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Ann. I. H. Poincaré – AN 28 (2011) 63–74
ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

Monotonicity constraints and supercritical Neumann problems

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Received 22 April 2010; received in revised form 24 September 2010; accepted 24 September 2010

Available online 20 October 2010

Abstract

We prove the existence of a positive and radially increasing solution for a semilinear Neumann problem on a ball. No growth conditions are imposed on the nonlinearity. The method introduces monotonicity constraints which simplify the existence of a minimizer for the associated functional. Special care must be employed to establish the validity of the Euler equation.

Résumé

On démontre l'existence d'une solution positive et radialement croissante pour un problème de Neumann semilinéaire sur une boule. Aucune restriction de croissance n'est imposée sur la nonlinéarité. La méthode indroduit des contraintes de monotonie qui simplifient la preuve de l'existence d'un minimum pour la fonctionnelle associée à l'équation. Une attention particulière est consacrée à la preuve de la validité de l'équation d'Euler.
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MSC: 35J60; 58E30

Keywords: Constrained variational problems; Supercritical elliptic equations; Neumann problem

1. Introduction and statement of the main result

In this paper we consider the Neumann problem

$$\begin{cases} -\Delta u + u = a\big(|x|\big) f(u) & \text{in } B_R, \\ u > 0 & \text{in } B_R, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial B_R, \end{cases} \tag{1}$$

where B_R is the ball of radius R centered at zero in $\mathbf{R}^N$, with $N \geqslant 2$.

The function f is assumed to behave superlinearly, and our principal scope is to prove, under qualitative assumptions on a, an existence result without any restriction on the growth of f. This generality usually prevents the use of

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[1] The author was supported by MIUR Project "Variational Methods and Nonlinear Differential Equations".

0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2010.10.003

variational methods; nevertheless we will show that the problem can still be treated variationally, and, more precisely, that a solution can be obtained by a variational principle on the set of radially *increasing* functions.

In trying to prove existence results for nonlinear elliptic problems, it is common wisdom to exploit the characteristic features of the equation under study to gain useful properties. For example, if a problem (equation and boundary conditions) is invariant under rotation, it is quite natural, as a first try, to work in the space of radially symmetric functions. Then the problem becomes essentially one-dimensional and, for instance, compactness properties are not only much easier to establish, but often hold under less restrictive assumptions. In this case everything works fine in the variational approach because of the Principle of Symmetric Criticality by Palais [8]. This means, loosely speaking, that constraining the problem to the space of radial functions does not prevent one to establish the validity of the Euler equation. One also says that this type of constraint is a *natural constraint*: constrained critical points (of the energy functional) are indeed free critical points.

This sort of arguments work generally well when the problem is invariant under the action of some group of symmetries, but may fail to be applicable in more general situations. For example, in some cases it is natural to expect a solution enjoying a certain property, but one cannot work directly in the set of functions satisfying that property. The reason for this is that the set to which one constrains the problem may not contain enough functions to prove that the differential of the energy functional vanishes at a supposedly "critical" point.

Problems like (1) have been studied extensively in the literature, and particularly in the case $a \equiv 1$ and $f(u) = u^{\frac{N+2}{N-2}}$, see for example [5,6,1,2,7,10] and the rich list of references provided by these papers. In contrast, nonautonomous problems, where a is not constant, and even with power nonlinearities, do not seem to have been deeply investigated in the supercritical case (with the exception of singularly perturbed problems, see for example [4] and the references therein). The only nonperturbative result that we are aware of is that of [3], which is concerned with the Neumann problem for the Hénon equation

$$-\Delta u + u = |x|^\alpha u^p.$$

In [3] the authors prove, by a shooting method, that the problem has a positive and radially *increasing* solution for every $p > 1$ and $\alpha > 0$. The proof of the existence result in [3] makes great use of the fact that the function $r \mapsto r^\alpha$ vanishes at $r = 0$. Also the particularity that all the functions (of r and u) involved are powers plays a relevant role.

If one neglects the technical aspects, the proof in [3] suggests that if the function $|x|^\alpha$ is replaced by some $a(|x|)$, *increasing* in $|x|$, and if the power nonlinearity is substituted by some more general $f(u)$, the qualitative structure of the problem should not be affected too much. In particular under some reasonable assumptions of this kind, one is led to conjecture that problem (1) should admit an increasing solution.

