Available online at www.sciencedirect.com
ScienceDirect
Ann. I. H. Poincaré – AN 28 (2011) 853–887
ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

Local well-posedness and blow-up in the energy space for a class of L^2 critical dispersion generalized Benjamin–Ono equations

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Received 27 May 2010; accepted 27 June 2011

Available online 2 August 2011

Abstract

We consider a family of dispersion generalized Benjamin–Ono equations (dgBO)

$$u_t - \partial_x |D|^\alpha u + |u|^{2\alpha} \partial_x u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R},$$

where $\widehat{|D|^\alpha u} = |\xi|^\alpha \widehat{u}$ and $1 \leqslant \alpha \leqslant 2$. These equations are critical with respect to the L^2 norm and global existence and interpolate between the modified BO equation ($\alpha = 1$) and the critical gKdV equation ($\alpha = 2$).

First, we prove local well-posedness in the energy space for $1 < \alpha < 2$, extending results in Kenig et al. (1991, 1993) [13,14] for the generalized KdV equations.

Second, we address the blow-up problem in the spirit of Martel and Merle (2000) [19] and Merle (2001) [22] concerning the critical gKdV equation, by studying rigidity properties of the dgBO flow in a neighborhood of the solitons. We prove that for α close to 2, solutions of negative energy close to solitons blow up in finite or infinite time in the energy space $H^{\frac{\alpha}{2}}$.

The blow-up proof requires both extensions to dgBO of monotonicity results for local L^2 norms by pseudo-differential operator tools and perturbative arguments close to the gKdV case to obtain structural properties of the linearized flow around solitons.
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Résumé

Nous considérons une famille d'équations de Benjamin–Ono à dispersion généralisée (dgBO)

$$u_t - \partial_x |D|^\alpha u + |u|^{2\alpha} \partial_x u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R},$$

où $\widehat{|D|^\alpha u} = |\xi|^\alpha \widehat{u}$ et $1 \leqslant \alpha \leqslant 2$. Ces équations sont critiques par rapport à la norme L^2 et à l'existence globale et peuvent être vues comme des interpolations entre l'équation de Benjamin–Ono généralisée critique ($\alpha = 1$) et l'équation de Korteweg–de Vries généralisée critique ($\alpha = 2$).

D'abord, nous montrons le caractère bien posé de ces équations dans l'espace d'énergie pour $1 < \alpha < 2$, étendant les résultats de Kenig et al. (1991, 1993) [13,14] pour les équations de Korteweg–de Vries généralisées.

Ensuite, nous étudions le phénomène d'explosion dans l'esprit de Martel et Merle (2000) [19] et Merle (2001) [22] concernant l'équation de gKdV critique, en étudiant les propriétés de rigidité du flot de dgBO dans un voisinage des solitons. Nous montrons

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doi:10.1016/j.anihpc.2011.06.005

que pour α proche de 2, les solutions d'énergie négative proches des solitons explosent en temps fini ou infini dans l'espace d'énergie $H^{\frac{\alpha}{2}}$.

La preuve de ce résultat d'explosion est basée d'une part sur l'adaptation à dgBO de résultats de monotonie de normes L^2 locales par des méthodes d'opérateurs pseudo-differentiels et d'autre part sur des arguments de perturbation pour obtenir des propriétés structurelles du flot linéarisé autour des solitons lorsque l'équation est proche de gKdV.
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1. Introduction

We consider the following dispersion generalized Benjamin–Ono equations (dgBO)

$$u_t - \partial_x |D|^\alpha u + |u|^{2\alpha} \partial_x u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \tag{1}$$

where $|D|^\alpha$ is such that $\widehat{|D|^\alpha u} = |\xi|^\alpha \widehat{u}$ and $1 \leqslant \alpha \leqslant 2$. Formally, the following three quantities are conserved for solutions

$$\int u(t,x)\,dx = \int u(0,x)\,dx, \tag{2}$$

$$M(t) = \int u^2(t,x)\,dx = M(0), \tag{3}$$

$$E(t) = \int \left(\left| |D|^{\frac{\alpha}{2}} u \right|^2 - \frac{|u|^{2\alpha+2}}{(\alpha+1)(2\alpha+1)} \right)(t,x)\,dx = E(0). \tag{4}$$

Recall the scaling and translation invariances of Eq. (1): if $u(t,x)$ is a solution of (1) then, for all $\lambda_0 > 0$, $x_0 \in \mathbb{R}$,

$$u_{\lambda_0,x_0}(t,x) = \lambda_0^{-\frac{1}{\alpha}} u\big(\lambda_0^{-(2+\frac{2}{\alpha})} t, \lambda_0^{-\frac{2}{\alpha}}(x-x_0)\big) \quad \text{is also a solution of (1).}$$

In particular, note that for any $\lambda_0 > 0$, $x_0 \in \mathbb{R}$, $\|u_{\lambda_0,x_0}\|_{L^2} = \|u\|_{L^2}$, which means that (1) is a family of L^2 critical equations interpolating between the critical Benjamin–Ono equation (also called modified Benjamin–Ono equation)

$$u_t - \partial_x |D| u + u^2 \partial_x u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \tag{5}$$

and the critical generalized Korteweg–de Vries equation

$$u_t + \partial_x^3 u + u^4 \partial_x u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{6}$$

1.1. Local well-posedness in the energy space

Recall that the local Cauchy problem is known to be well-posed in the energy space $H^{\frac{\alpha}{2}}$ both for the critical gKdV equation – see Kenig, Ponce and Vega [14] – and for the critical (BO) equation – see Kenig and Takaoka [15]. The first objective of this paper is to present a local Cauchy theory for (1) in the energy space $H^{\frac{\alpha}{2}}$ for $1 < \alpha < 2$.

Theorem 1 *(Local well-posedness in the energy space). Let $1 < \alpha < 2$ and $A > 0$. Let $u_0 \in H^{\frac{\alpha}{2}}$ be such that $\|u_0\|_{H^{\frac{\alpha}{2}}} \leqslant A$. Then there exists a unique solution $u \in C([0,T], H^{\frac{\alpha}{2}}) \cap Z_T$ of*

$$\begin{cases} u_t - \partial_x |D|^\alpha u \pm |u|^{2\alpha} \partial_x u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(t=0) = u_0, & x \in \mathbb{R}, \end{cases} \tag{7}$$

where $T = T(A) > 0$. Moreover, the map $u_0 \mapsto u \in C([0,T], H^{\frac{\alpha}{2}}) \cap Z_T$ is continuous.

Theorem 1 is proved by a contraction argument in Z_T, see the proof of Theorem 1 for the definition of this functional space. The linear estimates, mainly taken from [12] and [14], are gathered in Lemma 1.

Remark 1. Together with Theorem 1, we obtain in this paper a property of weak continuity of the flow of Eq. (1) in the energy space, see Theorem 3. See also [19] and [4] for the cases $\alpha = 1, 2$.

In this paper, by solutions of (1), we mean $H^{\frac{\alpha}{2}}$ solutions in the sense of Theorem 1. For such solutions, it follows from standard arguments that the two quantities $M(u(t))$ and $E(u(t))$ defined in (3), (4) are conserved as long as the solution exists (see also Remark 2).

1.2. Blow-up in finite or infinite time

The second objective of this paper is to study global well-posedness versus blow-up for Eqs. (1), i.e. in the focusing case. Recall that Eqs. (1) are critical with respect to global well-posedness in the following sense. For fixed $1 \leqslant \alpha \leqslant 2$, the power $2\alpha + 1$ of the nonlinearity in (1) is the smallest power for which blow-up is possible in the energy space, whereas from the critical Gagliardo–Nirenberg inequality

$$\int |u|^{2\alpha+2} \leqslant C_\alpha \left(\int \left|D^{\frac{\alpha}{2}} u\right|^2 \right) \left(\int u^2 \right)^{\alpha}, \tag{8}$$

it is a standard observation that small (in L^2) solutions of (1) are global and bounded from Theorem 1. Note that inequality (8) is easily proved using Fourier analysis and scaling arguments. See Proposition 1 for the value of the best constant in (8), related to soliton solutions of (1).

Following Martel and Merle [19] and Merle [22] concerning the critical gKdV equation, we look for blow-up solutions close to the soliton family, which we introduce now.

We call soliton any traveling wave solution $u(t,x) = Q_{\lambda_0}(x - x_0 - \lambda_0^{-2} t)$ of Eq. (1), with $\lambda_0 > 0$, $x_0 \in \mathbb{R}$, $Q_{\lambda_0}(x) = \lambda_0^{-\frac{1}{\alpha}} Q(\lambda_0^{-\frac{2}{\alpha}} x)$ and where Q solves:

$$|D|^\alpha Q + Q - \frac{1}{2\alpha+1} Q^{2\alpha+1} = 0, \quad Q \in H^{\frac{\alpha}{2}},\ Q > 0. \tag{9}$$

For the critical gKdV case ($\alpha = 2$), it follows from standard ODE arguments that there exists a unique (up to translations) solution of (9), which is

$$Q(x) = \frac{15^{1/4}}{\cosh^{1/2}(2x)}. \tag{10}$$

Moreover, Weinstein [28] proved that the function Q provides the best constant in estimate (8) for $\alpha = 2$.

For $1 \leqslant \alpha < 2$, existence of a positive even solution of (9) is known by variational arguments, see Weinstein [29,31] and Proposition 1 of the present paper. Such a solution is called a ground state of (9). Amick and Toland [1] proved uniqueness of the solution of the Benjamin–Ono equation (BO)

$$u_t - \partial_x |D| u + u \partial_x u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \tag{11}$$

but their argument does not adapt to (9). By a different and very general argument, Frank and Lenzmann [5] recently proved uniqueness of the ground state for a large class of equations $|D|^\alpha Q + Q - cQ^{\beta+1} = 0$ including (9) for all $\alpha \in [1,2)$.

