Available online at www.sciencedirect.com
ScienceDirect
Ann. I. H. Poincaré – AN 28 (2011) 75–90
ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

Finite-dimensionality in Tanaka theory

Boris Kruglikov

Department of Mathematics and Statistics, University of Tromsø, Tromsø, 9037, Norway

Received 7 April 2010; accepted 15 October 2010

Available online 20 October 2010

Abstract

In this paper we extend the Tanaka finiteness theorem and inequality for the number of symmetries to arbitrary distributions (differential systems) and provide several applications.

0. Introduction and main results

Let Δ be a regular completely non-holonomic distribution on a connected smooth manifold M, i.e. a vector subbundle of TM. This paper concerns the Lie algebra $\mathrm{sym}(\Delta)$ of its symmetries.

In [23] N. Tanaka introduced a graded nilpotent Lie algebra (GNLA) $\mathfrak{m}_x$ and its algebraic prolongation $\mathfrak{g}_x = \hat{\mathfrak{m}}_x$ at every point $x \in M$ to majorize the symmetry algebra of Δ (precise definitions follow in the next section). One of the purposes of this paper is to prove:

Theorem 1. *The Lie algebra* $\mathrm{sym}(\Delta)$ *of the symmetries of* Δ *satisfies*:

$$\dim \mathrm{sym}(\Delta) \leqslant \sup_M \dim \mathfrak{g}_x.$$

This statement was proven in [23] (corollary of Theorem 8.4) for strongly regular systems, i.e. such Δ that the GNLA $\mathfrak{m}_x$ does not depend on x. Generic distributions on high-dimensional manifolds (for instance, for rank 2 distributions starting from dimension 8) fail to satisfy this property, but we relax the assumption to usual regularity.

Remark 1. In fact, the equality in the above theorem, provided that all $\mathfrak{g}_x$ are finite-dimensional Lie algebras, is attained only in one case: when the distribution Δ is strongly regular and also (under certain assumptions) flat in the Tanaka sense (then it is locally isomorphic to the standard model of [23]).

Remark 2. If we assume $\dim \mathfrak{g}_x < \infty \ \forall x \in M$, then the above inequality refines to

$$\dim \mathrm{sym}(\Delta) \leqslant \inf_M \dim \mathfrak{g}_x. \tag{1}$$

E-mail address: boris.kruglikov@uit.no.

0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2010.10.001

Our approach here is to elaborate upon the Tanaka theory of symmetries for distributions and relate it to the Spencer theory of formal integrability for PDEs. Existence of this relation is natural since both theories are abstract versions and generalizations of the Cartan equivalence method [22,20,26].

Corollary 1. *Suppose for every $x \in M$ there exists no grading 0 derivation of the GNLA $\mathfrak{m}_x^{\mathbb{C}} = \bigoplus_{i<0} \mathfrak{g}_i^{\mathbb{C}}$, which has rank 1 in* $\mathrm{gl}(\mathfrak{g}_{-1}^{\mathbb{C}})$ *and acts trivially on $\mathfrak{g}_i^{\mathbb{C}}$, $i < -1$. Then the Lie algebra* $\mathrm{sym}(\Delta)$ *is finite-dimensional.*

The meaning of the condition in the theorem is that the complex characteristic variety of (the prolongation–projection of the Lie equation corresponding to) Δ is empty. When the distribution is strongly regular this assertion follows from [23] (Corollary 2 of Theorem 11.1).

This latter theorem and corollary concern finite-dimensionality of the Lie algebra $\mathfrak{g}_x$ and are purely algebraic. That is the reason it is "if and only if" statement. For just regular distributions (even strongly regular but non-flat) there are other reasons for decrease of the size of the symmetry algebra (for instance, abundance of the independent components of the curvature).

Our theorems imply a-priory knowledge of finite-dimensionality and size estimate for the algebra $\mathrm{sym}(\Delta)$. Here is one output (the derived distribution is defined in Section 1).

Theorem 2. *Consider a distribution Δ of rank n. Assume that rank of the derived distribution Δ_2 is greater than $\frac{n(n-1)}{2} + 2$ in the case $n > 2$. For $n = 2$ we assume that rank of the 2nd derived distribution Δ_3 is 5. Then the Lie algebra* $\mathrm{sym}(\Delta)$ *is finite-dimensional.*

Another application of Corollary 1 is investigation of distributions with infinite algebra of symmetries. In Section 8 we construct a model that, together with the operation of prolongation, allows to classify all rank 2 and 3 distributions with infinitely many independent symmetries.

1. Review of Tanaka theory

Given a distribution $\Delta \subset TM$, its *weak derived flag* $\{\Delta_i\}_{i>0}$ is given via the module of its sections by $\Gamma(\Delta_{i+1}) = [\Gamma(\Delta), \Gamma(\Delta_i)]$ with $\Delta_1 = \Delta$.

We will assume throughout this paper that our distribution is *completely non-holonomic*, i.e. $\Delta_i = TM$ for $i \geqslant \kappa$, and we also assume that the flag of Δ is *regular*, so that the ranks of Δ_i are constant (whence κ is constant as well).

The quotient sheaf $\mathfrak{m} = \bigoplus_{i<0} \mathfrak{g}_i$, $\mathfrak{g}_i = \Delta_{-i}/\Delta_{-i-1}$ (we let $\Delta_0 = 0$), has a natural structure of graded nilpotent Lie algebra at any point $x \in M$. The bracket on $\mathfrak{m}$ is induced by the commutator of vector fields on M. Δ is called *strongly regular* if the GNLA $\mathfrak{m} = \mathfrak{m}_x$ does not depend on the point $x \in M$.

The *growth vector* of $\Delta = \mathfrak{g}_{-1}$ is the sequence of dimensions[1] $(\dim \mathfrak{g}_{-1}, \dim \mathfrak{g}_{-2}, \ldots)$ depending on $x \in M$.

The Tanaka prolongation $\mathfrak{g} = \hat{\mathfrak{m}}$ is the graded Lie algebra with negative graded part $\mathfrak{m}$ and non-negative part defined successively by

$$\mathfrak{g}_k = \left\{ u \in \bigoplus_{i<0} \mathfrak{g}_{k+i} \otimes \mathfrak{g}_i^* \colon\ u\big([X,Y]\big) = \big[u(X), Y\big] + \big[X, u(Y)\big],\ X, Y \in \mathfrak{m} \right\}.$$

Since Δ is bracket-generating, the algebra $\mathfrak{m}$ is fundamental, i.e. $\mathfrak{g}_{-1}$ generates the whole GNLA $\mathfrak{m}$, and therefore the grading k homomorphism u is uniquely determined by the restriction $u : \mathfrak{g}_{-1} \to \mathfrak{g}_{k-1}$.

The space $\mathfrak{g} = \oplus \mathfrak{g}_i$ is naturally a graded Lie algebra, called the *Tanaka algebra* of Δ, and to indicate dependence on the point $x \in M$, we will write $\mathfrak{g} = \mathfrak{g}_x$ (also the value of a vector field Y at the point x will be denoted by Y_x).

Alternatively the above symbolic prolongation can be defined via Lie algebra cohomology with coefficients: $\mathfrak{g}_0 = H_0^1(\mathfrak{m}, \mathfrak{m})$, $\mathfrak{g}_1 = H_1^1(\mathfrak{m}, \mathfrak{m} \oplus \mathfrak{g}_0)$, etc., where the subscript indicates the grading [1]. The prolongation of $\mathfrak{m}$ is $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots$. In particular, if $\mathfrak{g}$ is finite-dimensional, then $H^1(\mathfrak{m}, \mathfrak{g}) = 0$ in accordance with [26].

In addition to introducing the Lie algebra $\mathfrak{g}$, which majorizes the symmetry algebra of Δ, the paper [23] contains the construction of an important ingredient to the equivalence problem – an absolute parallelism on the prolongation manifold of the structure, see also [15,27].

[1] In other sources, it is $(\dim \Delta_1, \dim \Delta_2, \ldots)$.

Distribution is locally flat if the structure functions of the absolute parallelism vanish. Then the distribution Δ is locally diffeomorphic to the standard model on the Lie group corresponding to $\mathfrak{m}$, see [23].

2. Lie algebra sheaf of a distribution

Instead of considering the global object (vector fields) $\mathrm{sym}(\Delta) \subset \mathscr{D}(M)$ let us study the more general structure of local symmetries. Namely we consider the Lie algebras sheaf (LAS) $\mathscr{L}$ of germs of vector fields preserving the distribution Δ. Regularity of the latter implies that $\mathscr{L}$ is a sub-sheaf of $\mathscr{D}_{\mathrm{loc}}(M) = \Gamma_{\mathrm{loc}}(TM)$.

Let $\mathscr{L}(x)$ be its stalk at $x \in M$ and $\mathscr{L}(x)^0$ the isotropy subalgebra. Note that in terms of the evaluation map $\mathrm{ev}_x : \mathscr{L}(x) \to T_xM$ the latter is $\mathrm{ev}_x^{-1}(0)$. The LAS $\mathscr{L}$ is transitive if ev_x is onto and in this case all stalks $\mathscr{L}(x)$ are isomorphic. In the opposite case (provided $\mathscr{L}$ is integrable and regular) the Frobenius theorem implies that M is foliated by the leaves of $\mathrm{ev}_x(\mathscr{L}(x))$; the stalks are constant along it.

We define the dimension of LAS to be

$$\dim \mathscr{L} = \sup_M \big(\mathrm{rank}[\mathrm{ev}_x] + \dim \mathscr{L}(x)^0\big).$$

Since globalization can only decrease objects (see Appendix A for precise conditions), we will have:

$$\dim \mathrm{sym}(\Delta) \leqslant \dim \mathscr{L}.$$

There are two decreasing filtrations of the stalk $\mathscr{L}(x)$ of the LAS $\mathscr{L}$ for every $x \in M$.

The first filtration is defined in the transitive case by the rule [20]: $\mathscr{L}(x)^i_* = \mathscr{L}(x)$ for $i < 0$, $\mathscr{L}(x)^0_* = \mathscr{L}(x)^0$ and

$$\mathscr{L}(x)^{i+1}_* = \big\{X \in \mathscr{L}(x)^i_* \colon \big[X, \mathscr{L}(x)\big] \subset \mathscr{L}(x)^i_*\big\} \quad \text{for } i \geqslant 0.$$

This definition works in the abstract setting and in general there are existence and realization theorems [20]. In our case this filtration is closely related to the jets. In fact, $\mathscr{L}(x)^i_* = \mathscr{L}(x) \cap \mu_x^{i+1} \cdot \mathscr{D}_{\mathrm{loc}}(M)$, where $\mu_x \subset C^\infty(M)$ is the maximal ideal. This latter definition of $\mathscr{L}(x)^i_*$ ($i \geqslant 0$) works nicely in the intransitive case as well.

