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Lagrangian flows for vector fields with anisotropic regularity

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Abstract

We prove quantitative estimates for flows of vector fields subject to anisotropic regularity conditions: some derivatives of some components are (singular integrals of) measures, while the remaining derivatives are (singular integrals of) integrable functions. This is motivated by the regularity of the vector field in the Vlasov–Poisson equation with measure density. The proof exploits an anisotropic variant of the argument in [20,14] and suitable estimates for the difference quotients in such anisotropic context. In contrast to regularization methods, this approach gives quantitative estimates in terms of the given regularity bounds. From such estimates it is possible to recover the well posedness for the ordinary differential equation and for Lagrangian solutions to the continuity and transport equations.

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1. Introduction

1.1. Ordinary differential equations with non smooth vector field

Given a smooth vector field $b : [0, T] \times \mathbb{R}^N \to \mathbb{R}^N$, the flow of b is the smooth map $X : [0, T] \times \mathbb{R}^N \to \mathbb{R}^N$ satisfying

$$\begin{cases} \dfrac{dX}{ds}(s, x) = b(s, X(s, x)), \quad s \in [0, T], \\ X(0, x) = x. \end{cases}$$

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In the last years much attention has been devoted to the study of flows associated to vector fields that are not smooth (in particular, less than Lipschitz in the space variable). In this context, the correct notion of flow is that of the *regular Lagrangian flow*, loosely speaking an "almost-everywhere flow which (almost) preserves the Lebesgue measure" (see Definition 3.1 for the precise definition).

Existence, uniqueness and stability of the regular Lagrangian flow have been proved by DiPerna and Lions [23] for Sobolev vector fields, and by Ambrosio [2] for vector fields with bounded variation, in both cases under suitable bounds on the divergence of the vector field. Both results make use of the connection with the well posedness of the continuity equation

$$\partial_t u + \operatorname{div}(bu) = 0,$$

which in turn is analyzed thanks to the theory of renormalized solutions. We address the interested reader to [5,6,12,18,22] for a detailed presentation of these results and for the further references.

1.2. Quantitative estimates for the ordinary differential equation

An alternative and more direct approach has been introduced in [20]. Many of the ODE results in [23] can be derived with simple a priori estimates, directly at the Lagrangian level, by studying a functional measuring an "integral logarithmic distance" between flows.

In detail, given two regular Lagrangian flows X and $\bar{X}$ associated to a vector field b, the idea is to consider the functional

$$\Phi_\delta(s) = \int \log\left(1 + \frac{|X(s,x) - \bar{X}(s,x)|}{\delta}\right) dx, \tag{1.1}$$

where $\delta > 0$ is a given parameter (which will be optimized in the course of the proof) and the integration is performed on a suitable compact set.

It is immediate to derive the following lower estimate, for a given $\gamma > 0$:

$$\Phi_\delta(s) \geq \int\limits_{\{|X-\bar{X}|\geq\gamma\}} \log\left(1 + \frac{\gamma}{\delta}\right) dx = \mathcal{L}^N(\{|X - \bar{X}| \geq \gamma\}) \log\left(1 + \frac{\gamma}{\delta}\right),$$

that is, the measure of the superlevels of the difference between two regular Lagrangian flows is upper estimated by

$$\frac{\Phi_\delta(s)}{\log\left(1 + \frac{\gamma}{\delta}\right)}. \tag{1.2}$$

A strategy for proving uniqueness is therefore deriving upper bounds on the functional $\Phi_\delta(s)$ which blow up in δ slower than $\log(1/\delta)$ as $\delta \to 0$.

Differentiating in time the functional and using the ordinary differential equation we obtain

$$\Phi'_\delta(s) \leq \int \frac{|b(X) - b(\bar{X})|}{\delta + |X - \bar{X}|} dx \leq \int \min\left\{\frac{2\|b\|_\infty}{\delta};\ \frac{|b(X) - b(\bar{X})|}{|X - \bar{X}|}\right\} dx. \tag{1.3}$$

In [20] it has been noted that the estimate of the difference quotients in terms of the maximal function

$$\frac{|b(X) - b(\bar{X})|}{|X - \bar{X}|} \lesssim MDb(X) + MDb(\bar{X}),$$

together with the *strong* estimate for the maximal function (2.7), imply an upper bound on $\Phi_\delta(s)$ independent of δ. This allowed in [20] the proof of existence, uniqueness, stability (with an effective rate), compactness, and mild regularity for the regular Lagrangian flow associated to a vector field with Sobolev regularity $W^{1,p}$, with $p > 1$. We note in passing that the rate obtained in these estimates has been recently proved to be sharp (see [1,30]).

The case $p = 1$ (and the more general case of vector fields with bounded variation) was left open in the above analysis due to the failure of the strong estimate (2.7): only the weak estimate (2.8) is available for $p = 1$. This case has been studied in [14] exploiting interpolation techniques in weak Lebesgue spaces. The weak estimate on the

second term in the minimum in (1.3) is interpolated with the (degenerating in δ) L^∞ estimate on the first term in the minimum. This gives an upper bound of the form

$$\Phi_\delta(s) \lesssim \|Db\|_{L^1} \log\left(\frac{1}{\delta}\right). \tag{1.4}$$

This estimate is on the critical scale discriminating uniqueness. Therefore we have to play with constants: up to an L^2-remainder, the L^1-norm of Db can be assumed to be arbitrarily small (we exploit here *equi-integrability* bounds on Db). This allows to re-gain smallness in (1.2) (notice that the L^2 part can be treated as in [20]).

For this reason the analysis in [14] is not able to address the case when Db is a measure (i.e., the case of a vector field with bounded variation). On the other hand, by considering *smooth maximal functions* instead of classical ones, and by exploiting more sophisticated tools from harmonic analysis, the case in which Db is a *singular integral* of an L^1 functional can be treated with the same strategy. This extends the case $b \in W^{1,1}$ and is relevant for some applications to nonlinear PDEs (see [9,10]). Results of existence, uniqueness, stability (with an effective rate), and compactness follows as in [20]. We refer to [6] and to the introduction of [14] for a more detailed presentation. See also [7,13,15–17,19,21,24].

1.3. A split case and the main result of the present paper

As mentioned above, the analysis in [14] is not able to include the case when Db is a measure; concerning the case of a singular integral of a measure, a counterexample in [23] shows that in general uniqueness may fail. However, in situations originating from models in mathematical physics, the vector field is endowed with a particular structure, and just *some* of the derivatives are singular integrals of measures, while the remaining derivatives are more regular.

For instance, the Vlasov–Poisson system

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E(t,x) \cdot \nabla_v f = 0, \\ E(t,x) = -\nabla_x U(t,x), \quad -\Delta_x U(t,x) = \omega\rho(t,x) = \omega \int f(t,x,v)\, dv, \end{cases}$$

where $\omega = \pm 1$, entails a (nonlinear) transport equation with vector field $b(t,x,v) = (v, E(t,x))$. If we look at the case when the space density ρ is a measure, it turns out that $D_x E$ is a singular integral of a measure, while all other derivatives of the vector field enjoy better regularity. However, we are not able to consider the case of f a measure in x, v, that has been studied in [31,27], since the characteristics are defined only almost everywhere. We mention however that some applications of this Lagrangian theory to the Vlasov–Poisson system with L^1 space density are presented in [9]. See also [4], where similar arguments have been applied to the study of the Vlasov–Poisson system, also exploiting the notion of maximal regular flow [3].

This motivates the setting of the present paper. We write $\mathbb{R}^N = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with coordinates x_1 and x_2, and split analogously the vector field according to $b = (b_1, b_2)$. Roughly speaking, we consider the case in which $D_1 b_2$ is a singular integral (in $\mathbb{R}^{n_1}$) of a measure, while $D_1 b_1$, $D_2 b_1$ and $D_2 b_2$ are singular integrals (in $\mathbb{R}^{n_1}$) of integrable functions:

$$Db = \begin{pmatrix} S * L^1 & S * L^1 \\ S * \mathcal{M} & S * L^1 \end{pmatrix}$$

(in fact our assumptions are slightly more general: see assumption **(R2)** in Section 4). Compared to [14], we are able to consider a situation in which some entries of the differential matrix Db are measures. (From a PDE point of view, related contexts have been considered in [25,26]).

The idea, analogous to the anisotropic regularization of [11,2], is to "weight" differently the two (groups of) directions, according to the different degrees of regularity. In our context, this can be done by considering, instead of (1.1), a functional depending on *two* parameters δ_1 and δ_2, with $\delta_1 \le \delta_2$, namely

$$\Phi_{\delta_1,\delta_2}(s) = \int \log\left(1 + \left|\left(\frac{|X_1(s,x) - \bar{X}_1(s,x)|}{\delta_1}, \frac{|X_2(s,x) - \bar{X}_2(s,x)|}{\delta_2}\right)\right|\right) dx. \tag{1.5}$$

Following the same strategy as before (estimate of the difference quotients and interpolation in the minimum in (1.3)), we derive the following bound, which replaces (1.4) in this context:

$$\Phi_{\delta_1,\delta_2}(s) \lesssim \left[\frac{\delta_1}{\delta_2}\|D_1 b_2\|_{\mathcal{M}} + \frac{\delta_2}{\delta_1}\|D_2 b_1\|_{L^1} + \|D_1 b_1\|_{L^1} + \|D_2 b_2\|_{L^1}\right] \log\left(\frac{1}{\delta_2}\right).$$

We need to gain some "smallness" in criterion (1.2). Observe that $\|D_2 b_1\|_{L^1}$, $\|D_1 b_1\|_{L^1}$ and $\|D_2 b_2\|_{L^1}$ can be assumed to be small, by the same equi-integrability argument as in [14]. This is however not the case for $\|D_1 b_2\|_{\mathcal{M}}$. But we can exploit the presence of the coefficient δ_1/δ_2 multiplying this term: both δ_1 and δ_2 have to be sent to zero, but we can do this with $\delta_1 \ll \delta_2$.

One relevant technical point in the proof is the estimate for the anisotropic difference quotients showing up when differentiating (1.5). We need an estimate of the form:

$$|f(x) - f(y)| \lesssim \left| \left(\frac{x_1 - y_1}{\delta_1}, \frac{x_2 - y_2}{\delta_2} \right) \right| \Big[U(x) + U(y) \Big]. \tag{1.6}$$

This is complicated by the fact that, as in the classical case, one expects to use a maximal function in x_1 and x_2 in order to estimate the difference quotients, but however this would not match (in terms of persistence of cancellations) with the presence of a singular integral in the variable x_1 only. This is resolved in Section 5 by the use of tensor products of maximal functions, and will result in the proof of (1.6) together with a bound of the form

$$\|U\| \leq \delta_1 \|D_1 f\| + \delta_2 \|D_2 f\|.$$

This is the plan how to obtain the proof of our main Theorem 6.1, containing the fundamental estimate for the distance between two regular Lagrangian flows associated to vector fields under the regularity assumption **(R2)**. As recalled in Section 6 we obtain as a corollary of Theorem 6.1 existence, uniqueness, stability (with an effective rate) and compactness for regular Lagrangian flows, and well posedness for Lagrangian solutions to the continuity and transport equations.

