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Ultra-parabolic H-measures and compensated compactness ☆

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Abstract

We present a generalization of compensated compactness theory to the case of variable and generally discontinuous coefficients, both in the quadratic form and in the linear, up to the second order, constraints. The main tool is the localization properties for ultra-parabolic H-measures corresponding to weakly convergent sequences.

Résumé

Nous présentons ici une généralisation de la théorie de la « compacité par compensation ». Le cas d'une forme quadratique et de contraintes différentielles avec coefficients variables, éventuellement discontinus en espace, est considéré. Ces contraintes différentielles peuvent être d'ordre un, mais aussi d'ordre deux. Notre outil principal est le principe de localisation pour les H-mesures ultra-paraboliques associées à des suites de fonctions faiblement convergentes.

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1. Introduction

Recall the classical results of the compensated compactness theory (see [5,10]). Suppose that Ω is an open subset of $\mathbb{R}^n$, and a sequence $u_r = (u_{1r}(x), \dots, u_{Nr}(x)) \in L^2(\Omega, \mathbb{R}^N)$, $r \in \mathbb{N}$, weakly converges to a vector-function $u(x)$ in $L^2(\Omega, \mathbb{R}^N)$. Assume that $a_{s\alpha k}$ are real constants for $s = 1, \dots, m$, $\alpha = 1, \dots, N$, $k = 1, \dots, n$, and the sequences of distributions

$$\sum_{\alpha=1}^{N}\sum_{k=1}^{n} a_{s\alpha k}\partial_{x_k} u_{\alpha r}, \quad s = 1, \dots, m,\ r \in \mathbb{N}, \tag{1.1}$$

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are strongly precompact in the space $H^{-1}_{loc}(\Omega) \doteq W^{-1}_{2,loc}(\Omega)$. Hereafter, we denote by $W^{-1}_{p,loc}(\Omega)$, $1 \leqslant p \leqslant \infty$ the locally convex space consisting of distributions $v \in \mathcal{D}'(\Omega)$ such that the distribution fv belongs to the Sobolev space $W^{-1}_p \doteq W^{-1}_p(\mathbb{R}^n)$ for all $f(x) \in C_0^\infty(\Omega)$. The topology in $W^{-1}_{p,loc}(\Omega)$ is generated by the family of semi-norms $u \to \|uf\|_{W^{-1}_p}$, $f(x) \in C_0^\infty(\Omega)$. Introduce the set

$$\Lambda = \left\{ \lambda \in \mathbb{R}^N \;\middle|\; \exists \xi \in \mathbb{R}^n,\ \xi \neq 0\colon\ \sum_{\alpha=1}^N \sum_{k=1}^n a_{s\alpha k} \lambda_\alpha \xi_k = 0,\ \forall s = 1, \dots, m \right\}.$$

Now, let $q(u) = \sum_{\alpha,\beta=1}^N q_{\alpha\beta} u_\alpha u_\beta$ be a quadratic functional on $\mathbb{R}^N$ such that $q(\lambda) \geqslant 0$ for all $\lambda \in \Lambda$, and $q(u_r) \rightharpoonup v$ weakly in the sense of distributions on Ω (in $\mathcal{D}'(\Omega)$).

Then, under the above assumptions,

$$q\big(u(x)\big) \leqslant v \quad \text{in } \mathcal{D}'(\Omega)$$

(the weak low semicontinuity). In particular, if $q(\lambda) = 0$ on Λ then $v = q(u)$.

In this paper we generalize this result to the case when the differential constraints may contain second order terms, while all the coefficients are variable and may be discontinuous. Thus, assume that a sequence $u_r(x)$ is bounded in $L^p_{loc}(\Omega, \mathbb{R}^N)$, $2 \leqslant p \leqslant \infty$ and converges weakly in $\mathcal{D}'(\Omega)$ to a vector-function $u(x)$ as $r \to \infty$. Let $d = p/(p-1)$ if $p < \infty$, and $d > 1$ if $p = \infty$. Assume that the sequences

$$\sum_{\alpha=1}^N \sum_{k=1}^n \partial_{x_k}(a_{s\alpha k} u_{\alpha r}) + \sum_{\alpha=1}^N \sum_{k,l=\nu+1}^n \partial_{x_k x_l}(b_{s\alpha kl} u_{\alpha r}), \quad s = 1, \dots, m \tag{1.2}$$

are pre-compact in the anisotropic Sobolev space $W^{-1,-2}_{d,loc}(\Omega)$, which will be defined later in Section 2. Here ν is an integer number between 0 and n, and the coefficients $a_{s\alpha k} = a_{s\alpha k}(x)$, $b_{s\alpha kl} = b_{s\alpha kl}(x)$ belong to the space $L^{2q}_{loc}(\Omega)$, $q = p/(p-2)$ ($q = 1$ in the case $p = \infty$) if $p > 2$, and to the space $C(\Omega)$ if $p = 2$. One example is given by $p = \infty$, $q = 1$ and corresponds to the case when the functions $u_r(x)$ are uniformly locally bounded.

We introduce the set Λ (here $i = \sqrt{-1}$):

$$\begin{aligned} \Lambda &= \Lambda(x) \\ &= \left\{ \lambda \in \mathbb{C}^N \;\middle|\; \exists \xi \in \mathbb{R}^n,\ \xi \neq 0\colon\ \sum_{\alpha=1}^N \left(i \sum_{k=1}^\nu a_{s\alpha k}(x)\xi_k - \sum_{k,l=\nu+1}^n b_{s\alpha kl}(x)\xi_k\xi_l \right) \lambda_\alpha = 0,\ \forall s = 1, \dots, m \right\}. \end{aligned} \tag{1.3}$$

Consider the quadratic form $q(x, u) = Q(x)u \cdot u$, where $Q(x)$ is a symmetric matrix with coefficients $q_{\alpha\beta}(x)$, $\alpha, \beta = 1, \dots, N$ and $u \cdot v$ denotes the scalar multiplication on $\mathbb{R}^N$. The form $q(x, u)$ can be extended as Hermitian form on $\mathbb{C}^N$ by the standard relation

$$q(x, u) = \sum_{\alpha,\beta=1}^N q_{\alpha\beta}(x) u_\alpha \overline{u_\beta},$$

where we denote by $\bar{u}$ the complex conjugation of $u \in \mathbb{C}$. We suppose that the coefficients $q_{\alpha\beta}(x) \in L^q_{loc}(\Omega)$ if $p > 2$, and $q_{\alpha\beta}(x) \in C(\Omega)$ if $p = 2$.

Now, let the sequence $q(x, u_r) \rightharpoonup v$ as $r \to \infty$ weakly in $\mathcal{D}'(\Omega)$. Since for each $\alpha, \beta = 1, \dots, N$ the sequences $u_{\alpha r}(x) u_{\beta r}(x)$ are bounded in $L^{p/2}_{loc}(\Omega)$ (here $p/2 = \infty$ for $p = \infty$) then, passing to a subsequence if necessary, we may claim that

$$u_{\alpha r}(x) u_{\beta r}(x) \underset{r\to\infty}{\rightharpoonup} \zeta_{\alpha\beta}(x)$$

weakly in $L^{p/2}_{loc}(\Omega)$ if $p > 2$ (hereafter, the weak convergence in $L^\infty_{loc}(\Omega)$ is understood in the sense of the weak-$*$ topology), and weakly in the space $M_{loc}(\Omega)$ of locally finite measures on Ω if $p = 2$. In view of the relation $\frac{1}{q} + \frac{2}{p} = 1$ this implies that

$$q(x,u_r) \underset{r\to\infty}{\rightharpoonup} \sum_{\alpha,\beta=1}^{N} q_{\alpha\beta}(x)\zeta_{\alpha\beta}(x)$$

weakly in $M_{loc}(\Omega)$ (weakly in $L^1_{loc}(\Omega)$ if $p>2$) and therefore

$$v(x) = \sum_{\alpha,\beta=1}^{N} q_{\alpha\beta}(x)\zeta_{\alpha\beta}(x).$$

In particular, $v = v(x) \in L^1_{loc}(\Omega)$ for $p > 2$ and $v \in M_{loc}(\Omega)$ for $p = 2$.

Our main result is the following

Theorem 1.1. *Assume that $q(x,\lambda) \geqslant 0$ for all $\lambda \in \Lambda(x)$, $x \in \Omega$. Then $q(x,u(x)) \leqslant v$ (in the sense of measures).*

To prove Theorem 1.1 we will use the techniques of H-measures. Let

$$F(u)(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i\xi\cdot x} u(x)\,dx, \quad \xi \in \mathbb{R}^n,$$

be the Fourier transformation extended as a unitary operator on the space $L^2(\mathbb{R}^n)$, let $S = S^{n-1} = \{\xi \in \mathbb{R}^n \mid |\xi| = 1\}$ be the unit sphere in $\mathbb{R}^n$.

The concept of an H-measure corresponding to some sequence of vector-valued functions bounded in $L^2(\Omega)$ was introduced by Tartar [11] and Gérard [4] on the basis of the following result. For $r \in \mathbb{N}$ let $U_r(x) = (U_r^1(x),\dots,U_r^N(x)) \in L^2(\Omega,\mathbb{R}^N)$ be a sequence weakly convergent to the zero vector.

Proposition 1.1. *(See [11, Theorem 1.1].) There exists a family of complex Borel measures $\mu = \{\mu^{\alpha\beta}\}_{\alpha,\beta=1}^N$ in $\Omega\times S$ and a subsequence of $U_r(x)$ (still denoted U_r) such that*

$$\langle \mu^{\alpha\beta}, \Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\rangle = \lim_{r\to\infty}\int_{\mathbb{R}^n} F\big(U_r^\alpha\Phi_1\big)(\xi)\overline{F\big(U_r^\beta\Phi_2\big)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi \tag{1.4}$$

for all $\Phi_1(x), \Phi_2(x) \in C_0(\Omega)$ and $\psi(\xi) \in C(S)$.

The family $\mu = \{\mu^{\alpha\beta}\}_{\alpha,\beta=1}^N$ is called the H-measure corresponding to $U_r(x)$.

