2, (3.25) and the remark after Lemma 3.6 that for some $C_0 > 0$ and some s_{13} large enough, for all $s \geqslant s_{13}$, $u_0 \in \mathcal{V}'_{\hat{u}_0}$, $a \in S_u \cap B(\hat{a}, \hat{\delta}')$,

$$\frac{1}{C_0} \leqslant \frac{\|v_{a,T}(s)\|_{L^2_\rho}}{\sum_{i=1}^N |l_i(s)|} \leqslant C_0,$$

clearly (1.17) follows from the following lemma:

Lemma 3.7. *There exist $C_0 > 0$ and s_{11} such that for all $s \geqslant s_{11}$, $u_0 \in \mathcal{V}_{10}$, $a \in S_u \cap B(\hat{a}, \delta_{10})$,*

$$\text{(i)} \quad \sum_{i=1}^N \left|l_i(s)\right| \geqslant \frac{1}{C_0 s}, \tag{3.29}$$

$$\text{(ii)} \quad -\frac{C_0}{s} \leqslant \sum_{i=1}^N l_i(s) < 0, \tag{3.30}$$

$$\text{(iii)} \quad \sum_{i=1}^N \left|l_i(s)\right| \leqslant \frac{C_0}{s}. \tag{3.31}$$

Therefore, our proof reduces to the proof of Lemma 3.7.

Proof of Lemma 3.7. The proof of this fact follows from the argument of Filippas and Liu for the proof of Proposition 3 in [8, p. 324] and Velázquez for the proof of Lemma 3.2 in [29, p. 458].

(i) Consider $u_0 \in \mathcal{V}_{10}$ and $a \in S_u \cap B(\hat{a}, \delta_{10})$. Thanks to (3.27), we know that for all $s \geqslant s_{10}(\frac{1}{2\beta N})$,

$$\left(\sum_{i=1}^N l_i^2(s)\right)' = 2\sum_{i=1}^N l_i(s) l_i'(s) \geqslant -c\left(\sum_{i=1}^N l_i^2(s)\right)^{\frac{3}{2}},$$

for some positive constant c independent of u_0, a and s. Integrating this inequality, we get the inequality (3.29).

(ii) Consider $u_0 \in \mathcal{V}_{10}$ and $a \in S_u \cap B(\hat{a}, \delta_{10})$. From (3.27), we know that for all $s \geqslant s_{10}(\frac{1}{2\beta N})$

$$\left(\sum_{i=1}^N l_i(s)\right)' \geqslant \frac{1}{2\beta} \sum_{i=1}^N \left|l_i(s)\right|^2 > 0. \tag{3.32}$$

Therefore, knowing that $\sum_{i=1}^N l_i(s) \to 0$ as $s \to +\infty$, we must have

$$\sum_{i=1}^N l_i(s) < 0 \quad \text{for all } s \geqslant s_{10}\left(\frac{1}{2\beta N}\right). \tag{3.33}$$

Then, by (3.32), we get

$$\forall s \geqslant s_{10}\left(\frac{1}{2\beta N}\right), \quad \left(\sum_{i=1}^N l_i(s)\right)' \geqslant c\left(\sum_{i=1}^N l_i(s)\right)^2,$$

for some $c > 0$, which gives (3.30) by integration.

(iii) Introducing

$$\eta(s) = \min_{1\leqslant i\leqslant N} l_i(s), \tag{3.34}$$

we see from (ii) of Lemma 3.6 that $\eta : s \mapsto \eta(s)$ is absolutely continuous on $[s_{10}(\varepsilon), +\infty)$. Hence, η is almost everywhere differentiable on $[s_{10}(\varepsilon), +\infty)$. Moreover, for any $\varepsilon > 0$ and almost every $s \geqslant s_{10}(\varepsilon)$, we have

$$\eta'(s) \geqslant \frac{1}{\beta}\eta^2(s) - \varepsilon\sum_{i=1}^{N}\big|l_i(s)\big|^2.$$

Since we trivially have

$$\eta(s) \leqslant -\frac{1}{N_*}\sum_{i=1}^{N}\big|l_i(s)\big| \quad \text{where } N_* = \max\big\{N, 2(N-1)\big\}, \tag{3.35}$$

it follows

$$\sum_{i=1}^{N}\big|l_i(s)\big|^2 \leqslant \left(\sum_{i=1}^{N}\big|l_i(s)\big|\right)^2 \leqslant N_*^2\eta^2(s).$$