The central point addressed by the present paper is the following. In absence of growth conditions on f, the variational approach seems useless, since the functional associated to the problem is not well defined on $H^1(B_R)$, and not even on $H^1_{rad}(B_R)$. But we are looking for radially increasing solutions. If we could confine ourselves to radially increasing functions only, then the action functional would turn out to be well defined on $H^1_{rad}(B_R)$, because nonnegative radially increasing H^1 functions are bounded. In this case it could be possible to recast the problem into a variational framework, which is so useful in dealing with subcritical equations. However, a major obstacle has to be faced: are there enough radially increasing functions to prove that the differential of the action functional, say at a constrained minimum point, vanishes? At first sight one is tempted to answer this question in the negative. On the contrary, we shall see that under suitable conditions, a variational approach in the set of radially increasing functions is possible, and does indeed lead to a solution of (1).

The precise setting is the following. Throughout the paper we make the following assumptions:

(H1) $a \in L^1(0, R)$ is increasing, not constant and $a(r) > 0$ a.e. in $[0, R]$;
(H2) $f \in C^1([0, +\infty))$, $f(0) = 0$ and $f'(0) = 0$;
(H3) $f'(t)t - f(t) > 0$ for all $t > 0$;
(H4) there exists $\mu > 2$ such that $f(t)t \geqslant \mu F(t) := \mu \int_0^t f(s)\,ds$ for all $t \in [0, +\infty)$.

Assumptions (H2)–(H4) are quite standard when one deals with elliptic problems from a variational point of view (though they can still be slightly relaxed). The point here is the absence of any upper bound on the growth of f.

Our main result is the following.

Theorem 1.1. *Assume that* (H1)–(H4) *hold. Then problem* (1) *admits at least one radially increasing solution.*

Remark 1.2. We have stated the result for nonlinearities of the form $a(|x|)f(u)$ mainly because the assumptions can be expressed in a very simple form. It is not difficult to show that the same result holds for nonlinearities like $f(|x|, u)$, provided (H1)–(H4) are suitably adapted.

Power nonlinearities satisfy (H2)–(H4). Therefore we obtain directly from the preceding theorem the following corollary.

Corollary 1.3. *Assume that $a \in L^1(0, R)$ is increasing, not constant and satisfies $a(r) > 0$ a.e. in $[0, R]$. Then for every $p > 1$ the problem*

$$\begin{cases} -\Delta u + u = a(|x|)u^p & \text{in } B_R, \\ u > 0 & \text{in } B_R, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial B_R, \end{cases} \tag{2}$$

admits at least one radially increasing solution.

We point out that, apart from the existence result, we believe that the *method* by which the result is proved is quite interesting, and should have a wider range of applicability than that shown here. In particular, variational methods on spaces of monotone functions do not seem to be a standard tool: the only example we are aware of, though in a different context, is the paper [9]. Some extensions to other types of equations will be the object of further research.

Notation. Open balls of center zero and radius r in $\mathbf{R}^N$ are denoted by B_r. For any radial function u, we write freely $u(x)$, with $x \in \overline{B}_R$, or $u(r)$, with $r \in [0, R]$. The symbol $\|u\|$ denotes the standard $H^1(B_R)$ norm, while $\|u\|_p$, with $p \in [1, +\infty]$, stands for the L^p norm on B_R. Finally, the scalar product in $H^1(B_R)$ is denoted by $\langle u, v\rangle$.

2. Minimizing over monotone functions

We begin by listing, for further reference, some properties of f and F that we will use frequently throughout the paper, and that can be deduced directly from (H2)–(H4).

The functions f and F are strictly increasing on $[0, +\infty)$ and positive on $(0, +\infty)$. As $t \to 0^+$, $f(t) = o(t)$ and $F(t) = o(t^2)$. Finally, there exists a constant $C > 0$ such that

$$f(t)t \geqslant Ct^\mu \quad \text{and} \quad F(t) \geqslant Ct^\mu \quad \forall t \geqslant 1. \tag{3}$$

Our ambient set is that of radially increasing functions. We define

$$M = \{u \in H^1_{rad}(B_R) \mid u(r) \geqslant 0,\ u(r) \leqslant u(s) \text{ for every } r, s \in [0, R],\ r \leqslant s\}.$$

Remark 2.1. Since we are working with radial functions in $H^1(B_R)$, we can assume that any function u is continuous on $(0, R]$, and since u is increasing and nonnegative, continuity can be guaranteed also at zero by defining $u(0) = \lim_{r\to 0^+} u(r)$, and we will tacitly use this fact throughout the paper.

The main advantage of working in the set M is that concentration phenomena are prevented, as is pointed out in the next statement.