2. Background material

This section is devoted to recalling some classical definitions and results from harmonic analysis. Most of the results below are stated without proofs, for which we refer to [28,29]. The proofs of the more specific results and additional comments can be found in [14].

2.1. Weak Lebesgue spaces and equi-integrability

We will denote by $\mathcal{L}^d$ the d-dimensional Lebesgue measure and by $B_r(x)$ the open ball or radius $r > 0$ centered at $x \in \mathbb{R}^d$, shortened to B_r in case the center of the ball is the origin of $\mathbb{R}^d$.

Definition 2.1. Let u be a measurable function on $\Omega \subset \mathbb{R}^d$. For $1 \leq p < \infty$, we set

$$|||u|||^p_{M^p(\Omega)} = \sup_{\lambda > 0} \{\lambda^p \mathcal{L}^d(\{x \in \Omega \,:\, |u(x)| > \lambda\})\}$$

and define the weak Lebesgue space $M^p(\Omega)$ as the space consisting of all such measurable functions $u : \Omega \to \mathbb{R}$ with $|||u|||_{M^p(\Omega)} < \infty$. For $p = \infty$, we set $M^\infty(\Omega) = L^\infty(\Omega)$.

Let us remark that the quantity $|||\cdot|||^p_{M^p(\Omega)}$ is *not* a norm, but just a quasinorm, therefore we have chosen the notation with the three vertical bars, different from the usual one for the norm.

The following lemma shows that we can interpolate M^1 and M^p, with $p > 1$, obtaining a bound on the L^1 norm, which depends logarithmically on the M^p quasinorm.

Lemma 2.2 *(Interpolation). Let $u : \Omega \mapsto [0, +\infty)$ be a nonnegative measurable function, where $\Omega \subset \mathbb{R}^d$ has finite measure. Then for every $1 < p < \infty$, we have the interpolation estimate*

$$||u||_{L^1(\Omega)} \leq \frac{p}{p-1} |||u|||_{M^1(\Omega)} \left[1 + \log \left(\frac{|||u|||_{M^p(\Omega)}}{|||u|||_{M^1(\Omega)}} \mathcal{L}^d(\Omega)^{1-\frac{1}{p}} \right) \right],$$

and analogously for $p = \infty$

$$||u||_{L^1(\Omega)} \leq |||u|||_{M^1(\Omega)} \left[1 + \log \left(\frac{||u||_{L^\infty(\Omega)}}{|||u|||_{M^1(\Omega)}} \mathcal{L}^d(\Omega) \right) \right].$$

We also recall the classical definition of equi-integrability.

Definition 2.3 *(Equi-integrability).* Let Ω be an open subset of $\mathbb{R}^d$. We say that a bounded family $\{\varphi_i\}_{i\in I} \subset L^1(\Omega)$ is equi-integrable if the following two conditions hold:

(i) For any $\varepsilon > 0$ there exists a Borel set $A \subset \Omega$ with finite measure such that $\int_{\Omega\setminus A} |\varphi_i|\,dx \le \varepsilon$ for any $i \in I$;
(ii) For any $\varepsilon > 0$ there exists $\delta > 0$ such that, for every Borel set $E \subset \Omega$ with $\mathcal{L}^d(E) \le \delta$, there holds $\int_E |\varphi_i|\,dx \le \varepsilon$ for any $i \in I$.

The Dunford–Pettis theorem ensures that a bounded family in $L^1(\Omega)$ is relatively compact for the weak L^1 topology if and only if it is equi-integrable. Also, a sequence $u_n \in L^1(\mathbb{R}^d)$ converges to u in $L^1(\mathbb{R}^d)$ if and only if it is equi-integrable and u_n converges to u locally in measure. The following lemma can be proved with elementary tools.

Lemma 2.4. *Consider a family $\{\varphi_i\}_{i\in I} \subset L^1(\Omega)$ which is bounded in $L^1(\Omega)$ and fix $1 < p \le \infty$. Then this family is equi-integrable if and only if for every $\varepsilon > 0$ there exists a constant C_ε and a Borel set $A_\varepsilon \subset \Omega$ with finite measure such that for every $i \in I$ one can write*

$$\varphi_i = \varphi_i^1 + \varphi_i^2,$$

with

$$\|\varphi_i^1\|_{L^1(\Omega)} \le \varepsilon \qquad \text{and} \qquad \operatorname{spt}(\varphi_i^2) \subset A_\varepsilon, \quad \|\varphi_i^2\|_{L^p(\Omega)} \le C_\varepsilon \qquad \text{for all } i \in I.$$

2.2. Singular integrals

We briefly summarize the classical Calderón–Zygmund theory of singular integrals.

Definition 2.5. We say that K is a *singular kernel* on $\mathbb{R}^d$ if

(1) $K \in \mathcal{S}'(\mathbb{R}^d)$ and $\hat{K} \in L^\infty(\mathbb{R}^d)$,
(2) $K|_{\mathbb{R}^d\setminus\{0\}} \in L^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\})$ and there exists a constant $A \ge 0$ such that

$$\int_{|x|>2|y|} |K(x-y) - K(x)|dx \le A$$

for every $y \in \mathbb{R}^d$.

We now state a classical result that allows the extension of (the convolution with) a singular kernel to an operator on L^p spaces.

Theorem 2.6 *(Calderón–Zygmund). Let K be a singular kernel and define*

$$Su = K * u \qquad \text{for } u \in L^2(\mathbb{R}^d)$$

in the sense of multiplication in the Fourier variable. Then for every $1 < p < \infty$ we have the strong estimate

$$\|Su\|_{L^p(\mathbb{R}^d)} \le C_{N,p}(A + ||\hat{K}||_{L^\infty})||u||_{L^p(\mathbb{R}^d)} \qquad u \in L^p \cap L^2(\mathbb{R}^d), \tag{2.1}$$

and for $p = 1$ the weak estimate

$$|||Su|||_{M^1(\mathbb{R}^d)} \le C_N(A + ||\hat{K}||_{L^\infty})||u||_{L^1(\mathbb{R}^d)} \qquad u \in L^1 \cap L^2(\mathbb{R}^d). \tag{2.2}$$

In addition, the operator S can be extended to the whole $L^p(\mathbb{R}^d)$ for any $1 < p < \infty$ with values in $L^p(\mathbb{R}^d)$, still satisfying (2.1). For $p = 1$, the operator S extends to the whole $L^1(\mathbb{R}^d)$ to an operator S^{M^1} with values in $M^1(\mathbb{R}^d)$, still

satisfying (2.2). However, a function in $M^1(\mathbb{R}^d)$ is in general not integrable, therefore it does not define a distribution. Notice that, for $u \in L^1(\mathbb{R}^d)$, we can define a tempered distribution $S^D \in \mathcal{S}'(\mathbb{R}^d)$ by the formula

$$\langle S^D u, \varphi \rangle = \langle u, \tilde{S}\varphi \rangle \qquad \text{for every } \varphi \in \mathcal{S}(\mathbb{R}^d), \tag{2.3}$$

where $\tilde{S}$ is the singular integral operator associated to the kernel $\tilde{K}(x) = K(-x)$. The same holds for u a finite measure in $\mathbb{R}^d$. The two operators S^{M^1} and S^D are different and cannot be identified. Since $\mathcal{F} : L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ is bounded, and by definition we have $\hat{K} \in L^\infty(\mathbb{R}^d)$, the definition in (2.3) is equivalent to the definition in Fourier variables

$$\widehat{S^D u} = \hat{K}\hat{u}.$$

We also recall a particular class of singular kernels:

Definition 2.7. A kernel K is a *singular kernel of fundamental type* in $\mathbb{R}^d$ if the following properties hold:

(1) $K|_{\mathbb{R}^d \setminus \{0\}} \in C^1(\mathbb{R}^d \setminus \{0\})$,
(2) There exists a constant $C_0 \geq 0$ such that

$$|K(x)| \leq \frac{C_0}{|x|^d} \qquad x \in \mathbb{R}^d \setminus \{0\}, \tag{2.4}$$

(3) There exists a constant $C_1 \geq 0$ such that

$$|\nabla K(x)| \leq \frac{C_1}{|x|^{d+1}} \qquad x \in \mathbb{R}^d \setminus \{0\}, \tag{2.5}$$

(4) There exists a constant $A_1 \geq 0$ such that

$$\left| \int\limits_{R_1 < |x| < R_2} K(x)dx \right| \leq A_1 \qquad \text{for every } 0 < R_1 < R_2 < \infty. \tag{2.6}$$

In particular, these conditions are sufficient to extend the function defined on $\mathbb{R}^d \setminus \{0\}$ to a singular kernel K on $\mathbb{R}^d$, unique up to addition of a multiple of a Dirac delta at the origin, and which satisfies the estimates in Definition 2.5.

2.3. Maximal functions

We now recall the classical maximal function.

Definition 2.8. Let $u \in L^1_{\text{loc}}(\mathbb{R}^d)$. The maximal function of u is defined as

$$Mu(x) = \sup_{\varepsilon > 0} \fint\limits_{B_\varepsilon(x)} |u(y)|dy$$

for every $x \in \mathbb{R}^d$.

The maximal function Mu is finite almost everywhere for $u \in L^p(\mathbb{R}^d)$, for every $1 \leq p \leq \infty$. For every $1 < p \leq \infty$ we have the strong estimate

$$\|Mu\|_{L^p(\mathbb{R}^d)} \leq C_{d,p} ||u||_{L^p(\mathbb{R}^d)}, \tag{2.7}$$

with only the weak estimate for $p = 1$

$$|||Mu|||_{M^1(\mathbb{R}^d)} \leq C_d ||u||_{L^1(\mathbb{R}^d)}. \tag{2.8}$$

2.4. The smooth maximal function and cancellations

Given two singular kernels of fundamental type K_1 and K_2, with bounded and smooth Fourier transform, we consider the associated singular integral operators S_1 and S_2. The composition $S_2 \circ S_1$ is still a singular integral operator S, associated to a singular kernel K characterized by $\widehat{K} = \widehat{K_2}\widehat{K_1}$. In general, composing two weak estimates (as in (2.2)) is not well defined. However, there are cancellations in the convolution $K_2 * K_1$ (that is, in the composition of the two singular integral operators), which allow us to define $S_2 \circ S_1$. A very important result is that we can compose a special class of maximal functions with a singular integral operator, yielding a composition operator that is bounded $L^1 \to M^1$ and $L^2 \to L^2$.

We consider a maximal function that is "smaller" than the classical maximal function, in order to allow cancellations with the singular integral operator. Here the absolute value is outside the integral, instead of inside. The result after taking smooth averages is a maximal function that is "smoother" than the classical maximal function.