Hence for $\varepsilon = \frac{1}{2\beta N_*^2}$ and for almost every $s \geqslant s_{10}(\varepsilon)$,

$$\eta'(s) \geqslant \eta^2(s)\left(\frac{1}{\beta} - \varepsilon N_*^2\right) \geqslant \frac{1}{2\beta}\eta^2(s).$$

Since $\eta(s) < 0$, from (3.34) and (3.33), we get by integration for all $s \geqslant s_{11} \equiv \max(s_{10}(\frac{1}{2\beta N}), s_{10}(\frac{1}{2\beta N_*^2}))$

$$\frac{-2\beta}{s} \leqslant \eta(s) < 0.$$

Using (3.35), we conclude the proof of (iii) of Lemma 3.7, which gives the conclusion of Proposition 3. □

Acknowledgements

The authors are very grateful to the reviewer for his carefull reading of their paper and his very useful comments and improvements.

Part of this work was carried out while the first and second authors were visiting the Ecole Normale Supérieure. The first and second authors wish to express their gratitude to DMA at ENS for its warm hospitality. They also thank the French Embassy in Tunisia and the Tunisian Ministry of Higher Education, Scientific Research and Technology for financing their research.

Appendix A. Proof of Lemma 2.5

Let $\bar{P}_i(v) = \sum_{l=2}^{i} P_l(v)$. Applying $\bar{P}_i$ on Eq. (1.6) satisfied by $v_{a,T}(y,s)$ ($v(y,s)$ for simplicity), multiplying by $\bar{P}_i v$ and integrating with respect to $\rho(y)\,dy$, we get

$$y_i'(s)y_i(s) = \int_{\mathbb{R}^N} \bar{P}_i(\mathcal{L}v)\bar{P}_i v\rho\,dy + \int_{\mathbb{R}^N} \bar{P}_i\big(f(v)\big)\bar{P}_i v\rho\,dy \equiv I_1 + I_2.$$

By definition of $\mathcal{L}$ and $\bar{P}_i$, we have

$$I_1 = \int_{\mathbb{R}^N} \bar{P}_i(\mathcal{L}v)\bar{P}_i v\rho\,dy = \sum_{l,j=2}^{i}\left(1-\frac{l}{2}\right)\int_{\mathbb{R}^N} P_l v P_j v\rho\,dy = \sum_{l=2}^{i}\left(1-\frac{l}{2}\right)\int_{\mathbb{R}^N}(P_l v)^2\rho\,dy,$$

from the orthogonality of the eigenspaces of $\mathcal{L}$ in L_ρ^2.

Since $y_i^2(s)=\sum_{l=2}^{i}\int_{\mathbb{R}^N}(P_l v)^2\rho\,dy$, it follows that

$$0\geqslant I_1\geqslant\left(1-\frac{i}{2}\right)y_i^2(s).$$

Using the Cauchy–Schwarz inequality and the fact that $|f(v)|\leqslant C|v|^2$, we have

$$I_2\leqslant C\left\|v^2(s)\right\|_{L^2_\rho}y_i(s).$$

So, we get

$$C\left\|v^2(s)\right\|_{L^2_\rho}\geqslant y_i'(s)\geqslant\left(1-\frac{i}{2}\right)y_i(s)-C\left\|v^2(s)\right\|_{L^2_\rho}. \tag{A.1}$$

Similarly, we have

$$x_+'(s)\geqslant\frac{1}{2}x_+(s)-C\left\|v^2(s)\right\|_{L^2_\rho}\quad\text{and}\quad\tilde z_i'(s)\leqslant\left(1-\frac{i+1}{2}\right)\tilde z_i(s)+C\left\|v^2(s)\right\|_{L^2_\rho}. \tag{A.2}$$