Lemma 2.2. *There exists a positive constant C, depending only on R and on the dimension, such that*

$$\|u\|_\infty \leqslant C\|u\| \quad \forall u \in M. \tag{4}$$

Proof. Fix $\rho \in (0, R)$. Since any $u \in M$ is nonnegative and increasing, we have

$$\|u\|_\infty = \|u\|_{L^\infty(B_R\setminus\overline{B}_\rho)} \leqslant C\|u\|_{H^1(B_R\setminus\overline{B}_\rho)} \leqslant C\|u\|,$$

by the continuity of the embedding of $H^1_{rad}(B_R \setminus \overline{B}_\rho)$ into $L^\infty(B_R \setminus \overline{B}_\rho)$. □

We now define a functional $I : M \to \mathbf{R}$ by

$$I(u) = \frac{1}{2}\int_{B_R} |\nabla u|^2\,dx + \frac{1}{2}\int_{B_R} u^2\,dx - \int_{B_R} a\big(|x|\big)F(u)\,dx$$
$$= \frac{1}{2}\|u\|^2 - \int_{B_R} a\big(|x|\big)F(u)\,dx.$$

Note that the last integral is well defined on M because $M \subset L^\infty(B_R)$. Clearly, working on M allows one to define I without growth conditions on f; on the other hand, it is by no means clear why constrained minimizer should solve the Euler equation (1).

In general, the functional I will not be bounded from below on M. To overcome this difficulty we use a standard procedure in Critical Point Theory, that consists in restricting I to a set where it becomes bounded from below. The set we use is the following version of "Nehari manifold": we define

$$\mathcal{N} = \left\{ u \in M \;\middle|\; u \not\equiv 0,\ \|u\|^2 = \int_{B_R} a\big(|x|\big)f(u)u\,dx \right\}.$$

This set has been used quite frequently in the literature. Under standard assumptions it can be proved that it is a manifold diffeomorphic to the unit sphere of H^1. In the present case some of these standard assumptions are missing, and are substituted by others. Moreover, since we are working on M, the set $\mathcal{N}$ does not look like the sphere of $H^1_{rad}(B_R)$. Hence, for completeness, we prove all the properties that we need.

Lemma 2.3. *The set $\mathcal{N}$ is not empty. In particular, for every $u \in M$, $u \not\equiv 0$, there exists $t > 0$ such that $tu \in \mathcal{N}$.*

Proof. Let $u \in M$, $u \not\equiv 0$. We show that there exists $t > 0$ such that $tu \in \mathcal{N}$, namely such that

$$t^2\|u\|^2 - \int_{B_R} a\big(|x|\big)f(tu)tu\,dx = 0. \tag{5}$$

Let $\gamma : [0, +\infty) \to \mathbf{R}$ be the function of t defined by the left-hand side of (5). We have to prove that γ vanishes at some positive t.

Now $f(t) = o(t)$ as $t \to 0^+$, and $u \in L^\infty(B_R)$. Therefore, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|tu(x)| \leqslant \delta$ implies $0 \leqslant f(tu(x)) \leqslant \varepsilon tu(x)$.

Hence, for every $0 \leqslant t \leqslant \frac{\delta}{\|u\|_\infty}$,

$$0 \leqslant \int_{B_R} a\big(|x|\big)f(tu)tu\,dx \leqslant \varepsilon t^2 \int_{B_R} a\big(|x|\big)u^2\,dx \leqslant \varepsilon t^2\|u\|_\infty^2\|a\|_1 = C\varepsilon t^2.$$

Therefore,

$$\gamma(t) \geqslant t^2\|u\|^2 - C\varepsilon t^2 > 0$$

for t and ε positive and small. We have proved that γ is positive in a right neighborhood of zero.

Next, since $u \geqslant 0$ and $u \not\equiv 0$, there exist $\alpha > 0$ and a set $E \subset B_R$ of positive measure such that $u(x) \geqslant \alpha$ for all $x \in E$. By (3), for $t \geqslant 1/\alpha$,

$$\int_{B_R} a\big(|x|\big)f(tu)tu\,dx \geqslant Ct^\mu \int_E a\big(|x|\big)u^\mu\,dx,$$

which is finite because $u \in L^\infty(B_R)$. Then,

$$\gamma(t) \leqslant t^2\|u\|^2 - Ct^\mu \int_E a\big(|x|\big)u^\mu\,dx \to -\infty$$

as $t \to +\infty$, because $\mu > 2$.

The function γ, being continuous, must then have a positive zero t_0. This means that $t_0 u \in \mathcal{N}$. $\square$

We now define

$$c = \inf_{u \in \mathcal{N}} I(u),$$

and we prove that c is a positive number. Note that on $\mathcal{N}$, the functional I reads

$$I(u) = \frac{1}{2} \int_{B_R} a(|x|) f(u) u \, dx - \int_{B_R} a(|x|) F(u) \, dx.$$

Lemma 2.4. *There results $c > 0$.*

Proof. We first prove that

$$\inf_{u \in \mathcal{N}} \|u\|^2 > 0. \tag{6}$$

To see this, we assume that there exists a sequence u_n of elements of $\mathcal{N}$ such that $\|u_n\| \to 0$ as $n \to \infty$, and we seek a contradiction.