The second filtration in the transitive case is defined as follows [23]: $\mathscr{L}(x)^i = \mathrm{ev}_x^{-1}(\Delta_{-i})$ for $i < 0$, $\mathscr{L}(x)^0$ as above and

$$\mathscr{L}(x)^{i+1} = \big\{X \in \mathscr{L}(x)^i \colon \big[X, \mathscr{L}(x)^{-1}\big] \subset \mathscr{L}(x)^i\big\} \quad \text{for } i \geqslant 0.$$

In particular, $\mathscr{L}(x)^{-i} = \mathscr{L}(x)$ for $i \geqslant \kappa$.

The two filtrations are related by the following diagram:

$$\begin{array}{ccccccccccc}
\mathscr{L}(x) & = \cdots = & \mathscr{L}(x)^{-1}_* & \supset & \mathscr{L}(x)^0_* & \supset & \mathscr{L}(x)^1_* & \supset & \mathscr{L}(x)^2_* & \supset \cdots \\
|| & & \cup & & || & & \cap & & \cap & \\
\mathscr{L}(x)^{-\kappa} & \supset \cdots \supset & \mathscr{L}(x)^{-1} & \supset & \mathscr{L}(x)^0 & \supset & \mathscr{L}(x)^1 & \supset & \mathscr{L}(x)^2 & \supset \cdots
\end{array}$$

In the intransitive case the situation with the second filtration is more complicated and can be resolved as follows. The non-positive terms keep the same value.

For $X \in \mathscr{L}(x)^0$ and $Y \in \mathscr{D}(M)$ the value $\Psi^1_X(Y) = [X, Y]_x \in T_xM$ depends on $Y_x \in T_xM$ only. Thus $\Psi^1_X : T_xM \to T_xM$ is a linear morphism and it maps Δ_i to Δ_i, $i > 0$. We define

$$\mathscr{L}(x)^1 = \big\{X \in \mathscr{L}(x)^0 \colon \Psi^1_X\big|_{\Delta_x} = 0\big\}.$$

Thus $X \in \mathscr{L}(x)^1$ are characterized by the property $\Psi^1_X : \Delta_i \to \Delta_{i-1}$.

For $X \in \mathscr{L}(x)^1$ and $Y, Z \in \mathscr{D}(M)$ the value $\Psi^2_X(Y, Z) = [[X, Y], Z]_x \in T_xM$ depends on $Y_x, Z_x \in T_xM$ provided $Y_x, Z_x \in \Delta_x$.

Indeed, since $\mathbb{R}$-linearity of Ψ^2_X is obvious, consider multiplication by a function $f \in C^\infty(M)$:

$$\big[[X, fY], Z\big] = f\big[[X, Y], Z\big] + X(f)[Y, Z] - Z(f)[X, Y] - X\big(Z(f)\big)Y + [X, Z](f)Y.$$

At the point x all the terms except the first in the r.h.s. vanish, implying the claim. The claim for the second argument is even easier, as the last term below vanishes at x:

$$\big[[X, Y], fZ\big] = f\big[[X, Y], Z\big] + [X, Y](f)Z.$$

Thus we have a linear morphism $\Psi^2_X : \Delta_x \otimes \Delta_x \to \Delta_x$, which we can also treat as a map $\Delta_x \to \Delta_x^* \otimes \Delta_x$, $Z \mapsto \Psi^2_X(\cdot, Z)$.

We define

$$\mathscr{L}(x)^2 = \big\{X \in \mathscr{L}(x)^1 \colon \Psi^2_X\big|_{\Delta_x \otimes \Delta_x} = 0\big\}.$$

The Jacobi identity implies that $\Psi^1_X|_{\Delta_2} = 0$ for $X \in \mathscr{L}(x)^2$ and more generally $\Psi^1_X : \Delta_i \to \Delta_{i-2}$ for such X and $i \geqslant 2$.

Continuing in the same way we define $\mathscr{L}(x)^i$ for $i > 0$ and we inductively get the multi-linear maps $\Psi^{i+1}_X : \bigotimes^{i+1} \Delta_x \to T_x M$, $X \in \mathscr{L}(x)^i$ by the formula

$$\Psi^{i+1}_X(Y_1, \dots, Y_{i+1}) = \big[\big[..\big[[X, Y_1], Y_2\big], ..\big], Y_{i+1}\big]_x.$$

In the next section we extend Ψ^{i+1}_X to various arguments and show that $\Psi^{i+1}_X : \bigotimes^{i+1} \Delta_x \to \Delta_x$. Then the next filtration term is

$$\mathscr{L}(x)^{i+1} = \big\{X \in \mathscr{L}(x)^i \colon \Psi^{i+1}_X\big|_{\Delta_x \otimes \cdots \otimes \Delta_x} = 0\big\}.$$

These two filtrations define the same topology on $\mathscr{L}(x)$ because

$$\mathscr{L}(x)^{i \cdot \kappa} \subset \mathscr{L}(x)^i_* \subset \mathscr{L}(x)^i, \quad i \geqslant 0. \tag{2}$$

3. Investigation of the second filtration

We need first to prove the following characterization of $X \in \mathscr{L}(x)^i$, $i \geqslant 0$. Let Z_{jl} be a frame around x, compatible with the weak flag, i.e. $Z_{1l}(x)$ is a basis of $\mathfrak{g}_{-1} = \Delta_x$, $Z_{2l}(x)$ produces a basis of $\mathfrak{g}_{-2} = \Delta_2/\Delta_1$, etc. Decompose a local symmetry $X = \sum f_{jl} Z_{jl}$.

Let us say that function $f \in C^\infty(M)$ belongs to $\mu^{k+1}_{\Delta,x}$ if $Y_1 \cdots Y_t(f)$ vanishes at x for all $Y_j \in \Gamma(\Delta)$, $t \leqslant k$ (for $k = 0$ we have: $\mu_{\Delta,x} = \mu_x$).

Lemma 1. *If for $i \geqslant 0$ $X \in \mathscr{L}(x)^i$, then $f_{jl} \in \mu^{i+j}_{\Delta,x}$, $1 \leqslant j \leqslant \kappa$.*

Proof. X is a local symmetry if for any $Y_{i_1} \in \Gamma(\Delta)$ the Lie bracket

$$[X, Y_{i_1}] = \sum \big(f_{jl}[Z_{jl}, Y_{i_1}] - Y_{i_1}(f_{jl}) Z_{jl}\big)$$

is also a section of Δ. This equality mod Δ_{j-1} implies the following decomposition

$$Y_{i_1}(f_{jl}) = \sum_{s \geqslant j-1} \alpha^{st}_{i_1 jl} f_{st}, \quad j > 1. \tag{3}$$

Differentiation of this implies

$$Y_{i_1} Y_{i_2}(f_{jl}) = \sum_{s \geqslant j-2} \alpha^{st}_{i_1 i_2 jl} f_{st}, \quad j > 2; \tag{4}$$

$$Y_{i_1} Y_{i_2}(f_{2l}) = \sum_{s,t} \alpha^{st}_{i_1 i_2 2l} f_{st} + \beta^t_{i_1 i_2 l} Y_{i_1}(f_{1t}). \tag{5}$$

Proceeding we get generally for $j > 1$,

$$Y_{i_1} \cdots Y_{i_r}(f_{jl}) = \sum_{s \geqslant j-r} \alpha^{st}_{i_1 \dots i_r jl} f_{st} + \sum_{k=1}^{r-j+1} \beta^{q_1 \dots q_k t}_{i_1 \dots i_r jl} Y_{q_1} \cdots Y_{q_k}(f_{1t}), \tag{6}$$

where we assume $\sum_A^B = 0$ if $A > B$; we can also assume that $(q_1 \dots q_k)$ is an ordered subset of $(i_1 \dots i_r)$ though it plays no role in what follows.

Consider at first the case $i = 0$. Then obviously $f_{jl}(x) = 0$, i.e. $f_{jl} \in \mu^1_x$. Substituting this into (3) we get $f_{jl} \in \mu^2_{\Delta,x}$ for $j \geqslant 2$. Next substitution into (4) yields $f_{jl} \in \mu^3_{\Delta,x}$ for $j \geqslant 3$, and continuing with (6) we get $f_{jl} \in \mu^j_{\Delta,x}$ for all j.

Now let us look at $i = 1$. By definition $f_{1l} \in \mu^2_{\Delta,x}$. Then (5) implies $f_{2l} \in \mu^3_{\Delta,x}$. Continuing with (6) we get $f_{jl} \in \mu^{j+1}_{\Delta,x}$.

Now the pattern is clear and the general claim $f_{jl} \in \mu^{i+j}_{\Delta,x}$ is easily obtained by induction. □

Lemma 2. *For $X \in \mathscr{L}(x)^i$ and $Y_1, \dots, Y_{i+1} \in \Delta_x$ we have*:

$$\Psi^{i+1}_X(Y_1, \dots, Y_{i+1}) \in \Delta_x.$$

Proof. Indeed Lemma 1 implies that for $X \in \mathscr{L}(x)^i$ and $Y_j \in \Gamma(\Delta)$,

$$\big[\big[..[X, Y_1]..., Y_i\big], Y_{i+1}\big] = \sum \pm Y_{i+1} \cdots Y_1(f_{1l}) Z_{1l} + \cdots,$$

where the omitted terms that are linear combinations of f_{jl}, $Y_{q_1}(f_{jl}), \dots, Y_{q_1} \cdots Y_{q_i}(f_{jl})$ and $Y_{q_1} \cdots Y_{q_{i+1}}(f_{tl})$ for $t > 1$, all of which vanish at x. Thus the result evaluated at x is a vector from Δ_x. □

Recall that $\Delta_s = 0$ for $s \leqslant 0$ and $\Delta_s = TM$ for $s \geqslant \kappa$. The formulae of the previous section for any $X \in \mathscr{L}(x)$ yield the maps ($j, s_v > 0$)

$$\Psi^j_X : \Gamma(\Delta_{s_1}) \otimes \cdots \otimes \Gamma(\Delta_{s_j}) \to \Gamma(\Delta_{s_1 + \cdots + s_j}).$$

Lemma 3. *If $X \in \mathscr{L}(x)^i$ and $Y_t \in \Gamma(\Delta_{s_t})$ for $1 \leqslant t \leqslant j \leqslant i + 1$, then*

$$\Psi^j_X(Y_1, \dots, Y_j)_x \in \Delta_{s_1 + \cdots + s_j - i}.$$

Proof. For $i = j$ and $s_1 = \cdots = s_j = 1$ this is just the definition as $\Delta_0 = 0$. Without loss of generality we can take decomposable $Y_t = [[..[V_{t1}, V_{t2}], ..], V_{ts_t}]_x$, $V_{t\alpha} \in \Gamma(\Delta)$, $1 \leqslant t \leqslant j$, $1 \leqslant \alpha \leqslant s_t$. Then as a result of the Jacobi identity $\Psi^j_X(Y_1, \dots, Y_j)$ decomposes into a linear combination of the terms

$$\big[\big[..\big[[X, V_{\alpha_1}], V_{\alpha_2}\big]..\big], V_{\alpha_r}\big], \quad r = s_1 + \cdots + s_j.$$

For $r \leqslant i$ this vanishes at x, i.e. the result belongs to Δ_0.