Definition 2.9 *(Smooth maximal function).* Given a family of functions $\{\rho^\nu\}_\nu \subset L^\infty_c(\mathbb{R}^d)$, for every function $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ we define the $\{\rho^\nu\}$-maximal function of u as

$$M_{\{\rho^\nu\}}(u)(x) = \sup_\nu \sup_{\varepsilon>0} \left| \int_{\mathbb{R}^d} \rho^\nu_\varepsilon(x-y)u(y)dy \right| = \sup_\nu \sup_{\varepsilon>0} \left|(\rho^\nu_\varepsilon * u)(x)\right|,$$

where as usual

$$\rho^\nu_\varepsilon(x) = \frac{1}{\varepsilon^d}\rho^\nu\left(\frac{x}{\varepsilon}\right).$$

In the case when u is a distribution, we take a smooth family $\{\rho^\nu\}_\nu \subset C^\infty_c(\mathbb{R}^d)$ and define in the distributional sense

$$M_{\{\rho^\nu\}}(u)(x) = \sup_\nu \sup_{\varepsilon>0} \left|\langle u, \rho^\nu_\varepsilon(x-\cdot)\rangle\right|.$$

The importance of this class of maximal functions is that it is possible to define the composition $M_{\{\rho^\nu\}}S$ with a singular integral operator, which is impossible with the usual maximal function. The following theorem has been proved in [14].

Theorem 2.10. *Let K be a singular kernel of fundamental type and let S be the associated singular integral operator. Let $\{\rho^\nu\}_\nu \subset L^\infty(\mathbb{R}^d)$ be a family of kernels such that*

$$\operatorname{spt}\rho^\nu \subset B_1 \quad \text{and} \quad \|\rho^\nu\|_{L^1(\mathbb{R}^d)} \le Q_1 \qquad \text{for every } \nu.$$

Assume that for every $\varepsilon > 0$ and every ν there holds

$$\left\|\left(\varepsilon^d K(\varepsilon\cdot)\right) * \rho^\nu\right\|_{C_b(\mathbb{R}^d)} \le Q_2 \qquad \text{for every } \varepsilon > 0 \text{ and for every } \nu.$$

Then we have the following estimates.

(1) *There exists a constant C_d, depending on the dimension d only, such that*

$$|||M_{\{\rho^\nu\}}(Su)|||_{M^1(\mathbb{R}^d)} \le C_d\Big(Q_2 + Q_1(C_0 + C_1 + ||\widehat{K}||_\infty\Big)||u||_{L^1(\mathbb{R}^d)}$$

for every $u \in L^1 \cap L^2(\mathbb{R}^d)$. If $\{\rho^\nu\} \subset C^\infty_c(\mathbb{R}^d)$, and u is a finite measure on $\mathbb{R}^d$, then the same estimate holds, where Su is defined as a distribution $S^D u$:

$$|||M_{\{\rho^\nu\}}(Su)|||_{M^1(\mathbb{R}^d)} \le C_d\Big(Q_2 + Q_1(C_0 + C_1 + ||\widehat{K}||_\infty\Big)||u||_{\mathcal{M}(\mathbb{R}^d)}.$$

(2) *If $Q_3 = \sup_\nu ||\rho^\nu||_{L^\infty(\mathbb{R}^d)}$ is finite, then there exists C_d, depending on the dimension d only, such that*

$$\|M_{\{\rho^\nu\}}(Su)\|_{L^2(\mathbb{R}^d)} \le C_d Q_3 \|\widehat{K}\|_\infty \|u\|_{L^2(\mathbb{R}^d)}$$

for every $u \in L^2(\mathbb{R}^d)$.

3. Regular Lagrangian flows

As mentioned in the Introduction, we will deal with flows of non-smooth vector fields. The adequate notion of flow in this context is that of the *regular Lagrangian flow*. Given a vector field $b(s,x):(0,T)\times\mathbb{R}^N\to\mathbb{R}^N$, we assume the following growth condition:

(R1) The vector field $b(s,x)$ can be decomposed as

$$\frac{b(s,x)}{1+|x|}=\tilde{b}_1(s,x)+\tilde{b}_2(s,x),$$

with

$$\tilde{b}_1\in L^1((0,T);L^1(\mathbb{R}^N)) \qquad \text{and} \qquad \tilde{b}_2\in L^1((0,T);L^\infty(\mathbb{R}^N)).$$

Given a vector field satisfying **(R1)**, we codify in the following definition of *regular Lagrangian flow* the notion of "almost everywhere flow which almost preserves the Lebesgue measure". We denote by L^0_{loc} the space of measurable functions endowed with the local convergence in measure, by $\mathcal{B}$ the space of bounded functions, and by $\log L_{\mathrm{loc}}$ the space of the locally logarithmically integrable functions.

Definition 3.1 *(Regular Lagrangian flow).* If b is a vector field satisfying **(R1)**, then for fixed $t\in[0,T)$, a map

$$X\in C([t,T]_s;L^0_{\mathrm{loc}}(\mathbb{R}^N_x))\cap\mathcal{B}([t,T]_s;\log L_{\mathrm{loc}}(\mathbb{R}^N_x))$$

is a regular Lagrangian flow in the renormalized sense relative to b starting at t if we have the following:

(1) The equation

$$\partial_s\big(\beta(X(s,x))\big)=\beta'(X(s,x))b(s,X(s,x))$$

holds in $\mathcal{D}'((t,T)\times\mathbb{R}^N)$, for every function $\beta\in C^1(\mathbb{R}^N;\mathbb{R})$ that satisfies $|\beta(z)|\le C(1+\log(1+|z|))$ and $|\beta'(z)|\le\frac{C}{1+|z|}$ for all $z\in\mathbb{R}^N$,

(2) $X(t,x)=x$ for $\mathcal{L}^N$-a.e. $x\in\mathbb{R}^N$,

(3) There exists a constant $L\ge 0$ such that $\int_{\mathbb{R}^N}\varphi(X(s,x))dx\le L\int_{\mathbb{R}^N}\varphi(x)dx$ for all measurable $\varphi:\mathbb{R}^N\to[0,\infty)$.

We will usually refer to the constant L in Definition 3.1(3) as the *compressibility constant* of the flow. We define the sublevel of the flow as

$$G_\lambda=\{x\in\mathbb{R}^N:|X(s,x)|\le\lambda \text{ for almost all } s\in[t,T]\}. \tag{3.1}$$

The following lemma gives an estimate for the decay of the superlevels of a regular Lagrangian flow.

Lemma 3.2. *Let $b:(0,T)\times\mathbb{R}^N\to\mathbb{R}^N$ be a vector field satisfying* **(R1)** *and let $X:[t,T]\times\mathbb{R}^N\to\mathbb{R}^N$ be a regular Lagrangian flow relative to b starting at time t, with compressibility constant L. Then for all $r,\lambda>0$*

$$\mathcal{L}^N(B_r\setminus G_\lambda)\le g(r,\lambda),$$

where the function g depends only on L, $\|\tilde{b}_1\|_{L^1((0,T);L^1(\mathbb{R}^N))}$ and $\|\tilde{b}_2\|_{L^1((0,T);L^\infty(\mathbb{R}^N))}$ and satisfies $g(r,\lambda)\downarrow 0$ for r fixed and $\lambda\uparrow\infty$.

Indeed the regular Lagrangian flow X has a logarithmic summability, and this clarifies the class of renormalization functions β considered in Definition 3.1(1). See [14] for the proof.

4. Regularity assumptions and the anisotropic functional

We wish to consider a regularity setting of the vector field $b(t,x)$ in which the (weak) regularity has a different character with respect to different directions in space. We split $\mathbb{R}^N$ as $\mathbb{R}^N = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with variables $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$. We denote by $D_1 = D_{x_1}$ the derivative with respect to the first n_1 variables x_1, and by $D_2 = D_{x_2}$ the derivative with respect to the last n_2 variables x_2. Accordingly, we denote $b = (b_1, b_2)(s, x_1, x_2)$. For $X(s, x_1, x_2)$ a regular Lagrangian flow associated to b we denote $X = (X_1, X_2)(s, x_1, x_2)$.

We are going to assume that $D_1 b_2$ is "less regular" than $D_1 b_1$, $D_2 b_1$, $D_2 b_2$: the derivative $D_1 b_2$ is a singular integral of a measure, whereas the other derivatives are singular integrals of L^1 functions. This is made precise as follows:

(R2) We assume that

$$Db = \begin{pmatrix} D_1 b_1 & D_2 b_1 \\ D_1 b_2 & D_2 b_2 \end{pmatrix} = \begin{pmatrix} \gamma^1 S^1 \mathfrak{p} & \gamma^2 S^2 \mathfrak{q} \\ \gamma^3 S^3 \mathfrak{m} & \gamma^4 S^4 \mathfrak{r} \end{pmatrix}, \tag{4.1}$$

where the sub-matrices have the representation

$$\begin{aligned}
&i, j \in \{1, \dots, n_1\}: && i \in \{1, \dots, n_1\}, \quad j \in \{n_1+1, \dots, n_2\}: \\
&(D_1 b_1)^i_j = \sum_{k=1}^m \gamma^{1i}_{jk}(s, x_2) S^{1i}_{jk} \mathfrak{p}^i_{jk}(s, x_1) && (D_2 b_1)^i_j = \sum_{k=1}^m \gamma^{2i}_{jk}(s, x_2) S^{2i}_{jk} \mathfrak{q}^i_{jk}(s, x_1) \\
&i \in \{n_1+1, \dots, n_2\}, \quad j \in \{1, \dots, n_1\}: && i \in \{n_1+1, \dots, n_2\}, \quad j \in \{n_1+1, \dots, n_2\}: \\
&(D_1 b_2)^i_j = \sum_{k=1}^m \gamma^{3i}_{jk}(s, x_2) S^{3i}_{jk} \mathfrak{m}^i_{jk}(s, x_1) && (D_2 b_2)^i_j = \sum_{k=1}^m \gamma^{4i}_{jk}(s, x_2) S^{4i}_{jk} \mathfrak{r}^i_{jk}(s, x_1).
\end{aligned}$$

In the above assumptions we have that:
- S^{1i}_{jk}, S^{2i}_{jk}, S^{3i}_{jk}, S^{4i}_{jk} are singular integral operators associated to singular kernels of fundamental type in $\mathbb{R}^{n_1}$,
- the functions $\mathfrak{p}^i_{jk}$, $\mathfrak{q}^i_{jk}$, $\mathfrak{r}^i_{jk}$ belong to $L^1((0,T); L^1(\mathbb{R}^{n_1}))$,
- $\mathfrak{m}^i_{jk} \in L^1((0,T); \mathcal{M}(\mathbb{R}^{n_1}))$,
- the functions $\gamma^{1,i}_{jk}$, $\gamma^{2,i}_{jk}$, $\gamma^{3,i}_{jk}$, $\gamma^{4,i}_{jk}$ belong to $L^\infty((0,T); L^q(\mathbb{R}^{n_2}))$ for some $q > 1$.

We have denoted by $L^1((0,T); \mathcal{M}(\mathbb{R}^{n_1}))$ the space of all functions $t \mapsto \mu(t, \cdot)$ taking values in the space $\mathcal{M}(\mathbb{R}^{n_1})$ of finite signed measures on $\mathbb{R}^{n_1}$ such that

$$\int_0^T \|\mu(t, \cdot)\|_{\mathcal{M}(\mathbb{R}^{n_1})}\, dt < \infty.$$

Remark 4.1. The assumption on the functions $\gamma^{1,i}_{jk}$, $\gamma^{2,i}_{jk}$, $\gamma^{3,i}_{jk}$, $\gamma^{4,i}_{jk}$ could be relaxed to $L^\infty((0,T); L^q_{\mathrm{loc}}(\mathbb{R}^{n_2}))$. This would require the use of a localized maximal function.