In [1] the new concept of parabolic H-measures was introduced. Here we need the more general variant of this concept recently developed in [6]. Suppose that $X \subset \mathbb{R}^n$ is a linear subspace, $X^\perp$ is its orthogonal complement, P_1, P_2 are orthogonal projections on X, $X^\perp$, respectively. We denote for $\xi \in \mathbb{R}^n$, $\tilde\xi = P_1\xi$, $\bar\xi = P_2\xi$, so that $\tilde\xi \in X$, $\bar\xi \in X^\perp$, $\xi = \tilde\xi + \bar\xi$.

Definition 1. Under the above notations we define the set

$$S_X = \{\xi \in \mathbb{R}^n \mid |\tilde\xi|^2 + |\bar\xi|^4 = 1\}$$

and the projection $\pi_X : \mathbb{R}^n \setminus \{0\} \to S_X$

$$\pi_X(\xi) = \frac{\tilde\xi}{(|\tilde\xi|^2 + |\bar\xi|^4)^{1/2}} + \frac{\bar\xi}{(|\tilde\xi|^2 + |\bar\xi|^4)^{1/4}}.$$

Obviously, S_X is a compact smooth manifold of codimension 1, in the case when $X = \{0\}$ or $X = \mathbb{R}^n$, it coincides with the unit sphere $S = \{\xi \in \mathbb{R}^n \mid |\xi| = 1\}$ and then $\pi_X(\xi) = \xi/|\xi|$ is the orthogonal projection on the sphere.

The following analogue of Proposition 1.1 holds.

Proposition 1.2. *There exist a family of complex Borel measures $\mu = \{\mu^{\alpha\beta}\}_{\alpha,\beta=1}^N$ in $\Omega \times S_X$ and a subsequence of $U_r(x)$ (still denoted by U_r) such that for all $\Phi_1(x), \Phi_2(x) \in C_0(\Omega)$, $\psi(\xi) \in C(S_X)$*

$$\langle \mu^{\alpha\beta}, \Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\rangle = \lim_{r\to\infty} \int_{\mathbb{R}^n} F(\Phi_1 U_r^\alpha)(\xi)\overline{F(\Phi_2 U_r^\beta)(\xi)}\psi(\pi_X(\xi))\,d\xi. \tag{1.5}$$

Besides, the matrix-valued measure μ is Hermitian and positive definite, that is, for each $\zeta = (\zeta_1, \dots, \zeta_N) \in \mathbb{C}^n$ the measure $\mu\zeta \cdot \zeta = \sum_{\alpha,\beta=1}^N \mu^{\alpha\beta}\zeta_\alpha\overline{\zeta_\beta} \geqslant 0$.

For completeness we give the proof of Proposition 1.2 in Appendix A.

Definition 2. The family $\mu^{\alpha\beta}$, $\alpha, \beta = 1, \dots, N$, is called the ultra-parabolic H-measure corresponding to a subspace $X \subset \mathbb{R}^n$ and a subsequence $U_r(x)$.

Remark 1.1. We can replace the function $\psi(\pi_X(\xi))$ in relation (1.5) by a function $\tilde{\psi}(\xi) \in C(\mathbb{R}^n)$, which equals $\psi(\pi_X(\xi))$ for large $|\xi|$. Indeed, since $\Phi_2(x)$ is a function with compact support, $\Phi_2 U_r^\beta \underset{r\to\infty}{\rightharpoonup} 0$ weakly in $L^2(\mathbb{R}^n)$ as well as in $L^1(\mathbb{R}^n)$. Therefore, $F(\Phi_2 U_r^\beta)(\xi) \xrightarrow[r\to\infty]{} 0$ point-wise and in $L^2_{loc}(\mathbb{R}^n)$ (in view of the bound $|F(\Phi_2 U_r^\beta)(\xi)| \leqslant \|\Phi_2 U_r^\beta\|_1 \leqslant \text{const}$). Taking into account that the function $\chi(\xi) = \tilde{\psi}(\xi) - \psi(\pi_X(\xi))$ is bounded and has a compact support, we conclude that

$$\overline{F(\Phi_2 U_r^\beta)(\xi)}\chi(\xi) \xrightarrow[r\to\infty]{} 0 \quad \text{in } L^2(\mathbb{R}^n).$$

This implies that

$$\lim_{r\to\infty} \int_{\mathbb{R}^n} F(\Phi_1 U_r^\alpha)(\xi)\overline{F(\Phi_2 U_r^\beta)(\xi)}\chi(\xi)\,d\xi = 0.$$

Therefore,

$$\begin{aligned}\lim_{r\to\infty} \int_{\mathbb{R}^n} F(\Phi_1 U_r^\alpha)(\xi)\overline{F(\Phi_2 U_r^\beta)(\xi)}\tilde{\psi}(\xi)\,d\xi &= \lim_{r\to\infty} \int_{\mathbb{R}^n} F(\Phi_1 U_r^\alpha)(\xi)\overline{F(\Phi_2 U_r^\beta)(\xi)}\psi(\pi_X(\xi))\,d\xi \\ &= \langle \mu^{\alpha\beta}, \Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\rangle,\end{aligned}$$

as required.

In Section 3 we establish the following localization principle for the ultra-parabolic H-measure corresponding to the subspace $X = \mathbb{R}^\nu$ and a subsequence of $U_r = u_r - u$.

Theorem 1.2 *(Localization principle). For each $s = 1, \dots, m$; $\beta = 1, \dots, N$*

$$\sum_{\alpha=1}^N P_{s\alpha}(x,\xi)\mu^{\alpha\beta} = 0,$$

where

$$P_{s\alpha}(x,\xi) = 2\pi i \sum_{k=1}^\nu a_{s\alpha k}(x)\xi_k - 4\pi^2 \sum_{k,l=\nu+1}^n b_{s\alpha kl}(x)\xi_k\xi_l.$$

Remark 1.2. Observe that the first order term in $P_{s\alpha}(x,\xi)$ contains only variables $\xi_1, \dots, \xi_\nu$ while the corresponding differential operators in (1.2) contain all first order derivatives ∂_{x_k}, $k = 1, \dots, n$.

Remark 1.3. The localization principle for "usual" H-measure (corresponding to the subspace $X = \{0\}$) yields $p_{s\alpha}(x,\xi)\mu^{\alpha\beta} = 0$, $s = 1,\dots,m$; $\beta = 1,\dots,N$, where

$$p_{s\alpha}(x,\xi) = \sum_{k,l=\nu+1}^{n} b_{s\alpha kl}(x)\xi_k\xi_l$$

are the principal symbols of the differential operators in (1.2). This easily follows from Theorem 1.2 with $\nu = 0$, see also [3, Lemma 2.10] in the case $p = 2$.

Remark also that for $\nu = n$, $p = 2$ the statement of Theorem 1.2 coincides with the classic localization principle by Tartar [11, Theorem 1.6]. As was demonstrated in [11], this localization principle allows to deduce the classic compensated compactness results.

Remark 1.4. Actually, our compensated compactness result is an easy consequence of Theorem 1.2. This is important that we use ultra-parabolic H-measures. The localization principle for "usual" H-measure (see Remark 1.3 above) yields the compensated compactness for the quadratic functionals nonnegative on the set

$$\Lambda(x) = \left\{\lambda \in \mathbb{C}^N \;\middle|\; \exists \xi \in \mathbb{R}^n,\ \xi \neq 0\colon \sum_{\alpha=1}^{N}\sum_{k,l=\nu+1}^{n} b_{s\alpha kl}(x)\xi_k\xi_l\lambda_\alpha = 0,\ \forall s = 1,\dots,m\right\}.$$

Obviously, $\Lambda(x) = \mathbb{C}^N$ in the case $\nu > 0$ and the assertion $q(x,u(x)) \leqslant v$ is trivial in this case.

The structure of this paper is following. In the next section we study some properties of ultra-parabolic H-measures and introduce anisotropic Sobolev spaces. Section 3 is devoted to the proofs of our main results. In Section 4 we give an application of the compensated compactness theory to a property of weak completeness for the set of generalized solutions to the semi-linear parabolic equation

$$L(u) = \partial_t u - \sum_{k,l=1}^{n} \partial_{x_k x_l}\big(a_{kl}(t,x)g(t,x,u)\big) = f.$$

Finally, in Appendix A we produce the proof of Proposition 1.2.

2. Preliminaries

Let the sequence $U_r = \{U_r^\alpha\}_{\alpha=1}^N$ converge weakly as $r \to \infty$ to the zero vector, let it be bounded in $L^p_{loc}(\Omega,\mathbb{R}^N)$, $p \geqslant 2$, and let $\mu = \{\mu^{\alpha\beta}\}_{\alpha,\beta=1}^N$ be an ultra-parabolic H-measure corresponding to this sequence. We define $\eta = \operatorname{Tr}\mu = \sum_{\alpha=1}^N \mu^{\alpha\alpha}$. As follows from Proposition 1.2, η is a locally finite nonnegative measure on $\Omega \times S_X$. We assume that this measure is extended on σ-algebra of η-measurable sets, and in particular that this measure is complete. We denote by γ the projection of η on Ω, that is, $\gamma(A) = \eta(A \times S_X)$ if the set $A \times S_X$ is η-measurable. Obviously, γ is a complete locally finite measure on Ω, $\gamma \geqslant 0$. Under the above assumptions the following statements hold.

Proposition 2.1.

