# On the energy exchange between resonant modes in nonlinear Schrödinger equations [☆]

# Echange d'énergie entre modes résonants dans une équation de Schrödinger non linéaire cubique

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**Abstract**

We consider the nonlinear Schrödinger equation

$$i\psi_t = -\psi_{xx} \pm 2\cos 2x |\psi|^2 \psi, \quad x \in S^1,\ t \in \mathbb{R}$$

and we prove that the solution of this equation, with small initial datum $\psi(0, x) = \varepsilon(A \exp(ix) + B \exp(-ix))$, will periodically exchange energy between the Fourier modes $e^{ix}$ and $e^{-ix}$ as soon as $A^2 \neq B^2$. This beating effect is described up to time of order $\varepsilon^{-5/2}$ while the frequency is of order $\varepsilon^2$. We also discuss some generalizations.
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**Résumé**

Nous considérons l'équation de Schödinger non linéaire

$$i\psi_t = -\psi_{xx} \pm 2\cos 2x |\psi|^2 \psi, \quad x \in S^1,\ t \in \mathbb{R}$$

et nous montrons la solution de cette équation ayant pour donnée initiale $\psi(0, x) = \varepsilon(A \exp(ix) + B \exp(-ix))$ avec $\varepsilon$ petit, va échanger périodiquement de l'énergie entre les modes de Fourier $e^{ix}$ et $e^{-ix}$ dès que $A^2 \neq B^2$. Cet effet de battement, dont la période est de l'ordre de $\varepsilon^{-2}$, est mis en évidence pour des temps de l'ordre de $\varepsilon^{-5/2}$. Nous présentons aussi quelques généralisations.
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## 1. Introduction

Let us consider the nonlinear Schrödinger equation (NLS) on the circle $S^1$

$$i\psi_t = -\psi_{xx} + V * \psi + g(x, \psi, \bar{\psi}) \tag{1.1}$$

where $V * \Psi$ denotes the convolution function between a potential $V : S^1 \to \mathbb{R}$ and the function $\Psi$. The nonlinear term $g$ is Hamiltonian in the sense that $g = \partial_{\bar{\psi}} G$ with $G$ analytic with respect to its three variables and $G(x, z, \bar{z}) \in \mathbb{R}$. We further assume that $G$ is globally at least of order three in $(z, \bar{z})$ at the origin in such a way that $g$ is effectively a nonlinear term.

The Fourier basis $\exp(ijx)$, $j \in \mathbb{Z}$ provides an orthonormal basis for $L^2(S^1)$ in which the linear operator $A = -\frac{\partial^2}{\partial x^2} + V *$ is diagonal. The corresponding eigenvalues of $A$ are given by the real numbers $\omega_j = j^2 + \hat{V}(j)$, $j \in \mathbb{Z}$, where $\hat{V}(j)$ are the Fourier coefficients of $V$:

$$\hat{V}(j) = \int_{-\pi}^{\pi} V(x) \exp(-ijx)\, dx \tag{1.2}$$

where $dx$ denotes the normalized Lebesgue measure on $S^1$.

Under a non-resonant condition on the frequencies $\omega_j$, it has been established by D. Bambusi, B. Grébert [1] (see also [3]) that, given a number $r \geqslant 3$, for small initial data in the $H^s$ norm,[1] say $\|\psi_0\|_s = \varepsilon$ with $s$ large enough, the solution to the NLS equation (1.1) remains small in the same Sobolev norm, $\|\psi(t)\|_s \leqslant 2\varepsilon$ for a large period of time, $|t| \leqslant \varepsilon^{-r}$. Furthermore the actions $I_j \equiv \xi_j \eta_j$, $j \in \mathbb{Z}$, are almost invariant during the same period of time. In [1], it is also proved that in the more natural case of a multiplicative potential,