Now, we state our second main result.

Theorem 2 *(Blow-up in finite or infinite time). There exists $\alpha_0 \in [1,2)$ such that for all $\alpha \in (\alpha_0, 2)$, the following holds. Let Q be the unique even positive solution of* (9) *which minimizes the constant C_α in* (8). *There exists $\beta_0 > 0$ such that if $u(t)$ is an $H^{\frac{\alpha}{2}}$ solution of* (1) *satisfying*

$$E\big(u(0)\big) < 0 \quad \text{and} \quad \int u^2(0) \leqslant \int Q^2 + \beta_0,$$

then $u(t)$ blows up in finite or infinite time in $H^{\frac{\alpha}{2}}$, i.e. there exists $0 < T \leqslant +\infty$ such that $\lim_{t\to T} \|u(t)\|_{H^{\frac{\alpha}{2}}} = +\infty$.

Note that from Gagliardo–Nirenberg's inequality with best constant (see Proposition 1) $E(u(0)) < 0$ implies that $\int u^2(0) > \int Q^2$. Therefore, we prove blow-up in finite or infinite time for any $u(0)$ such that $E(u(0)) < 0$, $\int Q^2 < \int u^2(0) \leqslant \int Q^2 + \beta_0$, which is a large class of initial data close to Q up to the invariances of the equation (see Lemma 9).

Theorem 2 extends to Eq. (1) the main result in [22]. We follow the same strategy based on rigidity properties of the nonlinear flow around solitons. First, we prove a nonlinear Liouville property around the soliton as a consequence of a linear Liouville property. See Section 5 where we extend the main results of [19]. Then, in Section 6, we prove blow-up in the sense of Theorem 2 by a contradiction argument, using the nonlinear Liouville property and the additional invariant $\int u(t) = \int u(0)$, as in [22].

Now, we discuss how techniques developed in [19] and [22] have to be adapted to Eq. (1) to prove Theorem 2.

1. Monotonicity properties of local L^2 quantities. These arguments were developed in [19] and [22] in order to study the variation in time of the L^2 norm of the solution in various regions of space (on the left or on the right, in some sense, to the soliton). For the critical gKdV equation, these monotonicity arguments are mainly based on the Kato identity and refined estimates on the nonlinear term in this identity. For Benjamin–Ono type equations, such localization arguments are subtle to adapt due to the nonlocal character of the linear operator. Such L^2 monotonicity arguments were developed in [11] to prove asymptotic stability of the solitons for the BO equation, but the arguments in [11] seem to work only for the operator $|D|$, i.e. for $\alpha = 1$ in (1). In the present paper, we extend these results to any $\alpha \in (1, 2)$ using tools from pseudo-differential calculus. Section 4 is devoted to these arguments.
2. Weak continuity of the flow. In addition to the local Cauchy theory, we need the weak continuity of the flow of (1) in several key limiting arguments. See Theorem 3.
3. Linear Liouville property. The proofs in [19] and [22] for the gKdV case depend crucially on a linear rigidity property of the ground state (hereafter called *linear Liouville property*). By perturbative arguments, we are able to extend the linear Liouville property for gKdV to (9) for $\alpha < 2$ close enough to 2 (i.e. when the model is close in some sense to the critical gKdV equation). See Proposition 3. In this paper, we rely on the simplified approach of [17].
   It follows from the arguments of this paper that Theorem 2 holds true for any $1 < \alpha < 2$ if the linear Liouville property is assumed. Indeed, it is the only part in the proof of Theorem 2 where we need perturbative arguments close to the gKdV case. In particular, the monotonicity arguments and the overall strategy work for any $1 < \alpha \leqslant 2$.

Remark finally that in addition to [19] and [22], two further works ([20] and [21]) provide refined information about the blow-up phenomenon for the critical gKdV equation close to the soliton family. Indeed, in [20], the soliton Q is found to be the universal blow-up profile in the context of Theorem 2. The proof is based on an additional rigidity property of the gKdV flow around solitons in a blow-up regime. Finally, [21] proves blow-up in finite time, together with an upper estimate on the blow-up rate, provided that the initial data has some space decay. However, note that the blow-up problem for the critical gKdV equations is not yet completely understood, in particular the question of the exact blow-up rate. The case of the nonlinear Schrödinger equation is by now much better known, see Merle and Raphaël [23–25] and references therein. For simplicity and brevity, we do not try here to extend results of [20] and [21] to dgBO equation.

1.3. Plan of the paper

The paper is organized as follows. In Section 2, we prove Theorem 1. In Section 3, we study the stationary problem (9) in the general case $1 \leqslant \alpha \leqslant 2$ and obtain further properties in the perturbative case where α is close to 2. In Section 4, we present L^2 monotonicity properties for the model (1) for all $\alpha \in (1, 2)$. In Section 5, we deal with solutions close to a (bounded) soliton and finally in Section 6, we prove Theorem 2, i.e. for α close to 2, blow-up in finite or infinite time for negative energy solutions close to solitons.

2. Local well-posedness in the energy space

2.1. Proof of Theorem 1

We denote the Fourier transform by $\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int e^{-ix\xi} f(x)\,dx$.

We introduce the group $W_\alpha(t)$ defined by

$$\mathcal{F}\big(W_\alpha(t) f\big)(\xi) = e^{it(|\xi|^\alpha \xi)} \hat{f}(\xi), \quad 1 < \alpha < 2.$$

Then, we claim (or recall) the following linear estimates (we use classical notation from [14]).

Lemma 1. *For $0 < T < 1$, there exists $C > 0$ such that, for all $u_0 \in L^2$, then*

$$\text{(i)} \qquad \sup_t \| W_\alpha(t)u_0 \|_{L^2} \leqslant C\|u_0\|_{L^2};$$

$$\text{(ii)} \qquad \| |D|^{\frac{\alpha}{2}} W_\alpha(t)u_0 \|_{L^\infty_x L^2_t} \leqslant C\|u_0\|_{L^2};$$

$$\text{(iii)} \qquad \text{for all } u_0 \in H^{\alpha/2},\ \| W_\alpha(t)u_0 \|_{L^\infty_x L^2_T} \leqslant CT^{1/2}\|u_0\|_{H^{\frac{\alpha}{2}}};$$

for $0 \leqslant \beta < \alpha/2$, there exists $\gamma > 0$ such that,

$$\text{(iv)} \qquad \| |D|^{\beta} W_\alpha(t)u_0 \|_{L^\infty_x L^2_T} \leqslant CT^{\gamma}\|u_0\|_{L^2};$$

$$\text{(v)} \qquad \text{for all } u_0 \in H^{\alpha/2},\ \| \partial_x W_\alpha(t)u_0 \|_{L^\infty_x L^2_T} \leqslant CT^{\gamma}\|u_0\|_{H^{\alpha/2}};$$

$$\text{(vi)} \qquad \text{for all } u_0 \in H^{\beta^+},\ \text{where } \beta^+ > \frac{3}{4} - \frac{\alpha}{4},\ \| W_\alpha(t)u_0 \|_{L^{2\alpha}_x L^\infty_T} \leqslant C\|u_0\|_{H^{\beta+}};$$

for all $h \in L^1_x L^2_T$,

$$\text{(vii)} \qquad \left\| |D|^{\alpha} \int_0^t W_\alpha(t-t')h(t')\,dt' \right\|_{L^\infty_x L^2_t} \leqslant C\|h\|_{L^1_x L^2_T};$$

$$\text{(viii)} \qquad \sup_{0<t<T} \left\| |D|^{\alpha/2} \int_0^t W_\alpha(t-t')h(t')\,dt' \right\|_{L^2_x} \leqslant C\|h\|_{L^1_x L^2_T};$$

for $0 \leqslant \beta < \alpha$, there exists $\gamma > 0$ such that,

$$\text{(ix)} \qquad \left\| |D|^{\beta} \int_0^t W_\alpha(t-t')h(t')\,dt' \right\|_{L^\infty_x L^2_T} \leqslant CT^{\gamma}\|h\|_{L^1_x L^2_T};$$

there exists $\gamma > 0$ such that,

$$\text{(x)} \qquad \sup_{0<t<T} \left\| \int_0^t W_\alpha(t-t')h(t')\,dt' \right\|_{L^2_x} \leqslant CT^{\gamma}\|h\|_{L^1_x L^2_T};$$

for all $h \in L^1_x L^2_T$ such that $\partial_x h \in L^1_x L^2_T$,

$$\text{(xi)} \qquad \left\| \int_0^t W_\alpha(t-t')\partial_x h(t')\,dt' \right\|_{L^{2\alpha}_x L^\infty_T} \leqslant C\|\partial_x h\|_{L^1_x L^2_T}.$$

Proof. (i) is the classical conservation law, and (ii) (sharp Kato smoothing effect) is proved in [13, Lemma 2.1].

By Sobolev embedding

$$\left| W_\alpha(t)u_0(x) \right|^2 \leqslant C \| W_\alpha(t)u_0 \|^2_{H^{\frac{\alpha}{2}}} \leqslant C\|u_0\|^2_{H^{\frac{\alpha}{2}}}. \tag{12}$$

Integrating (12) with respect t, we obtain (iii).