We will additionally assume that

(R3) $\qquad b \in L^p_{\mathrm{loc}}([0,T] \times \mathbb{R}^N) \qquad$ for some $p > 1$.

As mentioned in the Introduction, the proof of our main result will exploit an *anisotropic* functional (already provisionally introduced in (1.5)), which extends the functional (1.1) to the regularity setting under investigation. Let A be the constant $N \times N$ matrix

$$A = \mathrm{Diag}(\delta_1, \dots, \delta_1, \delta_2, \dots, \delta_2). \tag{4.2}$$

A acts on vectors in $\mathbb{R}^N$ by a dilation of a factor δ_1 on the first n_1 coordinates, and of a factor δ_2 on the last n_2 coordinates: $A(x_1, x_2) = (\delta_1 x_1, \delta_2 x_2)$.

Given $X(t,x_1,x_2)$ and $\overline{X}(t,x_1,x_2)$ regular Lagrangian flows associated to b and $\bar{b}$ respectively, we denote by G_λ and $\overline{G}_\lambda$ the sublevels of X and $\overline{X}$ defined as in (3.1). The proof of our main theorem (see Theorem 6.1) is based on the study of the following anisotropic functional:

$$\Phi_{\delta_1,\delta_2}(s) = \int\limits_{B_r\cap G_\lambda\cap\overline{G}_\lambda} \log\left(1+\left|A^{-1}\left[X(s,x_1,x_2)-\overline{X}(s,x_1,x_2)\right]\right|\right)dx. \tag{4.3}$$

5. Estimates of anisotropic difference quotients

In this section we first recall the classical estimate for the difference quotients of a BV function, and then recover an analogous "anisotropic" version of this result for vector fields in the regularity setting of **(R2)**. This will be a key tool in order to estimate the functional (4.3).

Lemma 5.1. *If $u\in BV(\mathbb{R}^d)$, then there exists an $\mathcal{L}^d$-negligible set $\mathcal{N}\subset\mathbb{R}^d$ such that*

$$|u(x)-u(y)|\le C_d|x-y|\Big((MDu)(x)+(MDu)(y)\Big)$$

for every $x,y\in\mathbb{R}^d\setminus\mathcal{N}$, where Du is the distributional derivative of u, represented by a measure.

It turns out that an analogous result holds for functions whose derivatives are singular integrals of measures. The following result has been proved in [14]. The smooth maximal function in Definition 2.9 plays an important role in this estimate.

Proposition 5.2. *Let $f\in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and assume that for every $j=1,\dots,d$ we have*

$$\partial_j f=\sum_{k=1}^m R_{jk}g_{jk}$$

in the sense of distributions, where R_{jk} are singular integral operators of fundamental type in $\mathbb{R}^d$ and $g_{jk}\in\mathcal{M}(\mathbb{R}^d)$ for $j=1,\dots,d$ and $k=1,\dots,m$, and $R_{jk}g_{jk}$ is defined in the sense of tempered distributions. Then there exists a nonnegative function $V\in M^1(\mathbb{R}^d)$ and an $\mathcal{L}^d$-negligible set $\mathcal{N}\subset\mathbb{R}^d$ such that for every $x,y\in\mathbb{R}^d\setminus\mathcal{N}$ there holds

$$|f(x)-f(y)|\le|x-y|\Big(V(x)+V(y)\Big),$$

where V is given by

$$V:=\mathcal{V}(R,g)=\sum_{j=1}^d\sum_{k=1}^m M_{\{\Upsilon^{\xi,j},\,\xi\in\mathbb{S}^{d-1}\}}(R_{jk}g_{jk})$$

and $\Upsilon^{\xi,j}$, for $\xi\in\mathbb{S}^{d-1}$ and $j=1,\dots,d$, is a family of smooth functions explicitly constructed in the course of the proof.

Remark 5.3. Theorem 2.10 implies that the operator $g\mapsto\mathcal{V}(R,g)$ is bounded $L^2\to L^2$ and $\mathcal{M}\to M^1$.

In the following three subsections we prove similar estimates in the anisotropic context.

5.1. Split regularity: the isotropic estimate

Given $\{\gamma^\nu(x_1)\}_\nu\subset C^\infty_c(\mathbb{R}^{n_1})$, $\{\rho^\sigma(x_2)\}_\sigma\subset C^\infty_c(\mathbb{R}^{n_2})$ and $u\in\mathcal{S}'(\mathbb{R}^N)$ we define

$$M_{\{\gamma^\nu\otimes\rho^\sigma\}}u(x)=\sup_{\varepsilon>0}\sup_{\nu,\sigma}|(\gamma^\nu(x_1)\rho^\sigma(x_2))_\varepsilon * u(x)|=\sup_{\varepsilon>0}\sup_{\nu,\sigma}\left|\left(\frac{1}{\varepsilon^N}\gamma^\nu\left(\frac{x_1}{\varepsilon^{n_1}}\right)\rho^\sigma\left(\frac{x_2}{\varepsilon^{n_2}}\right)\right)*u(x)\right|. \tag{5.1}$$

We first of all prove an isotropic estimate in a regularity context related to **(R2)**.

Lemma 5.4. *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a function such that for each $j = 1, \dots, N$ we have*

$$\partial_j f = \sum_{k=1}^{m} (R_{jk} g_{jk})(x_1) \gamma_{jk}(x_2), \tag{5.2}$$

where R_{jk} are singular integrals of fundamental type in $\mathbb{R}^{n_1}$, $g_{jk} \in \mathcal{M}(\mathbb{R}^{n_1})$ and $\gamma_{jk} \in L^q(\mathbb{R}^{n_2})$, for some $q > 1$. Then there exists a nonnegative function $V : \mathbb{R}^N \to [0, \infty)$ and an $\mathcal{L}^N$-negligible set $\mathcal{N} \subset \mathbb{R}^N$ such that for every $x, y \in \mathbb{R}^N \setminus \mathcal{N}$

$$|f(x) - f(y)| \le |x - y| \Big(V(x) + V(y) \Big).$$

The function V is given by

$$V := \mathcal{V}(R, \gamma, g) = \sum_{j=1}^{N} \sum_{k=1}^{m} M_{\{\Upsilon^{\xi,j} \otimes \bar{\Upsilon}^{\xi,j}\}}(\gamma_{jk} R_{jk} g_{jk}), \tag{5.3}$$

for suitable smooth compactly supported functions $\Upsilon^{\xi,j}$ and $\bar{\Upsilon}^{\xi,j}$, which will be introduced in the proof.

Proof. We adapt the proof of Proposition 5.2 to the current regularity setting. The difficulty is that a smooth maximal function in $\mathbb{R}^N$ composed with the singular kernel on $\mathbb{R}^{n_1}$ does not enjoy suitable bounds, and so we use a tensor product of smooth functions, as in (5.1).

Let $w = (w_1, w_2) \in \mathbb{R}^N$, and let $\{e_j\}_j$ be the standard basis for $\mathbb{R}^N$. We denote $\{w_1\}^j = (w_1, 1, \dots, 1) \cdot e_j$ and $\{w_2\}^j = (1, \dots, 1, w_2) \cdot e_j$. Define the families of functions

$$\begin{cases} \Upsilon^{\xi,j}(w_1) = h^1\left(\frac{\xi_1}{2} - w_1\right)\{w_1\}^j \\ \bar{\Upsilon}^{\xi,j}(w_2) = h^2\left(\frac{\xi_2}{2} - w_2\right)\{w_2\}^j, \end{cases}$$

where $h^i \in C_c^\infty(\mathbb{R}^{n_i})$ with $\int_{\mathbb{R}^{n_i}} h^i dx_i = 1$ and $\xi \in \mathbb{S}^{N-1}$. Let $h_r = \frac{1}{r^N} h^1(\frac{\cdot}{r}) h^2(\frac{\cdot}{r})$, set $r = |x - y|$, and write

$$f(x) - f(y) = \int_{\mathbb{R}^N} h_r\left(z - \frac{x+y}{2}\right)(f(x) - f(z))dz + \int_{\mathbb{R}^N} h_r\left(z - \frac{x+y}{2}\right)(f(z) - f(y))dz.$$

We assume that f, γ_{jk} and g_{jk} are smooth and compute the following:

$$\begin{aligned} &\int_{\mathbb{R}^N} h_r\left(z - \frac{x+y}{2}\right)(f(x) - f(z))dz \\ &\quad = -\sum_{j=1}^{N} \int_{\mathbb{R}^N} \int_0^1 h_r\left(z - \frac{x+y}{2}\right) \partial_j f(x + t(z-x))(z \cdot e_j - x \cdot e_j)\, dt dz. \end{aligned}$$

After the change of variable $-t(z - x) \mapsto w$ we get

$$\begin{aligned} &= \sum_{j=1}^{N} \int_{\mathbb{R}^N} \int_0^1 h_r\left(\frac{x-y}{2} - \frac{w}{t}\right) \partial_j f(x - w) \frac{w \cdot e_j}{t^{N+1}} dt dw \\ &= r \sum_{j=1}^{N} \int_{\mathbb{R}^N} \int_0^1 \frac{1}{t^N} h_r\left(\frac{x-y}{2} - \frac{w}{t}\right) \frac{w \cdot e_j}{tr} \partial_j f(x - w)\, dt dw \\ &= r \sum_{j=1}^{N} \sum_{k=1}^{m} \int_0^1 \left[\int_{\mathbb{R}^{n_1}} \frac{1}{t^{n_1}} h_r^1\left(\frac{x_1 - y_1}{2} - \frac{w_1}{t}\right) \left\{\frac{w_1}{tr}\right\}^j R_{jk} g_{jk}(x_1 - w_1)\, dw_1 \right. \end{aligned}$$

$$\times \int_{\mathbb{R}^{n_2}} \frac{1}{t^{n_2}} h_r^2\left(\frac{x_2-y_2}{2}-\frac{w_2}{t}\right)\left\{\frac{w_2}{tr}\right\}^j \gamma_{jk}(x_2-w_2)\,dw_2\Bigg]dt$$

$$= r\sum_{j=1}^{N}\sum_{k=1}^{m}\int_0^1 \left[\frac{1}{t^{n_1}} h_r^1\left(\frac{x_1-y_1}{2}-\frac{w_1}{t}\right)\left\{\frac{w_1}{tr}\right\}^j \underset{w_1}{*} R_{jk}g_{jk}(w_1)\right](x_1)$$

$$\times \left[\frac{1}{t^{n_2}} h_r^2\left(\frac{x_2-y_2}{2}-\frac{w_2}{t}\right)\left\{\frac{w_2}{tr}\right\}^j \underset{w_2}{*} \gamma_{jk}(w_2)\right](x_2)\,dt.$$