Letting C be the constant defined in (4), we have from Lemma 2.2

$$\begin{aligned} \|u_n\|^2 = \int_{B_R} a(|x|) f(u_n) u_n \, dx &\leqslant f(\|u_n\|_\infty) \|u_n\|_\infty \int_{B_R} a(|x|) \, dx \\ &\leqslant f(C\|u_n\|) C \|u_n\| \|a\|_1, \end{aligned}$$

and hence, since $f(t) = o(t)$ as $t \to 0$, from $\|u_n\| \to 0$ we obtain a contradiction. Therefore $\|u_n\|^2$ is bounded away from zero.

Now we turn to the level c. For every $u \in \mathcal{N}$ we have, using (H4),

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 - \int_{B_R} a(|x|) F(u) \, dx \\ &= \frac{1}{2}\|u\|^2 - \frac{1}{\mu}\|u\|^2 + \frac{1}{\mu}\|u\|^2 - \int_{B_R} a(|x|) F(u) \, dx \\ &= \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u\|^2 + \frac{1}{\mu} \int_{B_R} a(|x|) \big(f(u)u - \mu F(u)\big) \, dx \\ &\geqslant \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u\|^2. \end{aligned} \tag{7}$$

By (6), this concludes the proof. $\square$

We can now show that I has a minimizer on $\mathcal{N}$.

Lemma 2.5. *The level c is attained*: *there exists $u \in \mathcal{N}$ such that $I(u) = c$.*

Proof. Let u_n be a minimizing sequence for I on $\mathcal{N}$. By (7), we see that I is coercive on $\mathcal{N}$, and hence the sequence u_n is bounded in $H^1(B_R)$. Then, up to subsequences, we can assume that for a suitable $u \in H^1(B_R)$

- $u_n \rightharpoonup u$ in $H^1(B_R)$,
- $u_n \to u$ pointwise almost everywhere in B_R,

and since $u_n \in M$, we clearly have that also $u \in M$. Moreover, by Lemma 2.2, we see that the norms $\|u_n\|_\infty$ are uniformly bounded. Hence, by dominated convergence, as $n \to \infty$,

$$c + o(1) = I(u_n) = \frac{1}{2}\int_{B_R} a(|x|) f(u_n) u_n \, dx - \int_{B_R} a(|x|) F(u_n)\, dx$$
$$= \frac{1}{2}\int_{B_R} a(|x|) f(u) u \, dx - \int_{B_R} a(|x|) F(u)\, dx + o(1). \tag{8}$$

Since $c > 0$, this shows that $u \not\equiv 0$, so that $u \in M \setminus \{0\}$.

Now if $u \in \mathcal{N}$, then

$$\int_{B_R} a(|x|) f(u) u \, dx = \|u\|^2,$$

and the proof is complete, since the last equation shows that $I(u) = c$.

If, on the other hand, $u \notin \mathcal{N}$, then weak lower semicontinuity shows that necessarily

$$\|u\|^2 < \int_{B_R} a(|x|) f(u) u \, dx. \tag{9}$$

We now prove that this cannot hold, so that u does indeed belong to $\mathcal{N}$. We consider again the function $\gamma : [0, +\infty) \to \mathbf{R}$ defined by

$$\gamma(t) = t^2 \|u\|^2 - \int_{B_R} a(|x|) f(tu) tu \, dx.$$

Since $u \not\equiv 0$, we certainly have $\gamma(t) > 0$ for t positive and small, as in Lemma 2.3. Moreover, by (9), we see that $\gamma(1) < 0$, so that γ must have a zero $t_0 \in (0, 1)$.

Then the function $t_0 u$ is in $\mathcal{N}$. By (H3), the function

$$t \mapsto \frac{1}{2} f(t) t - F(t)$$

is strictly increasing for $t \geqslant 0$. Therefore, since $t_0 < 1$,

$$c \leqslant I(t_0 u) = \int_{B_R} a(|x|) \left(\frac{1}{2} f(t_0 u) t_0 u - F(t_0 u) \right) dx$$
$$< \int_{B_R} a(|x|) \left(\frac{1}{2} f(u) u - F(u) \right) dx = c,$$

by (8). This shows that (9) cannot hold, and therefore $u \in \mathcal{N}$ and is the required minimizer. □

3. The Euler equation

In the preceding section we have found a minimizer for I on $\mathcal{N}$. The fact that this minimizer solves the Euler equation is not obvious at all, since a priori there are not enough "test" functions to show that the first variation of I vanishes on $H^1_{rad}(B_R)$. This happens because we are working with increasing functions only. To be more precise, let $u \in \mathcal{N}$ be the minimizer found above. If u satisfies an estimate like $u'(r) \geqslant \delta > 0$ for all r in some interval $(a, b) \subset (0, R)$, then for every $\varphi \in C_0^\infty(a, b)$, the functions $u + \varepsilon\varphi$ (after radial projection on $\mathcal{N}$) are admissible variations when ε is small enough, and this is enough to show that u weakly solves the Euler equation. However, we do not have for the moment any lower bound on u', and in principle u might even be constant on certain subintervals of $(0, R)$. On these intervals, there would be little hope to prove that the Euler equation is satisfied.