(i) *As $r \to \infty$*

$$|U_r|^2 = \sum_{\alpha=1}^N \big|U_r^\alpha(x)\big|^2 \rightharpoonup \gamma$$

weakly in $M_{loc}(\Omega)$; if $p > 2$ then $\gamma \in L^{p/2}_{loc}(\Omega)$ (here we identify γ and the corresponding density $\tilde\gamma$ of γ with respect to the Lebesgue measure dx, so that $\gamma = \tilde\gamma(x)\,dx$), and $|U_r|^2 \rightharpoonup \gamma(x)$ weakly in $L^{p/2}_{loc}(\Omega)$;

(ii) *The H-measure μ is absolutely continuous with respect to η, more precisely, $\mu = H(x,\xi)\eta$, where $H(x,\xi) = \{h^{\alpha\beta}(x,\xi)\}_{\alpha,\beta=1}^N$ is a bounded η-measurable function taking values in the cone of positive definite Hermitian $N \times N$ matrices, besides $|h^{\alpha\beta}(x,\xi)| \leqslant 1$.*

Proof. By the Plancherel identity and relation (1.5) with $\psi \equiv 1$

$$\int_{\Omega} \Phi_1(x)\overline{\Phi_2(x)}|U_r|^2\,dx = \sum_{\alpha=1}^{N} \int_{\mathbb{R}^n} F\big(\Phi_1 U_r^\alpha\big)(\xi)\overline{F\big(\Phi_2 U_r^\alpha\big)(\xi)}\,d\xi \xrightarrow[r\to\infty]{} \big\langle \eta(x,\xi), \Phi_1(x)\overline{\Phi_2(x)}\big\rangle$$
$$= \big\langle \gamma, \Phi_1(x)\overline{\Phi_2(x)}\big\rangle.$$

Since any function $\Phi(x) \in C_0(\Omega)$ can be represented in the form $\Phi(x) = \Phi_1(x)\overline{\Phi_2(x)}$ (for instance, one can take $\Phi_1(x) = \Phi(x)$, $\Phi_2(x)$ being arbitrary function in $C_0(\Omega)$ equal to 1 on $\operatorname{supp}\Phi_1(x)$), we conclude that $|U_r|^2 \rightharpoonup \gamma$ as $r \to \infty$ weakly in $M_{loc}(\Omega)$. In the case $p > 2$ (here $p/2 = \infty$ if $p = \infty$) the sequence $|U_r|^2$ is bounded in $L^{p/2}_{loc}(\Omega)$, and we conclude that $\gamma \in L^{p/2}_{loc}(\Omega)$. The first assertion is proved.

To prove (ii), remark firstly that $\mu^{\alpha\alpha} \leqslant \eta$ for all $\alpha = 1,\dots,N$. Now, suppose that $\alpha,\beta \in \{1,\dots,N\}$, $\alpha \neq \beta$. By Proposition 1.2 for any compact set $B \subset \Omega \times S_X$ the matrix

$$\begin{pmatrix} \mu^{\alpha\alpha}(B) & \mu^{\alpha\beta}(B) \\ \overline{\mu^{\alpha\beta}(B)} & \mu^{\beta\beta}(B) \end{pmatrix}$$

is positive-definite; in particular,

$$\big|\mu^{\alpha\beta}(B)\big| \leqslant \big(\mu^{\alpha\alpha}(B)\mu^{\beta\beta}(B)\big)^{1/2} \leqslant \eta(B).$$

By regularity of measures $\mu^{\alpha\beta}$ and η this estimate is satisfied for all Borel sets B. This easily implies the inequality $\operatorname{Var}\mu^{\alpha\beta} \leqslant \eta$. In particular, the measures $\mu^{\alpha\beta}$ are absolutely continuous with respect to η, and by the Radon–Nykodim theorem $\mu^{\alpha\beta} = h^{\alpha\beta}(x,\xi)\eta$, where the densities $h^{\alpha\beta}(x,\xi)$ are η-measurable and, as follows from the inequalities $\operatorname{Var}\mu^{\alpha\beta} \leqslant \eta$, $|h^{\alpha\beta}(x,\xi)| \leqslant 1$ η-a.e. on $\Omega \times S_X$. We denote by $H(x,\xi)$ the matrix with components $h^{\alpha\beta}(x,\xi)$. Recall that the H-measure μ is positive definite. This means that for all $\zeta \in \mathbb{C}^N$

$$\mu\zeta\cdot\zeta = H(x,\xi)\zeta\cdot\zeta\eta \geqslant 0. \tag{2.1}$$

Hence $H(x,\xi)\zeta\cdot\zeta \geqslant 0$ for η-a.e. $(x,\xi) \in \Omega \times S_X$. Choose a countable dense set $E \subset \mathbb{C}^N$. Since E is countable, then it follows from (2.1) that for a set $(x,\xi) \in \Omega \times S_X$ of full η-measure $H(x,\xi)\zeta\cdot\zeta \geqslant 0$, $\forall\zeta \in E$, and since E is dense we conclude that actually $H(x,\xi)\zeta\cdot\zeta \geqslant 0$ for all $\zeta \in \mathbb{C}^N$. Thus, the matrix $H(x,\xi)$ is Hermitian and positive definite for η-a.e. (x,ξ). After an appropriate correction on a set of null η-measure, we can assume that the above property is satisfied for all $(x,\xi) \in \Omega \times S_X$, and also $|h^{\alpha\beta}(x,\xi)| \leqslant 1$ for all $(x,\xi) \in \Omega \times S_X$, $\alpha,\beta = 1,\dots,N$. The proof is complete. □

Corollary 2.1. *Suppose that the sequence $U_r = \{U_r^\alpha\}_{\alpha=1}^N$ is bounded in $L^p_{loc}(\Omega,\mathbb{R}^N)$, $p > 2$. Let $q = p/(p-2)$ (as usual we set $q = 1$ if $p = \infty$), and let $L^{2q}_0(\Omega)$ be the space of functions in $L^{2q}(\Omega)$ having compact supports. Then relation* (1.5) *still holds for all functions $\Phi_1(x), \Phi_2(x) \in L^{2q}_0(\Omega)$, $\psi(\xi) \in C(S_X)$.*

Proof. Let K be a compact subset of Ω and $\Phi_1(x), \Phi(x) \in L^{2q}(K)$. The functions from $L^{2q}(K)$ are supposed to be extended on Ω as zero functions outside of K. Using the Plancherel identity and the Hölder inequality (observe that $\frac{1}{2q} + \frac{1}{p} = \frac{1}{2}$), we get the following estimate

$$\bigg|\int_{\mathbb{R}^n} F\big(\Phi_1 U_r^\alpha\big)(\xi)\overline{F\big(\Phi_2 U_r^\beta\big)(\xi)}\psi\big(\pi_X(\xi)\big)\,d\xi\bigg| \leqslant \|\psi\|_\infty \big\|\Phi_1 U_r^\alpha\big\|_2 \big\|\Phi_2 U_r^\beta\big\|_2$$
$$\leqslant (C_K)^2\|\psi\|_\infty \cdot \|\Phi_1\|_{2q}\|\Phi_2\|_{2q}, \tag{2.2}$$

where $C_K = \sup_{r\in\mathbb{N}} \|U_r\|_{L^p(K)}$. On the other hand, by Proposition 2.1

$$\big|\big\langle \mu^{\alpha\beta}, \Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\big\rangle\big| = \big|\big\langle \eta, h^{\alpha\beta}(x,\xi)\Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\big\rangle\big| \leqslant \|\psi\|_\infty \int_{\Omega} \big|\Phi_1(x)\Phi_2(x)\big|\gamma(x)\,dx$$
$$\leqslant \|\psi\|_\infty \|\gamma\|_{L^{p/2}(K)} \|\Phi_1\|_{2q}\|\Phi_2\|_{2q} \tag{2.3}$$

(in the last estimate we used again the Hölder inequality). Estimates (2.2), (2.3) show that both sides of relation (1.5) are continuous with respect to $(\Phi_1, \Phi_2) \in (L^{2q}(K))^2$. Since (1.5) holds for $\Phi_1, \Phi_2 \in C_0(K)$ and the space $C_0(K)$ is dense in $L^{2q}(K)$, we claim that (1.5) holds for each $\Phi_1(x), \Phi_2(x) \in L^{2q}(K)$. To conclude the proof, it only remains to notice that K is an arbitrary compact subset of Ω. □

We will need in the sequel some results about Fourier multipliers in spaces L^d, $d > 1$. Recall (see for instance [2,9]) that a function $a(\xi) \in L^\infty(\mathbb{R}^n)$ is a Fourier multiplier in L^d if the pseudo-differential operator $\mathcal{A}$ with the symbol $a(\xi)$, defined as $F(\mathcal{A}u)(\xi) = a(\xi)F(u)(\xi)$, $u = u(x) \in L^2(\mathbb{R}^n) \cap L^d(\mathbb{R}^n)$ can be extended as a bounded operator on $L^d(\mathbb{R}^n)$, that is

$$\|\mathcal{A}u\|_d \leqslant C\|u\|_d, \quad \forall u \in L^2(\mathbb{R}^n) \cap L^d(\mathbb{R}^n),\ C = \text{const}.$$

We denote by M_d the set of Fourier multipliers in L^d.

Let X be a linear subspace of $\mathbb{R}^n$, and let $\pi_X : \mathbb{R}^n \to S_X$ be the projection indicated in Definition 1. Recall that for $\xi \in \mathbb{R}^n$ the notations $\tilde{\xi}$, $\bar{\xi}$ are used for orthogonal projections of ξ onto the spaces X and $X^\perp$, respectively: $\tilde{\xi} = P_1\xi$, $\bar{\xi} = P_2\xi$ (see Introduction).

The following proposition was proved in [6, Proposition 6].

Proposition 2.2. *The following functions are multipliers in spaces L^d for all $d > 1$:*

(i) $a_1(\xi) = \psi(\pi_X(\xi))$ *where* $\psi \in C^n(S_X)$;
(ii) $a_2(\xi) = \rho(\xi)(1 + |\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{-1/2}$, *where* $\rho(\xi) \in C^\infty(\mathbb{R}^n)$ *is a function such that* $0 \leqslant \rho(\xi) \leqslant 1$, $\rho(\xi) = 0$ *for* $|\tilde{\xi}|^2 + |\bar{\xi}|^4 \leqslant 1$, $\rho(\xi) = 1$ *for* $|\tilde{\xi}|^2 + |\bar{\xi}|^4 \geqslant 2$;
(iii) $a_3(\xi) = (1 + |\xi|^2)^{1/2}(1 + |\tilde{\xi}|^2 + |\bar{\xi}|^4)^{-1/2}$;
(iv) $a_4(\xi) = (1 + |\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}(1 + |\xi|^2)^{-1}$.

Now we define the anisotropic Sobolev space $W_d^{-1,-2}$.

