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On a model in radiation hydrodynamics

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Abstract

We consider a simplified model arising in radiation hydrodynamics based on the Navier–Stokes–Fourier system describing the macroscopic fluid motion, and a transport equation modeling the propagation of radiative intensity. We establish global-in-time existence for the associated initial–boundary value problem in the framework of weak solutions.

Keywords: Radiation hydrodynamics; Navier–Stokes–Fourier system; Weak solution

1. Introduction

The aim of *radiation hydrodynamics* is to incorporate the effects of radiation in the conventional hydrodynamics framework. There are numerous applications ranging from combustion and high-temperature hydrodynamics to models of gaseous stars in astrophysics. Various degrees of complexity of the mathematical models reflect the effect of coupling between the macroscopic description of the fluid and the statistical character of the motion of the massless photons. The reader may consult the monographs by Chandrasekhar [6], Mihalas and Weibel-Mihalas [34], Pomraning [38] for more information on the topic.

Following the recent studies by Buet and Després [5], Golse and Perthame [23], we consider a mathematical model, where the motion of the fluid is governed by the standard field equations of classical continuum fluid mechanics describing the evolution of *the mass density* $\varrho = \varrho(t,x)$, the *velocity field* $\vec{u} = \vec{u}(t,x)$, and the *absolute temperature* $\vartheta = \vartheta(t,x)$ as functions of the time t and the Eulerian spatial coordinate $x \in \Omega \subset \mathbb{R}^3$. The effect of radiation, represented by its quanta – massless particles called *photons* traveling at the speed of light c – is incorporated in the *radiative intensity* $I = I(t,x,\vec{\omega},\nu)$, depending on the direction vector $\vec{\omega} \in \mathcal{S}^2$, where $\mathcal{S}^2 \subset \mathbb{R}^3$ denotes the unit

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sphere, and the frequency $\nu \geqslant 0$. The collective effect of radiation is then expressed in terms of integral means with respect to the variables $\vec{\omega}$ and ν of quantities depending on I. In particular, the radiation energy E_R is given as

$$E_R(t,x) = \frac{1}{c}\int_{\mathcal{S}^2}\int_0^\infty I(t,x,\vec{\omega},\nu)\,\mathrm{d}\vec{\omega}\,\mathrm{d}\nu. \tag{1.1}$$

The time evolution of I is described by a transport equation with a source term depending on the absolute temperature, while the effect of radiation on the macroscopic motion of the fluid is represented by extra source terms in the momentum and energy equations evaluated in terms of I.

More specifically, the system of equations to be studied reads as follows:

Equation of continuity:

$$\partial_t \varrho + \mathrm{div}_x(\varrho\vec{u}) = 0 \quad \text{in } (0,T)\times\Omega. \tag{1.2}$$

Momentum equation:

$$\partial_t(\varrho\vec{u}) + \mathrm{div}_x(\varrho\vec{u}\otimes\vec{u}) + \nabla_x p(\varrho,\vartheta) = \mathrm{div}_x\,\mathbb{T} - \vec{S}_F \quad \text{in } (0,T)\times\Omega. \tag{1.3}$$

Energy balance equation:

$$\partial_t\left(\varrho\left(\frac{1}{2}|\vec{u}|^2 + e(\varrho,\vartheta)\right)\right) + \mathrm{div}_x\left(\varrho\left(\frac{1}{2}|\vec{u}|^2 + e(\varrho,\vartheta)\right)\vec{u}\right) + \mathrm{div}_x(p\vec{u} + \vec{q} - \mathbb{T}\vec{u}) = -S_E \quad \text{in } (0,T)\times\Omega. \tag{1.4}$$

Radiation transport equation:

$$\frac{1}{c}\partial_t I + \vec{\omega}\cdot\nabla_x I = S \quad \text{in } (0,T)\times\Omega\times(0,\infty)\times\mathcal{S}^2. \tag{1.5}$$

The symbol $p = p(\varrho,\vartheta)$ denotes the thermodynamic pressure and $e = e(\varrho,\vartheta)$ is the specific internal energy, interrelated through *Maxwell's equation*

$$\frac{\partial e}{\partial\varrho} = \frac{1}{\varrho^2}\left(p(\varrho,\vartheta) - \vartheta\frac{\partial p}{\partial\vartheta}\right). \tag{1.6}$$

Furthermore, $\mathbb{T}$ is the viscous stress tensor determined by *Newton's theological law*

$$\mathbb{T} = \mu\left(\nabla_x\vec{u} + \nabla_x^t\vec{u} - \frac{2}{3}\mathrm{div}_x\,\vec{u}\right) + \eta\,\mathrm{div}_x\,\vec{u}\mathbb{I}, \tag{1.7}$$

where the shear viscosity coefficient $\mu = \mu(\vartheta) > 0$ and the bulk viscosity coefficient $\eta = \eta(\vartheta) \geqslant 0$ are effective functions of the absolute temperature. Similarly, $\vec{q}$ is the heat flux given by *Fourier's law*

$$\vec{q} = -\kappa\nabla_x\vartheta, \tag{1.8}$$

with the heat conductivity coefficient $\kappa = \kappa(\vartheta) > 0$.

Finally,

$$S = S_{a,e} + S_s, \tag{1.9}$$

where

$$S_{a,e} = \sigma_a\big(B(\nu,\vartheta) - I\big), \qquad S_s = \sigma_s\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I(\cdot,\vec{\omega})\,\mathrm{d}\vec{\omega} - I\right), \tag{1.10}$$

and

$$S_E = \int_{\mathcal{S}^2}\int_0^\infty S(\cdot,\nu,\vec{\omega})\,\mathrm{d}\nu\,\mathrm{d}\vec{\omega}, \qquad \vec{S}_F = \frac{1}{c}\int_{\mathcal{S}^2}\int_0^\infty \vec{\omega}S(\cdot,\nu,\vec{\omega})\,\mathrm{d}\nu\,\mathrm{d}\vec{\omega}, \tag{1.11}$$

with the absorption coefficient $\sigma_a = \sigma_s(\nu, \vartheta) \geqslant 0$, and the scattering coefficient $\sigma_s = \sigma_s(\nu, \vartheta) \geqslant 0$. More restrictions on the structural properties of constitutive relations will be imposed in Section 2 below.

System (1.2)–(1.5) is supplemented with the boundary conditions:

No-slip, no-flux:

$$\vec{u}|_{\partial\Omega} = 0, \qquad \vec{q} \cdot \vec{n}|_{\partial\Omega} = 0. \tag{1.12}$$

Transparency:

$$I(t, x, \nu, \vec{\omega}) = 0 \quad \text{for } x \in \partial\Omega, \qquad \vec{\omega} \cdot \vec{n} \leqslant 0, \tag{1.13}$$

where $\vec{n}$ denotes the outer normal vector to $\partial\Omega$.

System (1.2)–(1.13) can be viewed as a toy model in radiation hydrodynamics, the physical foundations of which were described by Pomraning [38] and Mihalas and Weibel-Mihalas [34] in the framework of special relativity, see also [35,36] for a list of references and a review of related computational works in the relativistic framework. Similar systems have been investigated more recently in astrophysics and laser applications (in the relativistic and inviscid case) by Lowrie, Morel and Hittinger [30], Buet and Després [5], with a special attention to asymptotic regimes, see also Dubroca and Feugeas [10], Lin [28] and Lin, Coulombel and Goudon [29] for related numerical issues.