$$i\psi_t = -\psi_{xx} + V\psi + g(x, \psi, \bar{\psi}),$$

which corresponds to an asymptotically resonant case, $\omega_j \sim \omega_{-j}$ when $|j| \to \infty$, we can generically impose non-resonant conditions on the frequencies $\{\omega_{|j|}\}$ in such a way that the generalized actions $J_j = I_j + I_{-j}$ are almost conserved quantities. A priori nothing prevents $I_j$ and $I_{-j}$ from interacting. In this article, we exhibit a nonlinearity $g$ that incites this interaction and especially a beating effect. This is a nonlinear effect which is a consequence of the resonances in the linear part. We will see that not all nonlinearities incite the beating. For instance if $g$ does not depend on $x$ there is no beating for a long time. Typically, to obtain an interaction between the mode $j$ and the mode $-j$, the nonlinearity $g$ must contain oscillations of frequency $2j$ or a multiple of $2j$, i.e. $g$ must depend on $x$ as $\cos 2kjx$ or $\sin 2kjx$ for some $k \geqslant 1$. In what follows, we will focus on the totally resonant case $V = 0$ and a cubic nonlinearity $g = 2\cos 2x |\psi|^2 \psi$. Namely, we consider the following initial value problem:

$$\begin{cases} i\psi_t = -\psi_{xx} \pm 2\cos 2x |\psi|^2 \psi, & x \in S^1, \ t \in \mathbb{R}, \\ \psi(0, x) = \varepsilon(A\exp(ix) + B\exp(-ix)) \end{cases} \tag{1.3}$$

where $\varepsilon$ is a small parameter and $A, B \in \mathbb{C}$ with $|A|^2 + |B|^2 = 1$. By classical arguments based on the conservation of the energy and Sobolev embeddings, this problem has a unique global solution in all the Sobolev spaces $H^s$ for $s \geqslant 1$ and even for $s \geqslant 0$ using more refined technics (see for instance [2, Chapter 5]). Our result precises the behavior of the solution for time of order $\varepsilon^{-5/2}$. We write the solution $\psi$ in Fourier modes: $\psi(t, x) = \sum_{j \in \mathbb{Z}} \xi_j(t) e^{ijx}$.

**Theorem 1.1.** *For $\varepsilon$ small enough we have for $|t| \leqslant \varepsilon^{-5/2}$*

$$\begin{cases} |\xi_1(t)|^2 + |\xi_{-1}(t)|^2 = \varepsilon^2 + O(\varepsilon^{7/2}), \\ |\xi_1(t)|^2 - |\xi_{-1}(t)|^2 = \pm\varepsilon^2 |A^2 - B^2| \sin(2\varepsilon^2 t + \delta) + O(\varepsilon^{7/2}) \end{cases}$$

*where the phase $\delta$ is some explicit constant depending on $A$ and $B$ given by* (3.1).

---

[1] Here $H^s$ denotes the standard Sobolev Hilbert space on $S^1$ and $\|\cdot\|_s$ its associated norm.

In other words, the first estimate says that all the energy remains concentrated on the two Fourier modes $+1$ and $-1$ and the second estimate says that, as soon as $A^2 \neq B^2$, there is energy exchange, namely a beating effect, between these two modes. For instance when $\psi(0, x) = \varepsilon(\cos x + \sin x)$ and thus $A = \frac{1-i}{2}$ and $B = \frac{1+i}{2}$, we observe a total beating for $|t| \leqslant \varepsilon^{-5/2}$:

$$\left|\xi_1(t)\right|^2 = \varepsilon^2 \frac{1 \pm \sin 2\varepsilon^2 t}{2} + O\left(\varepsilon^{7/2}\right), \qquad \left|\xi_{-1}(t)\right|^2 = \varepsilon^2 \frac{1 \mp \sin 2\varepsilon^2 t}{2} + O\left(\varepsilon^{7/2}\right).$$

On the contrary when $A = \pm B$ no exchange of energy is observed for $|t| \leqslant \varepsilon^{-5/2}$. Notice that the sign in front of the nonlinearity does not affect the phenomena.

That the principal term of $g$ be cubic is certainly not necessary but convenient for the calculations. We see that the period of the beating depends on the nonlinearity but also on the value of all the initial actions (cf. Section 4).

The paper is organized as follows: in Section 2 we prove a normal form result for Eq. (1.3) that allows us to reduce the initial problem to the study of a finite dimensional classical system. The main theorem is proved in Section 3. Section 4 is devoted to generalizations and comments.