Definition 3. (See [6].) The space $W_d^{-1,-2}$ consists of distributions $u(x)$ such that

$$\left(1 + |\tilde{\xi}|^2 + |\bar{\xi}|^4\right)^{-1/2} F(u)(\xi) = F(v)(\xi), \quad v = v(x) \in L^d(\mathbb{R}^n).$$

This is a Banach space with the norm $\|u\| = \|v\|_d$.

Using Proposition 2.2(iii), (iv) one can easily prove that the space $W_d^{-1,-2}$ lays between the spaces W_d^{-1} and W_d^{-2}, that is the following statement holds (see [6, Proposition 7]):

Proposition 2.3. *For each $d > 1$ $W_d^{-1} \subset W_d^{-1,-2} \subset W_d^{-2}$ and the both embeddings are continuous.*

We also introduce the local space $W_{d,loc}^{-1,-2}(\Omega)$ consisting of distributions $u(x)$ such that $uf(x)$ belongs to $W_d^{-1,-2}$ for all $f(x) \in C_0^\infty(\Omega)$. The space $W_{d,loc}^{-1,-2}(\Omega)$ is a locally convex space with the topology generated by the family of semi-norms $u \mapsto \|uf\|_{W_d^{-1,-2}}$, $f(x) \in C_0^\infty(\Omega)$. Analogously, we define the spaces $W_{d,loc}^{-1}(\Omega)$, $W_{d,loc}^{-2}(\Omega)$. As it readily follows from Proposition 2.3, $W_{d,loc}^{-1} \subset W_{d,loc}^{-1,-2} \subset W_{d,loc}^{-2}$ and these embeddings are continuous.

We will need also the following statement, which is rather well known (see, for example, [6, Lemma 6]).

Lemma 2.1. *Let $U_r(x)$ be a sequence bounded in $L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and weakly convergent to zero; let $a(\xi)$ be a bounded function on $\mathbb{R}^n$ such that $a(\xi) \to 0$ as $|\xi| \to \infty$. Then $a(\xi)F(U_r)(\xi) \underset{r\to\infty}{\longrightarrow} 0$ in $L^2(\mathbb{R}^n)$.*

3. Localization principle and proof of Theorem 1.1

Suppose that the sequence $u_r(x)$ converges weakly to $u(x)$ in $L_{loc}^p(\Omega, \mathbb{R}^N)$, and the sequences of distributions

$$\sum_{\alpha=1}^{N}\sum_{k=1}^{n}\partial_{x_k}(a_{s\alpha k}u_{\alpha r})+\sum_{\alpha=1}^{N}\sum_{k,l=\nu+1}^{n}\partial_{x_kx_l}(b_{s\alpha kl}u_{\alpha r}),\quad r\in\mathbb{N},\ s=1,\dots,m,$$

are pre-compact in the anisotropic Sobolev space $W^{-1,-2}_{d,loc}(\Omega)$, where $d>1$ is indicated in the Introduction. We will also assume that $d\leqslant 2$. This assumption is not restrictive, because of the natural embeddings $W^{-1,-2}_{d,loc}(\Omega)\subset W^{-1,-2}_{d_1,loc}(\Omega)$ for each $d_1<d$. Let $U_r=u_r(x)-u(x)=(U_r^1,\dots,U_r^N)$, $U_r^\alpha=u_{\alpha r}(x)-u_\alpha(x)$. Then $U_r\rightharpoonup 0$ as $r\to\infty$ weakly in $L^2_{loc}(\Omega,\mathbb{R}^N)$. Therefore, after extraction of a subsequence (still denoted U_r), we can assume that the ultra-parabolic H-measure $\mu=\{\mu^{\alpha\beta}\}_{\alpha,\beta=1}^N$ corresponding to the subspace

$$X=\mathbb{R}^\nu=\bigl\{\xi=(\xi_1,\dots,\xi_\nu,0,\dots,0)\in\mathbb{R}^n\bigr\}$$

is well defined.

We are going to prove Theorem 1.2 (localization principle), asserting that for each $s=1,\dots,m$; $\beta=1,\dots,N$

$$\sum_{\alpha=1}^{N}P_{s\alpha}(x,\xi)\mu^{\alpha\beta}=0,$$

where

$$P_{s\alpha}(x,\xi)=2\pi i\sum_{k=1}^{\nu}a_{s\alpha k}(x)\xi_k-4\pi^2\sum_{k,l=\nu+1}^{n}b_{s\alpha kl}(x)\xi_k\xi_l.$$

Proof of Theorem 1.2. Since the coefficients $a_{s\alpha k}(x), b_{s\alpha kl}(x)$ belong to $L^{2q}_{loc}(\Omega)$, and $\frac{1}{2q}+\frac{1}{p}=\frac{1}{2}$, the sequences $a_{s\alpha k}U_r^\alpha$, $b_{s\alpha kl}U_r^\alpha$ converge to zero as $r\to\infty$ weakly in $L^2_{loc}(\Omega)$ and the sequences of distributions

$$\mathcal{L}_{sr}\doteq\sum_{\alpha=1}^{N}\sum_{k=1}^{n}\partial_{x_k}\bigl(a_{s\alpha k}U_r^\alpha\bigr)+\sum_{\alpha=1}^{N}\sum_{k,l=\nu+1}^{n}\partial_{x_kx_l}\bigl(b_{s\alpha kl}U_r^\alpha\bigr),\quad r\in\mathbb{N},\ s=1,\dots,m,$$

converge weakly to zero. Using the pre-compactness of these sequences in $W^{-1,-2}_{d,loc}(\Omega)$, we find that $\mathcal{L}_{sr}\to 0$ as $r\to\infty$ in $W^{-1,-2}_{d,loc}(\Omega)$. We choose $\Phi_1(x)\in C_0^\infty(\Omega)$ and consider the distributions

$$l_{sr}=\partial_{x_k}\bigl(a_{s\alpha k}\Phi_1U_r^\alpha-2b_{s\alpha kl}U_r^\alpha\partial_{x_l}\Phi_1\bigr)+\partial_{x_kx_l}\bigl(b_{s\alpha kl}\Phi_1U_r^\alpha\bigr).\tag{3.1}$$

To simplify the notation, we use here and below the conventional rule of summation over repeated indexes, and suppose that the coefficients $b_{s\alpha kl}$ are defined for all $k,l=1,\dots,n$ with $b_{s\alpha kl}=0$ if $\min(k,l)\leqslant\nu$. We can also assume that $b_{s\alpha kl}=b_{s\alpha lk}$ for $k,l=1,\dots,n$. Then, as it is easy to compute,

$$l_{sr}=\Phi_1\mathcal{L}_{sr}+a_{s\alpha k}U_r^\alpha\partial_{x_k}\Phi_1-b_{s\alpha kl}U_r^\alpha\partial_{x_kx_l}\Phi_1.\tag{3.2}$$

Since the coefficients $a_{s\alpha k}(x), b_{s\alpha kl}(x)$ belong to $L^{2q}_{loc}(\Omega)$, and $\frac{1}{2q}+\frac{1}{p}=\frac{1}{2}$, the sequences $a_{s\alpha k}U_r^\alpha\partial_{x_k}\Phi_1$, $b_{s\alpha kl}U_r^\alpha\partial_{x_kx_l}\Phi_1$ are bounded in $L^2(\mathbb{R}^n)$. Noticing that the function $\Phi_1(x)$ has a compact support, we see that these sequences are bounded also in $L^d(\mathbb{R}^n)$ for all $s=1,\dots,m$, and they weakly converge to zero as $r\to\infty$. Therefore, they converge to zero strongly in $W_d^{-1}(\mathbb{R}^n)$ and, in view of Proposition 2.3, also in $W_d^{-1,-2}(\mathbb{R}^n)$. By our assumptions, $\Phi_1\mathcal{L}_{sr}\to 0$ as $r\to\infty$ in $W_d^{-1,-2}(\mathbb{R}^n)$. Hence, it follows from the above limit relations and (3.2) that $l_{sr}\to 0$ as $r\to\infty$ in $W_d^{-1,-2}(\mathbb{R}^n)$. Applying the Fourier transformation to this relation and then multiplying by $(1+|\tilde\xi|^2+|\bar\xi|^4)^{-1/2}$, we arrive at

$$\begin{aligned}&\bigl(1+|\tilde\xi|^2+|\bar\xi|^4\bigr)^{-1/2}\bigl(2\pi i\xi_kF\bigl(a_{s\alpha k}\Phi_1U_r^\alpha\bigr)(\xi)-4\pi i\xi_kF\bigl(b_{s\alpha kl}U_r^\alpha\partial_{x_l}\Phi_1\bigr)(\xi)-4\pi^2\bar\xi_k\bar\xi_lF\bigl(b_{s\alpha kl}\Phi_1U_r^\alpha\bigr)(\xi)\bigr)\\&\quad=F(v_{sr})(\xi),\end{aligned}\tag{3.3}$$

where $v_{sr} \to 0$ as $r \to \infty$ in $L^d(\mathbb{R}^n)$. We take also into account that

$$\xi_k \xi_l F\big(b_{s\alpha kl}\Phi_1 U_r^\alpha\big)(\xi) = \sum_{k,l=\nu+1}^{n} \xi_k \xi_l F\big(b_{s\alpha kl}\Phi_1 U_r^\alpha\big)(\xi) = \bar{\xi}_k \bar{\xi}_l F\big(b_{s\alpha kl}\Phi_1 U_r^\alpha\big)(\xi).$$

By Proposition 2.2(ii), we have

$$a_2(\xi) = \rho(\xi)\big(1 + |\tilde{\xi}|^2 + |\bar{\xi}|^4\big)^{1/2}\big(|\tilde{\xi}|^2 + |\bar{\xi}|^4\big)^{-1/2} \in M_d.$$

Therefore, it follows from (3.3) that

$$\begin{aligned} &\rho(\xi)\big(|\tilde{\xi}|^2 + |\bar{\xi}|^4\big)^{-1/2}\big(2\pi i \xi_k F\big(a_{s\alpha k}\Phi_1 U_r^\alpha\big)(\xi) - 4\pi i \xi_k F\big(b_{s\alpha kl} U_r^\alpha \partial_{x_l}\Phi_1\big)(\xi) - 4\pi^2 \bar{\xi}_k \bar{\xi}_l F\big(b_{s\alpha kl}\Phi_1 U_r^\alpha\big)(\xi)\big) \\ &\quad = a_2(\xi) F(v_{sr})(\xi) = F(w_{sr})(\xi), \end{aligned} \tag{3.4}$$