The *existence* of local-in-time solutions and sufficient conditions for blow up of classical solutions in the non-relativistic inviscid case were obtained by Zhong and Jiang [41], see also the recent papers by Jiang and Wang [26,27] for a related one-dimensional "Euler–Boltzmann" type models. Moreover, a simplified version of the system has been investigated by Golse and Perthame [23], where global existence was proved by means of the theory of nonlinear semi-groups. To the best of our knowledge, similar results for *viscous* fluids are restricted to the one-dimensional geometry [1,13,14] (however see [10] for a simplified treatment of radiation in the diffusion regime in the physically relevant 3D-case).

Our goal in the present paper is to show that the existence theory for the Navier–Stokes–Fourier system developed in [18,12,16] and [17, Chapter 3] can be adapted to problem (1.2)–(1.13). As a complete proof of existence becomes rather involved and nowadays well understood (see [17, Chapter 3]), we focus only on the property of *weak sequential stability* for problem (1.2)–(1.13) in the framework of the weak solutions introduced in [12]. More specifically, we introduce a concept of finite energy weak solution in the spirit of [12] and show that any sequence $\{\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon\}_{\varepsilon>0}$ of solutions to problem (1.2)–(1.13), bounded in the natural energy norm, possesses a subsequence converging to another (weak) solution of the same problem. Such a property highlights the essential ingredients involved in the "complete" proof of existence that may be carried over by means of the arguments delineated in [17, Chapter 3].

In comparison with the standard Navier–Stokes–Fourier system studied in [17], problem (1.2)–(1.13) features a new principal difficulty due to the apparent discrepancy between the classical (non-relativistic) description of the fluid motion, and the behavior of photons traveling with the speed of light. In particular, in contrast with the Second law of thermodynamics, the associated entropy equation may contain a *negative* production term. This problem, related to the fact that, hypothetically, one might have $|\vec{u}| > c$, has already been observed by Buet and Després [5, Section 2.5]. On the other hand, non-negativity of the entropy production rate plays a crucial role in the approach developed in [12]; whence its adaptation to the present setting requires new ideas. Instead of introducing the radiation entropy, we keep the classical form of the entropy balance equation supplemented with the relevant "radiation" production term proportional to

$$\frac{1}{\vartheta}(\vec{u} \cdot \vec{S}_F - S_E),$$

see Section 2. As pointed out, this term may change sign and, accordingly, we have to establish its "weak continuity" with respect to ϑ, $\vec{u}$, and I contained in $\vec{S}_F$, S_E. Note that this is quite delicate as the velocity field $\vec{u}$ may develop uncontrolled *time oscillations* on the hypothetical vacuum zones where ϱ vanishes. In order to overcome this difficulty, we use higher regularity of the ω-averages of the radiative intensity discovered by Bardos et al. [2] and Golse et al. [24,25]. For further generalizations and a more complete list of references, see Bournaveas and Perthame [4].

The paper is organized as follows. In Section 2, we list the principal hypotheses imposed on constitutive relations, introduce the concept of weak solution to problem (1.2)–(1.13), and state the main result. Uniform bounds imposed on weak solutions by the data are derived in Section 3. The property of *weak sequential stability* of a bounded sequence of weak solutions is established in Section 4. Finally, we introduce a suitable approximation scheme and discuss the main steps of the proof of existence in Section 5.

2. Hypotheses and main results

The structural hypotheses imposed on constitutive relations are motivated by the general *existence theory* for the Navier–Stokes–Fourier system developed in [17, Chapter 3]. To a certain extent they can be viewed as a suitable compromise between the underlying physical properties of real fluids and the hypotheses required by the mathematical theory.

2.1. Constitutive equations

Motivated by Ruggeri and Trovato [40], we consider the pressure in the form

$$p(\varrho,\vartheta)=\vartheta^{5/2}P\left(\frac{\varrho}{\vartheta^{3/2}}\right)+\frac{a}{3}\vartheta^4,\quad a>0, \tag{2.1}$$

where $P:[0,\infty)\to[0,\infty)$ is a given function with the following properties:

$$P\in C^1[0,\infty),\quad P(0)=0,\quad P'(Z)>0\quad\text{for all } Z\geqslant 0, \tag{2.2}$$

$$0<\frac{\frac{5}{3}P(Z)-P'(Z)Z}{Z}<c\quad\text{for all } Z\geqslant 0, \tag{2.3}$$

$$\lim_{Z\to\infty}\frac{P(Z)}{Z^{5/3}}=p_\infty>0. \tag{2.4}$$

The reader may consult [12,17] for the physical background of hypotheses (2.1)–(2.4). Note that $\vartheta^{5/2}P(\varrho/\vartheta^{3/2})$ is a general form of the molecular pressure compatible with (1.6), satisfying the universal state equation of a monoatomic gas $p=\frac{2}{3}\varrho e$, see Eliezer et al. [15]. Hypothesis (2.2) reflects positive compressibility of the fluid, while the strangely looking condition (2.3) is in fact equivalent to positivity and boundedness of the specific heat at constant volume. Hypothesis (2.4) means that the fluid behaves like a Fermi gas in the degenerate area $\varrho\gg\vartheta^{3/2}$.

The component $\frac{a}{3}\vartheta^4$ represents the effect of "equilibrium" radiation pressure imposed on the fluid by the collective force of the part of photons that may be considered in thermal equilibrium with the fluid. As a matter of fact, since the radiative transfer equation is linear in I, we tacitly suppose that radiation is a sum of two contributions, where the radiative transfer equation (1.5), together with the sources S_E and $\vec{S}_F$ describes the "out-of-equilibrium" part of the radiation, with a temperature ϑ_r which is a priori distinct from the equilibrium temperature ϑ, while the equilibrium part is described by the ϑ^4 Stefan–Boltzmann correction to the gaseous equation of state. To motivate this kind of splitting, just recall that a difficult problem in high-temperature physics consists in the treatment of interfaces separating two media with different optical properties; one of them at Local Thermodynamical Equilibrium (LTE) and the other not. This is the case of stellar atmospheres in the astrophysical context, and specially in the studies of complicated radiative phenomena appearing at the surface of the sun, see Mihalas [33]. Another example appears in the ICF (Inertial Confinement Fusion) context, where intense laser beams attack a target producing thermonuclear fusion events through ablation fronts, see [8]. In numerical implementations, one needs to develop effective transmission conditions allowing to compute accurately the flow in both these regimes and it is therefore natural to consider the well-posedness problem for our composite model (cf. [11] for a study of the pure equilibrium system).

In accordance with Maxwell's equation (1.6), the specific internal energy e can be taken in the form

$$e(\varrho,\vartheta)=\frac{3}{2}\vartheta\left(\frac{\vartheta^{3/2}}{\varrho}\right)P\left(\frac{\varrho}{\vartheta^{3/2}}\right)+a\frac{\vartheta^4}{\varrho}, \tag{2.5}$$

whereas the associated specific entropy reads

$$s(\varrho,\vartheta)=M\left(\frac{\varrho}{\vartheta^{3/2}}\right)+\frac{4a}{3}\frac{\vartheta^3}{\varrho}, \tag{2.6}$$

with

$$M'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z^2} < 0.$$

The transport coefficients μ, η, and κ are continuously differentiable functions of the absolute temperature such that

$$0 < c_1(1+\vartheta) \leqslant \mu(\vartheta), \qquad \mu'(\vartheta) < c_2, \qquad 0 \leqslant \eta(\vartheta) \leqslant c(1+\vartheta), \tag{2.7}$$

$$0 < c_1\big(1+\vartheta^3\big) \leqslant \kappa(\vartheta) \leqslant c_2\big(1+\vartheta^3\big) \tag{2.8}$$

for any $\vartheta \geqslant 0$.