To prove (iv), we first write $u_0 = u_{0,1} + u_{0,2}$ where $\hat{u}_{0,1}(\xi) = \chi_{|\xi|\leqslant M}\hat{u}_0(\xi)$, for $M > 0$ to be chosen. Consider $\||D|^\beta W_\alpha(t)u_{0,2}\|_{L_x^\infty L_t^2}$ and let $v_{0,2}$ such that $\hat{v}_{0,2}(\xi) = \frac{|\xi|^\beta}{|\xi|^{\alpha/2}}\hat{u}_{0,2}(\xi)$, so that we have $|D|^{\alpha/2}W_\alpha(t)v_{0,2} = |D|^\beta W_\alpha(t)u_{0,2}$. Using (ii), we see that, since $\beta < \alpha/2$,

$$\big\||D|^\beta W_\alpha(t)u_{0,2}\big\|_{L_x^\infty L_T^2} \leqslant C\|v_{0,2}\|_{L^2} \leqslant CM^{\beta-\alpha/2}\|u_0\|_{L^2}. \tag{13}$$

Consider now $|D|^\beta u_{0,1}$. Then, $\||D|^\beta u_{0,1}\|_{H^{\alpha/2}} \leqslant CM^{\beta+\alpha/2}\|u_0\|_{L^2}$. From (iii),

$$\big\||D|^\beta W_\alpha(t)u_{0,1}\big\|_{L_x^\infty L_T^2} \leqslant CT^{1/2}M^{\beta+\alpha/2}\|u_0\|_{L^2}. \tag{14}$$

Choosing $M^\alpha = T^{-1/2}$, estimate (iv) follows from (13) and (14).

Writing $|D| = |D|^{1-\alpha/2}|D|^{\alpha/2}$ we use (iii) and (iv) and the fact that $1-\alpha/2 < \alpha/2$ to prove (v).

In [13, proof of Theorem 2.7, p. 332], it is proved that if $|\xi| \simeq 2^k$ (or $|\xi| \lesssim 1$ for $k = 0$) we have, for $\hat{u}_0$ with that support

$$\big\|W_\alpha(t)u_0\big\|_{L_x^2 L_T^\infty} \leqslant C2^{k(\alpha+1)/4}\|u_0\|_{L^2}. \tag{15}$$

Also in [12, Theorem 2.5], it is proved that

$$\big\|W_\alpha(t)u_0\big\|_{L_x^4 L_T^\infty} \leqslant C\big\||D|^{1/4}u_0\big\|_{L^2}. \tag{16}$$

Write now $\frac{1}{2\alpha} = \frac{\theta}{2} + \frac{1-\theta}{4}$ then $\theta = \frac{2}{\alpha} - 1 \in (0,1)$, by interpolation, we get

$$\big\|W_\alpha(t)u_0\big\|_{L_x^{2\alpha} L_T^\infty} \leqslant C2^{k(1-\theta)/4}2^{k(1+\alpha)\theta/4}\|u_0\|_{L^2} = C2^{k(3-\alpha)/4}\|u_0\|_{L^2} \tag{17}$$

which implies (vi). Note that (17) is more precise for $\hat{u}_0$ supported in $|\xi| \simeq 2^k$.

Estimates (vii) and (viii) are proved in a similar way as (3.8) and (3.7) in [14]. We omit their proofs.

Let $\theta \in C_0^\infty$, $\theta \equiv 1$ for $|\xi| \leqslant 1$, $\theta_M(\xi) = \theta(\xi/M)$, $\psi_M(\xi) = 1-\theta_M(\xi)$ where $M \geqslant 1$. Write $h = h_{1,M} + h_{2,M}$, where $\hat{h}_{1,M}(t,\xi) = \theta_M(\xi)\hat{h}(t,\xi)$. Write $\tilde{h}_{2,M}$ by $(\tilde{h}_{2,M})^\wedge(t,\xi) = \frac{|\xi|^\beta}{|\xi|^\alpha}\hat{h}_{2,M}(t,\xi)$. Thus $|D|^\beta\int_0^t W_\alpha(t-t')h_{2,M}(t')\,dt' = |D|^\alpha\int_0^t W_\alpha(t-t')\tilde{h}_{2,M}\,dt'$. Let $\hat{\eta}_M(\xi) = \frac{|\xi|^\beta}{|\xi|^\alpha}\psi_M(\xi)$. Using a dyadic partition of unity in frequency space and Bernstein inequality, we claim $\int|\eta_M| \leqslant CM^{\beta-\alpha}$.

Thus

$$\|\tilde{h}_{2,M}\|_{L_x^1 L_T^2} \leqslant CM^{\alpha-\beta}\|h\|_{L_x^1 L_T^2} \tag{18}$$

so that by (vii),

$$\left\||D|^\beta\int_0^t W_\alpha(t-t')h_{2,M}(t')\,dt'\right\|_{L_x^\infty L_T^2} \leqslant CM^{\alpha-\beta}\|h\|_{L_x^1 L_T^2}. \tag{19}$$

Next, we consider $|D|^\beta\int_0^t W_\alpha(t-t')h_{1,M}(t')\,dt'$. Then, let us define $\hat{\mu}_M(\xi) = |\xi|^\beta\langle\xi\rangle\theta_M(\xi)$ where $\langle\xi\rangle^2 = 1+|\xi|^2$. Then, $\|\mu_M\|_{L^2} \leqslant CM^{\beta+3/2}$. Moreover, for a fixed t, we have by Sobolev embedding,

$$\begin{aligned}
\left||D|^\beta\int_0^t W_\alpha(t-t')h_{1,M}(t')\,dt'\right| &\leqslant C\left\|\int_0^t W_\alpha(t-t')\langle D\rangle|D|^\beta h_{1,M}(t')\,dt'\right\|_{L_x^2} \\
&\leqslant C\int_0^T \big\|\mu_M * h(t')\big\|_{L_x^2}\,dt' \\
&\leqslant CM^{\beta+3/2}\int_0^T \big\|h(t')\big\|_{L_x^1}\,dt' = CM^{\beta+3/2}\|h\|_{L_x^1 L_T^1} \\
&\leqslant CT^{1/2}M^{\beta+3/2}\|h\|_{L_x^1 L_T^2}.
\end{aligned} \tag{20}$$

Hence, pick M so that $M^{\alpha+3/2} = T^{-1/2}$, (19) and (20) prove estimate (ix).

We obtain (x) by duality to the case $\beta = 0$ of (iv). Let $g \in L^1_x L^2_T$, $\|g\|_{L^1_x L^2_T} = 1$. Then

$$\int\int_0^T W_\alpha(t) u_0(x) \overline{g(t,x)}\, dx\, dt = \int\int_0^T u_0(x) \overline{W_\alpha(-t) g(t,x)}\, dt\, dx. \tag{21}$$

So estimate (iv) is equivalent to

$$\left\| \int_0^T W_\alpha\left(-t'\right) g\left(t', x\right) dt' \right\|_{L^2_x} \leqslant C T^\gamma \|g\|_{L^1_x L^2_T}. \tag{22}$$

Fix $0 < t < T$, let $g(t', x) = \chi_{[0,t]}(t') h(t, x)$, then

$$\left\| \int_0^t W_\alpha\left(-t'\right) h\left(t', x\right) dt' \right\|_{L^2_x} \leqslant C T^\gamma \|h\|_{L^1_x L^2_T}. \tag{23}$$

Apply now $W_\alpha(t)$ to the left-hand side, which is an isometry in L^2, to obtain (x).

Let P_k be a projection on frequencies $\simeq 2^k$ (or $\leqslant 1$ for $k = 0$), which is smooth on Fourier transform side. Consider

$$\begin{aligned} T_k h(x,t) &= \int_0^t W_\alpha\left(t - t'\right) P_k \partial_x h\left(\cdot, t'\right) dt'; \\ \tilde{T}_k h(x,t) &= \int_0^T W_\alpha\left(t - t'\right) P_k \partial_x h\left(\cdot, t'\right) dt'. \end{aligned} \tag{24}$$

By (vi), localization in frequencies and (viii) we have, for $\frac{3}{4} - \frac{\alpha}{4} < \beta^+ < \frac{\alpha}{2}$,

$$\begin{aligned} \|\tilde{T}_k h\|_{L^{2\alpha}_x L^\infty_T} &= \left\| W_\alpha(t) \int_0^T W_\alpha\left(-t'\right) P_k \partial_x h\left(\cdot, t'\right) dt' \right\|_{L^{2\alpha}_x L^\infty_T} \\ &\leqslant C 2^{k(\beta^+ - \alpha/2)} \left\| \int_0^T |D|^{\alpha/2} W_\alpha\left(-t'\right) P_k \partial_x h\left(\cdot, t'\right) dt' \right\|_{L^2_x} \\ &\leqslant C 2^{k(\beta^+ - \alpha/2)} \left\| \int_0^T |D|^{\alpha/2} W_\alpha\left(T - t'\right) P_k \partial_x h\left(\cdot, t'\right) dt' \right\|_{L^2_x} \\ &\leqslant C 2^{k(\beta^+ - \alpha/2)} \|\partial_x h\|_{L^1_x L^2_T}. \end{aligned} \tag{25}$$

Using the version of Christ and Kiselev's lemma in Molinet and Ribaud [26, Lemma 3], we obtain

$$\|T_k h\|_{L^{2\alpha}_x L^\infty_T} \leqslant C 2^{k(\beta^+ - \alpha/2)} \|\partial_x h\|_{L^1_x L^2_T}. \tag{26}$$

The sum of right side of (25) being convergent, (xi) follows. □

We are now ready for our well-posedness result in the energy space, Theorem 1.

Proof of Theorem 1. Let Z_T be the space defined by the maximum of the following norms,

$$\sup_{0 \leqslant t \leqslant T} \|u\|_{H^{\alpha/2}}, \quad \left\| |D|^\alpha u \right\|_{L^\infty_x L^2_T}, \quad T^{-\gamma} \|u\|_{L^\infty_x L^2_T}, \quad T^{-\gamma} \|\partial_x u\|_{L^\infty_x L^2_T}, \quad \|u\|_{L^{2\alpha}_x L^\infty_T},$$

for some $\gamma > 0$ to be chosen.