## 2. The normal form

Let us expand $\psi$ and $\bar{\psi}$ in Fourier modes:

$$\psi(t, x) = \sum_{j \in \mathbb{Z}} \xi_j(t) e^{ijx}, \qquad \bar{\psi}(t, x) = \sum_{j \in \mathbb{Z}} \eta_j(t) e^{-ijx}.$$

In this Fourier setting Eq. (1.3) reads as an infinite Hamiltonian system

$$\begin{cases} i\dot{\xi}_j = j^2 \xi_j + \frac{\partial P}{\partial \eta_j} & j \in \mathbb{Z}, \\ -i\dot{\eta}_j = j^2 \eta_j + \frac{\partial P}{\partial \xi_j} & j \in \mathbb{Z} \end{cases} \tag{2.1}$$

where the perturbation term is given by

$$P(\xi, \eta) = \pm 2 \int_{S^1} \cos 2x \left|\psi(x)\right|^4 dx$$

$$= \pm \sum_{\substack{j, l \in \mathbb{Z}^2 \\ \mathcal{M}(j, l) = \pm 2}} \xi_{j_1} \xi_{j_2} \eta_{l_1} \eta_{l_2}. \tag{2.2}$$

and $\mathcal{M}(j, l) = j_1 + j_2 - l_1 - l_2$ denotes the momentum of the multi-index $(j, l) \in \mathbb{Z}^4$. In the sequel we will develop the calculus with $\pm = +$ but all remains true, mutatis mutandis, with the minus sign.

Since the regularity is not an issue in this work, we will work in the phase space ($\rho \geqslant 0$)

$$\mathcal{A}_\rho = \left\{ (\xi, \eta) \in \ell^1(\mathbb{Z}) \times \ell^1(\mathbb{Z}) \,\Big|\, \left\|(\xi, \eta)\right\|_\rho := \sum_{j \in \mathbb{Z}} e^{\rho|j|} \left(|\xi_j| + |\eta_j|\right) < \infty \right\}$$

that we endow with the canonical symplectic structure $-i \sum_j d\xi_j \wedge d\eta_j$. Notice that this Fourier space corresponds to functions $\psi(z)$ analytic on a strip $|\text{Im } z| < \rho$ around the real axis.

According to this symplectic structure, the Poisson bracket between two functions $f$ and $g$ of $(\xi, \eta)$ is defined by

$$\{f, g\} = -i \sum_{j \in \mathbb{Z}} \frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial \eta_j} - \frac{\partial f}{\partial \eta_j} \frac{\partial g}{\partial \xi_j}.$$

In particular, if $(\xi(t), \eta(t))$ is a solution of (2.1) and $F$ is some regular Hamiltonian function, we have

$$\frac{d}{dt} F\left(\xi(t), \eta(t)\right) = \{F, H\}\left(\xi(t), \eta(t)\right)$$

where $H = H_0 + P = \sum_{j \in \mathbb{Z}} j^2 \xi_j \eta_j + P$ is the total Hamiltonian of the system.

We denote by $B_\rho(r)$ the ball of radius $r$ centered at the origin in $\mathcal{A}_\rho$. In the next proposition we put the Hamiltonian $H$ in normal form up to order 4:

**Proposition 2.1.** *There exists a canonical change of variable $\tau$ from $B_\rho(\varepsilon)$ into $B_\rho(2\varepsilon)$ with $\varepsilon$ small enough such that*