For $r > i$ we deduce from Lemma 1 for $X = \sum f_{jl} Z_{jl}$:

$$\big[\big[..\big[[X, V_{\alpha_1}], V_{\alpha_2}\big]..\big], V_{\alpha_r}\big]_x = \sum_{j \leqslant r-i} \pm V_{\alpha_r} \cdots V_{\alpha_1}(f_{jl} Z_{jl})_x \in \Delta_{r-i}.$$

Finally for $i = j - 1$ the claim follows from Lemma 2 and the extension argument of this proof. □

For $X \in \mathscr{L}(x)^i$ the multi-linear map $\Psi^j_X(Y_1, \dots, Y_j)$ of Lemma 3 is not $C^\infty(M)$-linear in Y_t, but considered with the values modulo $\Delta_{s_1 + \cdots + s_j - i - 1}$ it is. Thus it induces the map (all spaces evaluated at x)

$$\Psi^j_X : \Delta_{s_1} \otimes \cdots \otimes \Delta_{s_j} \to \mathfrak{g}_{i - s_1 - \cdots - s_j}$$

with the kernel $\sum_{t_1 + \cdots + t_j < s_1 + \cdots + s_j} \Delta_{t_1} \otimes \cdots \otimes \Delta_{t_j}$, whence the map

$$\Psi^j_X : \mathfrak{g}_{-s_1} \otimes \cdots \otimes \mathfrak{g}_{-s_j} \to \mathfrak{g}_{i - s_1 - \cdots - s_j}. \tag{7}$$

Remark 3. For $X \in \mathscr{L}(x)^i$ and $i \geqslant (j-1)\kappa$ the map

$$\Psi^j_X : \Delta_{s_1} \otimes \cdots \otimes \Delta_{s_j} \to \Delta_{s_1 + \cdots + s_j - i},$$

is tensorial, so in this case we do not need to quotient.

Lemma 4. *The second filtration respects the Lie brackets*:

$$\big[\mathscr{L}(x)^i, \mathscr{L}(x)^j\big] \subset \mathscr{L}(x)^{i+j}.$$

Proof. Let $X' \in \mathscr{L}(x)^i$, $X'' \in \mathscr{L}(x)^j$. If $i, j \leqslant 0$, then $X'_x \in \Delta_i$, $X''_x \in \Delta_j$ and so $[X', X'']_x \in \Delta_{i+j}$, i.e. $[X', X''] \in \mathscr{L}(x)^{i+j}$.

Consider $i, j \geqslant 0$. Then for $Y_1, \dots, Y_{i+j} \in \Gamma(\Delta)$ we have with some factors ν_t:

$$\big[..\big[\big[X', X''\big], Y_1\big].., Y_{i+j}\big] = \sum_{t;\, \sigma \in S_{i+j}} \nu_t \big[..\big[..\big[X', Y_{\sigma(1)}\big].., Y_{\sigma(t)}\big], \big[..\big[X'', Y_{\sigma(t+1)}\big].., Y_{\sigma(i+j)}\big]\big].$$

If a term in the above summation has $t \geqslant i$, then $i + j - t \leqslant j$ and for $Z = [..[X', Y_{\sigma(1)}].., Y_{\sigma(t)}]$ with $Z_x \in \Delta_{t-i}$ Lemma 3 implies: $[[..[X'', Y_{\sigma(t+1)}].., Y_{\sigma(i+j)}], Z]_x = 0$. If $t < i$, then a symmetric ($X' \leftrightarrows X''$) argument applies.

Let finally $i > 0$, $j < 0$, $i + j \geqslant 0$. Then

$$\big[..\big[\big[X', X''\big], Y_1\big].., Y_{i+j}\big] = \sum_{t, \sigma \in S_{i+j}} \nu_t \big[\big[..\big[X', Y_{\sigma(1)}\big].., Y_{\sigma(t)}\big], Z\big(X'', Y_{\sigma(t+1)}, \dots, Y_{\sigma(i+j)}\big)\big]$$

where $Z(..) = [..[X'', Y_{\sigma(t+1)}].., Y_{\sigma(i+j)}]$ has $Z_x \in \Delta_{i-t}$, so that the result follows from Lemma 3.

If $i + j < 0$, then the claim follows from Lemma 1. □

4. Formal Lie algebra of symmetries

In Section 2 we introduced two compatible filtrations (2). The corresponding *formal Lie algebra* is

$$L_x^\Delta = \lim_{i \to +\infty} \mathscr{L}(x)/\mathscr{L}(x)^i = \lim_{i \to +\infty} \mathscr{L}(x)/\mathscr{L}(x)^i_*.$$

This algebra has two gradations (jets and weighted jets[2]) corresponding to the above two filtrations. In fact, the first grading is

$$\mathrm{gr}_*\big(L_x^\Delta\big) = \bigoplus \bar{g}_i(x), \qquad \bar{g}_i(x) = \mathscr{L}(x)^{i-1}_*/\mathscr{L}(x)^i_* \quad (i \geqslant 0)$$

(the reason for the shift of indices will be clear in the next section).

The second grading is the following

$$\mathrm{gr}\big(L_x^\Delta\big) = \bigoplus \mathbf{g}_i(x), \qquad \mathbf{g}_i(x) = \mathscr{L}(x)^i/\mathscr{L}(x)^{i+1} \quad (i \geqslant -\kappa).$$

Notice the difference in the range of indices. Both gradings have the induced Lie bracket, so that we have two graded Lie structures (which might be different as Lie algebras from L_x^Δ).

We clearly have the inclusion of GNLAs $\bigoplus_{i<0} \mathbf{g}_i \subset \mathfrak{m}_x$, and now would like to elaborate upon (7) to understand the symbols $\mathbf{g}_i$, $i \geqslant 0$.

The Tanaka symbol space $\mathfrak{g}_0 \subset \sum_{i<0} \mathfrak{g}_i^* \otimes \mathfrak{g}_i$ is uniquely determined by its restriction to $\mathfrak{g}_{-1}$ and thus can be identified with its image $\mathfrak{g}_0 \hookrightarrow \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}$.

Similarly, $\mathfrak{g}_1 \subset (\sum_{j<0} \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_j^* \otimes \mathfrak{g}_j) \oplus (\sum_{i<-1} \mathfrak{g}_{i-1}^* \otimes \mathfrak{g}_i)$ can be identified with its image $\mathfrak{g}_1 \hookrightarrow \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0 \hookrightarrow \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}$.

In the general case, $\mathfrak{g}_i \hookrightarrow \bigotimes^{i+1} \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}$ is a monomorphism and we identify the symbol $\mathfrak{g}_i$ ($i \geqslant 0$) with its image.

Now notice that by (7) every $X \in \mathscr{L}(x)^i$ induces the linear map $\Psi_X^j : \bigotimes^j \mathfrak{g}_{-1} \to \mathfrak{g}_{i-j}$.

Lemma 5. *For $X \in \mathscr{L}(x)^i$ the element $\Psi_X^{i+1} \in \bigotimes^{i+1} \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}$ belongs to (the image of) $\mathfrak{g}_i$ ($i \geqslant 0$).*

[2] Weighted jets play a crucial role in Morimoto's approach to the equivalence problem [15].

Proof. For $i = 0$ the claim is obvious since $\Psi^1_X = \mathrm{ad}_X \in \mathrm{Der}_0(\mathfrak{m}_x)$.

For $i = 1$ the Jacobi identity implies the following symmetry of Ψ^2_X, with $Y, Z \in \Gamma(\Delta)$ so that $[Y, Z] \in \Gamma(\Delta_2)$:

$$\Psi^2_X(Y, Z) - \Psi^2_X(Z, Y) = \Psi^1_X\big([Y, Z]\big).$$

Moreover this formula holds true if we understand $Y \in \mathfrak{g}_j$, $Z \in \mathfrak{g}_l$, $[Y, Z] \in \mathfrak{g}_{j+l}$, which provides us the extension $\Psi^2_X \in \sum_{i<0} \mathfrak{g}_i^* \otimes \mathfrak{g}_{i+1}$ we seek. By the construction this element satisfies the Leibniz rule and so belongs to $\mathfrak{g}_1$. In other words, $\Psi^2_X|_{\Delta_x \otimes \Delta_x}$ belongs to the image of $\mathfrak{g}_1$ in $\mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}$.

For the general $i > 0$ the arguments are the same (induction), and the reason behind the claim of Lemma 5 is that both the element Ψ^{i+1}_X and the elements of $\mathfrak{g}_i$ are constructed on the same principle using the same formula (cf. introduction of the spaces $\mathfrak{g}_i$ for $i \geqslant 0$ in [23,26]). □

Now we are ready for our main technical result.

Theorem 3. *For all $i \in \mathbb{Z}$: $\mathbf{g}_i \subset \mathfrak{g}_i$. Thus we get the monomorphism of the graded Lie algebras*:

$$\mathrm{gr}\big(L^\Delta_x\big) = \bigoplus \mathbf{g}_i \hookrightarrow \bigoplus \mathfrak{g}_i = \mathfrak{g}_x$$

Proof. Consider the map

$$\mathscr{L}(x)^i \ni X \mapsto \Psi^{i+1}_X \in \mathfrak{g}_i, \quad i \geqslant 0.$$

Its kernel is $\mathscr{L}(x)^{i+1}$ and therefore we get the induced monomorphism $\mathbf{g}_i \to \mathfrak{g}_i$. Existence of this arrow for negative i is obvious.

Since the Lie bracket in both cases is induced by the commutator of vector fields and by Lemma 4 it respects the filtration, so it respects the gradation and the map is a homomorphism of Lie algebras. □

Notice though that the formal Lie algebra L^Δ_x is not a Lie subalgebra of $\mathfrak{g}_x$, as can be seen by studying the submaximal examples in [2].

Corollary 2. *At any point $x \in M$ we have* $\dim L^\Delta_x \leqslant \dim \mathfrak{g}_x$.

Thus for the LAS $\mathscr{L}$ we achieved our claim. To prove it for global/local[3] Lie algebra $\mathrm{sym}(\Delta)$ we shall study the first grading too.