Finally, we assume that σ_a, σ_s, B are continuous functions of ν, ϑ such that

$$0 \leqslant \sigma_a(\nu,\vartheta), \sigma_s(\nu,\vartheta) \leqslant c_1, \qquad 0 \leqslant \sigma_a(\nu,\vartheta)B(\nu,\vartheta) \leqslant c_2, \tag{2.9}$$

$$\sigma_a(\nu,\vartheta), \sigma_s(\nu,\vartheta), \sigma_a(\nu,\vartheta)B(\nu,\vartheta) \leqslant h(\nu), \quad h \in L^1(0,\infty), \tag{2.10}$$

and

$$\sigma_a(\nu,\vartheta), \sigma_s(\nu,\vartheta) \leqslant c\vartheta \tag{2.11}$$

for all $\nu \geqslant 0$, $\vartheta \geqslant 0$. Relations (2.9)–(2.11) represent a rather crude "cut-off" hypotheses neglecting the effect of radiation at large frequencies ν and low vales of the temperature ϑ. Note, however, that relations similar to (2.11) were derived by Ripoll et al. [39].

2.2. *Weak formulation*

In the weak formulation of the Navier–Stokes–Fourier system, it is customary to replace the equation of continuity (1.2) by its (weak) *renormalized* version [9] represented by a family of integral identities

$$\int_0^T\int_\Omega \Big(\big(\varrho + b(\varrho)\big)\partial_t\varphi + \big(\varrho + b(\varrho)\big)\vec{u}\cdot\nabla_x\varphi + \big(b(\varrho) - b'(\varrho)\varrho\big)\operatorname{div}_x\vec{u}\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t$$

$$= -\int_\Omega \big(\varrho_0 + b(\varrho_0)\big)\varphi(0,\cdot)\,\mathrm{d}x \tag{2.12}$$

satisfied for any $\varphi \in C_c^\infty([0,\infty)\times\overline{\Omega})$, and any $b \in C^\infty[0,\infty)$, $b' \in C_c^\infty[0,\infty)$. Note that (2.12) implicitly includes satisfaction of the initial condition

$$\varrho(0,\cdot) = \varrho_0.$$

Similarly, the momentum equation (1.3) is replaced by

$$\int_0^T\int_\Omega (\varrho\vec{u}\cdot\partial_t\varphi + \varrho\vec{u}\otimes\vec{u} : \nabla_x\varphi + p\operatorname{div}_x\varphi)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega \mathbb{T} : \nabla_x\varphi + \vec{S}_F\cdot\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega (\varrho\vec{u})_0\cdot\varphi(0,\cdot)\,\mathrm{d}x \tag{2.13}$$

for any $\varphi \in C_c^\infty([0,T)\times\Omega;\mathbb{R}^3)$. As the viscous stress contains first derivatives of the velocity $\vec{u}$, for (2.13) to make sense, the field $\vec{u}$ must belong to a certain Sobolev space with respect to the spatial variable. Here, we require that

$$\vec{u} \in L^2\big(0,T;W_0^{1,2}\big(\Omega;\mathbb{R}^3\big)\big), \tag{2.14}$$

where (2.14) already includes the no-slip boundary condition (1.12).

As a matter of fact, the total energy balance (1.4) is not suitable for the weak formulation since, at least according to the recent state-of-art, the term $\mathbb{S}\vec{u}$ is not controlled on the (hypothetical) vacuum zones of vanishing density. Following [17], we replace (1.4) by the internal energy equation

$$\partial_t(\varrho e) + \operatorname{div}_x(\varrho e\vec{u}) + \operatorname{div}_x\vec{q} = \mathbb{T} : \nabla_x\vec{u} - p\operatorname{div}_x\vec{u} + \vec{u}\cdot\vec{S}_F - S_E. \tag{2.15}$$

Furthermore, dividing (2.15) on ϑ and using Maxwell's relation (1.6), we may rewrite (2.16) as the entropy equation

$$\partial_t(\varrho s)+\operatorname{div}_x(\varrho s\vec{u})+\operatorname{div}_x\left(\frac{\vec{q}}{\vartheta}\right)=\frac{1}{\vartheta}\left(\mathbb{T}:\nabla_x\vec{u}-\frac{\vec{q}\cdot\nabla_x\vartheta}{\vartheta}\right)+\frac{1}{\vartheta}(\vec{u}\cdot\vec{S}_F-S_E). \tag{2.16}$$

Finally, similarly to [12], Eq. (2.16) is replaced in the weak formulation by an *inequality*, specifically,

$$\begin{aligned}&\int_0^T\int_\Omega(\varrho s\partial_t\varphi+\varrho\vec{u}\cdot\nabla_x\varphi+\vec{q}\vartheta\cdot\nabla_x\varphi)\,\mathrm{d}x\,\mathrm{d}t\\&\quad\leqslant-\int_\Omega(\varrho s)_0\varphi(0,\cdot)\,\mathrm{d}x-\int_0^T\int_\Omega\frac{1}{\vartheta}\left(\mathbb{S}:\nabla_x\vec{u}-\frac{\vec{q}\cdot\nabla_x\vartheta}{\vartheta}\right)\varphi\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_\Omega\frac{1}{\vartheta}(\vec{u}\cdot\vec{S}_F-S_E)\varphi\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{2.17}$$

for any $\varphi\in C_c^\infty([0,T)\times\overline{\Omega})$, $\varphi\geqslant 0$.

Since replacing *equation* (1.4) by *inequality* (2.17) would certainly result in a formally under-determined problem, system (2.12), (2.13), (2.17) must be supplemented with the total energy balance

$$\begin{aligned}&\int_\Omega\left(\frac{1}{2}\varrho|\vec{u}|^2+\varrho e(\varrho,\vartheta)+E_R\right)(\tau,\cdot)\,\mathrm{d}x+\int_0^\tau\int_{\partial\Omega\times\mathcal{S}^2,\,\vec{\omega}\cdot\vec{n}\geqslant 0}\int_0^\infty\vec{\omega}\cdot\vec{n}I(t,x,\vec{\omega},\nu)\,\mathrm{d}\nu\,\mathrm{d}\vec{\omega}\,\mathrm{d}S_x\,\mathrm{d}t\\&\quad=\int_\Omega\left(\frac{1}{2\varrho_0}\left|(\varrho\vec{u})_0\right|^2+(\varrho e)_0+E_{R,0}\right)\mathrm{d}x,\end{aligned} \tag{2.18}$$

where E_R is given by (1.1), and

$$E_{R,0}=\int_{\mathcal{S}^2}\int_0^\infty I_0(\cdot,\vec{\omega},\nu)\,\mathrm{d}\vec{\omega}\,\mathrm{d}\nu.$$

The transport equation (1.5) can be extended to the whole physical space $\mathbb{R}^3$ provided we set

$$\sigma_a(x,\nu,\vartheta)=1_\Omega\sigma_a(\nu,\vartheta),\qquad \sigma_s(x,\nu,\vartheta)=1_\Omega\sigma_s(\nu,\vartheta)$$

and take the initial distribution $I_0(x,\vec{\omega},\nu)$ to be zero for $x\in\mathbb{R}^3\setminus\Omega$. Accordingly, for any fixed $\vec{\omega}\in\mathcal{S}^2$, Eq. (1.5) can be viewed as a linear transport equation defined in $(0,T)\times\mathbb{R}^3$, with a right-hand side S. With the above mentioned convention, extending $\vec{u}$ to be zero outside Ω, we may therefore assume that both ϱ and I are defined on the whole physical space $\mathbb{R}^3$.