$$H \circ \tau = H_0 + Z_{4,1} + Z_{4,2} + Z_{4,3} + R_6 \tag{2.3}$$

*where*

(i) $H_0(\xi, \eta) = \sum_{j \in \mathbb{Z}} j^2 \xi_j \eta_j$.
(ii) $Z_4 = Z_{4,1} + Z_{4,2} + Z_{4,3}$ *is the (resonant) normal form at order* 4, *i.e.* $Z_4$ *is a polynomial of order* 4 *satisfying* $\{H_0, Z_4\} = 0$.
(iii) $Z_{4,1}(\xi, \eta) = 2(\xi_1 \eta_{-1} + \xi_{-1} \eta_1)(2 \sum_{p \in \mathbb{Z}, p \neq 1, -1} \xi_p \eta_p + (\xi_1 \eta_1 + \xi_{-1} \eta_{-1}))$ *is the effective Hamiltonian at order* 4.
(iv) $Z_{4,2}(\xi, \eta) = 4(\xi_2 \xi_{-1} \eta_{-2} \eta_1 + \xi_{-2} \xi_1 \eta_2 \eta_{-1})$.
(v) $Z_{4,3}$ *contains all the terms in* $Z_4$ *involving at most one mode of index* 1 *or* $-1$.
(vi) $R_6$ *is the remainder of order* 6, *i.e. a Hamiltonian satisfying* $\|X_{R_6}(z)\|_\rho \leqslant C\|z\|_\rho^5$ *for* $z = (\xi, \eta) \in B_\rho(\varepsilon)$.
(vii) $\tau$ *is close to the identity*: *there exists a constant* $C_\rho$ *such that* $\|\tau(z) - z\|_\rho \leqslant C_\rho \|z\|_\rho^2$ *for all* $z \in B_\rho(\varepsilon)$.

**Proof.** The proof uses the classical Birkhoff normal form procedure (see for instance [4] in a finite dimensional setting or [3] in the infinite dimensional one). Since the free frequencies are the square of the integers, we are in a totally resonant case and we are not facing the small denominator problems: a linear combination of integer numbers with integer coefficients is exactly 0 or its modulus equals at least 1. For convenience of the reader, we briefly recall the procedure. Let us search $\tau$ as time one flow of $\chi$ a polynomial Hamiltonian of order 4,

$$\chi = \sum_{\substack{j, l \in \mathbb{Z}^2 \\ \mathcal{M}(j,l) = \pm 2}} a_j \xi_{j_1} \xi_{j_2} \eta_{l_1} \eta_{l_2}.$$

Let $F$ be a Hamiltonian, one has by using the Taylor expansion of $F \circ \Phi_\chi^t$ between $t = 0$ and $t = 1$:

$$F \circ \tau = F + \{F, \chi\} + \int_0^1 (1 - t)\{\{F, \chi\}, \chi\} \circ \Phi_\chi^t \, dt.$$

Applying this formula to $H = H_0 + P$ we get

$$F \circ \tau = H_0 + P + \{H_0, \chi\} + \{P, \chi\} + \int_0^1 (1 - t)\{\{H, \chi\}, \chi\} \circ \Phi_\chi^t \, dt.$$

Therefore in order to obtain $H \circ \tau = H_0 + Z_4 + R_6$ we need to solve the homological equation

$$\{\chi, H_0\} + Z_4 = P \tag{2.4}$$

and then we define

$$R_6 = \{P, \chi\} + \int_0^1 (1 - t)\{\{H, \chi\}, \chi\} \circ \Phi_\chi^t \, dt. \tag{2.5}$$

For $j, l \in \mathbb{Z}^2$ we define the associated divisor by

$$\Omega(j, l) = j_1^2 + j_2^2 - l_1^2 - l_2^2.$$

The homological equation (2.4) is solved by defining

$$\chi = \sum_{\substack{j, l \in \mathbb{Z}^2 \\ \mathcal{M}(j,l) = \pm 2, \Omega(j,l) \neq 0}} \frac{1}{-i\Omega(j, l)} \xi_{j_1} \xi_{j_2} \eta_{l_1} \eta_{l_2} \tag{2.6}$$

and

$$Z_4 = \sum_{\substack{j, \ell \in \mathbb{Z}^2 \\ \mathcal{M}(j,\ell) = \pm 2, \Omega(j,\ell) = 0}} \xi_{j_1} \xi_{j_2} \eta_{\ell_1} \eta_{\ell_2}. \tag{2.7}$$

At this stage we remark that any formal polynomial

$$Q = \sum_{\substack{j,l \in \mathbb{Z}^2 \\ \mathcal{M}(j,\ell) = \pm 2}} a_{j,\ell} \xi_{j_1} \xi_{j_2} \eta_{\ell_1} \eta_{\ell_2}$$

is well defined and continuous (and thus analytic) on $\mathcal{A}_\rho$ as soon as the $a_{j\ell}$ form a bounded family. Namely, if $|a_{j,\ell}| \leqslant M$ for all $j, \ell \in \mathbb{Z}^2$ then

$$\big|Q(\xi,\eta)\big| \leqslant \frac{M}{16} \big\|(\xi,\eta)\big\|_0^4 \leqslant \frac{M}{16} \big\|(\xi,\eta)\big\|_\rho^4.$$