Fix $u_0 \in H^{\alpha/2}$, $\|u_0\|_{H^{\alpha/2}} \leqslant A$. For R, T to be determined, let $B_{R,T} = \{v \in Z_T,\ \|v\|_{Z_T} \leqslant R\}$. Let

$$\Phi_{u_0}(v) = W_\alpha(t)u_0 \pm \int_0^t W_\alpha(t-t')\big(|v|^{2\alpha}\partial_x v\big)(t')\,dt'. \tag{27}$$

We will show that, given A, we can find R, T such that $\Phi_{u_0}(v) : B_{R,T} \to B_{R,T}$ and is a contraction there. First, note that (i), (ii), (iii), (iv) (with $\beta = 1/2$), (v), (vi) show that $\|W_\alpha(t)u_0\|_{Z_T} \leqslant CA$, for some $\gamma > 0$.

Now, we work on the Duhamel term. It is easy to see that, using (vii), (viii), (ix) (with $\beta = 0$ and $\beta = 1$), (x), (xi), we have

$$\begin{aligned}\left\| \int_0^t W_\alpha(t-t')|v|^{2\alpha}\partial_x v\,dt' \right\|_{Z_T} &\leqslant C\big\{ \big\||v|^{2\alpha}\partial_x v\big\|_{L^1_x L^2_T} + \big\||v|^{2\alpha} v\big\|_{L^1_x L^2_T} \big\} \\ &\leqslant CT^\gamma \|v\|^{2\alpha}_{L^{2\alpha}_x L^\infty_T} \|v\|_{Z_T},\end{aligned} \tag{28}$$

for some $\gamma > 0$. We now choose $R = 2CA$ and T so that $C(2CA)^{2\alpha}T^\gamma \leqslant CA = \frac{1}{2}R$, which gives $\Phi_{u_0} : B_{R,T} \to B_{R,T}$.

For the contraction property, we estimate

$$\big||v|^{2\alpha}\partial_x v - |w|^{2\alpha}\partial_x w\big| \leqslant \big|\big(|v|^{2\alpha} - |w|^{2\alpha}\big)\partial_x w\big| + \big||v|^{2\alpha}\partial_x(v-w)\big| \tag{29}$$

since $||v|^{2\alpha} - |w|^{2\alpha}| \leqslant C|v-w|(|v|^{2\alpha-1} + |w|^{2\alpha-1})$ and $\alpha > 1$, this allows to conclude the proof (we argue similarly for $||v|^{2\alpha}v - |w|^{2\alpha}w|$). □

Remark 2. From Theorem 1, it follows that for any initial data in $H^{\frac{\alpha}{2}}$, we can define a maximal solution to the problem. Moreover, either this solution is globally defined or it blows up in finite time.

From the previous arguments and estimates, it is standard to obtain the property of persistence of regularity, i.e. if the initial data belongs to some H^s, for $s > \frac{\alpha}{2}$, then the maximal solution $u(t)$ of the equation belongs to H^s as long as it exists in $H^{\frac{\alpha}{2}}$. In particular, by density arguments and continuous dependence upon the initial data, we can approximate any $H^{\frac{\alpha}{2}}$ by smooth solutions in $C([0,T], H^{\frac{\alpha}{2}})$, which allows us to prove rigorously the conservation of mass and energy (3) and (4).

2.2. *Weak continuity of the flow*

Theorem 3. *Let* $1 < \alpha \leqslant 2$. *Let* $\{u_n\}_n$ *be a sequence of* $H^{\frac{\alpha}{2}}$ *solutions of* (7) *in* $[0,T]$; *assume that* $u_n(0) \rightharpoonup u_0$ *in* $H^{\frac{\alpha}{2}}$ *weak. Assume also that* (*without loss of generality*) $\|u_n(0)\|_{H^{\frac{\alpha}{2}}} \leqslant A$, $\|u_0\|_{H^{\frac{\alpha}{2}}} \leqslant A$, $T \leqslant T(A)$ *as in Theorem* 1. *Then, if* $u(t)$ *is the solution of* (7) *corresponding to* u_0, *we have*

$$\forall t \in [0,T], \quad u_n(t) \rightharpoonup u(t) \quad \text{in } H^{\frac{\alpha}{2}} \text{ weak.}$$

Note that for $\alpha = 1$, the result is proved in the final remark of [4] (see also [7]) and for $\alpha = 2$, it was proved by different arguments in [19].

Proof. For $1 < \alpha \leqslant 2$ we remark that a slight modification of the proof of Theorem 1 gives us the local well-posedness in $H^{\frac{\alpha'}{2}}$ for $1 < \alpha' < \alpha$. Then, the proof is identical to the one in Theorem 5 of [11], using this remark. □

3. Properties of the ground states and perturbation arguments

In this section, we first recall or prove general results about ground states for (1) for all $1 \leqslant \alpha \leqslant 2$, mainly by classical variational arguments. Then, we prove specific results for α close to 2 by perturbation of the well-known results for gKdV.

3.1. Construction and first properties of the ground states

Proposition 1. *Let $1 \leqslant \alpha \leqslant 2$. There exists a solution $Q \in H^{\frac{\alpha}{2}}(\mathbb{R}) \cap C^{\infty}(\mathbb{R})$ of* (9) *which satisfies the following properties*

(i) *First properties*: $Q > 0$ *on* $\mathbb{R}$, Q *is even,* $Q' < 0$ *on* $(0, +\infty)$.

(ii) *Variational properties. The infima*

$$J_1 = \inf\left\{\frac{(\int |D^{\frac{\alpha}{2}} v|^2)(\int v^2)^{\alpha}}{\int |v|^{2\alpha+2}}, \text{ for } v \in H^{\frac{\alpha}{2}}\right\}, \tag{30}$$

$$J_2 = \inf\left\{E(v), \text{ for } v \in H^{\frac{\alpha}{2}} \text{ such that } \int v^2 = \int Q^2\right\}, \tag{31}$$

are attained at Q $(J_2 = 0)$.

(iii) *Linearized operator*: *let* L *be the unbounded operator defined on* $L^2(\mathbb{R})$ *by*

$$Lv = |D|^{\alpha} v + v - Q^{2\alpha} v.$$

Then, L *has only one negative eigenvalue* μ_0, *associated to an even eigenfunction* $\chi_0 > 0$, $LQ' = 0$ *and the continuous spectrum of* L *is* $[1, +\infty)$. *Moreover, the following holds*

$$\inf\left\{(L\eta, \eta), \text{ for } \eta \in H^{\frac{\alpha}{2}} \text{ such that } \int \eta Q = 0\right\} = 0. \tag{32}$$

Finally, let $Q_\lambda(x) = \lambda^{-\frac{1}{\alpha}} Q(\lambda^{-\frac{2}{\alpha}} x)$ *for all* $\lambda > 0$ *and*

$$\Lambda Q = -\left(\frac{d}{d\lambda} Q_\lambda\right)_{\lambda=1} = \frac{1}{\alpha}\big(Q + 2xQ'\big) \quad \text{then} \quad L(\Lambda Q) = -2Q. \tag{33}$$

(iv) *Decay properties*:

$$\forall x \in \mathbb{R}, \quad Q(x) + \big(1 + |x|\big)\big|Q'(x)\big| + \big(1 + |x|^2\big)\big|Q''(x)\big| \leqslant \frac{C}{(1 + x^2)^{\frac{1}{2}(1+\alpha)}}. \tag{34}$$

Definition 1. An even positive solution of (9) in the sense of Proposition 1 is called a *ground state*.

Before proving the above proposition, we recall the following classical result.

Lemma 2. *Let* $1 \leqslant \alpha \leqslant 2$. *Let* $K(x)$ *be such that* $\hat{K}(\xi) = e^{-|\xi|^{\alpha}}$. *Then,* K *is a real and even function,* $K > 0$ *on* $\mathbb{R}$ *and* $K'(x) < 0$ *for* $x > 0$.

Proof. For $\alpha = 1, 2$, $K(x)$ is known explicitly. This result is not trivial for $1 < \alpha < 2$ but known in probabilistic literature: K is the law of stable distribution, special cases of distribution of class L (see Gnedenko and Kolmogorov [6, Theorem, p. 164]). Yamazato [33] proved the unimodality of distribution of class L, i.e. $K'(x) < 0$ for $x > 0$. □

Remark 3. It follows in particular from the previous lemma that the operator $|D|^{\alpha}$ for $1 \leqslant \alpha \leqslant 2$ satisfies properties $(\mathrm{L1})_{\alpha/2}$, (L2) and (L3) of [31].

We also recall the following identities satisfied by any solution of (9).

Lemma 3. *Let* $Q \in H^{\frac{\alpha}{2}}$ *be a solution of* (9). *Then,*

$$\int Q^2 = \alpha \int \big|D^{\frac{\alpha}{2}} Q\big|^2 = \frac{\alpha}{(2\alpha+1)(\alpha+1)} \int Q^{2\alpha+2}. \tag{35}$$

In particular,

$$E(Q) = \int \big|D^{\frac{\alpha}{2}} Q\big|^2 - \frac{1}{(2\alpha+1)(\alpha+1)} \int Q^{2\alpha+2} = 0.$$

Proof. Multiplying Eq. (9) by Q and integrating, we first find

$$\int \left|D^{\frac{\alpha}{2}} Q\right|^2 + \int Q^2 = \frac{1}{2\alpha+1} \int Q^{2\alpha+2}. \tag{36}$$

Second, note that by Plancherel and integration by parts, for all $u \in \mathcal{S}$, one has

$$\int \left(-|D|^\alpha u\right)(x u_x) = -\frac{\alpha-1}{2} \int \left|D^{\frac{\alpha}{2}} u\right|^2.$$

Thus, multiplying the equation of Q by xQ' and integrating, we obtain

$$(\alpha-1) \int \left|D^{\frac{\alpha}{2}} Q\right|^2 - \int Q^2 = -\frac{1}{(2\alpha+1)(\alpha+1)} \int Q^{2\alpha+2}. \tag{37}$$

Combining (36) and (37), we find (35). □

Sketch of the proof of Proposition 1. The existence of a solution Q of (9) satisfying (i), (ii) and (iii) follows from Weinstein's arguments [29–32] and Lemma 2. Property (iv) follows from Amick and Toland's arguments, see [1].