Definition 2.1. We say that $\varrho,\vec{u},\vartheta,I$ is a weak solution of problem (1.2)–(1.13) if

$$\begin{aligned}&\varrho\geqslant 0,\qquad \vartheta>0\quad\text{for a.a. }(t,x)\times\Omega,\qquad I\geqslant 0\quad\text{a.a. in }(0,T)\times\Omega\times\mathcal{S}^2\times(0,\infty),\\&\varrho\in L^\infty\big(0,T;L^{5/3}(\Omega)\big),\qquad \vartheta\in L^\infty\big(0,T;L^4(\Omega)\big),\\&\vec{u}\in L^2\big(0,T;W_0^{1,2}\big(\Omega;\mathbb{R}^3\big)\big),\qquad \vartheta\in L^2\big(0,T;W^{1,2}(\Omega)\big),\\&I\in L^\infty\big((0,T)\times\Omega\times\mathcal{S}^2\times(0,\infty)\big),\qquad I(t,\cdot)\in L^\infty\big(0,T;L^1\big(\Omega\times\mathcal{S}^2\times(0,\infty)\big)\big),\end{aligned}$$

and if $\varrho,\vec{u},\vartheta,I$ satisfy the integral identities (2.12), (2.13), (2.17), (2.18), together with the transport equation (1.5).

2.3. Main result

The main result of the present paper can be stated as follows.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Assume that the thermodynamic functions p, e, s satisfy hypotheses* (2.1)–(2.6)*, and that the transport coefficients μ, λ, κ, σ_a, and σ_s comply with* (2.7)–(2.11).

Let $\{\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon\}_{\varepsilon>0}$ be a family of weak solutions to problem (1.2)–(1.13) *in the sense of Definition* 2.1 *such that*

$$\varrho_\varepsilon(0,\cdot) \equiv \varrho_{\varepsilon,0} \to \varrho_0 \quad \text{in } L^{5/3}(\Omega), \tag{2.19}$$

$$\int_\Omega \left(\frac{1}{2}\varrho_\varepsilon|\vec{u}_\varepsilon|^2 + \varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon) + E_{R,\varepsilon}\right)(0,\cdot)\,\mathrm{d}x \equiv \int_\Omega \left(\frac{1}{2\varrho_{0,\varepsilon}}\left|(\varrho\vec{u})_{0,\varepsilon}\right|^2 + (\varrho e)_{0,\varepsilon} + E_{R,0,\varepsilon}\right)\mathrm{d}x \leqslant E_0, \tag{2.20}$$

$$\int_\Omega \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)(0,\cdot)\,\mathrm{d}x \equiv \int_\Omega (\varrho s)_{0,\varepsilon}\,\mathrm{d}x \geqslant S_0,$$

and

$$0 \leqslant I_\varepsilon(0,\cdot) \equiv I_{0,\varepsilon}(\cdot) \leqslant I_0, \qquad \left|I_{0,\varepsilon}(\cdot,\nu)\right| \leqslant h(\nu) \quad \text{for a certain } h \in L^1(0,\infty).$$

Then

$$\varrho_\varepsilon \to \varrho \quad \text{in } C_{\mathrm{weak}}\big([0,T]; L^{5/3}(\Omega)\big),$$
$$\vec{u}_\varepsilon \to \vec{u} \quad \text{weakly in } L^2\big(0,T; W_0^{1,2}\big(\Omega;\mathbb{R}^3\big)\big),$$
$$\vartheta_\varepsilon \to \vartheta \quad \text{weakly in } L^2\big(0,T; W^{1,2}(\Omega)\big),$$

and

$$I_\varepsilon \to I \quad \text{weakly-(*) in } L^\infty\big((0,T)\times\Omega\times\mathcal{S}^2\times(0,\infty)\big),$$

at least for suitable subsequences, where $\{\varrho, \vec{u}, \vartheta, I\}$ is a weak solution of problem (1.2)–(1.13).

Note that *compactness* is required only for the initial distribution of the densities $\{\varrho_{\varepsilon,0}\}_{\varepsilon>0}$ (see (2.19)), while the remaining initial data are only bounded in suitable norms. This is because the evolution of the density is governed by continuity equation (1.2) of hyperbolic character without any smoothing mechanism incorporated.

A major part of the rest of the paper is devoted to the proof of Theorem 2.1. It consists of three steps. First of all, we establish uniform estimates on the family $\{\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon\}_{\varepsilon>0}$ independent of $\varepsilon \to 0$. Secondly, we observe that the extra forcing terms in (2.13), (2.17) due to the effect of radiation are bounded in suitable Lebesgue norms. In particular, the analysis of the macroscopic variables ϱ_ε, $\vec{u}_\varepsilon$, ϑ_ε is essentially the same as in the case of the Navier–Stokes–Fourier system presented in [17]. Consequently, the proof of Theorem 2.1 reduces to the study of the transport equation (1.5) governing the time evolution of the radiation intensity I_ε. In the last part of the paper, we introduce an approximation scheme similar to that used in [17, Chapter 3] and sketch the main ideas of a complete proof of existence of global-in-time weak solutions to problem (1.2)–(1.13).

3. Uniform bounds

Uniform (*a priori*) bounds form the basis of the existence theory. They follow immediately from the total energy balance, entropy production equation, and other related physical principles. We follow the line of arguments similar to [12].

3.1. Energy estimates

As a direct consequence of the total energy balance (2.18), combined with hypotheses of Theorem 2.1, we obtain

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\sqrt{\varrho_\varepsilon}\vec{u}_\varepsilon\|_{L^2(\Omega;\mathbb{R}^3)} \leqslant c, \tag{3.1}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \left\|\varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon)\right\|_{L^1(\Omega)} \leqslant c, \tag{3.2}$$

and

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|E_{R,\varepsilon}\|_{L^1(\Omega)} \leqslant c. \tag{3.3}$$

Thus, as the internal energy contains the radiation component proportional to ϑ^4, we deduce from (3.2) that

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\vartheta_\varepsilon\|_{L^4(\Omega)} \leqslant c, \tag{3.4}$$

and, by virtue of hypotheses (2.1)–(2.4),

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\varrho_\varepsilon\|_{L^{5/3}(\Omega)} \leqslant c. \tag{3.5}$$

3.2. Estimates of the radiation intensity

At this stage we focus on the transport equation (1.5). Since the quantity I_ε is non-negative, we have

$$\frac{1}{c}\partial_t I_\varepsilon + \vec{\omega}\cdot\nabla_x I_\varepsilon \leqslant \sigma_s(\nu,\vartheta_\varepsilon)B(\nu,\vartheta_\varepsilon) + \sigma_a(\nu,\vartheta_\varepsilon)\frac{1}{4\pi}\int_{\mathcal{S}^2} I_\varepsilon(\cdot,\vec{\omega})\,\mathrm{d}\vec{\omega} \tag{3.6}$$

as the coefficients σ_s, σ_a are non-negative. Moreover, making use of the "cut-off" hypothesis (2.9), we deduce a uniform bound

$$0 \leqslant I_\varepsilon(t,x,\nu,\vec{\omega}) \leqslant c(T)\Bigl(1 + \sup_{x\in\Omega,\,\nu\geqslant 0,\,\vec{\omega}\in\mathcal{S}^2} I_{0,\varepsilon}\Bigr) \leqslant c(T)(1+I_0) \quad \text{for any } t\in[0,T]. \tag{3.7}$$