Furthermore the associated vector field is bounded (and thus smooth) from $\mathcal{A}_\rho$ to $\mathcal{A}_\rho$, namely

$$
\begin{aligned}
\big\|X_Q(\xi,\eta)\big\|_\rho &= \sum_{k \in \mathbb{Z}} e^{\rho|k|} \left( \left| \frac{\partial P}{\partial \xi_k} \right| + \left| \frac{\partial P}{\partial \eta_k} \right| \right) \\
&\leqslant 2M \sum_{k \in \mathbb{Z}} e^{\rho|k|} \sum_{\substack{j_1, \ell_1, \ell_2 \in \mathbb{Z} \\ \mathcal{M}(j_1, k, \ell_1, \ell_2) = \pm 2}} |\xi_{j_1} \eta_{\ell_1} \eta_{\ell_2}| + |\xi_{\ell_1} \xi_{\ell_2} \eta_{j_1}| \\
&\leqslant 2M e^{2\rho} \sum_{j_1, \ell_1, \ell_2 \in \mathbb{Z}} \big|\xi_{j_1} e^{\rho|j_1|} \eta_{\ell_1} e^{\rho|\ell_1|} \eta_{\ell_2} e^{\rho|\ell_2|}\big| + \big|\xi_{\ell_1} e^{\rho|\ell_1|} \xi_{\ell_2} e^{\rho|\ell_2|} \eta_{j_1} e^{\rho|j_1|}\big| \\
&\leqslant \frac{M}{2} e^{2\rho} \big\|(\xi,\eta)\big\|_\rho^3
\end{aligned}
\tag{2.8}
$$

where we used, $\mathcal{M}(j_1, k, \ell_1, \ell_2) = \pm 2 \Rightarrow |k| \leqslant 2 + |j_1| + |\ell_1| + |\ell_2|$.

Since there are no small divisors in this resonant case, $Z_4$ and $\chi$ are well defined on $\mathcal{A}_\rho$ and by construction $Z_4$ satisfies (ii). On the other hand, since $\chi$ is homogeneous of order 4, for $\varepsilon$ sufficiently small, the time one flow generated by $\chi$ maps the ball $B_\rho(\varepsilon)$ into the ball $B_\rho(2\varepsilon)$ and is close to the identity in the sense of assertion (vii).

Concerning $R_6$, by construction it is a Hamiltonian function which is of order at least 6. To obtain assertion (vi) it remains to prove that the vector field $X_{R_6}$ is smooth from $B_\rho(\varepsilon)$ into $\mathcal{A}_\rho$ in such a way we can Taylor expand $X_{R_6}$ at the origin. This is clear for the first term of (2.5), $\{P, \chi\}$. For the second, notice that $\{H, \chi\} = Z_4 - P + \{P, \chi\}$ which is a polynomial on $\mathcal{A}_\rho$ having bounded coefficients[2] and the same is true for $Q = \{\{H, \chi\}, \chi\}$. Therefore, in view of the previous paragraph, $X_Q$ is smooth. Now, since for $\varepsilon$ small enough $\Phi_\chi^t$ maps smoothly the ball $B_\rho(\varepsilon)$ into the ball $B_\rho(2\varepsilon)$ for all $0 \leqslant t \leqslant 1$, we conclude that $\int_0^1 (1-t)\{\{H, \chi\}, \chi\} \circ \Phi_\chi^t \, dt$ has a smooth vector field.