Denoting $\Upsilon_\varepsilon^{\xi,j}(w_1)=\frac{1}{\varepsilon^{n_1}}\Upsilon^{\xi,j}\left(\frac{w_1}{\varepsilon}\right)$ and $\bar\Upsilon_\varepsilon^{\xi,j}(w_2)=\frac{1}{\varepsilon^{n_2}}\bar\Upsilon^{\xi,j}\left(\frac{w_2}{\varepsilon}\right)$, this expression equals

$$r\sum_{j=1}^{N}\sum_{k=1}^{m}\int_0^1 [\Upsilon_{tr}^{\frac{x-y}{|x-y|},j} \underset{1}{*} R_{jk}g_{jk}](x_1)\,[\bar\Upsilon_{tr}^{\frac{x-y}{|x-y|},j} \underset{2}{*} \gamma_{jk}](x_2)\,dt,$$

and so

$$\left|\int_{\mathbb{R}^N} h_r\left(z-\frac{x+y}{2}\right)(f(x)-f(z))dz\right|$$

$$\le |x-y|\sum_{j=1}^{N}\sum_{k=1}^{m}\int_0^1 |[\Upsilon_{tr}^{\frac{x_1-y_1}{|x-y|},j} \underset{1}{*} R_{jk}g_{jk}](x_1)\,[\bar\Upsilon_{tr}^{\frac{x_2-y_2}{|x-y|}} \underset{2}{*} \gamma_{jk}](x_2)|\,dt$$

$$\le |x-y|\sum_{j=1}^{N}\sum_{k=1}^{m}\int_0^1 \sup_{\varepsilon>0}\sup_{\xi} |[\Upsilon_\varepsilon^{\xi,j} \underset{1}{*} R_{jk}g_{jk}](x_1)\,[\bar\Upsilon_\varepsilon^{\xi,j} \underset{2}{*} \gamma_{jk}](x_2)|\,dt$$

$$= |x-y|\sum_{j=1}^{N}\sum_{k=1}^{m} M_{\{\Upsilon^{\xi,j}\otimes\bar\Upsilon^{\xi,j}\}}(\gamma_{jk}R_{jk}g_{jk})(x) = |x-y|V(x).$$

This proves the statement in the smooth case. By a similar approximation argument as in [14], we conclude this holds for functions of the type in (5.2). □

5.2. *Split regularity: the anisotropic estimate*

We now modify Lemma 5.4 to obtain an estimate in which distances are measured "anisotropically" through the matrix A defined in (4.2). In the next lemma we will use the following notation:

$$\check g_{ij}(x_1) = g_{jk}(\delta_1 x_1), \qquad \check\gamma_{ij}(x_2) = \gamma_{ij}(\delta_2 x_2),$$

where with $g_{jk}(\delta_1 x_1)$ we denote the measure on $\mathbb{R}^{n_1}$ defined through

$$\langle g_{jk}(\delta_1 x_1), \varphi(x_1)\rangle = \delta_1^{-n_1}\langle g_{ij}(y_1), \varphi(y_1/\delta_1)\rangle, \qquad \varphi\in C_c^\infty(\mathbb{R}^{n_1}).$$

Moreover, $R_{jk}^{\delta_1}$ denotes the singular integral operator in $\mathbb{R}^{n_1}$ associated to the kernel $K_{jk}^{\delta_1}$, where

$$K_{jk}^{\delta_1}(x_1) = \delta_1^{n_1} K_{ij}(\delta_1 x_1). \tag{5.4}$$

Lemma 5.5. *Let $f:\mathbb{R}^N\to\mathbb{R}$ be a function in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ such that for each $j=1,\dots,N$ we have that $\partial_j f$ is as in (5.2). Let A be the matrix defined in (4.2). Then there exists a nonnegative function $U:\mathbb{R}^N\to[0,\infty)$, such that for $\mathcal{L}^N$-a.e. $x,y\in\mathbb{R}^N$,*

$$|f(x)-f(y)| \le |A^{-1}[x-y]|\Big(U(x)+U(y)\Big),$$

where (with the notation above)

$$U(x) = \mathcal{U}(R,\gamma,g)(x) = \sum_{j=1}^{N}\sum_{k=1}^{m}[M_{\{\Upsilon^{\xi,j}\otimes\bar{\Upsilon}^{\xi,j}\}}(R_{jk}^{\delta_1}\check{g}_{jk}\check{\gamma}_{jk}A_{jj})](A^{-1}x).$$

Proof. Define the following rescaled vector field. For each $z\in\mathbb{R}^N$, define

$$\check{f}(z) = f(Az).$$

Now $D\check{f}$ is related to Df by the following:

$$\partial_j\check{f}(z) = \partial_j f(Az)A_{jj} = \sum_{k=1}^{m}\gamma_{jk}(\delta_2 z_2)R_{jk}g_{jk}(\delta_1 z_1)A_{jj}.$$

We now apply Lemma 5.4. This gives the existence of a function $V\in M^1_{\mathrm{loc}}(\mathbb{R}^N)$ to estimate the difference quotient of $\check{f}$:

$$|\check{f}(z)-\check{f}(w)| \le |z-w|(V(z)+V(w)), \tag{5.5}$$

with V given by

$$V(z) = \mathcal{V}(R,\gamma,g) = \sum_{j=1}^{N}\sum_{k=1}^{m}M_{\{\Upsilon^{\xi,j}\otimes\bar{\Upsilon}^{\xi,j}\}}\Big(\gamma_{jk}(\delta_2 z_2)R_{jk}g_{jk}(\delta_1 z_1)\Big)A_{jj}. \tag{5.6}$$

With a change of variable we can verify that

$$(R_{jk}g_{jk})(\delta_1 z_1) = (R_{jk}^{\delta_1}\check{g}_{jk})(z_1).$$

Thus we can rewrite (5.6) as

$$V(z) = \sum_{j=1}^{N}\sum_{k=1}^{m}[M_{\{\Upsilon^{\xi,j}\otimes\bar{\Upsilon}^{\xi,j}\}}(R_{jk}^{\delta_1}\check{g}_{jk}\check{\gamma}_{jk}A_{jj})](z). \tag{5.7}$$

By letting $U(x) = V(A^{-1}x)$ the proof is concluded. □

Remark 5.6. In order to treat the case of a function with gradient given by the singular integral in $\mathbb{R}^N$ of a measure, that is

$$\partial_j f = \sum_{k=1}^{m}R_{jk}g_{jk}, \tag{5.8}$$

with R_{jk} singular integrals of fundamental type in $\mathbb{R}^N$ and $g_{jk}\in\mathcal{M}(\mathbb{R}^N)$, one should consider the function

$$U(x) = \mathcal{U}(R,g)(x) = \sum_{j=1}^{N}\sum_{k=1}^{m}[M_{\{\Upsilon^{\xi,j}\}}R_{jk}^{A}(g_{jk}(A\cdot))A_{jj})](A^{-1}x),$$

where R_{ij}^A is the singular integral operator corresponding to the kernel

$$K_{ij}^A(x) = |\det A|\,K_{ij}(Ax)$$

and A is the diagonal matrix defined in (4.2). This would however give a more singular estimate in Lemma 5.7 below, and would therefore be useless for the proof of Theorem 6.1.

On the other hand it is possible to treat the case $R_{ij}=\delta$ in (5.8), since the Dirac delta "does not see the dilation". This would correspond to the case of a vector field $b=(b_1,b_2)$ such that b_2 is BV in x_1 and $W^{1,1}$ in x_2, and b_1 is $W^{1,1}$ in both x_1 and x_2, the situation of [11]. This will be presented in [8].

5.3. Split regularity: operator bounds

We finally establish suitable estimates on the norms of the operator defined in Lemma 5.5.

Lemma 5.7. *Let $\mathcal{U}(R,\gamma,g)$ be as in Lemma 5.5. Then for any $1<p<\infty$ we have*

$$|||\mathcal{U}(R,\gamma,g)|||_{M^1(\Omega_r)} \leq C_{r,p,m}\left(\delta_1\sum_{j=1}^{n_1}\sum_{k=1}^{m}||\gamma_{jk}||_{L^p(\mathbb{R}^{n_2})}||g_{jk}||_{\mathcal{M}(\mathbb{R}^{n_1})}+\delta_2\sum_{j=n_1+1}^{N}\sum_{k=1}^{m}||\gamma_{jk}||_{L^p(\mathbb{R}^{n_2})}||g_{jk}||_{\mathcal{M}(\mathbb{R}^{n_1})}\right),$$

where $\Omega_r=B^1_r\times B^2_r\subset\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$, and

$$||\mathcal{U}(R,\gamma,g)||_{L^p(\mathbb{R}^N)}\leq C_p\left(\delta_1\sum_{j=1}^{n_1}\sum_{k=1}^{m}||\gamma_{jk}||_{L^p(\mathbb{R}^{n_2})}||g_{jk}||_{L^p(\mathbb{R}^{n_1})}+\delta_2\sum_{j=n_1+1}^{N}\sum_{k=1}^{m}||\gamma_{jk}||_{L^p(\mathbb{R}^{n_2})}||g_{jk}||_{L^p(\mathbb{R}^{n_1})}\right).$$

The constants $C_{r,p,m}$ and C_p also depends on the singular integral operators R_{jk} in (5.2) and on the space dimension. The first constant $C_{r,p,m}$ also depend on the integer m in (5.2).

Proof. Let us start with the estimate in M^1. We define $\check{B}^1_r=B^1_{r/\delta_1}$, $\check{B}^2_r=B^2_{r/\delta_2}$ and $\check{\Omega}_r=\check{B}^1_r\times\check{B}^2_r$. Consider first the measure of the superlevels of $U(x)$: changing variable via the linear transformation $z=A^{-1}x$ we obtain

$$\mathcal{L}^N(\{x\in\Omega_r:|U(x)|>\lambda\})=\mathcal{L}^N(\{x\in\Omega_r:|V(A^{-1}x)|>\lambda\})=\delta_1^{n_1}\delta_2^{n_2}\mathcal{L}^N(\{z\in\check{\Omega}_r:|V(z)|>\lambda\}),$$

where V is as before given by

$$V(z)=\delta_1\sum_{j=1}^{n_1}\sum_{k=1}^{m}[M_{\{\Upsilon^{\xi,j}\otimes\bar{\Upsilon}^{\xi,j}\}}(R^{\delta_1}_{jk}\check{g}_{jk}\check{\gamma}_{jk})](z)+\delta_2\sum_{j=n_1+1}^{N}\sum_{k=1}^{m}[M_{\{\Upsilon^{\xi,j}\otimes\bar{\Upsilon}^{\xi,j}\}}(R^{\delta_1}_{jk}\check{g}_{jk}\check{\gamma}_{jk})](z) \tag{5.9}$$

(compare with (5.7) and split the sum for $1\leq j\leq n_1$ and $n_1+1\leq j\leq n_1+n_2$).