$w_{sr} \to 0$ as $r \to \infty$ in $L^d(\mathbb{R}^n)$ for all $s = 1, \dots, m$. Since

$$\frac{\rho(\xi)|\bar{\xi}|^2}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}} \leqslant 1,$$

$$\frac{\rho(\xi)|\xi|}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}} \leqslant \rho(\xi)\frac{|\tilde{\xi}| + |\bar{\xi}|}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}} \leqslant 1 + \min\big(|\bar{\xi}|, |\bar{\xi}|^{-1}\big) \leqslant 2$$

(recall that $0 \leqslant \rho(\xi) \leqslant 1$, and $\rho(\xi) = 0$ for $|\tilde{\xi}|^2 + |\bar{\xi}|^4 \leqslant 1$), and $F(a_{s\alpha k}\Phi_1 U_r^\alpha)(\xi), F(b_{s\alpha kl}\Phi_1 U_r^\alpha)(\xi), F(b_{s\alpha kl} U_r^\alpha \partial_{x_l}\Phi_1)(\xi) \in L^2(\mathbb{R}^n)$, we see that $F(w_{sr})(\xi) \in L^2(\mathbb{R}^n)$, which implies that $w_{sr} \in L^2(\mathbb{R}^n)$ as well.

Since $b_{s\alpha kl} = 0$ for $k \leqslant \nu$,

$$\tilde{\xi}_k F\big(b_{s\alpha kl} U_r^\alpha \partial_{x_l}\Phi_1\big)(\xi) = \sum_{k=1}^{\nu} \xi_k F\big(b_{s\alpha kl} U_r^\alpha \partial_{x_l}\Phi_1\big)(\xi) = 0. \tag{3.5}$$

Now, observe that for each k the function

$$a(\xi) = \frac{\rho(\xi)\bar{\xi}_k}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}},$$

satisfies the assumption of Lemma 2.1. Indeed, this follows from the estimate

$$\big|a(\xi)\big| \leqslant \rho(\xi)\big(|\tilde{\xi}|^2 + |\bar{\xi}|^4\big)^{-1/4}\frac{|\bar{\xi}|}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/4}} \leqslant \rho(\xi)\big(|\tilde{\xi}|^2 + |\bar{\xi}|^4\big)^{-1/4}.$$

Since the sequences $a_{s\alpha k}\Phi_1 U_r^\alpha$, $b_{s\alpha kl} U_r^\alpha \partial_{x_l}\Phi_1$ are bounded in $L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and weakly converge to zero as $r \to \infty$, then by Lemma 2.1

$$\frac{\rho(\xi)}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}}\bar{\xi}_k F\big(a_{s\alpha k}\Phi_1 U_r^\alpha\big)(\xi) \xrightarrow[r\to\infty]{} 0 \quad \text{in } L^2\big(\mathbb{R}^n\big), \tag{3.6}$$

$$\frac{\rho(\xi)}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}}\bar{\xi}_k F\big(b_{s\alpha kl} U_r^\alpha \partial_{x_l}\Phi_1\big)(\xi) \xrightarrow[r\to\infty]{} 0 \quad \text{in } L^2\big(\mathbb{R}^n\big). \tag{3.7}$$

It follows from (3.5), (3.7) that

$$\frac{\rho(\xi)}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}}\xi_k F\big(b_{s\alpha kl} U_r^\alpha \partial_{x_l}\Phi_1\big)(\xi) \xrightarrow[r\to\infty]{} 0 \quad \text{in } L^2\big(\mathbb{R}^n\big). \tag{3.8}$$

Let $\Phi_2(x) \in C_0(\mathbb{R}^n)$, $\psi(\xi) \in C^n(S_X)$. Since the sequence $\Phi_2 U_r^\beta$ is bounded in $L^p(\Omega)$ and supported in the compact $\operatorname{supp}\Phi_2$, and $d' = d/(d-1) \leqslant p$, this sequence is also bounded in $L^2(\mathbb{R}^n) \cap L^{d'}(\mathbb{R}^n)$. By Proposition 2.2(i) for a fixed $\beta = 1, \dots, N$, $\overline{\psi(\pi_X(\xi))F(\Phi_2 U_r^\beta)(\xi)} = F(g_r)(\xi)$, where the sequence g_r is bounded in $L^2(\mathbb{R}^n) \cap L^{d'}(\mathbb{R}^n)$. We multiply (3.4) by $\psi(\pi_X(\xi))\overline{F(\Phi_2 U_r^\beta)(\xi)}$ and integrate the result over $\xi \in \mathbb{R}^n$. Passing then to the limit as $r \to \infty$ and taking into account relations (3.6), (3.8), we arrive at

$$\lim_{r\to\infty}\int_{\mathbb{R}^n}\frac{\rho(\xi)(2\pi i\tilde{\xi}_k F(a_{s\alpha k}\Phi_1 U_r^\alpha)(\xi)-4\pi^2\bar{\xi}_k\bar{\xi}_l F(b_{s\alpha kl}\Phi_1 U_r^\alpha)(\xi))}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\times\overline{F\big(\Phi_2 U_r^\beta\big)}\psi\big(\pi_X(\xi)\big)\,d\xi$$
$$=\lim_{r\to\infty}\int_{\mathbb{R}^n}F(w_{sr})(\xi)\overline{F(g_r)(\xi)}\,d\xi=\lim_{r\to\infty}\int_{\mathbb{R}^n}w_{sr}(x)\overline{g_r(x)}\,dx=0. \tag{3.9}$$

On the other hand, by relation (1.5), Remark 1.1 and Corollary 2.1 (in the case $p>2$), we see that

$$\lim_{r\to\infty}\int_{\mathbb{R}^n}\frac{\rho(\xi)(2\pi i\tilde{\xi}_k F(a_{s\alpha k}\Phi_1 U_r^\alpha)(\xi)-4\pi^2\bar{\xi}_k\bar{\xi}_l F(b_{s\alpha kl}\Phi_1 U_r^\alpha)(\xi))}{(|\tilde{\xi}|^2+|\bar{\xi}|^4)^{1/2}}\times\overline{F\big(\Phi_2 U_r^\beta\big)}\psi\big(\pi_X(\xi)\big)\,d\xi$$
$$=\big\langle\mu^{\alpha\beta},\big(2\pi i a_{s\alpha k}(x)\tilde{\xi}_k-4\pi^2 b_{s\alpha kl}(x)\bar{\xi}_k\bar{\xi}_l\big)\Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\big\rangle.$$

Then it follows from (3.9) that

$$\big\langle\mu^{\alpha\beta},P_{s\alpha}(x,\xi)\Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\big\rangle=0, \tag{3.10}$$

where

$$P_{s\alpha}(x,\xi)=2\pi i a_{s\alpha k}(x)\tilde{\xi}_k-4\pi^2 b_{s\alpha kl}(x)\bar{\xi}_k\bar{\xi}_l=2\pi i\sum_{k=1}^{\nu}a_{s\alpha k}(x)\xi_k-4\pi^2\sum_{k,l=\nu+1}^{n}b_{s\alpha kl}(x)\xi_k\xi_l.$$

We underline that the functions $P_{s\alpha}(x,\xi)\Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)$ are measurable and locally integrable with respect to the measure η. This is evident in the case $p=2$ (then $a_{s\alpha k}, b_{s\alpha kl}\in C(\Omega)$) while in the case $p>2$ this readily follows from Proposition 2.1, from the assumptions $a_{s\alpha k}, b_{s\alpha kl}\in L^{2q}_{loc}(\Omega)$, and from the inequality $\frac{1}{2q}+\frac{2}{p}<\frac{1}{q}+\frac{2}{p}=1$.

Since the functions $\Phi_1(x)\in C_0^\infty(\Omega)$, $\Phi_2(x)\in C_0(\Omega)$, $\psi(\xi)\in C^n(S_X)$ are arbitrary, we derive from (3.10) that $P_{s\alpha}(x,\xi)\mu^{\alpha\beta}=0$ for each $s=1,\dots,m$, $\beta=1,\dots,N$. The proof is complete. □

By Proposition 2.1 the H-measure μ admits the representation $\mu=H(x,\xi)\eta$, where $H(x,\xi)=\{h^{\alpha\beta}(x,\xi)\}_{\alpha,\beta=1}^N$ is a Hermitian matrix.

Corollary 3.1. *For η-a.e. $(x,\xi)\in\Omega\times S_X$ the image of $H(x,\xi)$ is contained in $\Lambda(x)$.*

Proof. By Theorem 1.2 $P_{s\alpha}(x,\xi)h^{\alpha\beta}(x,\xi)\eta=0$. This can be written as $P(x,\xi)H(x,\xi)=0$, where $P(x,\xi)$ is an $m\times N$ matrix with components $P_{s\alpha}$. Therefore, for η-a.e. $(x,\xi)\in\Omega\times S_X$, $\operatorname{Im}H(x,\xi)\subset\ker P(x,\xi)$. Now notice that if $\lambda=(\lambda_1,\dots,\lambda_N)\in\mathbb{C}^N$ belongs to $\ker P(x,\xi)$ then

$$\sum_{\alpha=1}^{N}\left(i\sum_{k=1}^{\nu}a_{s\alpha k}(x)2\pi\xi_k-\sum_{k,l=\nu+1}^{n}b_{s\alpha kl}(x)2\pi\xi_k 2\pi\xi_l\right)\lambda_\alpha=0$$

for all $s=1,\dots,m$. Remark that $2\pi\xi\neq0$ because of the inclusion $\xi\in S_X$. Hence, $\lambda\in\Lambda(x)$. We conclude that $\ker P(x,\xi)\subset\Lambda(x)$, and $\operatorname{Im}H(x,\xi)\subset\ker P(x,\xi)\subset\Lambda(x)$, as was to be proved. □

Now we are ready to prove our main Theorem 1.1.