We now show how these problems can be overcome. We begin with a necessary condition for minimizers.

Lemma 3.1. *Let u be a minimizer for I on $\mathcal{N}$. Assume that $v \in H^1_{rad}(B_R)$ (not necessarily in M) is such that $u + sv \in M$ for every s in some interval $[0, \varepsilon)$. Then*

$$\int_{B_R} \nabla u \nabla v \, dx + \int_{B_R} uv \, dx \geqslant \int_{B_R} a(|x|) f(u) v \, dx. \tag{10}$$

Proof. Since u does not vanish identically, the same holds for $u + sv$ for all s small enough, and we can then assume that this happens for every $s \in [0, \varepsilon)$.

Each function $u + sv$ is in M, and hence, by Lemma 2.3, for every $s \in [0, \varepsilon)$ there exists $t = t(s)$ such that

$$t(s)(u + sv) \in \mathcal{N}.$$

We now study some properties of $t(s)$.

To this aim we define a function $G : [0, \varepsilon) \times \mathbf{R} \to \mathbf{R}$ by

$$G(s, t) = t^2 \|u + sv\|^2 - \int_{B_R} a(|x|) f\big(t(u + sv)\big) t(u + sv) \, dx.$$

We have

$$G(0, 1) = \|u\|^2 - \int_{B_R} a(|x|) f(u) u \, dx = 0$$

because $u \in \mathcal{N}$. Next, G is of class C^1 and a simple computation, together with the fact that $u \in \mathcal{N}$, yields

$$\begin{aligned} \frac{\partial G}{\partial t}(0, 1) &= 2\|u\|^2 - \int_{B_R} a(|x|) f'(u) u^2 \, dx - \int_{B_R} a(|x|) f(u) u \, dx \\ &= \int_{B_R} a(|x|) f(u) u \, dx - \int_{B_R} a(|x|) f'(u) u^2 \, dx \\ &= \int_{B_R} a(|x|) u \big(f(u) - f'(u) u\big) \, dx < 0 \end{aligned}$$

because of (H3).

Then, a simple variant of the Implicit Function Theorem shows that there exist $\delta > 0$ and a function $t : [0, \delta) \to \mathbf{R}$ of class C^1 such that

$$G\big(s, t(s)\big) = 0 \quad \forall s \in [0, \delta),$$

and $t(0) = 1$. This means that

$$t(s)(u + sv) \in \mathcal{N} \quad \forall s \in [0, \delta).$$

Now define $H : [0, \delta) \to \mathbf{R}$ as

$$\begin{aligned} H(s) &= I\big(t(s)(u + sv)\big) \\ &= \frac{1}{2} t(s)^2 \|u + sv\|^2 - \int_{B_R} a(|x|) F\big(t(s)(u + sv)\big) \, dx. \end{aligned}$$

By construction, h is C^1 and has a local minimum at 0, so that $H'(0) \geqslant 0$.

Hence,

$$\begin{aligned} 0 \leqslant H'(0) &= t'(0)\|u\|^2 + \langle u, v\rangle - \int_{B_R} a(|x|) f(u) t'(0) u \, dx - \int_{B_R} a(|x|) f(u) v \, dx \\ &= t'(0)\left(\|u\|^2 - \int_{B_R} a(|x|) f(u) u \, dx\right) + \langle u, v\rangle - \int_{B_R} a(|x|) f(u) v \, dx. \end{aligned}$$

Since $u \in \mathcal{N}$, the first term in the last line vanishes, so that

$$\langle u, v\rangle \geqslant \int_{B_R} a(|x|) f(u) v \, dx,$$

and (10) is proved. □

Our scope is to show that (10) holds for every $v \in H^1_{rad}(B_R)$. The first step consists in gathering some more information about the minimizer u.