Let us sketch the proofs. (i): Follows by Theorem 3.2 in [31] and Remark 3.

(ii): As in [29,32], a suitable solution $Q(x)$ is obtained by minimizing the functional $j_1(v)$, defined for $v \in H^{\frac{\alpha}{2}}$ by

$$j_1(v) = \frac{(\int |D^{\frac{\alpha}{2}} v|^2)(\int v^2)^\alpha}{\int |v|^{2\alpha+2}}.$$

Note that by Theorem XIII.50 in [27], Lemma 2, Remark 3, and Lemma 6 in [31], for all $v \in H^{\frac{\alpha}{2}}$,

$$\left(|D|^\alpha |v|^*, |v|^*\right) \leqslant \left(|D|^\alpha v, v\right),$$

where v^* the symmetric decreasing rearrangement of v. Thus, in the minimization procedure, one can always assume that the minimization sequence is composed of nonnegative and even functions. It is not possible here to use the decay properties of H^1 radial functions as in [29], since such an argument is limited to space dimensions larger than or equal to 2. One rather uses the concentration–compactness approach [16] on a suitable continuous family of variational problems related to $j_1(v)$, as in [31].

Once a nonnegative, symmetric decreasing, minimizer ψ of j_1 is constructed, we verify that for some constants $a, b > 0$, $Q(x) = a\psi(bx)$ satisfies

$$|D|^\alpha Q + Q - \frac{1}{2\alpha+1} Q^{2\alpha+1} = 0, \quad Q \in H^{\frac{\alpha}{2}},\ Q > 0,\ Q' < 0 \text{ on } (0, +\infty),\ Q \text{ even},$$

and $j_1(Q) = \inf\{j_1(v) \text{ for } v \in H^{\frac{\alpha}{2}}\}$. By Lemma 3, we have $E(Q) = 0$. In particular, the definition of J_1 implies that for all $v \in H^{\frac{\alpha}{2}}$,

$$\frac{1}{(2\alpha+1)(\alpha+1)} \int |v|^{2\alpha+2} \leqslant \left(\frac{\int v^2}{\int Q^2}\right)^\alpha \int \left|D^{\frac{\alpha}{2}} v\right|^2, \tag{38}$$

which is the sharp Gagliardo–Nirenberg inequality in this context, and which means that if $\int v^2 \leqslant \int Q^2$, then $E(v) \geqslant 0$.

Note that also that for two different solutions Q and $\tilde{Q}$ of (9), both minimizers of j_1, we have $\|Q\|_{L^2} = \|\tilde{Q}\|_{L^2}$.

(iii): Exactly as in the proofs of Propositions 4.1 and 4.2 of [31] and Proposition 2.7 of [29], we obtain that

$$0 = \inf\left\{(Lv|v),\ \text{for } v \in H^{\frac{\alpha}{2}},\ \int vQ = 0\right\},$$

and $(LQ, Q) < 0$, so that there exists exactly one negative eigenvalue μ_0 of L, related an even eigenfunction χ_0 which can be taken to be positive. Moreover, the continuous spectrum of L is $[1, +\infty)$.

Finally, from the equation of $Q_\lambda(x + x_0) = \lambda^{-\frac{1}{\alpha}} Q(\lambda^{-\frac{2}{\alpha}}(x + x_0))$, we have

$$|D|^\alpha Q_\lambda(x + x_0) + \lambda^2 Q_\lambda(x + x_0) = \frac{1}{2\alpha+1} Q_\lambda^{2\alpha+1}(x + x_0).$$

Differentiating with respect to x_0 and taking $x_0 = 0$, $\lambda = 1$, we find $LQ' = 0$; differentiating with respect to λ, and taking $x_0 = 0$, $\lambda = 1$, we find $L(\Lambda Q) = -2Q$.

(iv): Proof of the decay property. For this part, we first recall the following facts from [3]. For a function $F : \mathbb{R} \to \mathbb{R}$, we denote by $f : \mathbb{R}^2_+ \to \mathbb{R}$ ($\mathbb{R}^2_+ = \mathbb{R} \times [0, +\infty)$) the extension

$$f(x, 0) = F(x) \quad \text{on } \mathbb{R}, \qquad \partial_x^2 f + \partial_y^2 f + \frac{1-\alpha}{y} \partial_y f = 0 \quad \text{on } \mathbb{R}^2_+.$$

Then, from [3], there exists a constant $C_\alpha > 0$ such that, on $\mathbb{R}$,

$$C_\alpha |D|^\alpha F = -\lim_{y \to 0^+} y^{1-\alpha} \partial_y f.$$

This generalizes a classical observation for $\alpha = 1$.

Next, following [1,2], if Q is solution of (9), and $q(x, y)$ is its extension to $\mathbb{R}^2_+$, then q satisfies

$$\begin{aligned}
&\partial_x^2 q + \partial_y^2 q + \frac{1-\alpha}{y} \partial_y q = 0 \quad \text{on } \mathbb{R}^2_+, \\
&\lim_{y \to 0^+} y^{1-\alpha} \partial_y q = C_\alpha \left(q - \frac{1}{2\alpha + 1} q^{2\alpha+1} \right) \quad \text{on } y = 0, \\
&\lim_{|x| \to +\infty} \left| q(x, 0) \right| = 0.
\end{aligned}$$

From [3] and [1,2], we are led to set

$$G_\alpha(x, y) = \left(\int \frac{dx'}{(1 + (x')^2)^{\frac{1+\alpha}{2}}} \right)^{-1} e^{\frac{1}{\alpha C_\alpha} y^\alpha} \int_0^{+\infty} e^{-\frac{1}{\alpha C_\alpha}(y+\omega)^\alpha} \frac{(y+\omega)^\alpha}{(x^2 + (y+\omega)^2)^{\frac{1+\alpha}{2}}} \, d\omega,$$

so that $q(x, y)$ satisfies on $\mathbb{R}^2_+$

$$q(x, y) = \frac{1}{2\alpha + 1} \int_{-\infty}^{+\infty} G_\alpha(x - z, y) q^{2\alpha+1}(z, y) \, dz.$$

From this expression, we get the decay estimate (34) following exactly the same arguments as in pp. 23–24 of [2] and using immediate estimates on G_α. $\square$

Now, we recall Frank and Lenzmann's recent uniqueness result.

Proposition 2 *(Uniqueness of the ground state and Kernel property). (See [5].) There exists a unique ground state of* (9). *Moreover,*

$$\operatorname{Ker}(L) = \operatorname{span}\{Q'\}.$$

Recall that the result in [5] is general and not restricted to the L^2 critical case.

Finally, we recall a direct consequence of the spectral theorem and Propositions 1 and 2.

Lemma 4. *For some constant* $\mu > 0$,

$$\forall v \in H^{\frac{\alpha}{2}}, \quad \int v\chi_0 = \int vQ' = 0 \quad \Rightarrow \quad (Lv, v) \geqslant \mu \|v\|^2_{H^1}. \tag{39}$$

3.2. Linear Liouville property by perturbation around the gKdV case

We have summarized in Proposition 1 standard results about the ground states of (9) which hold for any $1 \leqslant \alpha \leqslant 2$. To study the nonlinear flow of (1) around the solitons, we will also need the following fundamental rigidity property of the linearized flow around a ground state.

Definition 2 *(Linear Liouville property).* We say that L satisfies the *linear Liouville property* if any $H^{\frac{\alpha}{2}}$ bounded solution $w(t)$ of

$$w_t = \partial_x(Lw), \quad (t,x) \in \mathbb{R},$$

such that

$$\forall \epsilon > 0,\ \exists B > 0,\ \forall t \in \mathbb{R}, \quad \int_{|x|>B} \left|w(t,x)\right|^2 dx \leqslant \epsilon \tag{40}$$

is necessarily $w(t,x) \equiv c_0 Q'(x)$ for some $c_0 \in \mathbb{R}$.

The linear Liouville property was proved for $\alpha = 2$ in [19] and [17] by Virial type identities and the variational characterization of Q. Note that it is a stronger property than the Kernel property stated in Proposition 2. We are able to prove this property for $\alpha < 2$ sufficiently close to 2 by perturbation arguments.

Proposition 3. *There exists $\alpha_0 \in [1,2)$ such that for all $\alpha_0 \leqslant \alpha \leqslant 2$, the following properties hold.*

(i) *There exists a unique (positive, even) ground state solution $Q = Q_{[\alpha]} \in H^1$ of* (9) *and*

$$Q_{[\alpha]} \to Q_{[2]} \quad \text{as } \alpha \to 2^- \text{ in } H^1.$$

(ii) *Variational characterization of Q: $\forall u \in H^{\frac{\alpha}{2}}$,*

$$E(u) = 0, \quad \int u^2 = \int Q^2, \quad \int \left||D|^{\frac{\alpha}{2}} u\right|^2 = \int \left||D|^{\frac{\alpha}{2}} Q\right|^2 \quad \Rightarrow \quad u = \pm Q(. - x_0), \quad x_0 \in \mathbb{R}. \tag{41}$$

(iii) *The linear Liouville property holds true.*

Proof. The proof of Proposition 3 is perturbative. Let us denote by $Q_{[2]}$ the unique positive even solution of (9) given by (10).