Finally, hypothesis (2.10), together with (3.7), yield

$$\|S_{E,\varepsilon}\|_{L^\infty((0,T)\times\Omega)} + \|\vec{S}_{F,\varepsilon}\|_{L^\infty((0,T)\times\Omega;\mathbb{R}^3)} \leqslant c, \tag{3.8}$$

which, combined with hypothesis (2.11), implies

$$\left\|\frac{1}{\vartheta_\varepsilon}S_{E,\varepsilon}\right\|_{L^\infty((0,T)\times\Omega)} + \left\|\frac{1}{\vartheta_\varepsilon}\vec{S}_{F,\varepsilon}\right\|_{L^\infty((0,T)\times\Omega;\mathbb{R}^3)} \leqslant c. \tag{3.9}$$

3.3. Dissipative estimates

Since the viscosity coefficients satisfy (2.7), we get

$$\int_0^T\int_\Omega \frac{1}{\vartheta_\varepsilon}\mathbb{T}_\varepsilon : \nabla_x\vec{u}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t \geqslant c_1\left\|\nabla_x\vec{u}_\varepsilon + \nabla_x^t\vec{u}_\varepsilon - \frac{2}{3}\operatorname{div}_x\vec{u}_\varepsilon\mathbb{I}\right\|^2_{L^2((0,T)\times\Omega;\mathbb{R}^{3\times3})} \geqslant c_2\|\vec{u}_\varepsilon\|^2_{L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))},$$

where we have used a variant of the standard Korn's inequality.

On the other hand, since Ω is a bounded, in accordance with (3.9),

$$\left|\int_0^T\int_\Omega \frac{1}{\vartheta_\varepsilon}\vec{u}_\varepsilon\cdot\vec{S}_{F,\varepsilon}\,\mathrm{d}x\,\mathrm{d}t\right| \leqslant c\|\vec{u}_\varepsilon\|_{L^1((0,T)\times\Omega;\mathbb{R}^3)};$$

whence the entropy inequality (2.17) yields the uniform bounds

$$\|\vec{u}_\varepsilon\|_{L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))} \leqslant c, \tag{3.10}$$

$$\|\nabla_x\vartheta_\varepsilon\|_{L^2((0,T)\times\Omega;\mathbb{R}^3)} \leqslant c. \tag{3.11}$$

3.4. Pressure estimates

We start with a simple observation that estimates (3.5), (3.10) imply that the sequences $\{\varrho_\varepsilon \vec{u}_\varepsilon\}_{\varepsilon>0}$, $\{\varrho_\varepsilon \vec{u}_\varepsilon \otimes \vec{u}_\varepsilon\}_{\varepsilon>0}$ are bounded in the Lebesgue space $L^p((0,T)\times\Omega)$ for a certain $p>1$. Similarly, combining (3.4), (3.10), (3.11) we get

$$\{\mathbb{S}_\varepsilon\}_{\varepsilon>0} \quad \text{bounded in } L^p\big((0,T)\times\Omega; R^{3\times3}\big) \text{ for a certain } p>1.$$

Now, repeating step by step the arguments of [21], we observe that the quantities

$$\varphi(t,x) = \psi(t)\mathcal{B}\big[\varrho_\varepsilon^\omega\big], \quad \psi \in \mathcal{D}(0,T) \text{ for a sufficiently small parameter } \omega>0,$$

may be used as test functions in the momentum equation (2.13), where $\mathcal{B}[v]$ is a suitable branch of solutions to the boundary value problem

$$\operatorname{div}_x\big(\mathcal{B}[v]\big) = v - \frac{1}{|\Omega|}\int_\Omega v\,\mathrm{d}x, \qquad \mathcal{B}|_{\partial\Omega} = 0. \tag{3.12}$$

Note that the construction of the operator $\mathcal{B}$, described in detail in [22], is based on an integral representation formula due to Bogovskii [3].

The resulting estimate reads

$$\int_0^T\int_\Omega p(\varrho_\varepsilon,\vartheta_\varepsilon)\varrho_\varepsilon^\omega\,\mathrm{d}x\,\mathrm{d}t < c, \quad \text{with } c \text{ independent of } \varepsilon, \tag{3.13}$$

in particular,

$$\big\{p(\varrho_\varepsilon,\vartheta_\varepsilon)\big\}_{\varepsilon>0} \quad \text{is bounded in } L^p\big((0,T)\times\Omega\big) \text{ for a certain } p>1. \tag{3.14}$$

Note that an alternative way to obtain these estimates was proposed in [32].

The estimates deduced in this section represent, according to the recent state of art, the complete list of uniform (*a priori*) bounds available for solutions of our system.

4. Weak sequential stability

To begin, let us stress that the weak stability property for the studied system is a complex problem that requires a substantial amount of ingredients developed elsewhere. The sequential stability of the densities, for example, is based on the method for solving barotropic Navier–Stokes system proposed by Lions [31] and later developed in [20]. Compactness of the temperature requires tools from the theory of Young measures and depends heavily on the presence of the radiation pressure, see [17, Chapter 3] for details. On the other hand, as these steps are nowadays quite well developed and understood, we sketch the principal part of the proofs and focus mainly on the issues related to weak sequential stability of quantities related to radiation that require new ideas. In particular, we examine in detail the extra terms in the entropy balance equation (2.17).

4.1. Weak sequential stability of macroscopic thermodynamic quantities

In view of the uniform estimates on the radiation forcing terms S_E, $\vec{S}_F$ established in (3.8), (3.9), strong (pointwise) convergence of the macroscopic thermodynamic quantities $\{\varrho_\varepsilon\}_{\varepsilon>0}$, $\{\vartheta_\varepsilon\}_{\varepsilon>0}$ can be shown exactly as in [12].

To begin, we can use the uniform bounds established in Section 3 to observe that

$$\varrho_\varepsilon \to \varrho \quad \text{in } C_{\mathrm{weak}}\big([0,T]; L^{5/3}(\Omega)\big), \tag{4.1}$$

$$\vartheta_\varepsilon \to \vartheta \quad \text{weakly in } L^2\big(0,T; W^{1,2}(\Omega)\big), \tag{4.2}$$

and

$$\log(\vartheta_\varepsilon) \to \overline{\log(\vartheta)} \quad \text{weakly in } L^2\big((0,T)\times\Omega\big), \tag{4.3}$$

and

$$\vec{u}_\varepsilon \to \vec{u} \quad \text{weakly in } L^2\big(0,T;W_0^{1,2}\big(\Omega;\mathbb{R}^3\big)\big), \tag{4.4}$$

passing to suitable subsequences as the case may be. Moreover, since the (weak) time derivative $\partial_t \varrho_\varepsilon \vec{u}_\varepsilon$ of momenta can be expressed by means of momentum balance (2.13), we have

$$\varrho_\varepsilon \vec{u}_\varepsilon \to \varrho \vec{u} \quad \text{in } C_{\mathrm{weak}}\big([0,T];L^{5/4}\big(\Omega;\mathbb{R}^3\big)\big). \tag{4.5}$$

Since our system contains quantities depending on ϱ and ϑ in a nonlinear way, pointwise (a.a.) convergence of $\{\varrho_\varepsilon\}_{\varepsilon>0}$, $\{\vartheta_\varepsilon\}_{\varepsilon>0}$ must be established in order to perform the limit $\varepsilon \to 0$. This step is apparently easier to carry out for the temperature because of the uniform bounds available for $\nabla_x \vartheta_\varepsilon$.