Proof of Theorem 1.1. Since $H=H(x,\xi)\geqslant0$ there exists a unique Hermitian matrix $R=R(x,\xi)=H^{1/2}$ such that $R\geqslant0$ and $H=R^2$. By the known properties of Hermitian matrices $\ker R=\ker H$, which readily implies that $\operatorname{Im}R=\operatorname{Im}H$. By Corollary 3.1 we claim that $\operatorname{Im}R(x,\xi)\subset\Lambda(x)$ for η-a.e. $(x,\xi)\in\Omega\times S_X$. Now we represent the coefficients $q_{\alpha\beta}(x)$ of the quadratic form $q(x,u)$ as $q_{\alpha\beta}(x)=q^{(1)}_{\alpha\beta}(x)q^{(2)}_{\alpha\beta}(x)$, where for $j=1,2$, $q^{(j)}_{\alpha\beta}(x)\in L^{2q}_{loc}(\Omega)$ if $p>2$, and $q^{(j)}_{\alpha\beta}(x)\in C(\Omega)$ if $p=2$. For instance, we can set

$$q^{(1)}_{\alpha\beta}(x)=\big|q_{\alpha\beta}(x)\big|^{1/2}\operatorname{sign}q_{\alpha\beta}(x),\qquad q^{(2)}_{\alpha\beta}(x)=\big|q_{\alpha\beta}(x)\big|^{1/2}.$$

Taking into account Corollary 2.1, we find that for real $\Phi(x)\in C_0(\Omega)$

$$\int_\Omega \big(\Phi(x)\big)^2 q\big(x, U_r(x)\big)\,dx = \int_{\mathbb{R}^n} q^{(1)}_{\alpha\beta}(x)\Phi(x)U_r^\alpha(x) q^{(2)}_{\alpha\beta}(x)\Phi(x)U_r^\beta(x)\,dx$$
$$= \int_{\mathbb{R}^n} F\big(\Phi q^{(1)}_{\alpha\beta} U_r^\alpha\big)(\xi)\overline{F\big(\Phi q^{(2)}_{\alpha\beta} U_r^\beta\big)(\xi)}\,d\xi \underset{r\to\infty}{\longrightarrow} \big\langle \mu^{\alpha\beta}, \big(\Phi(x)\big)^2 q_{\alpha\beta}(x)\big\rangle$$
$$= \int_{\Omega\times S_X} \big(\Phi(x)\big)^2 q_{\alpha\beta}(x) h^{\alpha\beta}(x,\xi)\,d\eta(x,\xi). \tag{3.11}$$

Since $H = R^2$ then $h^{\alpha\beta}(x,\xi) = r_{\alpha j}\overline{r_{\beta j}}$, where $r_{ij} = r_{ij}(x,\xi)$, $i, j = 1, \dots, N$ are components of matrix R. Therefore,

$$q_{\alpha\beta}(x)h^{\alpha\beta} = q_{\alpha\beta}(x)r_{\alpha j}\overline{r_{\beta j}} = \sum_{j=1}^N Q(x)\operatorname{Re}_j \cdot \operatorname{Re}_j, \tag{3.12}$$

where $\{e_j\}_{j=1}^N$ is the standard basis in $\mathbb{C}^N$. Since $\operatorname{Re}_j \in \operatorname{Im} R \subset \Lambda(x)$ then it follows from the assumption of Theorem 1.1 that $Q(x)\operatorname{Re}_j \cdot \operatorname{Re}_j \geqslant 0$ for η-a.e. $(x,\xi) \in \Omega \times S_X$. In view of (3.12) we find that $q_{\alpha\beta}(x)h^{\alpha\beta}(x,\xi) \geqslant 0$ for η-a.e. $(x,\xi) \in \Omega \times S_X$. Now, it readily follows from (3.11) that

$$\lim_{r\to\infty} \int_\Omega \big(\Phi(x)\big)^2 q\big(x, U_r(x)\big)\,dx \geqslant 0 \tag{3.13}$$

for all real $\Phi(x) \in C_0(\Omega)$.

In view of the weak convergence $u_r \rightharpoonup u$, $q(x, u_r(x)) \rightharpoonup v$ as $r \to \infty$,

$$q\big(x, U_r(x)\big) = q\big(x, u_r(x)\big) + q\big(x, u(x)\big) - 2\operatorname{Re}\big(Q(x)u_r(x)\cdot u(x)\big) \rightharpoonup v - q\big(x, u(x)\big)$$

weakly in $M_{loc}(\Omega)$, and we derive from (3.13) that

$$\big\langle v - q\big(x, u(x)\big)\,dx, \big(\Phi(x)\big)^2\big\rangle \geqslant 0.$$

Since $(\Phi(x))^2$ is an arbitrary nonnegative function in $C_0(\Omega)$, this implies that $q(x, u(x)) \leqslant v$. The proof is complete. □

Corollary 3.2. *Suppose that $q(x,\lambda) = 0$ for all $\lambda \in \Lambda(x)$, $x \in \Omega$. Then $v = q(x, u(x))$, that is, the functional $u \to q(x,u)$ is weakly continuous.*

Proof. Applying Theorem 1.1 to the quadratic forms $\pm q(x,u)$, we obtain the inequalities $\pm v \geqslant \pm q(x, u(x))$, which readily imply that $v = q(x, u(x))$. □

Remark 3.1. In the particular case $\nu = n$ relations (1.2) are reduced to the requirement that the sequences of distributions

$$L_{sr} = \sum_{\alpha=1}^N \sum_{k=1}^n \partial_{x_k}\big(a_{s\alpha k}(x)u_{\alpha r}\big), \quad s = 1, \dots, m$$

are pre-compact in $W^{-1}_{d,loc}(\Omega)$. In applications to conservation laws, it usually happens that the sequences $u_{\alpha r}$ are bounded in $L^\infty_{loc}(\Omega)$ (so that $p = \infty$) while the sequences L_{sr} are bounded in $M_{loc}(\Omega)$. Since the space $M_{loc}(\Omega)$ is compactly embedded in $W^{-1}_{d,loc}(\Omega)$ for $d < n/(n-1)$ then condition (1.2) is satisfied.

In the case $\nu = 0$ the statement of Theorem 1.1 is a compensated compactness result under the second order constraints

$$L_{sr} = \sum_{\alpha=1}^N \sum_{k,l=1}^n \partial_{x_k x_l}\big(b_{s\alpha kl}(x)u_{\alpha r}\big), \quad s = 1, \dots, m,$$

which are required to be pre-compact in $W^{-2}_{d,loc}(\Omega)$. Observe also that in each of the cases $\nu = n, 0$ the set $\Lambda(x)$ may be defined as a subset of real space $\mathbb{R}^N$.

4. Some applications

We consider the parabolic operator

$$L(u) = \partial_t u - \sum_{k,l=1}^{n} \partial_{x_k x_l}\big(a_{kl}(t,x)g(t,x,u)\big), \quad u = u(t,x),\ (t,x) \in \Omega = (0,+\infty) \times V,$$

V being an open subset of $\mathbb{R}^n$. It is assumed that for $u = u(t,x)$

$$u, g(t,x,u) \in L^p_{loc}(\Omega), \quad 2 \leqslant p \leqslant \infty,$$

while

$$a_{kl} = a_{kl}(t,x) \in L^{2q}_{loc}(\Omega), \quad \text{where } q = p/(p-2),\ p > 2,$$

and

$$a_{kl} \in C(\Omega) \quad \text{if } p = 2.$$

The matrix $A(t,x) = \{a_{kl}(t,x)\}_{k,l=1}^n$ is supposed to be symmetric and strictly positive: $A(t,x)\xi \cdot \xi > 0$, $\forall \xi \in \mathbb{R}^n$, $\xi \neq 0$. The function $g(t,x,u)$ is a Caratheodory function on $\Omega \times \mathbb{R}$, non-strictly increasing with respect to the variable u.

Assume that the sequences $u_r(t,x)$, $g(t,x,u_r(t,x))$, $r \in \mathbb{N}$ are bounded in $L^p_{loc}(\Omega)$. Also suppose that $u_r \rightharpoonup u = u(t,x)$ as $r \to \infty$ weakly in $\mathcal{D}'(\Omega)$ while $f_r = L(u_r) \to f$ strongly in $W^{-1,-2}_{d,loc}(\Omega)$, where the latter space corresponds to the subspace $X = \{(\xi_0, 0, \ldots, 0)\} \subset \mathbb{R}^{n+1}$, here $(\xi_0, \xi_1, \ldots, \xi_n)$ are the dual variables (ξ_0 corresponds to the time variable t), and $d = p/(p-1)$ ($d > 1$ in the case $p = \infty$).