We are now interested in describing more explicitly the terms appearing in the expression (2.7) on base of the number of times that $\xi_{\pm 1}$ or $\eta_{\pm 1}$ appear as a factor. Let us define the resonant set

$$\mathcal{R} = \big\{ (j_1, j_2, j_3, j_4) \in \mathbb{Z}^4 \mid j_1 + j_2 - j_3 - j_4 = \pm 2 \text{ and } j_1^2 + j_2^2 - j_3^2 - j_4^2 = 0 \big\}
\tag{2.9}$$

in such a way (2.7) reads

$$Z_4 = \sum_{(j_1, j_2, j_3, j_4) \in \mathcal{R}} \xi_{j1} \xi_{j2} \eta_{j3} \eta_{j4}.
\tag{2.10}$$

Note that it is enough to deal with the case $j_1 + j_2 - j_3 - j_4 = 2$ because the other case $j_1 + j_2 - j_3 - j_4 = -2$ can be obtained by changing the signs of $j_1, j_2, j_3, j_4$. Thus let us study two cases: $j_1 = 1$ or $j_1 = -1$.

In the first case we have $j_2 = j_3 + j_4 + 1$ and therefore the quadratic condition implies $(1 + j_3)(1 + j_4) = 0$. Thus we conclude that $(j_1, j_2, j_3, j_4) = (1, p, -1, p)$ or $(j_1, j_2, j_3, j_4) = (1, p, p, -1)$. Thus when $j_1 = 1$, the only terms appearing in the expression for $Z_4$ in Eq. (2.10) must have the form $\xi_1 \xi_p \eta_{-1} \eta_p$ or $\xi_1 \xi_p \eta_p \eta_{-1}$.

In the second case, $j = -1$, we have $j_2 = j_3 + j_4 + 3$ which in turn implies $5 + 3(j_3 + j_4) + j_3 j_4 = 0$. By looking at the graph of the function $f(x) = -\frac{5+3x}{3+x}$ we find that the only acceptable cases appearing in Eq. (2.10) are $(j_1, j_2, j_3, j_4) = (-1, 1, -1, -1)$, $(-1, 2, -2, 1)$, $(-1, 2, 1, -2)$, $(-1, -8, -4, -7)$, $(-1, -8, -7, -4)$ or $(-1, -7, -5, -5)$.

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[2] Note that this is not true for $H_0$.

Noting that the resonant set $\mathcal{R}$ is invariant under permutations of $j_1$ with $j_2$ or $j_3$ with $j_4$ we obtain that

$$
\begin{aligned}
Z_4 = {}& \left( 4 \sum_{p \in \mathbb{Z}} \xi_p \eta_p - 2(\xi_1 \eta_1 + \xi_{-1} \eta_{-1}) \right) (\xi_1 \eta_{-1} + \xi_{-1} \eta_1) \\
& + 4(\xi_2 \xi_{-1} \eta_{-2} \eta_1 + \xi_{-2} \xi_1 \eta_2 \eta_{-1}) \\
& + 4(\xi_{-1} \xi_{-8} \eta_{-7} \eta_{-4} + \xi_1 \xi_8 \eta_7 \eta_4) + 2(\xi_{-1} \xi_{-7} \eta_{-5} \eta_{-5} + \xi_1 \xi_7 \eta_5 \eta_5) + \tilde{Z}_4
\end{aligned}
$$

where $\tilde{Z}_4$ is equal to the sum of terms of the form $\xi_{j1} \xi_{j2} \eta_{j3} \eta_{j4}$ with $(j_1, j_2, j_3, j_4) \in \mathcal{R}$ and satisfying the condition $|j_k| \neq 1$, for all $k = 1, 2, 3, 4$. $\quad\square$

## 3. Dynamical consequences

We denote by $I_p = \xi_p \eta_p$, $p \in \mathbb{Z}$, the actions, by $J_p = I_p + I_{-p}$ for $p \in \mathbb{N} \setminus \{0\}$ and $J_0 = I_0$, the generalized actions and by $J = \sum_{p \in \mathbb{Z}} I_p$. Notice that, when the initial condition $(\xi^0, \eta^0)$ of the Hamiltonian system (2.1) satisfies $\eta^0 = \bar{\xi}^0$ (and this is actually the case when $\xi^0$ and $\eta^0$ are the sequences of Fourier coefficients of $\psi_0$, $\bar{\psi}_0$) respectively, this reality property is conserved i.e. $\eta(t) = \bar{\xi}(t)$ for all $t$. As a consequence all the quantities $I_p$, $J_p$ and $J$ are real and positive. We also recall that the $L^2$ norm of $\psi(t, \cdot)$ is conserved by the NLS equation and thus $J(t) = J(0)$ for all $t$.