4.1.1. Pointwise convergence of temperature

Pointwise (a.a.) convergence of the sequence $\{\vartheta_\varepsilon\}_{\varepsilon>0}$ can be established essential by monotonicity arguments. The main problem are possible uncontrollable time oscillations in hypothetical zones of vacuum, here eliminated by the presence of radiation component in the entropy. More specifically, our goal is to show that

$$\int_0^T \int_\Omega \big(\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon) - \varrho_\varepsilon s(\varrho_\varepsilon,\vartheta)\big)(\vartheta_\varepsilon - \vartheta)\,\mathrm{d}x\,\mathrm{d}t \to 0 \quad \text{as } \varepsilon \to 0, \tag{4.6}$$

which, in accordance with hypothesis (2.6), implies the desired conclusion

$$\vartheta_\varepsilon \to \vartheta \quad \text{in } L^3\big((0,T)\times\Omega\big), \quad \text{in particular,} \quad \vartheta_\varepsilon \to \vartheta \quad \text{a.a. in } (0,T)\times\Omega. \tag{4.7}$$

In order to see (4.6), we first observe that

$$\int_0^T \int_\Omega \varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon)(\vartheta_\varepsilon - \vartheta)\,\mathrm{d}x\,\mathrm{d}t \to 0 \quad \text{as } \varepsilon \to 0.$$

Indeed this follows by means of Lions–Aubin compactness argument as

$$(\vartheta_\varepsilon - \vartheta) \to 0 \quad \text{weakly in } L^2\big(0,T;W^{1,2}(\Omega)\big),$$

and the (weak) time derivative $\partial_t(\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon))$ can be expressed by means of entropy inequality (2.17).

Consequently, it remains to show that

$$\int_0^T \int_\Omega \varrho_\varepsilon s(\varrho_\varepsilon,\vartheta)(\vartheta_\varepsilon - \vartheta)\,\mathrm{d}x\,\mathrm{d}t \to 0 \quad \text{as } \varepsilon \to 0. \tag{4.8}$$

To see (4.8), we combine the bounds imposed on $\partial_t b(\varrho_\varepsilon)$ by the renormalized equation (2.12), with the estimates on the temperature gradient (3.11), to deduce that

$$\nu_{t,x}[\varrho_\varepsilon,\vartheta_\varepsilon] = \nu_{t,x}[\varrho_\varepsilon] \otimes \nu_{t,x}[\vartheta_\varepsilon] \quad \text{for a.a. } t,x, \tag{4.9}$$

where the symbol $\nu[\varrho_\varepsilon,\vartheta_\varepsilon]$ denotes a Young measure associated to the family $\{\varrho_\varepsilon,\vartheta_\varepsilon\}_{\varepsilon>0}$, while $\nu[\varrho_\varepsilon]$, $\nu[\vartheta_\varepsilon]$ stand for Young measures generated by $\{\varrho_\varepsilon\}_{\varepsilon>0}$, $\{\vartheta_\varepsilon\}_{\varepsilon>0}$, respectively. In order to conclude, we use the following result frequently called *Fundamental theorem of the theory of Young measures* (see Pedregal [37, Chapter 6, Theorem 6.2]):

Theorem 4.1. *Let $\{\vec{v}_n\}_{n=1}^\infty$, $\vec{v}_n : Q \subset R^N \to R^M$ be a sequence of functions bounded in $L^1(Q;R^M)$, where Q is a domain in R^N.*

Then there exist a subsequence (not relabeled) and a parameterized family $\{\nu_y\}_{y\in Q}$ of probability measures on R^M depending measurably on $y \in Q$ with the following property:

For any Caratheodory function $\Phi = \Phi(y,z)$, $y \in Q$, $z \in R^M$ such that

$$\Phi(\cdot,\vec{v}_n) \to \overline{\Phi} \quad \text{weakly in } L^1(Q),$$

we have

$$\overline{\Phi}(y) = \int_{R^M} \psi(y,z)\,\mathrm{d}\nu_y(z) \quad \textit{for a.a. } y \in Q.$$

In virtue of Theorem 4.1, relation (4.9) implies (4.8). We have proved the strong (a.a.) pointwise convergence of the temperature claimed in (4.7). Note that this step leans heavily on the presence of the *radiative entropy*.

4.1.2. Pointwise convergence of density

Although the pointwise convergence of the family of densities $\{\varrho_\varepsilon\}_{\varepsilon>0}$ represents one of the most delicate questions of the existence theory for the compressible Navier–Stokes system, this step is nowadays well understood. The idea is to use the quantities

$$\varphi(t,x) = \psi(t)\phi(x)\nabla_x\Delta^{-1}\big[1_\Omega T_k(\varrho_\varepsilon)\big]$$

as test functions in the weak formulation of momentum equation (2.13). Similarly, we can let $\varepsilon \to 0$ in (2.13) and test the resulting expression on

$$\varphi(t,x) = \psi(t)\phi(x)\nabla_x\Delta^{-1}\big[1_\Omega \overline{T_k(\varrho_\varepsilon)}\big],$$

where $\psi \in C_c^\infty(0,T)$, $\phi \in C_c^\infty(\Omega)$, and T_k is a cut-off function,

$$T_k(z) = \min\{z,k\}.$$

In the limit for $\varepsilon \to 0$, this procedure yields the celebrated relation for the *effective viscous pressure* discovered by Lions [31], which reads in the present setting as

$$\int_0^T\int_\Omega \psi\phi\big(\overline{p(\varrho,\vartheta)T_k(\varrho)} - \overline{p(\varrho,\vartheta)}\;\overline{T_k(\varrho)}\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^T\int_\Omega \psi\phi\big(\overline{\mathbb{T}:\mathcal{R}\big[1_\Omega T_k(\varrho)\big]} - \mathbb{T}:\mathcal{R}\big[1_\Omega \overline{T_k(\varrho)}\big]\big)\,\mathrm{d}x\,\mathrm{d}t, \tag{4.10}$$

where the bars denote weak limits of the composed quantities, and where $\mathcal{R} = \mathcal{R}_{i,j} = \partial_{x_i}\Delta^{-1}\partial_{x_j}$ is a pseudo-differential operator with the symbol

$$\mathcal{R} = \frac{\xi\otimes\xi}{|\xi|^2}$$

(see [17, Section 3.6]). Note that the presence of the extra term $\vec{S}_F$ in (2.13) does not present any additional difficulty as

$$\int_0^T\int_\Omega \psi\vec{S}_{F,\varepsilon}\cdot\phi\nabla_x\Delta^{-1}\big[T_k(\varrho_\varepsilon)\big]\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T\int_\Omega \psi\vec{S}_F\cdot\phi\nabla_x\Delta^{-1}\big[\overline{T_k(\varrho)}\big]\,\mathrm{d}x\,\mathrm{d}t.$$

The following *commutator lemma* is in the spirit of Coifman and Meyer [7]:

Lemma 4.1. *Let $w \in W^{1,2}(\mathbb{R}^3)$ and $\vec{Z} \in L^p(\mathbb{R}^3;\mathbb{R}^3)$ be given, with $6/5 < p < \infty$.*

Then, for any $1 < s < 6p/(6+p)$,

$$\big\|\mathcal{R}[w\vec{Z}] - w\mathcal{R}[\vec{Z}]\big\|_{W^{\beta,s}(\mathbb{R}^3;\mathbb{R}^3)} \leqslant c\|w\|_{W^{1,2}(\mathbb{R}^3)}\|\vec{Z}\|_{L^p(\mathbb{R}^3;\mathbb{R}^3)},$$

where $0 < \beta = \frac{3}{s} - \frac{6+p}{6p}$, and $c = c(p)$ are positive constants.