To this aim, consider the problem

$$\begin{cases} -\Delta w + w = 0 & \text{in } B_R, \\ w = u & \text{on } \partial B_R. \end{cases} \tag{11}$$

It is well known that this problem has a unique solution φ, which is radial, strictly positive on B_R, and strictly radially increasing. The solution φ can be found for instance by solving

$$\min\{\|w\|^2 \mid w \in H^1(B_R),\ w_{|\partial B_R} = u_{|\partial B_R}\}$$

and its properties follow by a standard application of the strong maximum principle. Actually, problem (11) can be explicitly solved in terms of modified Bessel functions.

The function φ serves as a lower bound for the minimizer u.

Lemma 3.2. *Let u be a minimizer for I on $\mathcal{N}$, and let φ be the solution of* (11). *Then*

$$u(x) \geqslant \varphi(x) \quad \forall x \in \overline{B}_R. \tag{12}$$

In particular, u is strictly positive in $\overline{B}_R$.

Proof. We first note that for every $s \in [0, 1]$, the function

$$u + s(\varphi - u)^+$$

is in M. Indeed, $u(x) + s(\varphi(x) - u(x))^+ = \max(u(x), s\varphi(x) + (1 - s)u(x))$, and all the functions involved are radially increasing.

By Lemma 3.1,

$$\int_{B_R} \nabla u \nabla(\varphi - u)^+ \, dx + \int_{B_R} u(\varphi - u)^+ \, dx \geqslant \int_{B_R} a(|x|) f(u)(\varphi - u)^+ \, dx \geqslant 0. \tag{13}$$

Next, the definition of φ shows that $(\varphi - u)^+ \in H^1_0(B_R)$, and since φ solves problem (11),

$$\int_{B_R} \nabla\varphi \nabla(\varphi - u)^+ \, dx + \int_{B_R} \varphi(\varphi - u)^+ \, dx = 0.$$

Subtracting this from (13) we obtain

$$0 \leqslant \int_{B_R} \nabla(u - \varphi)\nabla(\varphi - u)^+ \, dx + \int_{B_R} (u - \varphi)(\varphi - u)^+ \, dx = -\|(\varphi - u)^+\|^2,$$

which means that $u \geqslant \varphi$ on $\overline{B}_R$. □

The next result is crucial in order to prove that u solves the Euler equation. In its statement and proof, we recall that we write freely $u(x)$, with $x \in \overline{B}_R$, or $u(r)$, with $r \in [0, R]$, depending on the convenience of the moment, and similarly for all radial functions. The context will always rule out any ambiguity.

Proposition 3.3. *Let u be a minimizer for I on $\mathcal{N}$. Then*

$$u'(r) > 0 \quad \text{a.e. in } (0, R). \tag{14}$$

Remark 3.4. The point here is the *strict* inequality. The function u, being increasing, obviously satisfies $u'(r) \geqslant 0$ wherever $u'(r)$ is defined.

Proof. Fix $\rho \in (0, R)$ such that ρ is a Lebesgue point for $u'(r)r^{N-1}$ and let λ be a *positive* real number to be determined later. Finally, let δ be a small positive number that will eventually tend to zero.

Now consider the piecewise linear function $v_\delta : [0, R] \to \mathbf{R}$ defined as

$$v_\delta(r) = \begin{cases} -1 & \text{if } r \in [0, \rho], \\ -1 + \frac{1+\lambda}{\delta}(r - \rho) & \text{if } r \in [\rho, \rho + \delta], \\ \lambda & \text{if } r \in [\rho + \delta, R]. \end{cases}$$

Clearly, the function $x \mapsto v_\delta(|x|)$, which we call simply v_δ, is in $H^1_{rad}(B_R)$.

The function v_δ is increasing, but is not in M, since it is not positive. However, for every nonnegative and small s, the function $u + sv_\delta$ belongs to M, because of Lemma 3.2.

Then, by Lemma 3.1,

$$\int_{B_R} \nabla u \nabla v_\delta \, dx + \int_{B_R} u v_\delta \, dx \geqslant \int_{B_R} a\big(|x|\big) f(u) v_\delta \, dx. \tag{15}$$

When $\delta \to 0^+$, the functions v_δ tend to

$$v_0(r) = \begin{cases} -1 & \text{if } r \in [0, \rho], \\ \lambda & \text{if } r \in (\rho, R], \end{cases}$$

pointwise and in every $L^p(B_R)$ with p finite.