To begin with we establish that apart from $I_1$ and $I_{-1}$ the others actions are almost constant:

**Lemma 3.1.** *Let* $\psi(t, \cdot) = \sum_{k \in \mathbb{Z}} \xi_k(t) e^{ikx}$ *be the solution of* (1.3) *then for* $\varepsilon$ *small enough and* $|t| \leqslant \varepsilon^{-5/2}$,

$$
I_p(t) \leqslant \varepsilon^4 \quad \text{for } p \in \mathbb{Z} \setminus \{-1, +1\}.
$$

**Proof.** Using Proposition 2.1 we have for all $p \in \mathbb{N}$

$$
\dot{J}_p = \{J_p, H\} = \{J_p, Z_{4,3}\} + \{J_p, R_6\}
$$

since by elementary calculations $\{J_p, Z_{4,1}\} = \{J_p, Z_{4,2}\} = 0$.

Let $T_\varepsilon$ be the maximal time such that for all $|t| \leqslant T_\varepsilon$

$$
J_p(t) \leqslant \varepsilon^4, \quad \text{for } p \neq 1.
$$

Since $J(t) = J(0) = \varepsilon^2$, we get for $\varepsilon$ small enough and $|t| \leqslant T_\varepsilon$

$$
\left| \xi_1(t) \right|, \ \left| \xi_{-1}(t) \right|, \ \left| \eta_1(t) \right|, \ \left| \eta_{-1}(t) \right| \leqslant \varepsilon \quad \text{while } \left| \xi_p(t) \right|, \ \left| \eta_p(t) \right| \leqslant \varepsilon^2 \quad \text{for } p \neq \pm 1.
$$

Therefore, from the definition of $Z_{4,3}$ and $R_6$ we deduce that for $|t| \leqslant T_\varepsilon$

$$
\{J_p, R_6\} = O(\varepsilon^7), \qquad \{J_p, Z_{4,3}\} = O(\varepsilon^7) \quad \text{for } p \neq 1,
$$

and thus there exists $C > 0$ such that for any $|t| \leqslant T_\varepsilon$

$$
|\dot{J}_p| \leqslant C \varepsilon^7 \quad \text{for } p \neq 1.
$$

Taking into account that $J_p(0) = 0$, we obtain

$$
\left| J_p(t) \right| \leqslant C |t| \varepsilon^7 \quad \text{for } p \neq 1
$$

for all $|t| \leqslant T_\varepsilon$ and we conclude by a classical bootstrap argument that $T_\varepsilon \geqslant C^{-1} \varepsilon^{-3} \geqslant \varepsilon^{-5/2}$ for $\varepsilon$ small enough. $\quad\square$

In order to prove Theorem 1.1 let us define some quadratic Hamiltonian functions ($p \in \mathbb{Z}$):

$$
\begin{aligned}
M_p &:= \xi_p \eta_p - \xi_{-p} \eta_{-p}, & J_p &:= \xi_p \eta_p + \xi_{-p} \eta_{-p}, \\
L_p &:= i(\xi_p \eta_{-p} - \xi_{-p} \eta_p), & K_p &:= \xi_p \eta_{-p} + \xi_{-p} \eta_p.
\end{aligned}
$$

One computes

$$
\begin{aligned}
\dot{M}_1 &= \{M_1, H\} = \{M_1, Z_{4,1} + Z_{4,2}\} + \{M_1, Z_{4,3} + R_6\} \\
&= 2JL_1 + L_1 K_2 - K_1 L_2 + \{M_1, Z_{4,3} + R_6\}
\end{aligned}
$$

and using Lemma 3.1 we get that for $|t| \leqslant \varepsilon^{-5/2}$,

$$\dot{M}_1 = 2JL_1 + O(\varepsilon^6)$$

and in the same way we verify

$$\dot{L}_1 = -2JM_1 + O(\varepsilon^6).$$

We can now compute the solution of the associated linear ODE (recall that $J$ is an invariant of the motion)