Applying Lemma 4.1 to the expression on the right-hand side of (4.10) and using (weak) compactness in time of $T_k(\varrho_\varepsilon)$ following from renormalized equation (2.12), we obtain

$$\overline{p(\varrho,\vartheta)T_k(\varrho)} - \left(\frac{4}{3}\mu(\vartheta)+\eta(\vartheta)\right)\overline{T_k(\varrho)\operatorname{div}_x\vec{u}} = \overline{p(\varrho,\vartheta)}\,\overline{T_k(\varrho)} - \left(\frac{4}{3}\mu(\vartheta)+\eta(\vartheta)\right)\overline{T_k(\varrho)}\operatorname{div}_x\vec{u}, \tag{4.11}$$

cf. [17, Section 3.7.4].

Now, introducing the functions

$$L_k(\varrho) = \int_1^\varrho \frac{T_k(z)}{z^2}\,\mathrm{d}z,$$

we deduce from renormalized equation (2.12) that

$$\int_0^T\int_\Omega \left(\overline{\varrho L_k(\varrho)}\partial_t\varphi + \overline{\varrho L_k(\varrho)}\vec{u}\nabla_x\varphi - \overline{T_k(\varrho)\operatorname{div}_x\vec{u}}\varphi\right)\mathrm{d}x\,\mathrm{d}t = -\int_\Omega \varrho_0 L_k(\varrho_0)\varphi(0,\cdot)\,\mathrm{d}x$$

for any $\varphi \in C_c^\infty([0,T)\times\overline{\Omega})$. It follows from (4.11) that

$$\mathbf{osc}_q[\varrho_\varepsilon\to\varrho]\big((0,T)\times\Omega\big) \equiv \sup_{k\geqslant 1}\left(\limsup_{\varepsilon\to 0}\int_0^T\int_\Omega \left|T_k(\varrho_\varepsilon)-T_k(\varrho)\right|^q\,\mathrm{d}x\,\mathrm{d}t\right) < \infty, \quad q = 8/3, \tag{4.12}$$

where **osc** is the oscillation defect measure introduced in [19]. In particular, relation (4.12) implies that the limit functions ϱ, $\vec{u}$ satisfy renormalized equation (2.12) (see [17, Lemma 3.8]); whence

$$\int_0^T\int_\Omega \left(\overline{\varrho L_k(\varrho)} - \varrho L_k(\varrho)\right)(\tau)\,\mathrm{d}x + \int_0^\tau\int_\Omega \left(\overline{T_k(\varrho)\operatorname{div}_x\vec{u}} - \overline{T_k(\varrho)}\operatorname{div}_x\vec{u}\right)\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^\tau\int_\Omega \left(T_k(\varrho)\operatorname{div}_x\vec{u} - \overline{T_k(\varrho)}\operatorname{div}_x\vec{u}\right)\mathrm{d}x\,\mathrm{d}t \quad \text{for any } \tau\in[0,T]. \tag{4.13}$$

Using once more (4.12), we can let $k\to\infty$ in (4.13) to obtain the desired conclusion

$$\overline{\varrho\log(\varrho)} = \varrho\log(\varrho),$$

in other words,

$$\varrho_\varepsilon \to \varrho \quad \text{in } L^1\big((0,T)\times\Omega\big), \tag{4.14}$$

see [17, Section 3.7.4] for details.

Relations (4.1)–(4.4), (4.7), (4.14), together with the uniform bounds established in Section 3, allow us to pass to the limit in the weak formulation of the Navier–Stokes–Fourier system introduced in Section 2.2, as soon as we show convergence of the sequence $\{I_\varepsilon\}_{\varepsilon>0}$. This will be accomplished in the forthcoming section.

4.2. Convergence of the radiation intensities

Our ultimate goal is to establish convergence of the quantities

$$\frac{1}{\vartheta_\varepsilon}\vec{u}_\varepsilon\cdot\vec{S}_{F,\varepsilon} = \frac{1}{c\vartheta_\varepsilon}\vec{u}_\varepsilon\cdot\int_0^\infty \sigma_a(\nu,\vartheta_\varepsilon)\left(\int_{\mathcal{S}^2}\vec{\omega}\big(B(\nu,\vartheta_\varepsilon)-I_\varepsilon\big)\,\mathrm{d}\vec{\omega}\right)\mathrm{d}\nu$$
$$+ \frac{1}{c\vartheta_\varepsilon}\vec{u}_\varepsilon\cdot\int_0^\infty \sigma_s(\nu,\vartheta_\varepsilon)\left(\int_{\mathcal{S}^2}\vec{\omega}\left(\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I_\varepsilon\,\mathrm{d}\vec{\omega}\right)-I_\varepsilon\right)\mathrm{d}\vec{\omega}\right)\mathrm{d}\nu$$

and

$$\frac{1}{\vartheta_\varepsilon} S_{E,\varepsilon} = \frac{1}{c\vartheta_\varepsilon} \int_0^\infty \sigma_a(\nu, \vartheta_\varepsilon) \left(\int_{\mathcal{S}^2} \big(B(\nu, \vartheta_\varepsilon) - I_\varepsilon\big) \,\mathrm{d}\vec{\omega} \right) \mathrm{d}\nu.$$

Since $\vartheta_\varepsilon \to \vartheta$ a.a. in $(0,T) \times \Omega$, and

$$\vec{u}_\varepsilon \to \vec{u} \quad \text{weakly in } L^2\big(0, T; W_0^{1,2}\big(\Omega; \mathbb{R}^3\big)\big)$$

the desired result follows from compactness of the velocity averages over the sphere $\mathcal{S}^2$ established by Golse et al. [24,25], see also Bournaveas and Perthame [4], and hypothesis (2.10). Specifically, we use the following result (see [24]):

Proposition 4.1. *Let $I \in L^q([0,T] \times R^{n+1} \times \mathcal{S}^2)$, $\partial_t I + \omega \cdot \nabla_x I \in L^q([0,T] \times R^{n+1} \times \mathcal{S}^2)$ for a certain $q > 1$. In addition, let $I_0 \equiv I(0, \cdot) \in L^\infty(R^{n+1} \times \mathcal{S}^2)$.*

Then

$$\tilde{I} \equiv \int_{\mathcal{S}^2} I(\cdot, \nu) \,\mathrm{d}\vec{\omega}$$

belongs to the space $W^{s,q}([0,T] \times R^{n+1})$ for any s, $0 < s < \inf\{1/q, 1 - 1/q\}$, and

$$\|\tilde{I}\|_{W^{s,q}} \leqslant c(I_0)\big(\|I\|_{L^q} + \|\partial_t I + \omega \cdot \nabla I\|_{L^q}\big).$$

As the radiation intensity I_ε satisfies the transport equation (1.5), by virtue of the cut-off hypotheses (2.9)–(2.11) where S is bounded in $L^q \cap L^\infty([0,T) \times \Omega \times R^1 \times S^2)$, a direct application of Proposition 4.1 yields the desired conclusion

$$\int_{\mathcal{S}^2} I_\varepsilon(\cdot, \nu) \,\mathrm{d}\vec{\omega} \to \int_{\mathcal{S}^2} I(\cdot, \nu) \,\mathrm{d}\vec{\omega} \quad \text{in } L^2\big((0,T) \times \Omega\big)$$

and

$$\int_{\mathcal{S}^2} \vec{\omega} I_\varepsilon(\cdot, \nu) \,\mathrm{d}\vec{\omega} \to \int_{\mathcal{S}^2} \vec{\omega} I(\cdot, \nu) \,\mathrm{d}\vec{\omega} \quad \text{in } L^2\big((0,T) \times \Omega\big)$$

for any fixed ν.