Remembering that $|||f(x_1,x_2)|||_{M^1_{x_1x_2}}\leq\big\||||f(x_1,x_2)|||_{M^1_{x_1}}\big\|_{L^1_{x_2}}$ we estimate for fixed $j=1,\dots,N$ as follows:

$$\begin{aligned}
&\delta_1^{n_1}\delta_2^{n_2}\left|\left|\left|\sum_{k=1}^{m}[M_{\{\Upsilon^{\xi,j}\otimes\bar{\Upsilon}^{\xi,j}\}}(R^{\delta_1}_{jk}\check{g}_{jk}\check{\gamma}_{jk})](z)\right|\right|\right|_{M^1(\check{\Omega}_r)}\\
&\leq C_m\,\delta_1^{n_1}\delta_2^{n_2}\sum_{k=1}^{m}\left|\left|\left|M_{\{\Upsilon^{\xi,j}\}}(R^{\delta_1}_{jk}\check{g}_{jk})\right|\right|\right|_{M^1(\check{B}^1_r)}\left\|M_{\{\bar{\Upsilon}^{\xi,j}\}}\check{\gamma}_{jk}\right\|_{L^1(\check{B}^2_r)}\\
&\leq C_m[\mathcal{L}^{n_2}(\check{B}^2_r)]^{1-1/p}\,\delta_1^{n_1}\delta_2^{n_2}\sum_{k=1}^{m}\left\|\check{g}_{jk}\right\|_{\mathcal{M}(\mathbb{R}^{n_1})}\left\|M_{\{\bar{\Upsilon}^{\xi,j}\}}\check{\gamma}_{jk}\right\|_{L^p(\check{B}^2_r)}\\
&\leq C_{r,p,m}\,\delta_2^{-n_2+n_2/p}\,\delta_1^{n_1}\delta_2^{n_2}\sum_{k=1}^{m}\left\|\check{g}_{jk}\right\|_{\mathcal{M}(\mathbb{R}^{n_1})}\left\|\check{\gamma}_{jk}\right\|_{L^p(\mathbb{R}^{n_2})}\\
&=C_{r,p,m}\sum_{k=1}^{m}\left\|g_{jk}\right\|_{\mathcal{M}(\mathbb{R}^{n_1})}\left\|\gamma_{jk}\right\|_{L^p(\mathbb{R}^{n_2})}.
\end{aligned}$$

In the above chain of inequalities we have used the fact that the norm of $R^{\delta_1}_{jk}$ as singular integral operator coincides with the norm of R_{jk} as singular integral operator.

Recalling (5.9) we immediately obtain the first inequality claimed in the lemma. The second one follows with a simile argument, using the continuity if the operator

$$\check{g}_{jk}\mapsto R^{\delta_1}_{jk}\check{g}_{jk}$$

from $L^p(\mathbb{R}^{n_1})$ into itself. □

6. The fundamental estimate for flows: main theorem and corollaries

Our main theorem is the following:

Theorem 6.1. *Let b and $\bar{b}$ be two vector fields satisfying assumption* **(R1)**, *and assume that b also satisfies assumptions* **(R2)** *and* **(R3)**. *Fix $t \in [0,T)$ and let X and $\bar{X}$ be regular Lagrangian flows starting at time t associated to b and $\bar{b}$ respectively, with compressibility constants L and $\bar{L}$. Then the following holds. For every $\gamma > 0$ and $r > 0$ and for every $\eta > 0$ there exist $\lambda > 0$ and $C_{\gamma,r,\eta} > 0$ such that*

$$\mathcal{L}^n\left(B_r \cap \{|X(s,\cdot) - \bar{X}(s,\cdot)| > \gamma\}\right) \leq C_{\gamma,r,\eta} ||b - \bar{b}||_{L^1((0,T)\times B_\lambda)} + \eta$$

for all $s \in [t,T]$. The constants λ and $C_{\gamma,r,\eta}$ also depend on:

- *The equi-integrability in $L^1((0,T);L^1(\mathbb{R}^{n_1}))$ of $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$, as well as the norm in $L^1((0,T);\mathcal{M}(\mathbb{R}^{n_1}))$ of $\mathfrak{m}$ (where $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$ and $\mathfrak{m}$ are associated to b as in* **(R2)**),
- *The norms of the singular integral operators S^i_{jk}, as well as the norms in $L^\infty((0,T);L^q(\mathbb{R}^{n_2}))$ of γ^i_{jk} (associated to b as in* **(R2)**),
- *The norm in $L^p((0,T)\times B_\lambda)$ of b,*
- *The $L^1((0,T);L^1(\mathbb{R}^N)) + L^1((0,T);L^\infty(\mathbb{R}^N))$ norms of the decompositions of b and $\bar{b}$ as in* **(R1)**,
- *The compressibility constants L and $\bar{L}$.*

From this fundamental estimate, the various corollaries regarding the well posedness of the regular Lagrangian flow and of Lagrangian solutions to the continuity and transport equations follow with the same proofs as in Sections 6 and 7 in [14]. In particular, we obtain:

- Uniqueness of the regular Lagrangian flow associated to a vector field satisfying **(R1)**, **(R2)** and **(R3)**,
- Stability (with an explicit rate) for a sequence X_n of regular Lagrangian flows associated to vector fields b_n, that converge in $L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^N)$ to a vector field satisfying **(R1)**, **(R2)** and **(R3)**, under the assumption that the decompositions of b_n in **(R1)** and the compressibility constants of X_n satisfy uniform bounds,
- Compactness for a sequence X_n of regular Lagrangian flows associated to vector fields b_n satisfying **(R1)**, **(R2)** and **(R3)** with suitable uniform bounds,
- Existence of a regular Lagrangian flow associated to a vector field satisfying **(R1)**, **(R2)** and **(R3)** and such that $[\operatorname{div} b]^- \in L^1((0,T);L^\infty(\mathbb{R}^N))$,
- If a vector field b satisfies **(R1)**, **(R2)** and **(R3)** and $\operatorname{div} b \in L^1((0,T);L^\infty(\mathbb{R}^N))$, then there exists a unique forward and backward regular Lagrangian flow associated to b, which satisfies the usual group property, and the Jacobian of the flow is well defined,
- Lagrangian solutions to the continuity and transport equations with a vector field b satisfying **(R1)**, **(R2)** and **(R3)** and $\operatorname{div} b \in L^1((0,T);L^\infty(\mathbb{R}^N))$ are well defined and stable.

7. Proof of the fundamental estimate (Theorem 6.1)

The proof of Theorem 6.1 makes use of the integral functional

$$\Phi_{\delta_1,\delta_2}(s) = \int\limits_{B_r\cap G_\lambda\cap\overline{G}_\lambda} \log\left(1 + \left|A^{-1}\left[X(s,x_1,x_2) - \overline{X}(s,x_1,x_2)\right]\right|\right) dx$$

already defined in (4.3). In the following proof we assume $\delta_1 \leq \delta_2$.

In order to improve the readability of the following (many) estimates, we will use the notation "$\lesssim$" to denote an estimate up to a constant only depending on absolute constants and on the bounds assumed in Theorem 6.1, and the notation "$\lesssim_\lambda$" to mean that the constant could also depend on the truncation parameter λ. We will however write explicitly the norm of the measure $\mathfrak{m}$, in order to make the reader aware of its role in the estimates.

Step 1: Differentiating Φ_{δ_1,δ_2}. We start by differentiating the integral functional with respect to time:

$$\Phi'_{\delta_1,\delta_2}(s) \leq \int\limits_{B_r\cap G_\lambda\cap\overline{G}_\lambda} \frac{|A^{-1}[b(s,X(s,x_1,x_2)) - \overline{b}(s,\overline{X}(s,x_1,x_2))]|}{1+|A^{-1}[X(s,x_1,x_2)-\overline{X}(s,x_1,x_2)]|}dx.$$

For simplicity, we drop the notation $X(s,x_1,x_2)$, setting $X(s,x_1,x_2)=X$ and $\overline{X}(s,x_1,x_2)=\overline{X}$. We estimate

$$\Phi'_{\delta_1,\delta_2}(s) \leq \int\limits_{B_r\cap G_\lambda\cap\overline{G}_\lambda} |A^{-1}[b(s,\overline{X}) - \overline{b}(s,\overline{X})]|dx + \int\limits_{B_r\cap G_\lambda\cap\overline{G}_\lambda} \frac{|A^{-1}[b(s,X) - b(s,\overline{X})]|}{1+|A^{-1}[X-\overline{X}]|}dx.$$

After a change in variable along the flow $\overline{X}$ in the first integral, and noting that $\delta_1 \leq \delta_2$, we further obtain

$$\begin{aligned}\Phi'_{\delta_1,\delta_2}(s) \leq{}& \frac{\overline{L}}{\delta_1}||b(s,\cdot)-\overline{b}(s,\cdot)||_{L^1(B_\lambda)}\\ &+ \int\limits_{B_r\cap G_\lambda\cap\overline{G}_\lambda} \min\left\{|A^{-1}[b(s,X)-b(s,\overline{X})]|, \frac{|A^{-1}[b(s,X)-b(s,\overline{X})]|}{|A^{-1}[X-\overline{X}]|}\right\}dx. \end{aligned} \tag{7.1}$$

Step 2: Decomposing the minimum. We consider the second element of the minimum. We have

$$A^{-1}[b(s,X)-b(s,\overline{X})] = \left(\frac{b_1(s,X)-b_1(s,\overline{X})}{\delta_1}, \frac{b_2(s,X)-b_2(s,\overline{X})}{\delta_2}\right),$$

and therefore

$$\frac{|A^{-1}[b(s,X)-b(s,\overline{X})]|}{|A^{-1}[X-\overline{X}]|} \lesssim \frac{1}{\delta_1}\frac{|b_1(s,X)-b_1(s,\overline{X})|}{|A^{-1}[X-\overline{X}]|} + \frac{1}{\delta_2}\frac{|b_2(s,X)-b_2(s,\overline{X})|}{|A^{-1}[X-\overline{X}]|}. \tag{7.2}$$

Step 3: Definition of the functions $U_{\mathfrak{p}}$, $U_{\mathfrak{q}}$, $U_{\mathfrak{m}}$ *and* $U_{\mathfrak{r}}$. We aim at estimating the difference quotients in (7.2). We apply Lemma 5.5 and (with a slight extension of the notation) we obtain that

$$\frac{|b_1(s,x)-b_1(s,\bar{x})|}{|A^{-1}[x-\bar{x}]|} \leq \mathcal{U}(S^1,S^2,\gamma^1,\gamma^2,\mathfrak{p},\mathfrak{q})(x) + \mathcal{U}(S^1,S^2,\gamma^1,\gamma^2,\mathfrak{p},\mathfrak{q})(\bar{x}) =: U_{\mathfrak{p},\mathfrak{q}}(x) + U_{\mathfrak{p},\mathfrak{q}}(\bar{x})$$

and

$$\frac{|b_2(s,x)-b_2(s,\bar{x})|}{|A^{-1}[x-\bar{x}]|} \leq \mathcal{U}(S^3,S^4,\gamma^3,\gamma^4,\mathfrak{m},\mathfrak{r})(x) + \mathcal{U}(S^3,S^4,\gamma^3,\gamma^4,\mathfrak{m},\mathfrak{r})(\bar{x}) =: U_{\mathfrak{m},\mathfrak{r}}(x) + U_{\mathfrak{m},\mathfrak{r}}(\bar{x})$$

for a.e. x and $\bar{x}\in\mathbb{R}^N$ and $s\in[t,T]$.