Hence, as $\delta \to 0^+$,

$$\int_{B_R} u v_\delta \, dx \to -\int_{B_\rho} u \, dx + \lambda \int_{B_R \setminus B_\rho} u \, dx$$

and, similarly,

$$\int_{B_R} a\big(|x|\big) f(u) v_\delta \, dx \to -\int_{B_\rho} a\big(|x|\big) f(u) \, dx + \lambda \int_{B_R \setminus B_\rho} a\big(|x|\big) f(u) \, dx.$$

Furthermore, denoting by σ the area of the unit sphere in $\mathbf{R}^N$,

$$\int_{B_R} \nabla u \nabla v_\delta \, dx = \sigma \int_0^R u'(r) v_\delta'(r) r^{N-1} \, dr = \sigma \frac{1+\lambda}{\delta} \int_\rho^{\rho+\delta} u'(r) r^{N-1} \, dr.$$

Since ρ is a Lebesgue point for $u'(r)r^{N-1}$, as $\delta \to 0^+$ we have

$$\int_{B_R} \nabla u \nabla v_\delta \, dx \to \sigma(1 + \lambda)\rho^{N-1} u'(\rho).$$

We can now let $\delta \to 0^+$ in (15), to conclude that for almost every $\rho \in (0, R)$,

$$\sigma(1+\lambda)\rho^{N-1}u'(\rho) - \int_{B_\rho} u \, dx + \lambda \int_{B_R \setminus B_\rho} u \, dx \geqslant -\int_{B_\rho} a\big(|x|\big) f(u) \, dx + \lambda \int_{B_R \setminus B_\rho} a\big(|x|\big) f(u) \, dx,$$

which we write as

$$\begin{aligned} (1+\lambda)\rho^{N-1}u'(\rho) \geqslant{} & \int_0^\rho u r^{N-1} \, dr - \lambda \int_\rho^R u r^{N-1} \, dr \\ & - \int_0^\rho a(r) f(u) r^{N-1} \, dr + \lambda \int_\rho^R a(r) f(u) r^{N-1} \, dr. \end{aligned} \tag{16}$$

Now the functions a and u are increasing, and the function $t \mapsto \frac{f(t)}{t}$ is strictly increasing by (H3). All these functions are nonnegative. Therefore

$$a(r) f\big(u(r)\big) = a(r)\frac{f(u(r))}{u(r)}u(r) \leqslant a(\rho)\frac{f(u(\rho))}{u(\rho)}u(r) \quad \text{for a.e. } r \in [0, \rho],$$

while

$$a(r) f\big(u(r)\big) = a(r)\frac{f(u(r))}{u(r)}u(r) \geqslant a(\rho)\frac{f(u(\rho))}{u(\rho)}u(r) \quad \text{for a.e. } r \in [\rho, R].$$

Hence we obtain

$$-\int_0^\rho a(r) f(u) r^{N-1}\,dr \geqslant -a(\rho)\frac{f(u(\rho))}{u(\rho)}\int_0^\rho u(r) r^{N-1}\,dr \tag{17}$$

and

$$\lambda\int_\rho^R a(r) f(u) r^{N-1}\,dr \geqslant \lambda a(\rho)\frac{f(u(\rho))}{u(\rho)}\int_\rho^R u(r) r^{N-1}\,dr. \tag{18}$$

The key remark now is that since the function a is not constant, at least one of the two preceding inequalities must be strict. Indeed, if they were both equalities, then we should have

$$a(r)\frac{f(u(r))}{u(r)} = a(\rho)\frac{f(u(\rho))}{u(\rho)} \quad \text{for almost every } r \in [0, R].$$

If u is not constant, this is impossible because $f(t)/t$ is strictly increasing. If u is constant, then a must be constant too, against assumption (H1).

Inserting (17) and (18) into (16), and keeping in mind the preceding remark we see that for almost every $\rho \in (0, R)$,

$$(1+\lambda)\rho^{N-1}u'(\rho) > \int_0^\rho u r^{N-1}\,dr - \lambda\int_\rho^R u r^{N-1}\,dr + a(\rho)\frac{f(u(\rho))}{u(\rho)}\left(-\int_0^\rho u r^{N-1}\,dr + \lambda\int_\rho^R u r^{N-1}\,dr\right). \tag{19}$$

We now choose $\lambda > 0$ such that

$$\int_0^\rho u r^{N-1}\,dr = \lambda\int_\rho^R u r^{N-1}\,dr,$$

which is possible since both integrals are positive, by Lemma 3.2. Then (19) becomes

$$(1+\lambda)\rho^{N-1}u'(\rho) > 0 \quad \text{for almost every } \rho \in (0, R).$$

Since λ is positive, this is the required inequality. □

We now show that the estimate provided by the preceding proposition is enough to show that the "first variation" of the functional I vanishes on a dense subset of $H^1_{rad}(B_R)$.