5. Lie equation associated to a distribution

We will use here jet-spaces and the geometric theory of PDE, for which we refer the reader to [21,12,11]. Consider the Lie equation $\mathcal{E} = \mathfrak{Lie}(K)$ of a geometric structure K, see [13,10]. For instance, if K is a tensorial field, then this is the equation for its symmetries, i.e. vector fields X satisfying $L_X(K) = 0$.

In the case of our current interest $K = \Delta$ and the Lie equation, considered as the submanifold in jets, is

$$\mathcal{E} = \big\{ j^1_x X\colon\ L_X(\Delta)_x \subset \Delta_x \ \big|\ x \in M \big\} \subset J^1(TM).$$

As is customary in the theory of overdetermined systems, finding compatibility conditions on the solutions of $\mathcal{E}$ binds to applying the prolongation–projection method. Namely the prolongation $\mathcal{E}^{(k-1)} \subset J^k$ is determined by the original equations and their derivatives up to order k. It can happen that some projections $\pi_{k,l} : \mathcal{E}^{(k-1)} \to \mathcal{E}^{(l-1)}$ are not surjective (i.e. there is a differential corollary of lower order), then we take the image as the new system of equations and apply the prolongations and projections again. The procedure is finite due to Cartan–Kuranishi theorem [14] and the output is formally integrable, meaning that it possesses a formal series solution through every regular point (or local analytic solution for analytic $\mathcal{E}$).

[3] This means over a small neighborhood, fixed for all vector fields.

Calculation of prolongation–projection in general is a difficult task, and $\mathfrak{Lie}(\Delta)$ is not an exception. Hopefully, we can guess an equation squeezed in between $\mathcal{E}$ and its k-th derived $\pi_{k,1}(\mathcal{E}^{(k-1)}) \subset J^1$. This is so because of the obvious fact that if k-jet of X preserves Δ at x, then $j^1_x X$ preserves its weak derived flag up to length k.

Thus instead of studying symmetries of Δ we can equally well study symmetries of the derived flag $\mathcal{W}_\Delta = \{\Delta_i\}_{i>0}$. Denote by $\mathcal{E}_\Delta$ the corresponding Lie equation

$$\mathfrak{Lie}(\mathcal{W}_\Delta) = \big\{ j^1_x X\colon\ L_X(\Delta_i)_x \subset (\Delta_i)_x\ \forall i > 0 \,\big|\, x \in M \big\} \subset J^1(TM).$$

Denote by $\bar{\mathcal{E}}$ the result of the prolongation–projection. This consists of the subsets (submanifolds with singularities)

$$\bar{\mathcal{E}}_i = \lim_{j\to\infty} \pi_{j+1,i}\big(\mathcal{E}^{(j)}_\Delta\big) \subset J^i(TM).$$

Symbols of this equation are the vector spaces (we use linearity of $\bar{\mathcal{E}}$ which simplifies the general formulae)

$$\bar{g}_i = \mathrm{Ker}(\pi_{i,i-1} : \bar{\mathcal{E}}_i \to \bar{\mathcal{E}}_{i-1}).$$

Clearly these are subspaces of the symbols g_i of the original Lie equation $\mathcal{E}_\Delta$:

$$\bar{g}_i \subset g_i \subset S^i T^*_x M \otimes T_x M. \tag{8}$$

In particular, $\bar{g}_0$ is the tangent to the orbit of the symmetry group action (the whole $T_x M$ in the transitive case).

Ultimately the infinite jets $\bar{\mathcal{E}}_\infty$ correspond to L^Δ_x (another descriptions is this: $\bar{\mathcal{E}}_\infty$ consists of formal vector fields preserving all differential invariants of Δ). We shall relate the Spencer symbols $\bar{g}_{i+1}$ to the Tanaka symbols $\mathfrak{g}_i$, $i \geqslant 0$.

Consider the decreasing filtration $\mathscr{L}(x)^i \cap \mathscr{L}(x)^j_*$ of the space $\mathscr{L}(x)^i$. It produces the grading (the isomorphisms below are not natural and respect only the linear structure)

$$\mathscr{L}(x)^i \simeq \bigoplus_{j\geqslant 0} \frac{\mathscr{L}(x)^i \cap \mathscr{L}(x)^j_*}{\mathscr{L}(x)^i \cap \mathscr{L}(x)^{j+1}_*} = \bigoplus_{j=0}^{i-1} \frac{\mathscr{L}(x)^i \cap \mathscr{L}(x)^j_*}{\mathscr{L}(x)^i \cap \mathscr{L}(x)^{j+1}_*} \oplus \bigoplus_{j\geqslant i} \bar{g}_{j+1}.$$

This implies for $i \geqslant 0$

$$\mathbf{g}_i = \mathscr{L}(x)^i / \mathscr{L}(x)^{i+1} \simeq \bigoplus_{j\leqslant i} h_{ij}, \tag{9}$$

where $h_{ij} = (\mathscr{L}(x)^i \cap \mathscr{L}(x)^j_*)/(\mathscr{L}(x)^i \cap \mathscr{L}(x)^{j+1}_* + \mathscr{L}(x)^{i+1} \cap \mathscr{L}(x)^j_*)$. Note that $h_{ij} = 0$ if either $i < j$ or $i \geqslant (j+1)\kappa$ $(j \geqslant 0)$.

Similarly the decreasing filtration $\mathscr{L}(x)^i \cap \mathscr{L}(x)^j_*$ of the space $\mathscr{L}(x)^j_*$ yields

$$\bar{g}_{j+1} = \mathscr{L}(x)^j_* / \mathscr{L}(x)^{j+1}_* \simeq \bigoplus_{i\geqslant j} h_{ij}. \tag{10}$$

As a by-product of calculations in the previous sections, we can interpret the vector space h_{ij} as a subspace in

$$\sum_{s_1+\cdots+s_{j+1}=i+t} \mathfrak{g}^*_{-s_1} \otimes \cdots \otimes \mathfrak{g}^*_{-s_{j+1}} \otimes \mathfrak{g}_{-t} \quad (t, s_\nu \geqslant 1).$$

Theorem 4. *The Spencer symbols $\bar{g}_i$ of the equation $\bar{\mathcal{E}}$ are related to the Tanaka symbols $\mathbf{g}_i$ via* (9)–(10)*. This yields a* (*noncanonical*) *monomorphism of $\bigoplus_{j>0} \bar{g}_j$ into $\bigoplus_{i\geqslant 0} \mathfrak{g}_i$.*

The claim of the theorem follows from the inclusions $\mathbf{g}_i(x) \subset \mathfrak{g}_i(x)$ of Theorem 3. These are strict and the only case, when we have equalities for all x is a homogeneous distribution (with some mild assumptions even Tanaka flat).

Indeed, equalities everywhere mean that the Lie equation of the GNLA $\mathfrak{m}_x$ is formally integrable, and the prolongation–projection does not decrease its symbols. In particular, $\bar{g}_0(x) = T_x M$, the pseudogroup of symmetries is transitive, so that the distribution is strongly regular and the result follows. This justifies Remark 1.

Corollary 3. *For a regular distribution* Δ (*not necessarily strongly regular*)

$$\sum_{i>0} \dim \bar{g}_i \leqslant \sum_{j\geqslant 0} \dim \mathfrak{g}_j.$$

This implies that if the map $j^\infty_x : \mathscr{L}(x)^0 \to J^\infty_x(TM)$ is injective (we justify the assumption in the next section), then

$$\dim \mathscr{L}(x) \leqslant \sum_{j=-\kappa}^{\infty} \dim \mathfrak{g}_j(x),$$

and so $\dim \mathscr{L} \leqslant \sup_M \sum_j \dim \mathfrak{g}_j$.

6. Proof of Theorem 1 and beyond

We can suppose that $\dim \mathfrak{g}_x$ is finite at every point $x \in M$, because else the inequality in Theorem 1 is trivial.

Lemma 6. *The function* $x \mapsto \dim \mathfrak{g}_x$ *is upper semi-continuous.*

Proof. Indeed, $\mathfrak{g}_x$ is obtained from $\mathfrak{m}_x$ by certain linear algebra rules (in [23,26] the positive grades $\mathfrak{g}_i$ are defined successively, but this can be easily modified to obtain $\mathfrak{g}_+$ via $\mathfrak{m}$ at once). Since ranks of matrices can only drop in the limit process, the result follows. □

Thus $\dim \mathfrak{g}_x$ attains a maximum in any compact domain $\bar{U}$.

Let N be the number such that $\mathfrak{g}_i(x) = 0$ for all $i \geqslant N$ and $x \in \bar{U}$. Then (9) + (10) imply that $\bar{g}_i(x) = 0$ for all $i > N$ and $x \in \bar{U}$. In fact, the whole prolongation–projection process is not required, but as the condition that X preserves the GNLA $\mathfrak{m}_x$ structure is obtained from the original $\mathfrak{Lie}(\Delta)$ in a finite number of steps and $\mathfrak{g}_x$ is obtained via $\mathfrak{m}_x$ by algebraic prolongation we conclude: In finite number of steps of prolongation–projection the Lie equation becomes of finite type at all points $x \in \bar{U}$.

Consequently by the results of Theorem 8 from Appendix A, there are no symmetries in $\bar{U}$ flat at some point. Since $\bar{U} \subset M$ is arbitrary, we conclude the result for the whole M and hence the map

$$j^\infty_x : \operatorname{sym}(\Delta) \to \bar{\mathcal{E}}^\infty_x$$

is injective for every point $x \in M$.

Thus results of Section 5 imply the inequality

$$\dim \operatorname{sym}(\Delta) \leqslant \dim \mathscr{L}(x) \quad \forall x \in M$$

and consequently we prove inequality (1) of Remark 2, which implies in turn Theorem 1. □

Notice that in the case when the weak derived flag $\{\Delta_i\}$ is regular, i.e. all $\Delta_i \subset TM$ are the sub-bundles, another proof can be given along the lines of Tanaka's original ideas by constructing a canonical co-frame. Indeed, the Tanaka theory was essentially generalized to this context in [15], see also [27] for an alternative approach.

This weaker regularity is not granted a priori for distributions Δ with the largest symmetry algebra, but a-posteriori we conclude that most symmetric distributions are even strongly regular.

Remark 4. If the distribution is not strongly regular, then the leaves on M through a typical point x (obtained by fixing the invariants) have codimension $r = n - \operatorname{rank}[\mathrm{ev}_x] > 0$ and we refine inequality (1) to

$$\dim \operatorname{sym}(\Delta) \leqslant \inf_M \dim \mathfrak{g}_x - r.$$

Proof of Corollary 1. We use Corollary 2 of Theorem 11.1 from [23], which states that if the subalgebra

$$\mathfrak{h}_0 = \{v \in g_0 \colon [v, g_r] = 0 \ \forall r < -1\}$$

is of finite type as the subalgebra of $\mathrm{gl}(\mathfrak{g}_{-1})$, then $\mathfrak{g}$ is finite-dimensional.