It is immediate from the definition of the operator $\mathcal{U}$ that it is subadditive in its entries. Therefore we can further estimate

$$U_{\mathfrak{p},\mathfrak{q}}(x) = \mathcal{U}(S^1,S^2,\gamma^1,\gamma^2,\mathfrak{p},\mathfrak{q})(x) \leq \mathcal{U}(S^1,\gamma^1,\mathfrak{p})(x) + \mathcal{U}(S^2,\gamma^2,\mathfrak{q})(x) =: U_{\mathfrak{p}}(x) + U_{\mathfrak{q}}(x)$$

and

$$U_{\mathfrak{m},\mathfrak{r}}(x) = \mathcal{U}(S^3,S^4,\gamma^3,\gamma^4,\mathfrak{m},\mathfrak{r})(x) \leq \mathcal{U}(S^3,\gamma^3,\mathfrak{m})(x) + \mathcal{U}(S^4,\gamma^4,\mathfrak{r})(x) =: U_{\mathfrak{m}}(x) + U_{\mathfrak{r}}(x)$$

for a.e. $x\in\mathbb{R}^N$, implying that

$$\frac{|b_1(s,x)-b_1(s,\bar{x})|}{|A^{-1}[x-\bar{x}]|} \leq U_{\mathfrak{p}}(x) + U_{\mathfrak{q}}(x) + U_{\mathfrak{p}}(\bar{x}) + U_{\mathfrak{q}}(\bar{x}) \tag{7.3}$$

and

$$\frac{|b_2(s,x)-b_2(s,\bar{x})|}{|A^{-1}[x-\bar{x}]|} \leq U_{\mathfrak{m}}(x) + U_{\mathfrak{r}}(x) + U_{\mathfrak{m}}(\bar{x}) + U_{\mathfrak{r}}(\bar{x}) \tag{7.4}$$

for a.e. x and $\bar{x}\in\mathbb{R}^N$ and $s\in[t,T]$.

Step 4. Splitting of the quotient. Let $\Omega = (t,\tau)\times B_r \cap G_\lambda \cap \overline{G}_\lambda \subset \mathbb{R}^{N+1}$. We return to the estimate in (7.1) of Step 1. For any $\tau \in [t,T]$ we integrate this expression over $s\in(t,\tau)$, and recall (7.2) to get

$$\begin{aligned}\Phi_{\delta_1,\delta_2}(\tau) &\lesssim \frac{\overline{L}}{\delta_1}||b(s,\cdot)-\overline{b}(s,\cdot)||_{L^1((t,\tau)\times B_\lambda)}\\ &\quad+\int_\Omega \min\left\{|A^{-1}[b(s,X)-b(s,\overline{X})]|, \frac{1}{\delta_1}\frac{|b_1(s,X)-b_1(s,\overline{X})|}{|A^{-1}[X-\overline{X}]|}+\frac{1}{\delta_2}\frac{|b_2(s,X)-b_2(s,\overline{X})|}{|A^{-1}[X-\overline{X}]|}\right\}dxds\\ &= \frac{\overline{L}}{\delta_1}||b(s,\cdot)-\overline{b}(s,\cdot)||_{L^1((t,\tau)\times B_\lambda)}+\widetilde{\Phi}_{\delta_1,\delta_2}(\tau). \end{aligned}\tag{7.5}$$

We analyze the term $\widetilde{\Phi}_{\delta_1,\delta_2}(\tau)$. Using the estimates in (7.3) and (7.4) in Step 3, we can write

$$\begin{aligned}\widetilde{\Phi}_{\delta_1,\delta_2}(\tau) &\lesssim \int_\Omega \min\left\{|A^{-1}[b(s,X)-b(s,\overline{X})]|, \frac{1}{\delta_1}\frac{|b_1(s,X)-b_1(s,\overline{X})|}{|A^{-1}[X-\overline{X}]|}\right\}dxds\\ &\quad+\int_\Omega \min\left\{|A^{-1}[b(s,X)-b(s,\overline{X})]|, \frac{1}{\delta_2}\frac{|b_2(s,X)-b_2(s,\overline{X})|}{|A^{-1}[X-\overline{X}]|}\right\}dxds\\ &\le \int_\Omega \min\left\{|A^{-1}[b(s,X)-b(s,\overline{X})]|, \frac{1}{\delta_1}\left((U_{\mathfrak{p}}+U_{\mathfrak{q}})(s,X)+(U_{\mathfrak{p}}+U_{\mathfrak{q}})(s,\overline{X})\right)\right\}dxds\\ &\quad+\int_\Omega \min\left\{|A^{-1}[b(s,X)-b(s,\overline{X})]|, \frac{1}{\delta_2}\left((U_{\mathfrak{m}}+U_{\mathfrak{r}})(s,X)+((U_{\mathfrak{m}}+U_{\mathfrak{r}}))(s,\overline{X})\right)\right\}dxds. \end{aligned}\tag{7.6}$$

Step 5. Decomposition of the functions $U_{\mathfrak{p}}$, $U_{\mathfrak{q}}$ and $U_{\mathfrak{r}}$. We further decompose the functions $U_{\mathfrak{p}}$, $U_{\mathfrak{q}}$ and $U_{\mathfrak{r}}$ exploiting the equi-integrability of $\mathfrak{p}$, $\mathfrak{q}$ and $\mathfrak{r}$.

We apply the equi-integrability Lemma 2.4 in L^1+L^q, with the same $1<q\le\infty$ as in the assumption on the functions γ in **(R2)**. Given $\varepsilon>0$, we find $C_\varepsilon>0$, a Borel set $A_\varepsilon\subset(0,T)\times\mathbb{R}^{n_1}$ with finite measure and decompositions

$$\begin{aligned}\mathfrak{p}^i_{jk} &= (\mathfrak{p}^i_{jk})^1+(\mathfrak{p}^i_{jk})^2=:\mathfrak{p}^1+\mathfrak{p}^2,\\ \mathfrak{q}^i_{jk} &= (\mathfrak{q}^i_{jk})^1+(\mathfrak{q}^i_{jk})^2=:\mathfrak{q}^1+\mathfrak{q}^2\end{aligned}$$

and

$$\mathfrak{r}^i_{jk} = (\mathfrak{r}^i_{jk})^1+(\mathfrak{r}^i_{jk})^2=:\mathfrak{r}^1+\mathfrak{r}^2,$$

so that

$$\begin{aligned}&\|\mathfrak{p}^1\|_{L^1((0,T)\times\mathbb{R}^{n_1})}\le\varepsilon,\quad \|\mathfrak{q}^1\|_{L^1((0,T)\times\mathbb{R}^{n_1})}\le\varepsilon,\quad \|\mathfrak{r}^1\|_{L^1((0,T)\times\mathbb{R}^{n_1})}\le\varepsilon,\\ &\|\mathfrak{p}^2\|_{L^q((0,T)\times\mathbb{R}^{n_1})}\le C_\varepsilon,\quad \|\mathfrak{q}^2\|_{L^q((0,T)\times\mathbb{R}^{n_1})}\le C_\varepsilon,\quad \|\mathfrak{r}^2\|_{L^q((0,T)\times\mathbb{R}^{n_1})}\le C_\varepsilon,\end{aligned}$$

and

$$\operatorname{spt}(\mathfrak{p}^2)\subset A_\varepsilon,\quad \operatorname{spt}(\mathfrak{q}^2)\subset A_\varepsilon,\quad \operatorname{spt}(\mathfrak{r}^2)\subset A_\varepsilon.$$

We then decompose the functions $U_{\mathfrak{p}}$, $U_{\mathfrak{q}}$ and $U_{\mathfrak{r}}$ from Step 3 as

$$\begin{aligned}U_{\mathfrak{p}} &= \mathcal{U}(S^1,\gamma^1,\mathfrak{p})\le\mathcal{U}(S^1,\gamma^1,\mathfrak{p}^1)+\mathcal{U}(S^1,\gamma^1,\mathfrak{p}^2)=:U^1_{\mathfrak{p}}+U^2_{\mathfrak{p}},\\ U_{\mathfrak{q}} &= \mathcal{U}(S^2,\gamma^2,\mathfrak{q})\le\mathcal{U}(S^2,\gamma^2,\mathfrak{q}^1)+\mathcal{U}(S^2,\gamma^2,\mathfrak{q}^2)=:U^1_{\mathfrak{q}}+U^2_{\mathfrak{q}}\end{aligned}$$

and

$$U_{\mathfrak{r}} = \mathcal{U}(S^4,\gamma^4,\mathfrak{r})\le\mathcal{U}(S^4,\gamma^4,\mathfrak{r}^1)+\mathcal{U}(S^4,\gamma^4,\mathfrak{r}^2)=:U^1_{\mathfrak{r}}+U^2_{\mathfrak{r}}.$$

Applying Lemma 5.7 to $U^1_{\mathfrak{p}}$ and $U^2_{\mathfrak{p}}$ we get

$$\begin{aligned} |||U^1_{\mathfrak{p}}|||_{M^1((0,T)\times B_\lambda)} &\lesssim_\lambda \delta_1 ||\gamma^1||_{L^\infty((0,T);L^q(\mathbb{R}^{n_2}))} ||\mathfrak{p}^1||_{L^1((0,T)\times\mathbb{R}^{n_1}))} \lesssim_\lambda \delta_1\varepsilon, \\ \|U^2_{\mathfrak{p}}\|_{L^q((0,T)\times B_\lambda)} &\lesssim \delta_1 ||\gamma^1||_{L^\infty((0,T);L^q(\mathbb{R}^{n_2}))} ||\mathfrak{p}^2||_{L^q((0,T)\times\mathbb{R}^{n_1})} \lesssim \delta_1 C_\varepsilon. \end{aligned} \tag{7.7}$$

We have a similar estimate for $U_{\mathfrak{q}}$ and $U_{\mathfrak{r}}$:

$$\begin{aligned} |||U^1_{\mathfrak{q}}|||_{M^1((0,T)\times B_\lambda)} &\lesssim_\lambda \delta_2\varepsilon, \qquad |||U^1_{\mathfrak{r}}|||_{M^1((0,T)\times B_\lambda)} \lesssim_\lambda \delta_2\varepsilon, \\ \|U^2_{\mathfrak{q}}\|_{L^q((0,T)\times B_\lambda)} &\lesssim \delta_2 C_\varepsilon, \qquad \|U^2_{\mathfrak{r}}\|_{L^q((0,T)\times B_\lambda)} \lesssim \delta_2 C_\varepsilon. \end{aligned} \tag{7.8}$$

Note that we cannot apply such a decomposition to $U_{\mathfrak{m}}$, since it is defined as the operator $\mathcal{U}$ acting on a measure rather than integrable function. We only have the bound

$$|||U_{\mathfrak{m}}|||_{M^1((0,T)\times B_\lambda)} \lesssim_\lambda \delta_1 ||\mathfrak{m}||_{L^1((0,T);\mathcal{M}(\mathbb{R}^{n_1}))}. \tag{7.9}$$