(i) Let $\alpha_n \to 2$ be an increasing sequence and for all n, let $Q_{[\alpha_n]}$ be a solution of (9) given by Proposition 1. First, we claim that $\lim_{n\to+\infty} Q_{[\alpha_n]} = Q_{[2]}$. Indeed, from (38) applied to a given function w, we obtain $\int Q_{[\alpha]}^2 \leqslant C$. Then, by Lemma 3, $\|Q_{[\alpha_n]}\|_{H^{\frac{\alpha_n}{2}}} \leqslant C$, and using the equation of $Q_{[\alpha_n]}$, it follows that $Q_{[\alpha_n]} \in H^1$ and $\|Q_{[\alpha_n]}\|_{H^1} \leqslant C$. In particular, there exists $V \in H^1$, a weak limit in H^1 of a subsequence of $Q_{[\alpha_n]}$, still denoted by $Q_{[\alpha_n]}$. It is easy to see that $V \neq 0$, using Lemma 3. Indeed, since

$$\int Q_{[\alpha_n]}^2 \leqslant C \|Q_{[\alpha_n]}\|_{L^\infty}^{2\alpha} \int Q_{[\alpha_n]}^2,$$

it follows that $Q_{[\alpha_n]}(0) = \|Q_{[\alpha_n]}\|_{L^\infty} \geqslant c_1 > 0$ and since weak H^1 convergence implies uniform convergence on compact sets, we obtain $V(0) \neq 0$.

Moreover, we easily check that V satisfies Eq. (9) with $\alpha = 2$ and thus by uniqueness, we deduce $V = Q_{[2]}$.

To obtain the strong convergence, we just observe that

$$\limsup_{n\to+\infty} \int Q_{[\alpha_n]}^2 \leqslant \int Q_{[2]}^2$$

follows from the following consequence of Lemma 3

$$\begin{aligned}\left[(\alpha_n+1)(2\alpha_n+1)\right]^{-1}\left(\int Q_{[\alpha_n]}^2\right)^{\alpha_n} &= j_{1,[\alpha_n]}(Q_{[\alpha_n]}) \\ &\leqslant j_{1,[\alpha_n]}(Q_{[2]}) \to j_{1,[2]}(Q_{[2]}) = [15]^{-1}\left(\int Q_{[2]}^2\right)^2.\end{aligned}$$

This gives L^2 strong convergence. To obtain H^1 convergence, we just use the equation of $Q_{[\alpha_n]}$ and interpolation argument.

Let us give a direct proof of the uniqueness of the ground state for α close to 2. We consider two sequences $Q_{[\alpha_n]}$ and $\widetilde{Q}_{[\alpha_n]}$ of solutions of (9) as in Proposition 1. By the first observation, we have $Q_{[\alpha_n]} \to Q_{[2]}$ and $\widetilde{Q}_{[\alpha_n]} \to Q_{[2]}$ in $H^1(\mathbb{R})$. Moreover, by the equation satisfied by $Q_{[\alpha_n]}$ and $\widetilde{Q}_{[\alpha_n]}$, we have

$$\left\| |D|^{\alpha_n}(Q_{[\alpha_n]} - \widetilde{Q}_{[\alpha_n]})\right\|_{L^2} \leqslant C\|Q_{[\alpha_n]} - \widetilde{Q}_{[\alpha_n]}\|_{L^2}. \tag{42}$$

Let

$$w_n = \frac{Q_{[\alpha_n]} - \widetilde{Q}_{[\alpha_n]}}{\|Q_{[\alpha_n]} - \widetilde{Q}_{[\alpha_n]}\|_{H^1}}.$$

By (42), the sequence w_n is bounded in $H^{\frac{3}{2}}$ (say $\alpha_n > 3/2$). A more precise computation using the equations of $Q_{[\alpha_n]}$ and $\widetilde{Q}_{[\alpha_n]}$ shows that the function w_n satisfies

$$\|L_{[\alpha_n]}w_n\|_{H^1} = \left\| |D|^{\alpha_n} w_n + w_n - Q_{[\alpha_n]}^{2\alpha_n} w_n\right\|_{H^1} \leqslant C\|Q_{[\alpha_n]} - \widetilde{Q}_{[\alpha_n]}\|_{L^2}$$

where we observe a special cancellation. Using this estimate, the bound of the sequence (w_n) in $H^{\frac{3}{2}}$ and standard Fourier analysis, we find

$$\lim_{n\to+\infty}(L_{[2]}w_n, w_n)_{L^2} = 0.$$

It is known that (39) holds for $\alpha = 2$, moreover, it can be rewritten as

$$\forall v \in H^{\frac{\alpha}{2}}, \quad (L_{[2]}v, v) \geqslant \frac{\mu}{2}\|v\|_{H^1}^2 - C\left(\int v\chi_0\right)^2 - C\left(\int vQ'\right)^2.$$

By parity properties, we observe $\int w_n Q'_{[2]} = 0$. By the previous equation, and (39), we have

$$\int w_n \chi_{0,[2]} = \frac{1}{\mu_0}(L_{[2]}\chi_0, w_n) = \frac{1}{\mu_0}(L_{[\alpha_n]}\chi_0, w_n) + o(1) = \frac{1}{\mu_0}(\chi_0, L_{[\alpha_n]}w_n) + o(1),$$

and thus $\lim_{n\to+\infty}\int w_n\chi_{0,[2]} = 0$. Since $\|w_n\|_{H^1} = 1$, we find a contradiction for n large enough.

Therefore, there exists $\alpha_0 \in [1, 2)$ so that there is one and only one solution of (9) satisfying the properties of Proposition 1. See [5] for a general proof.

(ii) Variational characterization. It follows from the arguments of the proof of Proposition 1. Indeed, for such a function u, $|u|$ is a minimizer of J_1 and satisfies the same equation as Q. By the uniqueness result of (i), it follows that $|u|$ is a translation of Q. Thus, u being continuous, it is a translation of Q or $-Q$.

Using a similar argument and possibly taking α_0 closer to 2, we can prove directly that $\mathrm{Ker}(L_{[\alpha]}) = \mathrm{span}\{Q'_{[\alpha]}\}$ for $\alpha \in [\alpha_0, 2]$.

(iii) Now, we prove the linear Liouville property for α close to 2. The proof is by contradiction and similar to (i), using a compactness argument. For the sake of contradiction, we assume that there exists an increasing sequence $\alpha_n \to 2$ and functions $w_n(t,x)$ satisfying

$$(w_n)_t = (L_{[\alpha_n]}w_n)_x,$$
$$w_n(t) \not\equiv a_n(t)Q'_{[\alpha_n]}, \quad \sup_{t\in\mathbb{R}}\left\|w_n(t)\right\|_{H^{\frac{\alpha_n}{2}}} \leqslant C_n,$$
$$\forall\epsilon > 0,\ \exists B_n(\epsilon) > 0,\ \forall t \in \mathbb{R}, \quad \int\limits_{|x|>B_n(\epsilon)} \left|w_n(t,x)\right|^2 dx \leqslant \epsilon.$$

We introduce several auxiliary functions defined from w_n. First, set

$$\tilde{w}_n(t) = w_n(t) - \frac{\int Q'_{[\alpha_n]}w_n(t)}{\int (Q'_{[\alpha_n]})^2}Q'_{[\alpha_n]},$$

satisfying

$$(\tilde{w}_n)_t = (L_{[\alpha_n]}\tilde{w}_n)_x + \delta_n(t) Q'_{[\alpha_n]},$$
$$\tilde{w}_n(t) \not\equiv 0, \quad \sup_{t\in\mathbb{R}} \|\tilde{w}_n(t)\|_{H^{\frac{\alpha_n}{2}}} \leqslant C'_n, \quad \int \tilde{w}_n(t) Q'_{[\alpha_n]} = 0,$$
$$\forall \epsilon > 0,\ \exists B_n(\epsilon) > 0,\ \forall t \in \mathbb{R}, \quad \int_{|x|>B_n(\epsilon)} |\tilde{w}_n(t,x)|^2\,dx \leqslant \epsilon.$$

Moreover, using monotonicity arguments on $\tilde{w}_n(t)$ as in Section 4 of the present paper and Lemma 4 in [17], we find ($\alpha_n > 3/2$)

$$\forall x_0 > 1,\ \forall t \in \mathbb{R}, \quad \int_{|x|>x_0} |\tilde{w}_n(t,x)|^2\,dx \leqslant \sup_{t\in\mathbb{R}} \|\tilde{w}_n(t)\|_{L^2}^2 \frac{C}{|x_0|^{\frac{3}{2}}}.$$

In particular, by Fubini, we obtain

$$\forall t \in \mathbb{R}, \quad \int |x| |\tilde{w}_n(t)|^2 \leqslant C \sup_{t\in\mathbb{R}} \|\tilde{w}_n(t)\|_{L^2}^2.$$

Multiplying the equation of $\tilde{w}_n$ by $x\tilde{w}_n$ and using the argument of Lemma 3, we find, for $C > 0$,

$$\frac{d}{dt} \int x(\tilde{w}_n(t))^2 \leqslant -C \||D|^{\frac{\alpha}{2}} \tilde{w}_n(t)\|_{L^2}^2 + C' \|\tilde{w}_n(t)\|_{L^2}^2,$$

and thus, for all $t \in \mathbb{R}$, $\int_t^{t+1} \||D|^{\frac{\alpha}{2}} \tilde{w}_n(t)\|_{L^2}^2 \leqslant C \sup_{t\in\mathbb{R}} \|\tilde{w}_n(t)\|_{L^2}^2$. Therefore, from standard arguments, using the equation of $\tilde{w}_n$,

$$\sup_{t\in\mathbb{R}} \|\tilde{w}_n(t)\|_{H^{\frac{\alpha_n}{2}}} \leqslant C \sup_{t\in\mathbb{R}} \|\tilde{w}_n(t)\|_{L^2},$$

for a constant $C > 0$ independent of n.