The subalgebra $\mathfrak{h}_0 \subset \mathrm{gl}(\mathfrak{g}_{-1})$ is of finite type iff its complex characteristic variety is empty [9].
Thus all Tanaka algebras $\mathfrak{g}_x$, $x \in M$, are finite-dimensional and the result follows. □

We can introduce the characteristic variety (the set of covectors satisfying the defining relation is homogeneous and we projectivize it)

$$\mathrm{Char}(\Delta) = \mathbb{P}\big\{p \in g_{-1}^* \setminus \{0\}\colon \exists q \in g_{-1} \setminus \{0\},\ p \otimes q \in \mathfrak{h}_0\big\} \subset \mathbb{P}\Delta^*.$$

Working over $\mathbb{C}$ (this part of the theory is algebraic, and complexification makes no problem) we obtain $\mathrm{Char}^{\mathbb{C}}(\Delta) \subset \mathbb{P}^{\mathbb{C}}\Delta^*$. The criterion of the theorem reformulates now as follows:

$$\mathrm{Char}^{\mathbb{C}}(\Delta) = \emptyset \quad \Longrightarrow \quad \dim \mathrm{sym}(\Delta) < \infty.$$

If the set $\mathrm{Char}^{\mathbb{C}}(\Delta)$ is empty, then the complex characteristic variety of the differential closure $\bar{\mathcal{E}}$ of the Lie equation $\mathcal{E}_\Delta$ is empty as well, but the reverse implication is not generally true though it holds for a Tanaka flat distribution Δ.

Remark 5. One should be cautious as $\mathrm{Char}^{\mathbb{C}}(\bar{\mathcal{E}})$ is usually smaller than the initial complex characteristic variety $\mathrm{Char}^{\mathbb{C}}(\mathcal{E}_\Delta)$.

Example. Consider the Tanaka flat rank 2 distribution in $\mathbb{R}^6$ of growth $(2, 1, 2, 1)$ and maximal symmetry algebra $\mathfrak{p}_6$ of dimension 11. In [1] (see also [5]) it was shown that it corresponds to the Monge equation

$$\mathcal{E}_{1,3}\colon \quad y' = \big(z'''\big)^2,$$

namely $\Delta = \langle \mathcal{D}_x = \partial_x + z_3^2\partial_y + z_1\partial_z + z_2\partial_{z_1} + z_3\partial_{z_2}, \partial_{z_3}\rangle$ in the standard jet-coordinates in mixed jets $J^{1,3}(\mathbb{R}, \mathbb{R}\times\mathbb{R}) \supset \mathcal{E}_{1,3} \simeq \mathbb{R}^6(x, y, z, z_1, z_2, z_3)$.

Moreover its algebra of symmetries $\mathrm{sym}(\Delta)$ is isomorphic to the Tanaka algebra $\mathfrak{g} = \mathfrak{p}_6$ with $\mathfrak{g}_0 \simeq \mathbb{R}^3$, $\mathfrak{g}_1 \simeq \mathbb{R}^2$, $\mathfrak{g}_2 = 0$, and it has the following basis corresponding to elements of pure grade in $\mathfrak{g}$:

$$\begin{aligned}
\mathfrak{g}_{-4}\colon\quad & Z_0 = \partial_z,\\
\mathfrak{g}_{-3}\colon\quad & Z_1 = x\partial_z + \partial_{z_1}, \qquad Y_0 = \partial_y,\\
\mathfrak{g}_{-2}\colon\quad & Z_2 = \frac{x^2}{2}\partial_z + x\partial_{z_1} + \partial_{z_2},\\
\mathfrak{g}_{-1}\colon\quad & Z_3 = \frac{x^3}{3!}\partial_z + \frac{x^2}{2}\partial_{z_1} + x\partial_{z_2} + \partial_{z_3} + 2z_2\partial_y, \qquad S_0 = \partial_x,\\
\mathfrak{g}_0\colon\quad & Z_4 = \frac{x^4}{4!}\partial_z + \frac{x^3}{3!}\partial_{z_1} + \frac{x^2}{2}\partial_{z_2} + x\partial_{z_3} + 2(xz_2 - z_1)\partial_y,\\
& S_1 = x\partial_x + \frac{5}{2}z\partial_z + \frac{3}{2}z_1\partial_{z_1} + \frac{1}{2}z_2\partial_{z_2} - \frac{1}{2}z_3\partial_{z_3},\\
& R = y\partial_y + \frac{1}{2}z\partial_z + \frac{1}{2}z_1\partial_{z_1} + \frac{1}{2}z_2\partial_{z_2} + \frac{1}{2}z_3\partial_{z_3},\\
\mathfrak{g}_1\colon\quad & Z_5 = \frac{x^5}{5!}\partial_z + \frac{x^4}{4!}\partial_{z_1} + \frac{x^3}{3!}\partial_{z_2} + \frac{x^2}{2}\partial_{z_3} + 2\left(\frac{x^2}{2}z_2 - xz_1 + z\right)\partial_y,\\
& S_2 = x^2\partial_x + 9z_2^2\partial_y + 5xz\partial_z + (5z + 3xz_1)\partial_{z_1} + (8z_1 + xz_2)\partial_{z_2} + (9z_2 - xz_3)\partial_{z_3}.
\end{aligned}$$

If we take the classes of these fields in $\mathscr{L}(x)^i_*/\mathscr{L}(x)^{i+1}_*$ at 0 we get:

$$\bar{g}_0 = \big\langle [S_0], [Y_0], [Z_0], [Z_1], [Z_2], [Z_3]\big\rangle, \qquad \bar{g}_1 = \big\langle [S_1], [S_2], [R], [Z_4], [Z_5]\big\rangle.$$

However this corresponds to the differential closure (via prolongation–projection) $\bar{\mathcal{E}}_\Delta$, while the original Lie equation $\mathcal{E}_\Delta = \mathfrak{Lie}(\Delta)$ is bigger. In particular, the element $[Z_5] = dz \otimes \partial_y \in \bar{g}_1 \subset g_1$ is of rank 1. Further prolongation–projections of $\mathcal{E}_\Delta$ yield $\bar{g}_2 = 0$ and kill this characteristic.

7. Finite-dimensionality of the symmetry algebra

Theorem 2 for $n = 2$ was essentially established in [1] (but in that paper we restricted to the strongly regular case) by showing that $\mathfrak{h}_0 = 0$ provided the growth vector is $(2, 1, 2, \ldots)$, and even in a more general case $(2, 1, \ldots, 1, 2, \ldots)$.

Another proof is as follows. We can work over $\mathbb{C}$ as this does not change the dimensions of graded components. Suppose there is a nonzero element of rank 1 $\omega = p \otimes \zeta \in \mathfrak{h}_0 \subset \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}$. Let ξ be a complement to ζ in $\mathfrak{g}_{-1}$. Then ω acts trivially on $\mathfrak{g}_{-2} = \mathbb{R} \cdot [\zeta, \xi]$ iff $p(\zeta) = 0$. Furthermore ω acts trivially on $\mathfrak{g}_i$, $i < -2$, iff the operator $\mathrm{ad}_\zeta : \mathfrak{g}_{i+1} \to \mathfrak{g}_i$ is zero. Consequently $\dim \mathfrak{g}_i = 1$ for $i < -1$.

Proof of Theorem 2 for $n > 2$. Again, working over $\mathbb{C}$ and taking an element of rank 1 $\omega = p \otimes \zeta \in \mathfrak{h}_0 \subset \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}$ we observe that ω acts trivially on $\mathfrak{g}_{-2}$ iff $\mathrm{ad}_\zeta|_{\mathrm{Ann}(p)} = 0$. As ζ can belong to $\mathrm{Ann}(p)$, this imposes $(n-2)$ restrictions. Thus in the case the characteristic variety is non-empty, dimension of $\mathfrak{g}_{-2} = \mathrm{ad}(\Lambda^2 \mathfrak{g}_{-1})$ does not exceed $\frac{n(n-1)}{2} - (n-2) = \frac{(n-1)(n-2)}{2} + 1$.

Thus distributions with growth starting $(3, 3, \ldots)$, $(4, 5, \ldots)$, $(4, 6, \ldots)$, etc., have finite-dimensional symmetry algebras. □

The a-priory knowledge of finite-dimensionality of the symmetry algebra can be enhanced by estimation of its maximal size in many cases. For distributions of rank 2 it is done in [5,1].

It can be also done for distributions Δ with free truncated GNLAs. Assume that $\mathfrak{m}_x$ is the free Lie algebra of step k, i.e. the only constraints are the Jacobi identity and the requirement $\mathfrak{g}_{-k-1} = 0$. Then obviously $\mathfrak{g}_0 = \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}$. Moreover by [24] $\mathfrak{g}_1 = 0$ provided $k > 2$, $n > 2$ or $k > 3$, $n = 2$. This implies ([24] concerned only homogeneous distributions, but the result holds for any Δ with truncated free $\mathfrak{m}$ due to [23] or Theorem 1):

$$\dim \mathrm{sym}(\Delta) \leqslant \ell_n(k) + n^2.$$

Here $\ell_n(k) = \dim \mathfrak{m}_x$ can be calculated recursively or via the Möbius function by [19]

$$k\ell_n(k) = n^k - \sum_{m|k, m<k} m\ell_n(m) = \sum_{m|k} \mu(m) n^{k/m}.$$

For the exceptional cases we have: $\dim \mathrm{sym}(\Delta) \leqslant 2n^2 + n$, provided $k = 2$, $n > 2$ and the maximal symmetric case is given by $B_n = \mathfrak{o}(2n+1)$ [26,4].

For $k = 3$, $n = 2$ $\dim \mathrm{sym}(\Delta) \leqslant 14$, the maximal dimension being realized only in the case of $\mathrm{Lie}(G_2)$ [2].

The estimate of Theorem 2 is sharp as there exist infinite-dimensional Lie algebras acting as symmetries on distributions with growth vector beginning $(2, 1, 1, \ldots)$, $(3, 1, \ldots)$, $(3, 2, \ldots)$, $(4, 1, \ldots)$, $(4, 2, \ldots)$, $(4, 3, \ldots)$, $(4, 4, \ldots)$, etc., as discussed in the next section.

8. Infinite algebras of symmetries

As in the rest of the paper we assume Δ totally non-holonomic. Notice however that if the bracket-closure $\Delta_\infty \neq TM$, and $\mathcal{F}$ are the leaves of Δ_∞, then existence of one symmetry transversal to $\mathcal{F}$ implies existence of an infinite-dimensional space of such symmetries.