Theorem 4.1. *Under the above assumptions, $L(u) = f$ in $\mathcal{D}'(\Omega)$. In addition, the sequence $g(t,x,u_r(t,x))$ converges to $g(t,x,u(t,x))$ as $r \to \infty$ strongly in $L^1_{loc}(\Omega)$.*

Proof. Let $u_{1r} = u_r(t,x)$, $u_{2r} = g(t,x,u_r(t,x))$. Passing to a subsequence if necessary, we can assume that $u_{2r}(t,x) \rightharpoonup \tilde{u}_2 = \tilde{u}_2(t,x)$ weakly as $r \to \infty$. Then the sequence (u_{1r}, u_{2r}) converges weakly to $(\tilde{u}_1, \tilde{u}_2) \in L^p_{loc}(\Omega, \mathbb{R}^2)$ with $\tilde{u}_1 = u(t,x)$. Further, it satisfies the condition that the sequence of distributions

$$f_r = \partial_t u_{1r} - \sum_{k,l=1}^{n} \partial_{x_k x_l}\big(a_{kl}(t,x)u_{2r}\big)$$

is pre-compact in $W^{-1,-2}_{d,loc}(\Omega)$. In accordance with (1.3), we define the set $\Lambda = \Lambda(t,x)$:

$$\Lambda = \big\{(\lambda_1, \lambda_2) \in \mathbb{C}^2 \ \big|\ \exists(\xi_0, \xi) \in \big(\mathbb{R} \times \mathbb{R}^n\big) \setminus \{0\},\ i\xi_0\lambda_1 + \big(A(t,x)\xi \cdot \xi\big)\lambda_2 = 0\big\}.$$

Since $(A(t,x)\xi \cdot \xi) > 0$ for $\xi \neq 0$ then $\Lambda = \{(\lambda_1, \lambda_2) \in \mathbb{C}^2 \mid \operatorname{Re} \lambda_1 \overline{\lambda_2} = 0\}$. Therefore, the quadratic functional $q = q(u) = (u_1\overline{u_2} + u_2\overline{u_1})/2$ is zero for $u = \lambda \in \Lambda$. By Corollary 3.2 (observe that all the assumptions of this corollary are satisfied) we claim that

$$q(u_{1r}, u_{2r}) = u_{1r}u_{2r} \underset{r\to\infty}{\rightharpoonup} q(\tilde{u}_1, \tilde{u}_2) = \tilde{u}_1\tilde{u}_2 \tag{4.1}$$

weakly in $M_{loc}(\Omega)$. Since the function $g(t,x,u)$ increases with respect to u then for every $k, r \in \mathbb{N}$, $(g(t,x,u_k) - g(t,x,u_r))(u_k - u_r) \geqslant 0$, where $u_k = u_k(t,x)$, $u_r = u_r(t,x)$. Passing in this inequality to the weak limit as $k \to \infty$ and taking into account that by (4.1)

$$u_k g(t,x,u_k) = u_{1k}u_{2k} \underset{k\to\infty}{\rightharpoonup} \tilde{u}_1\tilde{u}_2,$$

we find that

$$
\begin{aligned}
0 &\leqslant \big(g(t,x,u_k)-g(t,x,u_r)\big)(u_k-u_r)\\
&= u_k g(t,x,u_k)-u_k g(t,x,u_r)-g(t,x,u_k)u_r+u_r g(t,x,u_r)\\
&\underset{k\to\infty}{\rightharpoonup} \tilde u_1\tilde u_2-\tilde u_1 g(t,x,u_r)-\tilde u_2 u_r+u_r g(t,x,u_r)\\
&= \big(\tilde u_2-g(t,x,u_r)\big)(\tilde u_1-u_r).
\end{aligned}
$$

Therefore, $(\tilde u_2-g(t,x,u_r))(\tilde u_1-u_r)\geqslant 0$. By (4.1) again

$$\big(\tilde u_2-g(t,x,u_r)\big)(\tilde u_1-u_r)=\tilde u_1\tilde u_2+u_{1r}u_{2r}-\tilde u_1 u_{2r}-\tilde u_2 u_{1r}\underset{r\to\infty}{\rightharpoonup}0$$

weakly in $M_{loc}(\Omega)$. Since the sequence $(\tilde u_2-g(t,x,u_r))(\tilde u_1-u_r)\geqslant 0$ is nonnegative, we claim that it converges to zero strongly: $(\tilde u_2-g(t,x,u_r))(\tilde u_1-u_r)\to 0$ in $L^1_{loc}(\Omega)$.

Extracting again a subsequence (still denoted by u_r), we may suppose that the Young measure $\nu_{t,x}$ corresponding to this subsequence is well defined. Recall that a Young measure $\nu_{t,x}$ on Ω is a weakly measurable map $(t,x)\to\nu_{t,x}$ of Ω into the space $\mathrm{Prob}(\mathbb{R})$ of probability measures on $\mathbb{R}$. The weak measurability means that for each bounded continuous function $p(\lambda)$ the function $(t,x)\to\int p(\lambda)\,d\nu_{t,x}(\lambda)$ is Lebesgue measurable on Ω. It is known (see, for example, [8]) that the Young measure corresponding to u_r satisfies the property that whenever the sequence $\psi(t,x,u_r(t,x))$ converges weakly in $L^1_{loc}(\Omega)$ for a Caratheodory function $\psi(t,x,\lambda)$, its weak limit is the function

$$\bar\psi(t,x)=\int\psi(t,x,\lambda)\,d\nu_{t,x}(\lambda).$$

Moreover, $\nu_{t,x}(\lambda)=\delta(\lambda-u(t,x))$, where $\delta(\lambda-u)$ is the Dirac mass at u, if and only if $u_r\to u$ in $L^1_{loc}(\Omega)$. Since $u_r\rightharpoonup\tilde u_1=u(t,x)$, $g(t,x,u_r)\rightharpoonup\tilde u_2(t,x)$ weakly in $L^1_{loc}(\Omega)$, and $(\tilde u_2-g(t,x,u_r))(\tilde u_1-u_r)\to 0$ even strongly in $L^1_{loc}(\Omega)$, then these limit functions admit the representations:

$$\tilde u_1=\int\lambda\,d\nu_{t,x}(\lambda),\qquad \tilde u_2=\int g(t,x,\lambda)\,d\nu_{t,x}(\lambda),\qquad 0=\int\big(\tilde u_2-g(t,x,\lambda)\big)(\tilde u_1-\lambda)\,d\nu_{t,x}(\lambda).$$

It follows from these equalities that for a.e. $(t,x)\in\Omega$

$$u(t,x)\int g(t,x,\lambda)\,d\nu_{t,x}(\lambda)=\tilde u_1(t,x)\tilde u_2(t,x)=\int\lambda g(t,x,\lambda)\,d\nu_{t,x}(\lambda).$$

It is reduced to the equality

$$\int\big(\lambda-u(t,x)\big)g(t,x,\lambda)\,d\nu_{t,x}(\lambda)=0,$$

and since $\int(\lambda-u(t,x))\,d\nu_{t,x}(\lambda)=0$, we arrive at the relation

$$
\begin{aligned}
&\int\big(\lambda-u(t,x)\big)\big(g(t,x,\lambda)-g\big(t,x,u(t,x)\big)\big)\,d\nu_{t,x}(\lambda)\\
&\quad=\int\big(\lambda-u(t,x)\big)g(t,x,\lambda)\,d\nu_{t,x}(\lambda)-g\big(t,x,u(t,x)\big)\int\big(\lambda-u(t,x)\big)\,d\nu_{t,x}(\lambda)=0
\end{aligned}
\tag{4.2}
$$

for a.e. $(t,x)\in\Omega$. Taking into account the fact that the function $g(t,x,\lambda)$ is non-decreasing with respect to λ, we derive from (4.2) that for a.e. $(t,x)\in\Omega$, $g(t,x,\lambda)=g(t,x,u(t,x))$ on $\operatorname{supp}\nu_{t,x}$. Therefore,

$$\tilde u_2=\int g(t,x,\lambda)\,d\nu_{t,x}(\lambda)=g\big(t,x,u(t,x)\big)$$

almost everywhere in Ω. Hence, in the limit as $r\to\infty$

$$L(u_r)\rightharpoonup L(u)=\partial_t u-\sum_{k,l=1}^n\partial_{x_kx_l}\big(a_{kl}(t,x)g(t,x,u)\big)\quad\text{in }\mathcal{D}'(\Omega).$$

Since $L(u_r)=f_r\rightharpoonup f$ as $r\to\infty$ in $\mathcal{D}'(\Omega)$, we conclude that $L(u)=f$. Besides, the image of $\nu_{t,x}$ under the map

$u \to g(t,x,u)$ coincides with the Dirac measure $\delta(\lambda - g(t,x,u(t,x)))$:

$$\tilde{\nu}_{t,x}(\lambda) \doteq \big(g(t,x,\cdot)^* \nu_{t,x}\big)(\lambda) = \delta\big(\lambda - g\big(t,x,u(t,x)\big)\big).$$

It is easy to see that $\tilde{\nu}_{t,x}(\lambda)$ is the Young measure corresponding to the sequence $g(t,x,u_r(t,x))$. Since this Young measure coincides with $\delta(\lambda - g(t,x,u(t,x)))$, we conclude that the sequence $g(t,x,u_r(t,x))$ converges to $g(t,x,u(t,x))$ strongly in $L^1_{loc}(\Omega)$. Finally, observe that the limit function does not depend on the prescribed above choice of a subsequence. Therefore, $g(t,x,u_r(t,x))$ also converges strongly to $g(t,x,u(t,x))$ for the original sequence u_r. The proof is complete. □

Remark 4.1. In the case when the function $g(t,x,u)$ is strictly monotone we deduce from Theorem 4.1 the strong pre-compactness property for weak solutions of the equation $L(u) = f = f(t,x) \in W^{-1,-2}_{loc}(\Omega)$, which satisfy the equation in $\mathcal{D}'(\Omega)$. Notice that for entropy solutions of this equation the strong pre-compactness property follows from general results of [6,7].

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Appendix A. The proof of Proposition 1.2

Denote

$$I_r^{\alpha\beta}(\Phi_1,\Phi_2,\psi) = \int_{\mathbb{R}^n} F\big(\Phi_1 U_r^\alpha\big)(\xi)\overline{F\big(\Phi_2 U_r^\beta\big)(\xi)}\psi\big(\pi_X(\xi)\big)\,d\xi$$

and observe that, by the Buniakovskii inequality and the Plancherel identity,

$$\big|I_r^{\alpha\beta}\big| \leqslant \|\Phi_1\|_\infty \|\Phi_2\|_\infty \|\psi\|_\infty \cdot \big\|U_r^\alpha\big\|_{L^2(K)} \big\|U_r^\beta\big\|_{L^2(K)},$$

where $K \subset \Omega$ is a compact containing supports of Φ_1 and Φ_2. In view of the weak convergence of sequences U_r^α in $L^2(K)$ these sequences are bounded in $L^2(K)$. Therefore, for some constant C_K we have $\|U_r^\alpha\|^2_{L^2(K)} \leqslant C_K$ for all $r \in \mathbb{N}$, $\alpha = 1,\dots,N$. Hence,

$$\big|I_r^{\alpha\beta}(\Phi_1,\Phi_2,\psi)\big| \leqslant C_K \|\Phi_1\|_\infty \|\Phi_2\|_\infty \|\psi\|_\infty \tag{A.1}$$

and the sequences $I_r^{\alpha\beta}$ are bounded. Let D be a countable dense set in $(C_0(\Omega))^2 \times C(S_X)$. Using the standard diagonal process, we can extract a subsequence U_r (we keep the notation U_r for this subsequence) such that