Let t_n be such that $\|\tilde{w}_n(t_n)\|_{L^2} \geqslant \frac{1}{2} \sup_{t\in\mathbb{R}} \|\tilde{w}_n(t)\|_{L^2}$ and set

$$\bar{w}_n(t,x) = \frac{\tilde{w}_n(t_n + t, x)}{\sup_{t\in\mathbb{R}} \|\tilde{w}_n(t)\|_{L^2}},$$

so that we have

$$(\bar{w}_n)_t = (L_{[\alpha_n]}\bar{w}_n)_x + \bar{\delta}_n(t) Q'_{[\alpha_n]},$$
$$\|\bar{w}_n(0)\|_{L^2} \geqslant \frac{1}{2}, \quad \sup_{t\in\mathbb{R}} \|\bar{w}_n(t)\|_{H^{\frac{\alpha_n}{2}}} \leqslant C, \quad \int \bar{w}_n(t) Q'_{[\alpha_n]} = 0,$$
$$\bar{\delta}_n(t) = \frac{1}{\int (Q'_{[\alpha_n]})^2} \int \bar{w}_n L_{[\alpha_n]}(Q''_{[\alpha_n]}),$$
$$\forall x_0 > 1,\ \forall t \in \mathbb{R}, \quad \int_{|x|>x_0} |\bar{w}_n(t,x)|^2\,dx \leqslant \frac{C}{|x_0|^{\frac{3}{2}}}.$$

Finally, we set

$$\hat{w}_n(t) = \bar{w}_n(t) - Q'_{[\alpha_n]} \int_0^t \bar{\delta}_n(s)\,ds,$$

so that

$$(\hat{w}_n)_t = (L_{[\alpha_n]}\hat{w}_n)_x,$$
$$\|\hat{w}_n(0)\|_{L^2} \geqslant \frac{1}{2}, \quad \|\hat{w}_n(0)\|_{H^{\frac{\alpha_n}{2}}} \leqslant C, \quad \int \hat{w}_n(0) Q'_{[\alpha_n]} = 0,$$
$$\forall x_0 > 1, \quad \int_{|x|>x_0} |\hat{w}_n(0,x)|^2\,dx \leqslant \frac{C}{|x_0|^{\frac{3}{2}}}.$$

We are now able to pass to the strong limit in H^{1^-}, for any $0 < 1^- < 1$.

$$\hat{w}_n(0) \to \hat{w}_0 \not\equiv 0,$$

and we define the solution $\hat{w}(t)$ of

$$(\hat{w})_t = (L_{[2]}\hat{w})_x, \qquad \hat{w}(0) = \hat{w}_0.$$

By well-posedness argument in H^{1^-}, we have $\hat{w}_n(t) \to \hat{w}(t)$ in H^{1^-}. Moreover,

$$\bar{\delta}_n(t) \to \bar{\delta}(t) = \frac{1}{\int (Q'_{[2]})^2} \int \hat{w}(t) L_{[2]}\big(Q''_{[2]}\big).$$

Set $\bar{w}(t) = \hat{w}(t) + Q'_{[2]} \int_0^t \bar{\delta}(s)\,ds$. Then

$$\forall t \in \mathbb{R}, \quad \bar{w}_n(t) \to \bar{w}(t) \text{ in } H^{1^-},$$
$$\bar{w}_t = (L_{[2]}\bar{w})_x + \bar{\delta} Q'_{[2]},$$
$$\bar{w}(0) \not\equiv 0, \quad \int \bar{w}(0) Q'_{[2]} = 0,$$
$$\forall t \in \mathbb{R},\ \forall x_0 > 1, \quad \int\limits_{|x|>x_0} \big|\bar{w}(t,x)\big|^2\,dx \leqslant \frac{C}{|x_0|^{\frac{3}{2}}}.$$

But the existence of such a $\bar{w}$ is a contradiction with Theorem 1 in [17], i.e. the linear Liouville property for the gKdV case (see also [19]). □

4. Modulation and monotonicity for solutions close to solitons

In this section, we consider $1 \leqslant \alpha \leqslant 2$ and Q is the ground state solution of (9).

4.1. Modulation

Lemma 5 *(Modulation of a solution close to the family of solitons). There exist $C, \epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$, if $u(t)$ is an $H^{\frac{\alpha}{2}}$ solution of* (1) *such that for $t_1 < t_2$ and $\lambda_0(t) > 0$, $\rho_0(t) \in \mathbb{R}$, defined on $[t_1, t_2]$,*

$$\forall t \in [t_1, t_2], \quad \big\|u(t) - Q_{\lambda_0(t)}\big(. - \rho_0(t)\big)\big\|_{H^{\frac{\alpha}{2}}} < \epsilon, \tag{43}$$

then there exist $\lambda(t) > 0$, $\rho(t) \in C^1([t_1, t_2])$ such that

$$\eta(t, y) = \lambda^{\frac{1}{\alpha}}(t) u\big(t, \lambda^{\frac{2}{\alpha}}(t) y + \rho(t)\big) - Q(y) \tag{44}$$

satisfies

$$\forall t \in [t_1, t_2], \quad \int Q'(y)\eta(t, y)\,dy = \int \chi_0(y)\eta(t, y)\,dy = 0, \quad \big\|\eta(t)\big\|_{H^{\frac{\alpha}{2}}} \leqslant C\epsilon, \tag{45}$$

$$\left|\frac{\lambda_0(t)}{\lambda(t)}\right| + \big|\rho_0(t) - \rho(t)\big| \leqslant C\epsilon. \tag{46}$$

Moreover, setting

$$s = \int_0^t \frac{dt'}{\lambda^{2+\frac{2}{\alpha}}(t')}, \qquad \Lambda\eta = \frac{1}{\alpha}(\eta + 2y\eta_y),$$

the function $\eta(s, x)$ is a solution of

$$\eta_s - \partial_y(L\eta) = \frac{\lambda_s}{\lambda}\Lambda Q + \left(\frac{\rho_s}{\lambda^{\frac{2}{\alpha}}} - 1\right) Q' + \frac{\lambda_s}{\lambda}\Lambda\eta + \left(\frac{\rho_s}{\lambda^{\frac{2}{\alpha}}} - 1\right)\eta_y - \partial_y\big(\mathcal{R}(\eta)\big),$$

$$\text{where} \quad \mathcal{R}(\eta) = \frac{1}{2\alpha+1}|Q+\eta|^{2\alpha}(Q+\eta) - \frac{1}{2\alpha+1} Q^{2\alpha+1} - Q^{2\alpha}\eta,$$

and the following holds

$$\left|\frac{\rho_s(s)}{\lambda^{\frac{2}{\alpha}}(s)}-1\right|+\left|\frac{\lambda_s(s)}{\lambda(s)}\right|\leqslant C\left(\int\frac{\eta^2(s,y)}{1+y^2}\,dy\right)^{\frac{1}{2}}\leqslant C\big\|\eta(s)\big\|_{L^2}. \tag{47}$$

Sketch of the proof of Lemma 5. This result is completely proved for $\alpha=2$ in [18]. For $1\leqslant\alpha<2$, the proof is exactly the same. In particular, the existence of the modulation parameters $(\lambda(t),\rho(t))$ such that (45) hold is based on the implicit function theorem.

Then, the equation of $\eta(t)$, $\lambda(t)$ and $\rho(t)$ is easily obtained from the equation of $u(t)$, and the estimates (47) on λ_s, ρ_s follow from the equation of η multiplied by χ_0 and Q'. Indeed, let us first introduce

$$v(t,y)=\lambda^{\frac{1}{\alpha}}(t)u\big(t,\lambda^{\frac{2}{\alpha}}y+\rho(t)\big).$$

Then, $v(t,y)$ satisfies

$$\lambda^{\frac{2\alpha+2}{\alpha}}v_t-\partial_y\big(|D|^\alpha v\big)+|v|^{2\alpha}\partial_y v-\lambda^{\frac{2\alpha+2}{\alpha}}\frac{\lambda_t}{\lambda}\Lambda v-\lambda^{\frac{2\alpha+2}{\alpha}}\frac{\rho_t}{\lambda^{\frac{2}{\alpha}}}\partial_y v=0.$$

Using the new time variable s, since $\lambda^{\frac{2\alpha+2}{\alpha}}\,ds=dt$,

$$v_s-\partial_y\left(|D|^\alpha v+v-\frac{1}{1+2\alpha}|v|^{2\alpha}v\right)=\frac{\lambda_s}{\lambda}\Lambda v+\left(\frac{\rho_s}{\lambda^{\frac{2}{\alpha}}}-1\right)\partial_y v.$$

Now, expanding $v=Q+\eta$ and using the equation of Q, we find

$$\eta_s-\partial_y(L\eta)=\frac{\lambda_s}{\lambda}\Lambda Q+\left(\frac{\rho_s}{\lambda^{\frac{2}{\alpha}}}-1\right)Q'+\frac{\lambda_s}{\lambda}\Lambda\eta+\left(\frac{\rho_s}{\lambda^{\frac{2}{\alpha}}}-1\right)\eta_y$$
$$-\partial_y\left(\frac{1}{2\alpha+1}|Q+\eta|^{2\alpha}(Q+\eta)-\frac{1}{2\alpha+1}Q^{2\alpha+1}-Q^{2\alpha}\eta\right).$$