$$\begin{cases} \dot{M}_1 = 2JL_1, \\ \dot{L}_1 = -2JM_1 \end{cases}$$

to conclude that for $|t| \leqslant \varepsilon^{-5/2}$

$$M_1(t) = M_1(0)\cos 2Jt + L_1(0)\sin 2J(0)t + O(\varepsilon^{7/2}).$$

In Theorem 1.1 we have chosen $\psi(0, x) = \varepsilon(A\exp(ix) + B\exp(-ix))$ which corresponds to $\xi_1(0) = A\varepsilon$, $\eta_1(0) = \overline{A}\varepsilon$, $\xi_{-1}(0) = B\varepsilon$ and $\eta_{-1}(0) = \overline{B}\varepsilon$. Therefore $J = J_1(0) = (|A|^2 + |B|^2)\varepsilon^2 = \varepsilon^2$, $M_1(0) = (|A|^2 - |B|^2)\varepsilon^2$ and $L_1(0) = -2\operatorname{Im}\{A\overline{B}\}\varepsilon^2$ which implies

$$\begin{aligned} M_1(t) &= (|A|^2 - |B|^2)\varepsilon^2 \cos\bigl(2(|A|^2 + |B|^2)\varepsilon^2 t\bigr) - 2\operatorname{Im}\{A\overline{B}\}\varepsilon^2 \sin(2\varepsilon^2 t) + O(\varepsilon^{9/4}) \\ &= \varepsilon^2 |A^2 - B^2| \sin(2\varepsilon^2 t + \delta) + O(\varepsilon^{9/4}) \end{aligned}$$

with $\delta$ determined by the equations:

$$\begin{cases} \sin\delta = (|A|^2 - |B|^2)/|A^2 - B^2|, \\ \cos\delta = -2\operatorname{Im}\{A\overline{B}\}/|A^2 - B^2|. \end{cases} \tag{3.1}$$

We finally remark that choosing the minus sign in front of the nonlinearity will lead to the following linear system

$$\begin{cases} \dot{M}_1 = -2JL_1, \\ \dot{L}_1 = 2JM_1 \end{cases}$$

which again gives the desired result.

## 4. Generalizations and comments

– The same type of result remains true when we add a higher order term to the nonlinearity, i.e. considering the equation

$$i\psi_t = -\psi_{xx} \pm 2\cos 2x |\psi|^2 \psi + O(|\psi|^4), \quad x \in S^1, \ t \in \mathbb{R}.$$

– We can prove a similar result when changing the nonlinearity in such a way that we still privilege the modes 1 and $-1$. The game is to conserve an effective Hamiltonian at order 4 (see Proposition 2.1). For instance $2\cos 2x$ can be replaced by $a\cos 2x + b\sin 2x$ but not by $\cos 4x$ which generates an effective Hamiltonian only at order 6. We can also choose to privilege another couple of modes $p$ and $-p$ by choosing $a\cos 2px + b\sin 2px$. In that case we have also to adapt the initial datum.

– If we choose a nonlinearity that does not depend on $x$, for instance the standard cubic nonlinearity $g = \pm|\psi|^2\psi$, then we can prove that there is no beating effect between any modes for $|t| \leqslant \varepsilon^{-3}$ since $Z_4$ only depends on the actions in that case (independently of the sign in front of the nonlinearity).

– The beating frequency, namely $2J$, depends on all the modes initially excited. For instance if $\psi_0 = \varepsilon(\cos x + \sin x) + \varepsilon^2 \cos qx$ ($q \neq 1$) then we can still prove that there is no energy exchanges between the mode $p$, for $|p| \neq 1$, and modes 1 and $-1$, that there is the same beating effect between modes 1 and $-1$, nevertheless the beating frequency is slightly changed: $2J = 2\varepsilon^2 + \varepsilon^4$.

– As stated in the introduction, when adding a linear potential – a multiplicative one or a convolution one – we can choose the potential in order to avoid resonances between the different blocks of modes $p$ and $-p$ (see [1] for multiplicative potentials or [3] for convolution potentials). In that case the same result can be proved and actually in an easier way since we avoid exchanges between the blocks for arbitrary long time.

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