In order to estimate $\|v^2\|_{L^2_\rho}$, we follow an idea of Filippas and Kohn in their paper [7]. Using Proposition 2.2, we have: $\forall u_0\in\mathcal{V}_2$, $a\in S_u\cap B(\hat a,\delta_2)$, $\varepsilon>0$, and $\delta>0$, $\exists s_2(\varepsilon)$ such that $\forall s\geqslant s_2(\varepsilon)$

$$\left\|v^2(s)\right\|^2_{L^2_\rho}=\int\limits_{|y|\leqslant\delta^{-1}}v^4(y,s)\rho(y)\,dy+\int\limits_{|y|\geqslant\delta^{-1}}v^4(y,s)\rho(y)\,dy\leqslant\varepsilon^2\left\|v(s)\right\|^2_{L^2_\rho}+\delta^k J_i^2(s)$$

where $k\in\mathbb{N}$ (later, we will choose k large and depending on m). Then,

$$\left\|v^2(s)\right\|_{L^2_\rho}\leqslant\varepsilon\left\|v(s)\right\|_{L^2_\rho}+\delta^{\frac{k}{2}}J_i(s)\leqslant\varepsilon(x_++y_i+\tilde z_i)(s)+\delta^{\frac{k}{2}}J_i(s),\quad\forall s\geqslant s_2(\varepsilon). \tag{A.3}$$

In order to get an estimation on J_i, we multiply Eq. (1.6) by $v^3(y,s)|y|^k\rho(y)$ and integrate by parts. Writing $\mathcal{L}(v)=\frac{1}{\rho}\nabla(\rho\nabla v)+v$, we then have

$$\frac{1}{4}\frac{d}{ds}\left(\int\limits_{\mathbb{R}^N}v^4(y,s)|y|^k\rho\,dy\right)=J_i^2(s)+I_3+I_4, \tag{A.4}$$

where, using the fact that $|f(v)|\leqslant C|v|$,

$$I_3=\int\limits_{\mathbb{R}^N}f\big(v(y,s)\big)v^3(y,s)|y|^k\rho\,dy\leqslant CJ_m^2(s),$$

and

$$\begin{aligned}I_4&=\int\limits_{\mathbb{R}^N}\nabla\big(\rho(y)\nabla v(y,s)\big)v^3(y,s)|y|^k\rho(y)\,dy\\&=-\frac{k}{8}J_m^2(s)+\frac{k(k-1)}{4}\int\limits_{\mathbb{R}^N}v^4(y,s)|y|^{k-2}\rho(y)\,dy-3\int\limits_{\mathbb{R}^N}v^2(y,s)\left|\nabla v(y,s)\right|^2|y|^k\rho(y)\,dy\\&\leqslant-\frac{k}{8}J_i^2(s)+\frac{k(k-1)}{4}\int\limits_{\mathbb{R}^N}v^4(y,s)|y|^{k-2}\rho(y)\,dy.\end{aligned}$$

Then, going back to (A.4) and using Cauchy–Schwarz inequality, we obtain

$$\frac{1}{4}\frac{d}{ds}\big(J_i^2(s)\big)\leqslant-\left(\frac{k}{8}-1-C\right)J_i^2(s)+\frac{k(k-1)}{4}J_i(s)\left(\int\limits_{\mathbb{R}^N}v^4(y,s)|y|^{k-4}\rho(y)\,dy\right)^{\frac{1}{2}}. \tag{A.5}$$

Now, using the uniform convergence to zero of v over the compact subsets of $\mathbb{R}^N$ when $s\to+\infty$ given by Proposition 2.2, we have

$$\left(\int_{\mathbb{R}^N} v^4(y,s)|y|^{k-4}\rho(y)\,dy\right)^{\frac{1}{2}} \leqslant \left(\int_{|y|\leqslant\delta^{-1}} v^4(y,s)|y|^{k-4}\rho(y)\,dy\right)^{\frac{1}{2}} + \left(\int_{|y|\geqslant\delta^{-1}} v^4(y,s)|y|^{k-4}\rho(y)\,dy\right)^{\frac{1}{2}}$$
$$\leqslant \varepsilon\delta^{2-\frac{k}{2}}\left\|v(s)\right\|_{L^2_\rho} + \delta^2 J_m(s).$$

Thus replacing in (A.5), we get the following integral inequality satisfied by J_m:

$$J_i'(s) \leqslant -\theta J_i(s) + \varepsilon'\left\|v(s)\right\|_{L^2_\rho}, \tag{A.6}$$

where

$$\theta = \theta(k,\delta) = \frac{k}{2} - 2 - 2C - \frac{k(k-1)}{2}\delta^2 \quad \text{and} \quad \varepsilon' = \varepsilon'(\varepsilon,\delta,k) = \frac{1}{2}k(k-1)\varepsilon\delta^{2-\frac{k}{2}}.$$