8.1. Distributions of rank $n = 2$

If a totally non-holonomic distribution with $\mathrm{rank}(\Delta) = 2$ has infinite-dimensional symmetry algebra then its growth vector starts $(2, 1, 1, \ldots)$.

Recall that a Cauchy characteristic of a distribution is a symmetry tangent to this distribution. The following statement is due to E. Cartan [3] (see [1] for another proof; alternatively the vector field ζ is the one constructed in the beginning of Section 7).

Theorem 5. *The growth vector of a rank* 2 *distribution* Δ *is* $(2, 1, 1, \ldots)$ *if and only if there exists a vector field* $\zeta \in \Gamma(\Delta)$ *which is a Cauchy characteristic for the distribution* Δ_2. *In this, and only in this case,* Δ *is locally the prolongation of another rank* 2 *distribution* $\bar{\Delta}$.

Here the (geometric) prolongation of a rank 2 distribution $\bar\Delta$ on $\bar M$ is the projectivization $M=\mathbb{P}\bar\Delta=\{l_x\subset\bar\Delta_x\colon x\in\bar M\}$ with natural projection $\pi: M\to\bar M$ and the distribution $\Delta=\pi_*^{-1}(l_{\pi(x)})$. Locally in $\bar M$ for $\bar\Delta=\langle U,V\rangle$ and $t\in S^1=\mathbb{R}^1/2\pi\mathbb{Z}$ we have: $M=\bar M\times S^1$ and $\Delta=\langle\cos t\cdot U+\sin t\cdot V,\partial_t\rangle$.

If the distribution $\bar\Delta$ has a preferred (vertical) section $V\in\Gamma(\bar\Delta)$ [this means that the symmetries preserve it], then the (affine) prolongation $\Delta=\bar\Delta^{(1)}$ is simply $\Delta=\langle U+tV,\partial_t\rangle$ on $M=\bar M\times\mathbb{R}^1$, where t is the coordinate on $\mathbb{R}^1$. This coincides with (Spencer) prolongation in the geometric theory of PDEs (see [12,17]).

When $\Delta=\bar\Delta^{(1)}$, the operation $\Delta\mapsto\bar\Delta$ is called de-prolongation. In the case of Theorem 5, de-prolongation is the quotient of the first derived distribution by the Cauchy characteristic

$$(\bar M,\bar\Delta)=(M,\Delta_2)/\zeta.$$

The algebras $\mathrm{sym}(\bar\Delta)$ and $\mathrm{sym}(\bar\Delta^{(1)})$ are isomorphic [indeed any symmetry on $\bar M$ induces the action on t and thus lifts to M]. Therefore Theorem 2 and a sequence of de-prolongations yield the following important statement (it gives a-posteriori transitivity of the symmetry group action; with transitivity imposed a priori – as an additional assumption – the claim follows from Theorem 7.1 of [16]).

Theorem 6. *A regular germ of a rank* 2 *non-holonomic distribution* Δ *has infinite-dimensional symmetry algebra if and only if* Δ *is equivalent to the Cartan distribution* $\mathcal{C}_k$ *on jet-space* $J^k(\mathbb{R},\mathbb{R})$, $k=\dim M-2$.

Recall that distributions Δ with strong growth vector $(2,1,1,\dots,1)$ are called Goursat distributions. Their regular points can be characterized by the condition that the growth vector is $(2,1,1,\dots,1)$ in both the weak and the strong sense. Near such points the normal form

$$\mathcal{C}_k=\mathrm{Ann}\{dy_i-y_{i+1}\,dx\mid 0\leqslant i<k\}\subset TJ^k(\mathbb{R},\mathbb{R})$$

is provided by the von Weber–Cartan theorem [25,3], see also [8].

8.2. Distributions of higher rank

We indicate a local construction to produce a distribution with infinite-dimensional (intransitive) symmetry algebra from any distribution.

For simplicity let's start with the case, when we extend a rank 2 distribution. Locally any such distribution can be represented as the Cartan distribution $\mathcal{C}_k$ for the Monge system, i.e. an underdetermined ODE $\mathcal{E}\subset J^k(\mathbb{R},\mathbb{R}^m)$ given by $(m-1)$ equations.

Then we define $M=\mathcal{E}\times_{\mathbb{R}}J^l(\mathbb{R},\mathbb{R})\subset J^{k,l}(\mathbb{R},\mathbb{R}^m\times\mathbb{R})$ and the distribution is $\Delta=(\mathcal{C}_k|_{\mathcal{E}})\times_{\mathbb{R}}\mathcal{C}_l$. In local coordinates if the equations in $\mathcal{E}$ are $u^i_k=\psi^i(x,u^j,\dots,u^j_{k-1},v,\dots,v_k)$, $i,j=1,\dots,m-1$ ($u^i=u^i(x)$, $v=u^m(x)$ are unknowns, $u^j_1=u^j_x$, $v_1=v_x$, $v_2=v_{xx}$, etc.), then $\mathcal{C}_k|_{\mathcal{E}}=\langle\mathcal{D}'_x=\partial_x+u^i_1\partial_{u^i}+\dots+\psi^i\partial_{u^i_{k-1}}+v_1\partial_v+\dots+v_k\partial_{v_{k-1}},\partial_{v_k}\rangle$ and $\mathcal{C}_l=\langle\mathcal{D}''_x=\partial_x+w_1\partial_w+w_2\partial_{w_1}+\dots,\partial_{w_l}\rangle$, so that

$$\Delta=\big\langle\mathcal{D}_x=\partial_x+u^i_1\partial_{u^i}+v_1\partial_v+w_1\partial_w+\dots,\partial_{v_k},\partial_{w_l}\big\rangle.$$

It is obvious that prolongation $\hat X$ of the vector field $X=f(w)\partial_w$ is a symmetry of Δ for any function $f(w)$.

Similarly a general regular rank n distribution is realized locally as the Cartan distribution $\mathcal{C}_k|_{\mathcal{E}}$ of an overdetermined PDE system $\mathcal{E}\subset J^k(\mathbb{R}^s,\mathbb{R}^m)$. Then the distribution $\Delta=(\mathcal{C}_k|_{\mathcal{E}})\times_{\mathbb{R}^s}\mathcal{C}_l$ on $M=\mathcal{E}\times_{\mathbb{R}^s}J^l(\mathbb{R}^s,\mathbb{R})\subset J^{k,l}(\mathbb{R}^s,\mathbb{R}^m\times\mathbb{R})$ has an infinite algebra of symmetries.

We can shrink the 2nd factor to a symmetric equation $\mathcal{R}\subset J^l(\mathbb{R}^s,\mathbb{R})$, for instance taking $\mathcal{R}=J^l(\mathbb{R},\mathbb{R})$ given by the equations $w_{x_2}=\dots=w_{x_s}=0$, and still have infinitely many symmetries $\hat X$, $X=f(w)\partial_w$ for the distribution $\Delta=(\mathcal{C}_k|_{\mathcal{E}})\times_{\mathbb{R}^s}(\mathcal{C}_l|_{\mathcal{R}})$.

Remark 6. For $l=0$ we get extension of the distribution via the Cauchy characteristic, i.e. locally $(M,\Delta)=(\bar M,\bar\Delta)\times(\mathbb{R},\mathbb{R})$.

This construction allows realizing all cases not prohibited by Theorem 2 as distributions with infinite-dimensional symmetry algebras.

For rank 3 distributions all infinite primitive symmetry algebras that occur are Lie transformations[4] of $J^k(\mathbb{R},\mathbb{R}^2)$, while for rank 4 there appear new real primitive algebras from the list of [16]: the symmetries of the Cartan distribution of $J^k(\mathbb{R},\mathbb{R}^3)$, $J^1(\mathbb{R}^2,\mathbb{R})$ and $J^k(\mathbb{C},\mathbb{C})_{\mathbb{R}}$.

8.3. Distributions of rank $n=3$

Let us describe in more details the case of $\operatorname{rank}(\Delta)=3$. If the growth vector is $(3,1,\dots)$, then the bracket $\Lambda^2\mathfrak{g}_{-1}\to\mathfrak{g}_{-2}$ has a kernel $v\in\mathfrak{g}_{-1}$, and this corresponds to a Cauchy characteristic vector field, so that $(M,\Delta)\simeq(\bar M,\bar\Delta)\times(\mathbb{R},\mathbb{R})$.

Consider the growth $(3,2,\dots)$. Then Δ contains a rank 2 sub-distribution Π, with $\Pi_2\subset\Delta$. If $\Pi_2=\Delta$, then $\operatorname{sym}(\Delta)=\operatorname{sym}(\Pi)$. The growth vector of Π starts $(2,1,2,\dots)$ and so $\operatorname{sym}(\Pi)$ is finite-dimensional.

Thus infinite algebras $\operatorname{sym}(\Delta)$ correspond to integrable Π. In this case the characteristic variety is real and $\operatorname{Char}^{\mathbb{C}}=\operatorname{Char}$ equals the one point set $\mathbb{P}(\Pi^\perp)$, where $\Pi^\perp$ is the annihilator of $\Pi\subset\mathfrak{g}_{-1}$.

Moreover let $p\in\Pi^\perp\cap\mathfrak{g}_{-1}^*\setminus 0$. Then the kernel bundle over characteristic variety $\mathcal{K}=\{q\in\mathfrak{g}_{-1}\colon\ p\otimes q\in\mathfrak{h}_0\}$ is either 1- or 2-dimensional subspace of Π.

I: $\dim\mathcal{K}=1$. Here $\operatorname{sym}(\Delta)$ is the symmetry of the flag $(\mathcal{K},\Pi,\Delta)$. Moreover this triple extends (locally) to a complete flag $\mathfrak{F}$ of subspaces $(\mathcal{K},\Pi,\Delta,[\mathcal{K},\Delta],\Delta_2,\dots)$ in TM, invariant under $\operatorname{sym}(\Delta)$.

Remark 7. For $n=2$ the infinite symmetry algebra also leaves invariant the complete flag $\mathfrak{F}=(\langle\zeta\rangle,\Delta,\Delta_2,\dots)$, see Section 8.1.

Now $\mathcal{K}$ is the Cauchy characteristic space of the distribution $\Delta^\dagger=[\mathcal{K},\Delta]$, so we can pass to the (local) quotient $\bar M=M/\mathcal{K}$, $\bar\Delta=\Delta^\dagger/\mathcal{K}$. Any symmetry of Δ descends to a symmetry of $\bar\Delta$, and the latter contains a preferred direction $\Pi/\mathcal{K}$. It is not necessarily the kernel direction $\bar{\mathcal{K}}$ of $\bar\Delta$, but it belongs to the corresponding 2-distribution $\bar\Pi$.