$$I_r^{\alpha\beta}(\Phi_1,\Phi_2,\psi) \xrightarrow[r\to\infty]{} I^{\alpha\beta}(\Phi_1,\Phi_2,\psi) \tag{A.2}$$

for all triples $(\Phi_1,\Phi_2,\psi) \in D$. By estimate (A.1) we see that sequences $I_r^{\alpha\beta}(\Phi_1,\Phi_2,\psi)$ are uniformly continuous with respect to $(\Phi_1,\Phi_2,\psi) \in (C_0(\Omega))^2 \times C(S_X)$ and since D is dense in $(C_0(\Omega))^2 \times C(S_X)$, we conclude that limit relation (A.2) holds for all $\Phi_1(x), \Phi_2(x) \in C_0(\Omega)$, $\psi(\xi) \in C(S_X)$. Passing in (A.1) to the limit as $r \to \infty$, we derive that for all $\Phi_1(x), \Phi_2(x) \in C_0(\Omega)$, $\psi(\xi) \in C(S_X)$

$$\big|I^{\alpha\beta}(\Phi_1,\Phi_2,\psi)\big| \leqslant C_K \|\Phi_1\|_\infty \|\Phi_2\|_\infty \|\psi\|_\infty, \tag{A.3}$$

with $K = \operatorname{supp}\Phi_1 \cup \operatorname{supp}\Phi_2$. Now, we observe that

$$I_r^{\alpha\beta}(\Phi_1,\Phi_2,\psi) = \big(\Phi_1 U_r^\alpha, \mathcal{A}\big(\Phi_2 U_r^\beta\big)\big)_2, \tag{A.4}$$

where $\mathcal{A}$ is a pseudo-differential operator on $L^2 = L^2(\mathbb{R}^n)$ with symbol $\overline{\psi(\pi_X(\xi))}$, and $(\cdot,\cdot)_2$ is the scalar product in L^2. Let $\mathcal{B}$ be a pseudo-differential operator on L^2 with symbol $\Phi_2(x)$, and let $\omega(x) \in C_0(\mathbb{R}^n)$ be a function such that $\omega(x) \equiv 1$ on $\operatorname{supp} \Phi_2$. Then

$$\mathcal{A}\big(\Phi_2 U_r^\beta\big) = \mathcal{A}\mathcal{B}\big(\omega U_r^\beta\big) = \mathcal{B}\mathcal{A}\big(\omega U_r^\beta\big) + [\mathcal{A}, \mathcal{B}]\big(\omega U_r^\beta\big). \tag{A.5}$$

By [6, Lemma 2] the operator $[\mathcal{A}, \mathcal{B}]$ is compact on L^2 and since $\omega U_r^\beta \rightharpoonup 0$ as $r \to \infty$ weakly in L^2, we claim that $[\mathcal{A}, \mathcal{B}](\omega U_r^\beta) \to 0$ as $r \to \infty$ strongly in L^2. Since the sequence $\Phi_1 U_r^\alpha$ is bounded in L^2, we conclude that $(\Phi_1 U_r^\alpha, [\mathcal{A}, \mathcal{B}](\omega U_r^\beta))_2 \to 0$ as $r \to \infty$. It follows from this limit relation and (A.4), (A.5) that

$$\lim_{r\to\infty} \big(\Phi_1 U_r^\alpha, \mathcal{B}\mathcal{A}\big(\omega U_r^\beta\big)\big)_2 = \lim_{r\to\infty} I_r^{\alpha\beta}(\Phi_1, \Phi_2, \psi) = I^{\alpha\beta}(\Phi_1, \Phi_2, \psi).$$

Taking into account that

$$\big(\Phi_1 U_r^\alpha, \mathcal{B}\mathcal{A}\big(\omega U_r^\beta\big)\big)_2 = \int_{\mathbb{R}^n} \Phi_1(x)\overline{\Phi_2(x)} U_r^\alpha(x) \overline{\mathcal{A}\big(\omega U_r^\beta\big)(x)}\, dx,$$

we find that

$$I^{\alpha\beta}(\Phi_1, \Phi_2, \psi) = \tilde{I}^{\alpha\beta}(\Phi_1 \overline{\Phi_2}, \psi),$$

where $\tilde{I}^{\alpha\beta}(\Phi, \psi)$ is a bilinear functional on $C_0(\Omega) \times C(S_X)$ for each $\alpha, \beta = 1, \dots, N$. Taking in the above relation $\Phi_1 = \Phi(x)/\sqrt{|\Phi(x)|}$ (we set $\Phi_1(x) = 0$ if $\Phi(x) = 0$), $\Phi_2 = \sqrt{|\Phi(x)|}$, where $\Phi(x) \in C_0(\Omega)$, we find with the help of (A.3) that

$$\begin{aligned} \big|\tilde{I}^{\alpha\beta}(\Phi, \psi)\big| &= \big|I^{\alpha\beta}(\Phi_1, \Phi_2, \psi)\big| \leqslant C_K \|\Phi_1\|_\infty \|\Phi_2\|_\infty \|\psi\|_\infty \\ &= C_K \|\Phi\|_\infty \|\psi\|_\infty, \quad K = \operatorname{supp} \Phi. \end{aligned}$$

This estimate shows that the functionals $\tilde{I}^{\alpha\beta}(\Phi, \psi)$ are continuous on $C_0(\Omega) \times C(S_X)$. Now, we observe that for nonnegative $\Phi(x)$ and $\psi(\xi)$ the matrix $\tilde{I} \doteq \{\tilde{I}^{\alpha\beta}(\Phi, \psi)\}_{\alpha,\beta=1}^N$ is Hermitian and positive definite. Indeed, taking $\Phi_1(x) = \Phi_2(x) = \sqrt{\Phi(x)}$, we find

$$\tilde{I}^{\alpha\beta}(\Phi, \psi) = I^{\alpha\beta}(\Phi_1, \Phi_1, \psi) = \lim_{r\to\infty} \int_{\mathbb{R}^n} F\big(\Phi_1 U_r^\alpha\big)(\xi) \overline{F\big(\Phi_1 U_r^\beta\big)(\xi)} \psi\big(\pi_X(\xi)\big)\, d\xi. \tag{A.6}$$

For $\zeta = (\zeta_1, \dots, \zeta_N) \in \mathbb{C}^N$ we have, in view of (A.6),

$$\tilde{I}\zeta \cdot \zeta = \sum_{\alpha,\beta=1}^N \tilde{I}^{\alpha\beta}(\Phi, \psi)\zeta_\alpha \overline{\zeta_\beta} = \lim_{r\to\infty} \int_{\mathbb{R}^n} \big|F(\Phi_1 V_r)(\xi)\big|^2 \psi\big(\pi_X(\xi)\big)\, d\xi \geqslant 0,$$

where $V_r(x) = \sum_{\alpha=1}^N U_r^\alpha \zeta_\alpha$. The above relation proves that the matrix $\tilde{I}$ is Hermitian and positive definite.

We see that for any $\zeta \in \mathbb{C}^n$ the bilinear functional $\tilde{I}(\Phi, \psi)\zeta \cdot \zeta$ is continuous on $C_0(\Omega) \times C(S_X)$ and nonnegative, that is, $\tilde{I}(\Phi, \psi)\zeta \cdot \zeta \geqslant 0$ whenever $\Phi(x) \geqslant 0$, $\psi(\xi) \geqslant 0$. It is rather well known (see for example [11, Lemma 1.10]), that such a functional is represented by integration over some unique locally finite nonnegative Borel measure $\mu = \mu_\zeta(x, \xi) \in M_{loc}(\Omega \times S_X)$:

$$\tilde{I}(\Phi, \psi)\zeta \cdot \zeta = \int_{\Omega \times S_X} \Phi(x)\psi(\xi)\, d\mu_\zeta(x, \xi).$$

As a function of the vector ζ, μ_ζ is a measure valued Hermitian form. Therefore,

$$\mu_\zeta = \sum_{\alpha,\beta=1}^N \mu^{\alpha\beta} \zeta_\alpha \overline{\zeta_\beta} \tag{A.7}$$

with measure valued coefficients $\mu^{\alpha\beta} \in M_{loc}(\Omega \times S_X)$, which can be expressed as follows

$$\mu^{\alpha\beta} = [\mu_{e_\alpha + e_\beta} + i\mu_{e_\alpha + ie_\beta}]/2 - (1 + i)(\mu_{e_\alpha} + \mu_{e_\beta})/2,$$

where $e_1, \dots, e_N$ is the standard basis in $\mathbb{C}^N$, and $i^2 = -1$.

By (A.7)

$$\tilde{I}(\Phi,\psi)\zeta\cdot\zeta=\sum_{\alpha,\beta=1}^{l}\langle\mu^{\alpha\beta},\Phi(x)\psi(\xi)\rangle\zeta_\alpha\overline{\zeta_\beta}$$

and since

$$\tilde{I}(\Phi,\psi)\zeta\cdot\zeta=\sum_{\alpha,\beta=1}^{l}\tilde{I}^{\alpha\beta}(\Phi,\psi)\zeta_\alpha\overline{\zeta_\beta},$$

then, comparing the coefficients, we find that

$$\langle\mu^{\alpha\beta},\Phi(x)\psi(\xi)\rangle=\tilde{I}^{\alpha\beta}(\Phi,\psi). \tag{A.8}$$

In particular,

$$\langle\mu^{\alpha\beta},\Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\rangle=I^{\alpha\beta}(\Phi_1,\Phi_2,\psi)=\lim_{r\to\infty}\int_{\mathbb{R}^n}F\big(\Phi_1U_r^\alpha\big)(\xi)\overline{F\big(\Phi_2U_r^\beta\big)(\xi)}\psi\big(\pi_X(\xi)\big)\,d\xi.$$

To complete the proof, observe that for each $\zeta\in\mathbb{C}^N$ the measure

$$\sum_{\alpha,\beta=1}^{N}\mu^{\alpha\beta}\zeta_\alpha\overline{\zeta_\beta}=\mu_\zeta\geqslant 0.$$

Hence, μ is Hermitian and positive definite.

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