Given $i \geqslant 2$, by first choosing $k = k(i)$ large, then $\delta = \delta(i)$ small, we can always make $\theta \geqslant \frac{i+1}{2} - 1$. Note that by a trivial induction on i, it is possible to choose $k(i)$ as an increasing sequence of i. Note also that choosing ε small enough, we can make ε' as small as we want. Hence, from (A.6), we obtain

$$J_i'(s) \leqslant \left(1 - \frac{i+1}{2}\right)J_i(s) + \varepsilon'\left\|v(s)\right\|_{L^2_\rho} \leqslant \left(1 - \frac{i+1}{2}\right)J_i(s) + \varepsilon'(x_+ + y_i + \tilde{z}_i)(s), \quad \forall s \geqslant s_2(\varepsilon). \tag{A.7}$$

Finally, using (A.3), (A.1) and (A.2), we have for all $s \geqslant s_2(\varepsilon)$:

$$\begin{cases} y_i'(s) \geqslant \left(1 - \frac{i}{2}\right)y_i(s) - C\varepsilon(x_+ + y_i + \tilde{z}_i)(s) - C\delta^{\frac{k}{2}}J_i(s), \\ \tilde{z}_i'(s) \leqslant \left(1 - \frac{i+1}{2}\right)\tilde{z}_i(s) + C\varepsilon(x_+ + y_i + \tilde{z}_i)(s) + C\delta^{\frac{k}{2}}J_i(s), \end{cases}$$

as $z_i = \tilde{z}_i + J_i$. Taking $\hat{\varepsilon} > 0$, then by choosing k large enough, δ small enough and finally ε small enough, we can make $\delta^{\frac{k}{2}}C \leqslant \varepsilon C + \varepsilon' < \hat{\varepsilon}$. Therefore, we get the desired result for some $s_3(\hat{\varepsilon}) \geqslant s_2(\varepsilon)$ and some neighborhood $\mathcal{V}_3 \subset \mathcal{V}_2$ of initial data $\hat{u}_0$.

References

[1] S. Alinhac, Blowup for Nonlinear Hyperbolic Equations, Birkhäuser Boston Inc., Boston, 1995.

[2] S. Alinhac, A minicourse on global existence and blowup of classical solutions to multidimensional quasilinear wave equations, in: Journées "Équations aux Dérivées Partielles", Forges-Les-Eaux, 2002, Univ. Nantes, Nantes, 2002, Exp. No. I, p. 33.

[3] J.M. Ball, Remarks on blow-up and nonexistence theorems for nonlinear evolution equations, Quart. J. Math. Oxford Ser. (2) 28 (112) (1977) 473–486.

[4] J. Bricmont, A. Kupiainen, Universality in blow-up for nonlinear heat equations, Nonlinearity 7 (2) (1994) 539–575.

[5] C. Fermanian Kammerer, H. Zaag, Boundedness up to blow-up of the difference between two solutions to a semilinear heat equation, Nonlinearity 13 (4) (2000) 1189–1216.

[6] C. Fermanian Kammerer, F. Merle, H. Zaag, Stability of the blow-up profile of nonlinear heat equations from the dynamical system point of view, Math. Ann. 317 (2000) 347–387.

[7] S. Filippas, R.V. Kohn, Refined asymptotics for the blow-up of $u_t - \Delta u = u^p$, Comm. Pure Appl. Math. XLV (1992) 821–869.

[8] S. Filippas, W. Liu, On the blowup of multidimensional semilinear heat equation, Ann. Inst. H. Poincaré Anal. Non Linéaire 10 (3) (1993) 313–344.

[9] H. Fujita, On the blowing-up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 13 (1966) 109–124.

[10] Y. Giga, R.V. Kohn, Asymptotically self-similar blow-up of semilinear heat equations, Comm. Pure Appl. Math. 38 (3) (1985) 297–319.

[11] Y. Giga, R.V. Kohn, Characterizing blowup using similarity variables, Indiana Univ. Math. J. 36 (1) (1987) 1–40.

[12] Y. Giga, R.V. Kohn, Nondegeneracy of blow-up for semilinear heat equations, Comm. Pure Appl. Math. 42 (6) (1989) 845–884.

[13] Y. Giga, S. Matsui, S. Sasayama, Blow up rate for semilinear heat equations with subcritical nonlinearity, Indiana Univ. Math. J. 53 (2) (2004) 483–514.