The passage $\Delta\mapsto\bar\Delta$ is de-prolongation of rank 3 distributions in the same sense as in Section 8.1, namely this operation is inverse to the prolongation defined as follows (this applies only for $\operatorname{Char}(\Delta)\neq\emptyset$, so that $\Pi\subset\Delta$ is integrable, etc.).

Prolongation I_a. This works in the case $\bar\Delta$ has a preferred direction $\ell=\langle Y\rangle$ different from $\bar{\mathcal{K}}$ [if not, then the Lie algebra $\operatorname{sym}(\bar\Delta)$ shrinks to the stabilizer]. Namely M is the space of all 2-planes $P\subset\bar\Delta$ through ℓ transversal to $\bar{\mathcal{K}}$ (this is the affine version; compact version is without transversality). If $\pi:M\to\bar M$ is the projection, then $\Delta=\pi_*^{-1}(P)$.

Locally if $\bar\Delta=\langle X,Y,Z\rangle$ with $X\in\bar{\mathcal{K}}$ and $Y\in\ell\subset\bar\Pi$, then $M=\bar M\times\mathbb{R}(t)$, $\Delta=\langle Y,\partial_t,Z+tX\rangle$. In these notations $\Pi=\langle Y,\partial_t\rangle$, $\partial_t\in\mathcal{K}$. This is inverse to the above de-prolongation when $\dim\mathcal{K}=1$.

Prolongation I_b. $\bar{\mathcal{K}}$ is clearly a preferred direction, so we can prolong along it: M is the space of all 2-planes $P\subset\bar\Delta$ through $\bar{\mathcal{K}}$ different from $\bar\Pi$ (the affine version; compact version – no conditions).

Again $\Delta=\pi_*^{-1}(P)$ and locally $\Delta=\langle X,\partial_t,Z+tY\rangle$. Here $\Pi=\langle X,\partial_t\rangle=\mathcal{K}$, so this is not inverse to the above de-prolongation.

II: $\dim\mathcal{K}=2$. Here $\Pi=\mathcal{K}$ is the space of Cauchy characteristics for rank 5 distribution Δ_2, and we can reduce (M,Δ) to a rank 3 distribution $\bar\Delta=\Delta_2/\Pi$ on a lower-dimensional manifold $\bar M=M/\Pi$. This is de-prolongation in the same sense as in Section 8.1, while the prolongation (Π integrable) mimics the Spencer prolongation on $J^k(\mathbb{R},\mathbb{R}^2)$.

Prolongation II. Here M is the space of lines $L\subset\Delta$ transversal to Π (the affine version; compact version – no restrictions), $\pi:M\to\bar M$ the natural projection, then $\Delta=\pi_*^{-1}(L)$.

Locally if $\bar\Delta=\langle X,Y,Z\rangle$ with $\Pi=\langle X,Y\rangle$, then $M=\bar M\times\mathbb{R}^2(u,v)$ and $\Delta=\langle\partial_u,\partial_v,Z+uX+vY\rangle$. Clearly $\mathcal{K}=\langle\partial_u,\partial_v\rangle$.

Notice that de-prolongations in both cases induce the injective map $\pi_*:\operatorname{sym}(\Delta)\hookrightarrow\operatorname{sym}(\bar\Delta)$ [prolongations I_a can shrink the symmetry algebra, though in a controllable way].

[4] In their natural representation $k=0$ and $k=0,1$ in the next cases to be primitive, but the algebra does not change with k due to Lie–Bäcklund theorem.

Thus de-prolongations can be applied until the distribution gets a Cauchy characteristic or its length (degree of non-holonomy) becomes $\kappa = 1$. Thus we obtain the following normal form [for $n = 3$ infinitely-symmetric models can have functional moduli, but these are gone for most symmetric models in the sense of functional dimension].

Theorem 7. *A regular germ of a rank* 3 *non-holonomic distribution* Δ *has infinite-dimensional symmetry algebra if and only if* Δ *is obtained from the product* $\Pi \times \mathbb{R}$*, where* Π *is some rank* 2 *distribution, by the operations of prolongations* (*of type* I *or* II).

Here we include the case when Π is the trivial distribution $(\mathbb{R}^2, \mathbb{R}^2)$, in which case prolongation II of $\Pi \times \mathbb{R}$ yields $J^k(\mathbb{R}, \mathbb{R}^2)$.

Example I. Consider the Monge equation $\mathcal{E}_{m,n} = \{y_m = z_n^2\}$ as a submanifold in the mixed jets (subscripts count the derivatives by x)

$$J^{m,n}\big(\mathbb{R}, \mathbb{R}^2\big) \simeq \mathbb{R}^{m+n+3}(x, y, y_1, \dots, y_m, z, z_1, \dots, z_n)$$

and let $M = \mathcal{E}_{m,n} \times_{\mathbb{R}} J^l(\mathbb{R}, \mathbb{R}) \subset J^{m,n,l}(\mathbb{R}, \mathbb{R}^3)$ with the natural rank 3 distribution $\Delta = \mathcal{C}_{m,n} \times_{\mathbb{R}} \mathcal{C}_l = \langle \mathcal{D}_x, \partial_{z_n}, \partial_{w_l} \rangle$,

$$\mathcal{D}_x = \partial_x + y_1\partial_y + \dots + z_n^2\partial_{y_{m-1}} + z_1\partial_z + \dots + z_n\partial_{z_{n-1}} + w_1\partial_w + \dots + w_n\partial_{w_{l-1}}.$$

De-prolongation I results in $\bar{M} = \mathbb{R}^{m+n+l+2}(x, \{y_i\}_0^{m-1}, \{z_i\}_0^n, \{w_i\}_0^{l-1})$ with $\bar{\Delta} = \langle \mathcal{D}_x, \partial_{z_n}, \partial_{w_{l-1}} \rangle$. Further de-prolongations I bring the distribution to $\hat{\Delta} = \langle \mathcal{D}_x, \partial_{z_n}, \partial_w \rangle = \mathcal{C}_{m,n} \times \mathbb{R}$ on $\mathcal{E}_{m,n} \times \mathbb{R}(w)$.

The general symmetry of this distribution is $X_0 = Y_{\mathcal{E}} + f \cdot \partial_w$, where $Y_{\mathcal{E}}$ is the general symmetry of $\mathcal{E}_{m,n}$, $m \leqslant n$ (it depends on $2n + 5$ parameters for $m = 1$, $n > 2$ or $2n + 4$ parameters for $m > 1$, see [1]) and $f \in C^\infty(\mathcal{E}_{m,n} \times \mathbb{R})$.

In order for this symmetry to allow prolongation I_a, it shall preserve the vertical line $\mathfrak{v} = \langle \partial_{z_n} \rangle$. This is so for the first component $Y_{\mathcal{E}}$ [1], and it holds for $f \cdot \partial_w$ iff $f_{z_n} = 0$. The prolongation is $X_1 = Y_{\mathcal{E}} + f \cdot \partial_w + \mathcal{D}_x(f) \cdot \partial_{w_1}$. Now this preserves $\mathfrak{v}$ iff $f_{z_{n-1}} = 0$, $f_{y_{m-1}} = 0$, and the prolongation is $X_2 = Y_{\mathcal{E}} + f \cdot \partial_w + \mathcal{D}_x(f) \cdot \partial_{w_1} + \mathcal{D}_x^2(f) \cdot \partial_{w_2}$.

Continuing in this way we obtain that the symmetries of (M, Δ) are $X_l = Y_{\mathcal{E}} + \sum_{k \leqslant l} \mathcal{D}_x^k(f) \cdot \partial_{w_k}$ and $f = f(x, \{y_i\}_0^{m-l}, \{z_i\}_0^{n-l}, w)$ is any smooth function of $2 + \max\{0, m - l + 1\} + \max\{0, n - l + 1\}$ arguments.

Example II. For the mixed jets $J^{m,n}(\mathbb{R}, \mathbb{R}^2) = J^m(\mathbb{R}, \mathbb{R}) \times_{\mathbb{R}} J^n(\mathbb{R}, \mathbb{R})$ with canonical Cartan distribution $\mathcal{C}_{m,n}$ of rank 3, de-prolongation II reduces it to $J^{0,n-m}(\mathbb{R}, \mathbb{R}^2) = \mathbb{R} \times J^{n-m}(\mathbb{R}, \mathbb{R})$, $m \leqslant n$. The infinite algebra of symmetries for this model is described in [1].

Some other results on distributions with infinitely many symmetries (on graded nilpotent Lie groups) can be found in [6,18].

Acknowledgements

I am grateful to Ian Anderson, Keizo Yamaguchi and Igor Zelenko for useful discussions during the mini-workshop organized by Ian Anderson in Utah State University in November 2009. I would also like to thank IHES for hospitality in October 2010.

Appendix A. Estimate of the solution space size

In the theory of formal integrability [7,21] a PDE (system) is considered as a submanifold $\mathcal{E}_k$ in jets $J^k\pi$ of a bundle $\pi : E \to M$ (if different orders are involved, $\mathcal{E}$ is a collection of submanifolds [11]), and its prolongations defined geometrically are submanifolds, possibly with singularities, in higher jets $\mathcal{E}_i \subset J^i\pi$.

The (tangent to the) fiber of the projection $\pi_{i,i-1} : \mathcal{E}_i \to \mathcal{E}_{i-1}$ is the symbol space $g_i \subset S^iT^* \otimes N$. Then the following estimate on the size of the solution space holds formally (i.e. the solutions are viewed as formal series at x and the r.h.s. is evaluated at the same point):

$$\dim \mathrm{Sol}(\mathcal{E}) \leqslant \sum_0^\infty \dim g_i. \tag{11}$$

Of course, this inequality is meaningful only in the case the r.h.s. is finite (such $\mathcal{E}$ are said to have finite type at x).

If the equation $\mathcal{E}$ is formally integrable and regular ($\mathcal{E}_i$ are smooth and dimensions of g_i are constant), the same estimate holds locally and globally (but in the latter case the l.h.s. can shrink by other reasons).

But often for a regular but formally integrable overdetermined PDE $\mathcal{E}$ the result of prolongation–projection $\bar{\mathcal{E}}$ (which is formally integrable in regular points by Cartan–Kuranishi theorem [14]) is not everywhere regular (e.g. for Lie equations). We would like to prove inequality (11) in the presence of singularities under some mild assumptions.

To be more precise, assume that the original equation $\mathcal{E}_k$ is regular (we restrict to pure order systems, but the result can be generalized), consider the nested sequence $\mathcal{E}_i^l = \pi_{l,i}(\mathcal{E}_k^{(l-k)})$ and let $\bar{\mathcal{E}}_i = \bigcap_l \mathcal{E}_i^l$. Then the co-filtration $\bar{\mathcal{E}}_i$ represents a formally integrable equation away from singularities.

We restrict to linear equations $\mathcal{E}$, as our main application here – the Lie equation for symmetries $\mathfrak{Lie}(K)$ – is linear.