We further split the minima according to this decomposition:

$$\begin{aligned} \widetilde{\Phi}_{\delta_1,\delta_2}(\tau) \lesssim & \int_\Omega \min\left\{ |A^{-1}[b(s,X)-b(s,\overline{X})]|, \frac{1}{\delta_2}(U_{\mathfrak{m}}(s,X)+U_{\mathfrak{m}}(s,\overline{X}))\right\} dxds \\ &+ \int_\Omega \min\left\{ |A^{-1}[b(s,X)-b(s,\overline{X})]|, \frac{1}{\delta_2}(U^1_{\mathfrak{r}}(s,X)+U^1_{\mathfrak{r}}(s,\overline{X}))\right\} dxds \\ &+ \int_\Omega \min\left\{ |A^{-1}[b(s,X)-b(s,\overline{X})]|, \frac{1}{\delta_2}(U^2_{\mathfrak{r}}(s,X)+U^2_{\mathfrak{r}}(s,\overline{X}))\right\} dxds \\ &+ \int_\Omega \min\left\{ |A^{-1}[b(s,X)-b(s,\overline{X})]|, \frac{1}{\delta_1}((U^1_{\mathfrak{p}}+U^1_{\mathfrak{q}})(s,X)+(U^1_{\mathfrak{p}}+U^1_{\mathfrak{q}})(s,\overline{X}))\right\} dxds \\ &+ \int_\Omega \min\left\{ |A^{-1}[b(s,X)-b(s,\overline{X})]|, \frac{1}{\delta_1}((U^2_{\mathfrak{p}}+U^2_{\mathfrak{q}})(s,X)+(U^2_{\mathfrak{p}}+U^2_{\mathfrak{q}})(s,\overline{X}))\right\} dxds \\ =& \int_\Omega \varphi_1(s,X,\overline{X}) + \int_\Omega \varphi_2(s,X,\overline{X}) + \int_\Omega \varphi_3(s,X,\overline{X}) + \int_\Omega \varphi_4(s,X,\overline{X}) + \int_\Omega \varphi_5(s,X,\overline{X}). \end{aligned} \tag{7.10}$$

Step 6. Estimating the functions φ_j. Let $\Omega' = (t,\tau)\times B_\lambda \subset \mathbb{R}^{N+1}$. We estimate the first element of each minima in L^p: changing variables along the flows we obtain

$$\|\varphi_j(s,X,\overline{X})\|_{L^p(\Omega)} \le \frac{L^{1/p}+\overline{L}^{1/p}}{\delta_1}\|b\|_{L^p(\Omega')} \lesssim \frac{1}{\delta_1} \tag{7.11}$$

for every $j=1,\dots,5$.

We now consider the second elements of the minima. Let us start with φ_1. Changing variable along the flows and using (7.9) we obtain

$$|||\varphi_1(s,X,\overline{X})|||_{M^1(\Omega)} \le \frac{1}{\delta_2}\left|\left|\left|U_{\mathfrak{m}}(s,X)+U_{\mathfrak{m}}(s,\overline{X})\right|\right|\right|_{M^1(\Omega)} \lesssim \frac{1}{\delta_2}|||U_{\mathfrak{m}}|||_{M^1(\Omega')} \lesssim_\lambda \frac{\delta_1}{\delta_2}||\mathfrak{m}||_{L^1((0,T);\mathcal{M}(\mathbb{R}^{n_1}))}. \tag{7.12}$$

Consider φ_2. Using (7.8) we obtain

$$|||\varphi_2(s,X,\overline{X})|||_{M^1(\Omega)} \le \frac{1}{\delta_2}|||U^1_{\mathfrak{r}}(s,X)+U^1_{\mathfrak{r}}(s,\overline{X})|||_{M^1(\Omega)} \lesssim \frac{1}{\delta_2}|||U^1_{\mathfrak{r}}|||_{M^1(\Omega')} \lesssim_\lambda \varepsilon. \tag{7.13}$$

For φ_3 and φ_5 we neglect the first element of the minimum, since we have directly an estimate on the $L^1(\Omega)$ norm. Using (7.8) we obtain

$$\|\varphi_3(s,X,\overline{X})\|_{L^1(\Omega)} \le \frac{1}{\delta_2}\|U^2_{\mathfrak{r}}(s,X)+U^2_{\mathfrak{r}}(s,\overline{X})\|_{L^1(\Omega)} \lesssim \frac{1}{\delta_2}||U^2_{\mathfrak{r}}||_{L^1(\Omega')} \lesssim_\lambda C_\varepsilon. \tag{7.14}$$

Similarly, using (7.7) and (7.8), we estimate φ_5 as follows:

$$||\varphi_5(s,X,\overline{X})||_{L^1(\Omega)} \le \frac{1}{\delta_1}||(U^2_{\mathfrak{p}}+U^2_{\mathfrak{q}})(s,X)+(U^2_{\mathfrak{p}}+U^2_{\mathfrak{q}})(s,\overline{X})||_{L^1(\Omega)} \lesssim \frac{1}{\delta_1}||(U^2_{\mathfrak{p}}+U^2_{\mathfrak{q}})||_{L^1(\Omega')} \lesssim_\lambda \frac{\delta_2}{\delta_1}C_\varepsilon. \tag{7.15}$$

Finally, using (7.7) and (7.8), we estimate φ_4:

$$\begin{aligned}|||\varphi_4(s,X,\overline{X})|||_{M^1(\Omega)} &\le \frac{1}{\delta_1}|||(U^1_{\mathfrak{p}}+U^1_{\mathfrak{q}})(s,X)+(U^1_{\mathfrak{p}}+U^1_{\mathfrak{q}})(s,\overline{X})|||_{M^1(\Omega)} \lesssim \frac{1}{\delta_1}|||(U^1_{\mathfrak{p}}+U^1_{\mathfrak{q}})|||_{M^1(\Omega')}\\ &\lesssim_\lambda \frac{\delta_1\varepsilon+\delta_2\varepsilon}{\delta_1} \lesssim_\lambda \frac{\delta_2}{\delta_1}\varepsilon.\end{aligned} \tag{7.16}$$

Step 7. Interpolation. We now apply the Interpolation Lemma 2.2 to estimate the $L^1(\Omega)$ norms of φ_1, φ_2 and φ_4.

Using (7.11) and (7.12) we obtain

$$\|\varphi_1(s,X,\overline{X})\|_{L^1(\Omega)} \lesssim_\lambda \frac{\delta_1}{\delta_2}\|\mathfrak{m}\|\left[1+\log\left(\frac{\delta_2}{\delta_1^2\|\mathfrak{m}\|}\right)\right]. \tag{7.17}$$

Proceeding similarly and using (7.11), (7.13) and (7.16) we obtain

$$||\varphi_2(s,X,\overline{X})||_{L^1(\Omega)} \lesssim_\lambda \varepsilon\left[1+\log\left(\frac{1}{\delta_1\varepsilon}\right)\right] \tag{7.18}$$

and

$$||\varphi_4(s,X,\overline{X})||_{L^1(\Omega)} \lesssim_\lambda \frac{\delta_2}{\delta_1}\varepsilon\left[1+\log\left(\frac{1}{\delta_2\varepsilon}\right)\right]. \tag{7.19}$$

Finally, we sum all the terms in (7.10). Using (7.17), (7.18), (7.14), (7.19) and (7.15), and setting $\frac{\delta_1}{\delta_2}=\alpha$, we get:

$$\begin{aligned}\Phi_{\delta_1,\delta_2}(\tau) \lesssim_\lambda{}& \frac{1}{\delta_1}||b(s,\cdot)-\overline{b}(s,\cdot)||_{L^1(B_\lambda\times(t,\tau))}+\alpha\|\mathfrak{m}\|\left[1+\log\left(\frac{1}{\delta_1\alpha\|\mathfrak{m}\|}\right)\right]+\varepsilon\left[1+\log\left(\frac{1}{\delta_1\varepsilon}\right)\right]+C_\varepsilon\\ &+\frac{\varepsilon}{\alpha}\left[1+\log\left(\frac{1}{\delta_2\varepsilon}\right)\right]+\frac{1}{\alpha}C_\varepsilon.\end{aligned} \tag{7.20}$$

Step 8. The final estimate. By definition of Φ_{δ_1,δ_2}, given $\gamma>0$ we estimate

$$\begin{aligned}\Phi_{\delta_1,\delta_2}(\tau) &\ge \int\limits_{B_r\cap\{|X(\tau,x)-\overline{X}(\tau,x)|>\gamma\}\cap G_\lambda\cap\overline{G}_\lambda} \log\left(1+\frac{\gamma}{\delta_2}\right)dx\\ &= \log\left(1+\frac{\gamma}{\delta_2}\right)\mathcal{L}^N\Big(B_r\cap\{|X(\tau,x)-\overline{X}(\tau,x)|>\gamma\}\cap G_\lambda\cap\overline{G}_\lambda\Big).\end{aligned} \tag{7.21}$$

This implies that

$$\mathcal{L}^N(B_r\cap\{|X(\tau,x)-\overline{X}(\tau,x)|>\gamma\}) \le \frac{\Phi_{\delta_1,\delta_2}(\tau)}{\log\left(1+\frac{\gamma}{\delta_2}\right)}+\mathcal{L}^N(B_r\setminus G_\lambda)+\mathcal{L}^N(B_r\setminus\overline{G}_\lambda). \tag{7.22}$$

Combining (7.20) and (7.22) we obtain

$$\begin{aligned}&\mathcal{L}^N(B_r\cap\{|X(\tau,x)-\overline{X}(\tau,x)|>\gamma\})\\ &\le C_\lambda\left\{\frac{\frac{1}{\delta_1}\|b-\overline{b}\|_{L^1}}{\log\left(1+\frac{\gamma}{\delta_2}\right)}+\frac{\alpha\|\mathfrak{m}\|\left[1+\log\left(\frac{1}{\delta_1\alpha\|\mathfrak{m}\|}\right)\right]}{\log\left(1+\frac{\gamma}{\delta_2}\right)}+\frac{\varepsilon\left[1+\log\left(\frac{1}{\delta_1\varepsilon}\right)\right]}{\log\left(1+\frac{\gamma}{\delta_2}\right)}+\frac{\frac{\varepsilon}{\alpha}\left[1+\log\left(\frac{1}{\delta_2\varepsilon}\right)\right]}{\log\left(1+\frac{\gamma}{\delta_2}\right)}\right.\end{aligned}$$

$$+\frac{\frac{1}{\alpha}C_\varepsilon}{\log\left(1+\frac{\gamma}{\delta_2}\right)}+\frac{C_\varepsilon}{\log\left(1+\frac{\gamma}{\delta_2}\right)}\Big\}+\mathcal{L}^N(B_r\setminus G_\lambda)+\mathcal{L}^N(B_r\setminus \overline{G}_\lambda)$$
$$=: 1)+2)+3)+4)+5)+6)+7)+8). \tag{7.23}$$

Fix $\eta>0$. By Lemma 3.2, we can choose $\lambda>0$ large enough so that $7)+8)\le 2\eta/7$. Choose α small enough so that $2)\le\eta/7$. Then choose $\varepsilon<\alpha^2$ small enough so that $3)+4)\le 2\eta/7$, since these terms are uniformly bounded as $\delta_1,\delta_2\to 0$ and for all $\varepsilon>0$.

Now λ and ε (and therefore C_ε) are fixed. Also α is fixed, but δ_1 and δ_2 are free to be chosen so long as the ratio equals α. Hence, we now choose δ_2 small enough, in particular depending on C_ε, so that $5)+6)\le 2\eta/7$. This fixes all parameters.

Setting

$$C_{\gamma,r,\eta}=\frac{C_\lambda}{\delta_1\log(1+\frac{\gamma}{\delta_2})}$$

we have proved our statement.

Conflict of interest statement

No conflicts of interest.

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