Consequently,

$$\frac{1}{\vartheta_\varepsilon} \vec{u}_\varepsilon \cdot F_{s,\varepsilon} \to \frac{1}{\vartheta} \vec{u} \cdot F_s$$

and, similarly,

$$\frac{1}{\vartheta_\varepsilon} S_{E,\varepsilon} \to \frac{1}{\vartheta} S_E$$

as required. Note that strong (a.a. pointwise) convergence of the ω-averages is needed as $\vec{u}_\varepsilon$ may fail to converge strongly on hypothetical vacuum zones, meaning on the part of Ω where the limit density ϱ vanishes.

Theorem 2.1 has been proved.

5. Approximations, global-in-time existence

We conclude the paper by proposing an approximation scheme to be used to prove existence of global-in-time weak solutions to problem (1.2)–(1.13). The scheme is essentially the same as in [17, Chapter 3], the extra terms are put in { }.

- The continuity equation (1.2) is replaced by an "artificial viscosity" approximation

$$\partial_t \varrho + \operatorname{div}_x(\varrho \vec{u}) = \{d \Delta \varrho\}, \quad d > 0, \tag{5.1}$$

to be satisfied on $(0, T) \times \Omega$, and supplemented by the homogeneous Neumann boundary conditions

$$\nabla_x \varrho \cdot \vec{n}|_{\partial\Omega} = 0. \tag{5.2}$$

The initial distribution of the approximate densities is given through

$$\varrho(0, \cdot) = \varrho_{0,\delta}, \tag{5.3}$$

where

$$\varrho_{0,\delta} \in C^{2,\nu}(\overline{\Omega}), \qquad \nabla_x \varrho_{0,\delta} \cdot \vec{n}|_{\partial\Omega} = 0, \qquad \inf_{x\in\Omega} \varrho_{0,\delta}(x) > 0, \tag{5.4}$$

with a positive parameter $\delta > 0$.

- The momentum equation is replaced by a Faedo–Galerkin approximation:

$$\begin{aligned}
&\int_0^T \int_\Omega \Big((\varrho\vec{u})\partial_t \varphi + (\varrho\vec{u} \otimes \vec{u}) : \nabla_x \varphi + \big(p + \big\{\delta\big(\varrho^\Gamma + \rho^2\big)\big\}\big) \operatorname{div}_x \varphi \Big) \\
&\quad = \int_0^T \int_\Omega \big(\big\{ d(\nabla_x \rho \nabla_x \vec{u}) \big\} \cdot \varphi + \mathbb{T}_\delta : \nabla_x \varphi - S_F \varphi \big) \mathrm{d}x \, \mathrm{d}t - \int_\Omega (\rho \vec{u})_0 \cdot \varphi \, \mathrm{d}x,
\end{aligned} \tag{5.5}$$

to be satisfied for any test function $\varphi \in C_c^1([0, T), X_n)$, where

$$X_n \subset C^{2,\nu}\big(\overline{\Omega}; R^3\big) \subset L^2\big(\Omega; R^3\big) \tag{5.6}$$

is a finite-dimensional space of functions satisfying the no-slip boundary conditions

$$\varphi|_{\partial\Omega} = 0. \tag{5.7}$$

The space X_n is endowed with the Hilbert structure induced by the scalar product of the Lebesque space $L^2(\Omega; R^3)$.

We set

$$\mathbb{T}_\delta = \mathbb{T}_\delta(\vartheta, \nabla_x \vec{u}) = \big(\mu(\vartheta) + \delta\vartheta\big)\left(\nabla_x \vec{u} + \nabla_x^T \vec{u} - \frac{2}{3} \operatorname{div}_x \vec{u}\mathbb{I}\right) + \lambda(\vartheta) \operatorname{div}_x \vec{u}\mathbb{I}. \tag{5.8}$$

- We replace the entropy equation (1.4) by a modified internal energy balance

$$\begin{aligned}
&\partial_t \big(\varrho e + \{\delta\varrho\vartheta\}\big) + \operatorname{div}_x \big(\big(\varrho e + \{\delta\varrho\vartheta\}\big)\vec{u}\big) - \operatorname{div}_x \nabla_x \mathcal{K}_\delta \\
&\quad = \mathbb{T}_\delta(\vartheta, \nabla_x \vec{u}) : \nabla_x \vec{u} - p \operatorname{div}_x \vec{u} + \left\{ d\delta\big(\Gamma|\varrho|^{\Gamma-2} + 2\big)|\nabla_x \rho|^2 + \delta \frac{1}{\vartheta^2} - d\vartheta^5 \right\} - S_E + \vec{u} S_F
\end{aligned} \tag{5.9}$$

to be satisfied in $(0, T) \times \Omega$, together with no-flux boundary conditions

$$\nabla_x \vartheta \cdot \vec{n}|_\partial = 0. \tag{5.10}$$

The initial conditions read

$$\varrho(e + \delta\vartheta)(0, \cdot) = \varrho_{0,\delta}\big(e(\varrho_{0,\delta}, \vartheta_{0,\delta}) + \delta\vartheta_{0,\delta}\big), \tag{5.11}$$

where the (approximate) temperature distribution satisfies

$$\vartheta_{0,\delta} \in C^1(\overline{\Omega}), \qquad \nabla_x \vartheta_{0,\delta} \cdot \vec{n}|_{\partial\Omega} = 0, \qquad \inf_{x\in\Omega} \vartheta_{0,\delta}(x) > 0. \tag{5.12}$$

By $\mathcal{K}_\delta$ we mean

$$\mathcal{K}_\delta(\vartheta) = \int_1^\vartheta \kappa_\delta \,\mathrm{d}z, \qquad \kappa_\delta = \kappa + \delta\left(\vartheta^\Gamma + \frac{1}{\vartheta}\right). \tag{5.13}$$

- We add the equation for the radiative transfer

$$\frac{1}{c}\partial_t I + \vec{\omega}\cdot\nabla_x I = S \quad \text{in } (0,T)\times\Omega\times(0,\infty)\times\mathcal{S}^2, \tag{5.14}$$

together with the transparency condition (1.13).

Given a family of approximate solutions $\{\varrho_{d,\delta}, \vec{u}_{d,\delta}, \vartheta_{d,\delta}, I_{d,\delta}\}_{d>0,\delta>0}$, we may construct a weak solution of system (1.2)–(1.13) letting successively $d\to 0$, $\delta\to 0$ and using compactness arguments delineated in the previous part of this paper. The reader may consult [17, Chapter 3] for all technical details. The approximate solutions can be constructed by means of a fixed point argument applied to the couple $\vec{u}$, I, similarly to [17, Chapter 3, Section 3.4].

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