Our main result states that if in finite number of steps of prolongation–projection we get finite type at all points (including singularities), then we conclude (11) even before arriving to formally integrable equation.

Denote by $\bar{g}_i$ the fiber of the projection $\pi_{i,i-1} : \bar{\mathcal{E}}_i \to \bar{\mathcal{E}}_{i-1}$ (can be empty). For linear equations the symbol $\bar{g}_i$ depends on $x \in M$.

Theorem 8. *Let $\mathcal{E}_k \subset J^k(\pi)$ be a linear equation in a bundle over a connected manifold M. The space* $\mathrm{Sol}(\mathcal{E})$ *of global solutions over M is a linear space, and its dimension is well defined. Suppose that there exists a number l such that the projection $\pi_{i,i-1} : \mathcal{E}_i \to \mathcal{E}_{i-1}$ is injective for $i = l$ (and hence for $i \geqslant l$), where $\mathcal{E}_i = \mathcal{E}_k^{(i-k)}$. Then*

$$\dim \mathrm{Sol}(\mathcal{E}) \leqslant \inf_M \sum_0^\infty \dim \bar{g}_i(x). \tag{12}$$

Proof. The result follows from the following claim (with the imposed assumptions): for every point $x \in M$ the map $j_x^\infty : \mathrm{Sol}(\mathcal{E}) \to J_x^\infty \pi$ is injective. Indeed, in this case $\mathrm{Sol}(\mathcal{E})$ is a subset of $\bar{\mathcal{E}}_x$, which has dimension bounded by $\sum_0^\infty \dim \bar{g}_i(x)$. What the claim states is that there exists no solutions flat at the given point.

To prove the claim consider a relatively compact connected neighborhood $U'' \subset \mathcal{E}_{l+1}$ and let $U' = \pi_{l+1,l}(U'') \subset \mathcal{E}_l$, $U = \pi_{l,l-1}(U') \subset \mathcal{E}_{l-1}$. For any $x_l \in U'$ and $x_{l-1} = \pi_{l,l-1}(x_l) \in U$ we have the horizontal n-plane $\Pi(x_{l-1}) = d\pi_{l,l-1}(\mathcal{C}(x_l)) \subset \mathcal{C}(x_{l-1}) \subset T_{x_{l-1}} J^{l-1}\pi$, where $\mathcal{C}(\cdot)$ is the canonical Cartan distribution in jets (see [12]).

If U is sufficiently small, then $\pi_{l,l-1} : U' \to U$ is a homeomorphism. Indeed, it is bijective (the equation is linear, the fiber g_l of $\pi_{l,l-1}$ is zero) and is a continuous map of a compact into a Hausdorff space. Moreover the slope of $\Pi(x_{l-1})$ (in canonical coordinates) is determined by x_l. Variation of slopes is governed by $x_{l+1} = \pi_{l+1,l}^{-1}(x_l) \cap U''$, and $\pi_{l+1,l} : U'' \to U'$ is a homeomorphism as well. Consequently this distribution of planes has Lipschitz dependence on the point $x_{l-1} \in \mathcal{E}_{l-1}$ (and is C^1 in regular points).

Suppose there are two local solutions of $\mathcal{E}$ with infinite tangency at $x \in M$. Let x_{l-1} be their $(l-1)$-jet at x, and denote by $\Upsilon, \Upsilon' \subset \mathcal{E}_{l-1}$ the corresponding integral manifolds (jet-lifts) through x_{l-1}.

Consider a curve $\sigma \subset M$ through x and denote $M_\sigma = \pi_{l-1}^{-1}(\sigma) \cap U$. Then $\gamma_\sigma = \Upsilon \cap M_\sigma$ and $\gamma'_\sigma = \Upsilon' \cap M_\sigma$ are integral curves of the Lipschitz line distribution $l_\sigma = \Pi \cap TM_\sigma$.

As in the usual uniqueness theorem for ODEs this implies our claim. Actually if $\gamma_\sigma(t)$, $\gamma'_\sigma(t)$ are parametrized integral curves, $t \in I$ and $\tau = |I|$ is the length of the interval, then for C^0-max norms (in any coordinates) we have from the integral version of the ODE $\dot{\gamma} = l_\sigma \circ \gamma$:

$$\|\gamma_\sigma - \gamma'_\sigma\| \leqslant c\tau \|\gamma_\sigma - \gamma'_\sigma\|$$

with some constant c depending only on Π and σ. Shrinking I if necessary we obtain $c\tau < 1$ and so locally $\gamma_\sigma = \gamma'_\sigma$. As σ is arbitrary, we get locally $\Upsilon = \Upsilon'$. As U is connected, this implies coincidence over the whole neighborhood $\pi_l(U)$ and further over the whole domain of the solutions (ultimately M). □

A version of this theorem holds for non-linear $\mathcal{E}$ (then dim in the l.h.s. means the Hausdorff dimension, the infimum in the r.h.s. is taken over the equation $\bar{\mathcal{E}}$ and the symbols $\bar{g}_i$ are the linear envelopes of the $\pi_{i,i-1}$-vertical tangent cone to $\bar{\mathcal{E}}_i$ at the singular point x_i), but more care shall be taken about regularity of the original equation $\mathcal{E}$.

Let us notice that the statement obviously holds true in the case, when the singularities in the differential closure $\bar{\mathcal{E}}$ of $\mathcal{E}$ have codim $\geqslant 2$.

In the opposite case we have the following counter-example (thanks to Valentin Lychagin).

Example. Consider the ODE $\mathcal{E}$: $x^3y'(x) = y$. Then $g_0 = \mathbb{R}$ and $g_i = 0$ for $i > 0$ and all points $x \neq 0$. Near all such points the equation is regular and the solution space is one-dimensional. However the space of global solutions $\mathrm{Sol}(\mathcal{E})$ is two-dimensional with the basis $\chi(x)e^{-1/x^2}$, $\chi(-x)e^{-1/x^2}$, where $\chi(x)$ is the Heaviside function.

Notice however that $\mathcal{E}$ is neither regular, nor of finite type at 0, and the jet-lift does not resolve this singularity. In addition, the codimension of the singularity is 1.

Remark 8. Thus a generalization of (12) could fail if one wants to disregard the singularities and change the r.h.s. to $\mathrm{ess.sup}_M \sum_0^\infty \dim \bar{g}_i(x)$.

A modification of the above example yields an equation $\mathcal{E}$ having finite type at all regular points but with infinite-dimensional $\mathrm{Sol}(\mathcal{E})$.

References

[1] I.M. Anderson, B. Kruglikov, Rank 2 distributions of Monge equations: symmetries, equivalences, extensions, arXiv:0910.5946 [math.DG], 2009.
[2] E. Cartan, Les systèmes de Pfaff, à cinq variables et les équations aux dérivées partielles du second ordre, Ann. Sci. Éc. Norm. Super. (3) 27 (1910) 109–192.
[3] E. Cartan, Sur l'équivalence absolue de certains systèmes d'équations différentielles et sur certaines familles de courbes, Bull. Soc. Math. France 42 (1914) 12–48.
[4] A. Čap, K. Neusser, On automorphism groups of some types of generic distributions, Differential Geom. Appl. 27 (2009) 769–779.
[5] B. Doubrov, I. Zelenko, On local geometry of nonholonomic rank 2 distributions, J. Lond. Math. Soc. 80 (2009) 545–566.
[6] B. Doubrov, O. Radko, Graded nilpotent Lie algebras of infinite type, arXiv:1001.0379 [math.DG], 2010.
[7] H. Goldschmidt, Integrability criteria for systems of nonlinear partial differential equations, J. Differential Geom. 1 (3) (1967) 269–307.
[8] A. Giaro, A. Kumpera, C. Ruiz, Sur la lecture correcte d'un résultat d'Élie Cartan, C. R. Acad. Sci. Paris Sér. A–B 287 (4) (1978) A241–A244.
[9] V. Guillemin, D. Quillen, S. Sternberg, The classification of the irreducible complex algebras of infinite type, J. Anal. Math. 18 (1967) 107–112.
[10] S. Kobayashi, Transformation Groups in Differential Geometry, Springer-Verlag, 1972.
[11] B. Kruglikov, V.V. Lychagin, Geometry of differential equations, in: D. Krupka, D. Saunders (Eds.), Handbook on Global Analysis, Elsevier Sci., 2008, pp. 725–771.
[12] I.S. Krasilschik, V.V. Lychagin, A.M. Vinogradov, Geometry of Jet Spaces and Differential Equations, Gordon and Breach, 1986.
[13] A. Kumpera, D. Spencer, Lie Equations, vol. 1: General Theory, Princeton University Press, University of Tokyo Press, 1972.
[14] M. Kuranishi, On E. Cartan's prolongation theorem of exterior differential systems, Amer. J. Math. 79 (1957) 1–47.
[15] T. Morimoto, Geometric structures on filtered manifolds, Hokkaido Math. J. 22 (3) (1993) 263–347.
[16] T. Morimoto, N. Tanaka, The classification of the real primitive infinite Lie algebras, J. Math. Kyoto Univ. 10 (2) (1970) 207–243.
[17] R. Montgomery, M. Zhitomirskii, Geometric approach to Goursat flags, Ann. Inst. H. Poincaré Anal. Non Lineáire 18 (4) (2001) 459–493.
[18] A. Ottazzi, B. Warhurst, Contact and 1-quasiconformal maps on Carnot groups, arXiv:1001.3817 [math.DG], 2010.
[19] J.-P. Serre, Lie algebras and Lie groups, Lectures given at Harvard University, 1964.
[20] I.M. Singer, S. Sternberg, On the infinite groups of Lie and Cartan, J. Anal. Math. 15 (1965) 1–114.
[21] D.C. Spencer, Overdetermined systems of linear partial differential equations, Bull. Amer. Math. Soc. (N.S.) 75 (1969) 179–239.
[22] S. Sternberg, Lectures on Differential Geometry, Prentice Hall, Englewood Cliffs, 1964.
[23] N. Tanaka, On differential systems, graded Lie algebras and pseudo-groups, J. Math. Kyoto Univ. 10 (1) (1970) 1–82.
[24] B. Warhurst, Tanaka prolongation of free Lie algebras, Geom. Dedicata 130 (2007) 59–69.
[25] E. von Weber, Zur Invariantentheorie der Systeme Pfaff'scher Gleichungen, Berichte Ges. Leipzig Math. Phys. Classe L (1898) 207–229.
[26] K. Yamaguchi, Differential systems associated with simple graded Lie algebras, Adv. Stud. Pure Math. 22 (1993) 413–494.
[27] I. Zelenko, On Tanaka's prolongation procedure for filtered structures of constant type, SIGMA 5 (2009) 094, 21 pp.