Proposition 3.5. *Let u be a minimizer for I on $\mathcal{N}$. Then*

$$\int_{B_R} \nabla u \nabla v\,dx + \int_{B_R} uv\,dx = \int_{B_R} a\big(|x|\big) f(u) v\,dx \quad \textit{for every } v \in C^1_{rad}(\overline{B}_R).$$

Proof. Let

$$E_k = \left\{ r \in (0, R) \;\middle|\; u'(r) \text{ exists and } u'(r) > \frac{1}{k} \right\}, \quad k = 1, 2, \ldots$$

and note that

$$\{r \in (0, R) \mid u'(r) \text{ exists and } u'(r) > 0\} = \bigcup_k E_k, \quad \operatorname{meas}\Bigl([0, R] \setminus \bigcup_k E_k\Bigr) = 0 \tag{20}$$

by the preceding proposition.

Denote by χ_{E_k} be the characteristic function of the set E_k.

Now take any $v \in C^1_{rad}(\overline{B}_R)$, and define, for every $k = 1, 2, \ldots$, a function $v_k : [0, R] \to \mathbf{R}$ by

$$v_k(r) = v(0) + \int_0^r v'(s)\chi_{E_k}(s)\, ds.$$

Of course, $v_k \in W^{1,\infty}(0, R)$ and

$$v_k'(r) = v'(r)\chi_{E_k}(r)$$

for almost every $r \in [0, R]$.

Now for a fixed k, we let $\varepsilon \in \mathbf{R}$ be such that

$$|\varepsilon| < \min\left(\frac{u(0)}{1 + \|v\|_\infty}, \frac{1}{k(1 + \|v'\|_\infty)}\right) \tag{21}$$

and for such ε's we claim that

$$u + \varepsilon v_k \in M \setminus \{0\}.$$

To see this, we use (21): we first note that for every $r \in [0, R]$,

$$u(r) + \varepsilon v_k(r) \geqslant u(0) - |\varepsilon|\|v\|_\infty > u(0) - \frac{\|v\|_\infty}{1 + \|v\|_\infty} u(0) > 0.$$

Next, for almost every $r \in (0, R)$, we have

$$(u + \varepsilon v_k)'(r) = u'(r) + \varepsilon v'(r)\chi_{E_k}(r).$$

If $r \in E_k$, then

$$(u + \varepsilon v_k)'(r) = u'(r) + \varepsilon v'(r)\chi_{E_k}(r) = u'(r) + \varepsilon v'(r) > \frac{1}{k} - |\varepsilon|\|v'\|_\infty > \frac{1}{k} - \frac{\|v'\|_\infty}{k(1 + \|v'\|_\infty)} > 0.$$

If $r \notin E_k$, then

$$(u + \varepsilon v_k)'(r) = u'(r),$$

and we know from Proposition 3.3, that $u'(r) > 0$ almost everywhere.

It follows that $(u + \varepsilon v_k)'(r) > 0$ for almost every $r \in (0, R)$, and then the function $u + \varepsilon v_k$ is also increasing.

Hence, by Lemma 3.1,

$$\int_0^R u' v_k' r^{N-1}\, dr + \int_0^R u v_k r^{N-1}\, dr \geqslant \int_0^R a(r) f(u) v_k r^{N-1}\, dr, \quad \forall k = 1, 2, \ldots. \tag{22}$$

We now let $k \to \infty$. To this aim we note that

$$\bigl|v_k(r)\bigr| \leqslant \bigl|v(0)\bigr| + \int_0^r \bigl|v'(s)\bigr|\bigl|\chi_{E_k}(s)\bigr|\, ds \leqslant \bigl|v(0)\bigr| + R\bigl\|v'\bigr\|_\infty$$

and $\|v_k'\|_\infty \leqslant \|v'\|_\infty$. By the Ascoli–Arzelà Theorem, $v_k \to v$ uniformly on $[0, R]$. Moreover, since $E_k \subset E_{k+1}$ for every k, and by (20),

$$v_k'(r) \to v'(r) \quad \text{a.e. in } (0, R).$$

By dominated convergence, we can let $k \to \infty$ in (22) to obtain

$$\int_0^R u'v'r^{N-1}\,dr + \int_0^R uvr^{N-1}\,dr \geqslant \int_0^R a(r)f(u)vr^{N-1}\,dr.$$

Since v is arbitrary, this means

$$\int_{B_R} \nabla u \nabla v\,dx + \int_{B_R} uv\,dx = \int_{B_R} a\big(|x|\big)f(u)v\,dx \quad \text{for every } v \in C^1_{rad}(\overline{B}_R),$$

and the proof is complete. □

End of the proof of Theorem 1.1. By the previous result one easily deduces, with a density argument, that

$$\int_{B_R} \nabla u \nabla v\,dx + \int_{B_R} uv\,dx = \int_{B_R} a\big(|x|\big)f(u)v\,dx \quad \text{for every } v \in H^1_{rad}(B_R),$$

namely that $u \in M$ weakly solves problem (1). □

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