[14] M.A. Herrero, J.J.L. Velázquez, Flat blow-up in one-dimensional semilinear heat equations, Differential Integral Equations 5 (1992) 973–997.

[15] M.A. Herrero, J.J.L. Velázquez, Comportement générique au voisinage d'un point d'explosion pour des solutions d'équations paraboliques unidimensionnelles, C. R. Math. Acad. Sci. Paris 314 (I) (1992) 201–203.

[16] M.A. Herrero, J.J.L. Velázquez, Generic behaviour of one-dimensional blow-up patterns, Ann. Sc. Norm. Super. Pisa Cl. Sci. 19 (3) (1992) 381–450.

[17] M.A. Herrero, J.J.L. Velázquez, Blow-up behaviour of one-dimensional semilinear parabolic equations, Ann. Inst. H. Poincaré 10 (2) (1993) 131–189.
[18] M.A. Herrero, J.J.L. Velázquez, Generic behaviour near blow-up points for a N-dimensional semilinear heat equation, in preparation.
[19] T. Kato, Perturbation Theory for Linear Operators, Springer, Berlin, 1995, reprint of the 1980 edition.
[20] H.A. Levine, Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $Pu_t = -Au + F(u)$, Arch. Ration. Mech. Anal. 51 (1973) 371–386.
[21] H. Lindblad, C.D. Sogge, On existence and scattering with minimal regularity for semilinear wave equations, J. Funct. Anal. 130 (2) (1995) 357–426.
[22] H. Matano, F. Merle, On nonexistence of type II blowup for a supercritical nonlinear heat equation, Comm. Pure Appl. Math. 57 (11) (2004) 1494–1541.
[23] F. Merle, Solution of a nonlinear heat equation with arbitrarily given blow-up points, Comm. Pure Appl. Math. 45 (3) (1992) 263–300.
[24] F. Merle, H. Zaag, Optimal estimates for blowup rate and behavior for nonlinear heat equations, Comm. Pure Appl. Math. 51 (2) (1998) 139–196.
[25] F. Merle, H. Zaag, A Liouville theorem for vector-valued nonlinear heat equations and applications, Math. Ann. 316 (1) (2000) 103–137.
[26] F. Merle, H. Zaag, Existence and universality of the blow-up profile for the semilinear wave equation in one space dimension, J. Funct. Anal. 253 (1) (2007) 43–121.
[27] N. Nouaili, $\mathcal{C}^{1,\alpha}$ regularity of the blow-up curve at non characteristic points for the one dimensional semilinear wave equation, Comm. Partial Differential Equations 33 (2008) 1540–1548.
[28] J.J.L. Velázquez, Higher dimensional blow-up for semilinear parabolic equations, Comm. Partial Differential Equations 17 (9–10) (1992) 1567–1596.
[29] J.J.L. Velázquez, Classification of singularities for blowing up solutions in higher dimensions, Trans. Amer. Math. Soc. 338 (1) (1993) 441–464.
[30] J.J.L. Velázquez, Estimates on the $(n-1)$-dimensional Hausdorff measure of the blow-up set for a semilinear heat equation, Indiana Univ. Math. J. 42 (2) (1993) 445–476.
[31] J.J.L. Velázquez, Blow up for semilinear parabolic equations, in: Recent Advances in Partial Differential Equations, El Escorial, 1992, in: RAM Res. Appl. Math., vol. 30, Masson, Paris, 1994, pp. 131–145.
[32] F.B. Weissler, Single point blow-up for a semilinear initial value problem, J. Differential Equations 55 (2) (1984) 204–224.
[33] H. Zaag, On the regularity of the blow-up set for semilinear heat equations, Ann. Inst. H. Poincaré Anal. Non Linéaire 19 (5) (2002) 505–542.
[34] H. Zaag, One dimensional behavior of singular N dimensional solutions of semilinear heat equations, Comm. Math. Phys. 225 (3) (2002) 523–549.
[35] H. Zaag, Regularity of the blow-up set and singular behavior for semilinear heat equations, in: Mathematics & Mathematics Education, Bethlehem, 2000, World Sci. Publishing, River Edge, NJ, 2002, pp. 337–347.
[36] H. Zaag, Determination of the curvature of the blow-up set and refined singular behavior for a semilinear heat equation, Duke Math. J. 133 (3) (2006) 499–525.