To prove (47), we multiply the above equation by χ_0 and then by Q' and we use the orthogonality conditions (45). Indeed, using decay properties of χ_0 and Q' (proved as in Proposition 1(iv) and $(\Lambda Q,\chi_0)=-\frac{1}{\mu_0}(\Lambda Q,L\chi_0)=\frac{2}{\mu_0}(Q,\chi_0)\neq 0$, $(Q',\chi_0)=0$, $(\Lambda Q,Q')=0$, we obtain

$$\left|\frac{\lambda_s}{\lambda}\right|+\left|\frac{\rho_s}{\lambda^{\frac{2}{\alpha}}}-1\right|\leqslant C\left(\int\frac{\eta^2}{1+y^2}\,dy\right)^{\frac{1}{2}}+C\left(\left|\frac{\lambda_s}{\lambda}\right|+\left|\frac{\rho_s}{\lambda^{\frac{2}{\alpha}}}-1\right|\right)\|\eta\|_{L^2},$$

and for ϵ_0 small enough, we obtain (47). □

4.2. Monotonicity argument on $u(t)$

This section contains the main new argument of this paper, i.e. the extension to Eq. (1) of the L^2 monotonicity arguments proved in [19,22] for the gKdV equation and in [11] for the BO equation. With respect to the gKdV case, the difficulty comes from the nonlocal character of the operator in (1). Note that in [11], using special symmetry arguments and harmonic extensions, we could overcome the difficulty created the nonlocal operator $|D|$. For the general case of Eq. (1) with $1<\alpha<2$, we can prove similar results using pseudo-differential operators tools. This is the objective of this section.

Using the standard notation $\langle x\rangle^2=1+x^2$, we set, for $\frac{1}{2}<r\leqslant\frac{1}{2}(\alpha+1)$ to be chosen later

$$\varphi(x)=\int_{-\infty}^{x}\frac{ds}{\langle s\rangle^{2r}},\qquad \phi(x)=\frac{1}{\langle s\rangle^{r}}=\sqrt{\varphi'}.$$

For $A>1$ to be chosen, let

$$\varphi_A(x)=\varphi\left(\frac{x}{A}\right).$$

We now claim the following L^2 monotonicity results.

Proposition 4. *Let $r \in (\frac{1}{2}, \frac{1}{2}(\alpha+1)]$ and $0 < \mu < 1$. Under the assumptions of Lemma 5, assuming in addition*

$$\forall t \in [t_1, t_2], \quad \lambda(t) \leqslant 2 \tag{48}$$

for $\epsilon_0 = \epsilon_0(\mu, r)$ small enough and $A = A(\mu, r)$ large enough, there exists $C_0 = C(\mu, r, A) > 0$ such that for all $x_0 > 1$,

(i) *Monotonicity on the right of the soliton*:

$$\begin{aligned} &\int u^2(t_2, x)\varphi_A(x - \rho(t_2) - x_0)\,dx \\ &\quad \leqslant \int u^2(t_1, x)\varphi_A\big(x - \rho(t_1) - \mu\big(\rho(t_2) - \rho(t_1)\big) - x_0\big)\,dx + \frac{C_0}{x_0^{2r-1}}. \end{aligned} \tag{49}$$

(ii) *Monotonicity on the left of the soliton*:

$$\begin{aligned} &\int u^2(t_2, x)\varphi_A\big(x - \rho(t_2) + \mu\big(\rho(t_2) - \rho(t_1)\big) + x_0\big)\,dx \\ &\quad \leqslant \int u^2(t_1, x)\varphi_A\big(x - \rho(t_1) + x_0\big)\,dx + \frac{C_0}{x_0^{2r-1}}. \end{aligned} \tag{50}$$

The case $\alpha = 1$ is treated in [11] by different techniques. For $\alpha = 2$, the error term is in fact exponential in x_0. See e.g. [19].

Proof. Let $u(t)$ be a solution of (1) under the assumptions of Lemma 5. By standard regularization arguments (density arguments and continuous dependence of the solution of (1) upon the initial data), we may assume that $u(t)$ is smooth (see Remark 2). We prove (49). Estimate (50) is then deduced from (49), L^2-norm conservation and the symmetry $x \to -x$, $t \to -t$ of the equation.

For $0 < \mu < 1$, $x_0 > 1$ and any $t \in [t_1, t_2]$, $x \in \mathbb{R}$, set

$$\tilde{x} = x - x_0 - \rho(t) - \mu\big(\rho(t_2) - \rho(t)\big), \quad M_\varphi(t) = M_{\varphi,A,x_0,t_2}(t) = \frac{1}{2}\int u^2(t, x)\varphi_A(\tilde{x})\,dx.$$

By direct computations, we have the following generalization of the well-known Kato identity [10]

$$\begin{aligned} \frac{d}{dt}M_\varphi(t) &= \frac{\mu - 1}{2}\rho_t \int u^2 \varphi_A'(\tilde{x})\,dx + \int u_t u \varphi_A(\tilde{x})\,dx \qquad (51) \\ &= \frac{\mu - 1}{2}\rho_t \int u^2 \varphi_A'(\tilde{x})\,dx - \int \big(\partial_x\big(-|D|^\alpha u\big) + |u|^{2\alpha} u_x\big) u \varphi_A(\tilde{x})\,dx \\ &= \frac{\mu - 1}{2}\rho_t \int u^2 \varphi_A'(\tilde{x})\,dx + \int \big(-|D|^\alpha u\big)\big(u_x \varphi_A(\tilde{x}) + u\varphi_A'(\tilde{x})\big)\,dx \\ &\quad + \frac{1}{2(\alpha+1)} \int |u|^{2\alpha+2} \varphi_A'(\tilde{x})\,dx. \qquad (52) \end{aligned}$$

Two terms in the right-hand side of (52) are treated by the following two lemmas.

Lemma 6. *Let $\alpha \in [1, 2]$, and $r \in (\frac{1}{2}, \frac{1}{2}(\alpha+1)]$. There exists $C > 0$ such that, for all $u \in \mathcal{S}$,*

$$\int \big(-|D|^\alpha u\big) u_x \varphi(x) \leqslant -\frac{(\alpha - 1)}{2} \int \big(|D|^{\frac{\alpha}{2}}(\phi u)\big)^2 + C \int u^2 \varphi'(x)\,dx.$$

Lemma 7. *Let $\alpha \in [1, 2]$, and $r \in (\frac{1}{2}, \frac{1}{2}(\alpha+1)]$. There exists $C > 0$ such that, for all $u \in \mathcal{S}$,*

$$\int \big(-|D|^\alpha u\big) u \varphi'(x)\,dx \leqslant -\int \big(|D|^{\frac{\alpha}{2}}(\phi u)\big)^2 + C \int u^2 \varphi'(x)\,dx.$$

Assuming Lemmas 6–7, we finish the proof of the proposition. First, note that from Lemmas 6 and 7, by changing variables ($x' = x/A$), we find for any $u \in \mathcal{S}$,

$$\int \big(-|D|^{\alpha} u\big) u_x \varphi_A(x) \leqslant -\frac{(\alpha-1)}{2} \int \big(|D|^{\frac{\alpha}{2}}\big(u\sqrt{\varphi_A'}\big)\big)^2 + \frac{C}{A^{\alpha}} \int u^2 \varphi_A'(x)\,dx, \tag{53}$$

$$\int \big(-|D|^{\alpha} u\big) u \varphi_A'(x)\,dx \leqslant -\int \big(|D|^{\frac{\alpha}{2}}\big(u\sqrt{\varphi_A'}\big)\big)^2 + \frac{C}{A^{\alpha}} \int u^2 \varphi_A'(x)\,dx. \tag{54}$$

By (52), (53), (54), we find

$$M_{\varphi}'(t) \leqslant -\frac{1}{2}\left(\rho_t(1-\mu) - \frac{C}{A^{\alpha}}\right) \int u^2(t)\varphi_A'(\tilde{x})\,dx + \frac{1}{2(\alpha+1)} \int |u|^{2\alpha+2} \varphi_A'(\tilde{x})\,dx.$$

Note that from (47) for ϵ_0 small enough

$$\frac{1}{\lambda^2}\left|\frac{\rho_s}{\lambda^{\frac{2}{\alpha}}} - 1\right| = \left|\rho_t - \frac{1}{\lambda^2}\right| \leqslant \frac{1}{10}\frac{1}{\lambda^2}.$$

In particular, since $\lambda < 2$, $\rho_t > 1/5$. Choosing A large enough, we find

$$M_{\varphi}'(t) \leqslant -\frac{1-\mu}{4}\rho_t \int u^2(t)\varphi_A'(\tilde{x})\,dx + \frac{1}{2(\alpha+1)} \int |u|^{2\alpha+2} \varphi_A'(\tilde{x})\,dx.$$

The constant $A > 0$ is now fixed.

Now, we estimate the nonlinear term as in [19], using the decomposition (44) and the decay of Q (34). Let a_0 to be fixed later. We decompose the nonlinear term as follows

$$\int |u|^{2\alpha+2} \varphi_A'(\tilde{x})\,dx = \mathbf{I} + \mathbf{II},$$

where

$$\mathbf{I} = \int_{|x-\rho(t)|>a_0} |u|^{2\alpha+2} \varphi_A'(\tilde{x})\,dx \quad \text{and} \quad \mathbf{II} = \int_{|x-\rho(t)|<a_0} |u|^{2\alpha+2} \varphi_A'(\tilde{x})\,dx.$$

On the one hand

$$\begin{aligned}
\mathbf{I} &\leqslant \big\|u(t)\big\|_{L^\infty(|x-\rho(t)|>a_0)}^{2\alpha} \int u^2 \varphi_A'(\tilde{x}) \\
&\leqslant C\big(\|Q_{\lambda(t)}\|_{L^\infty(|x|>a_0)}^{2\alpha} + \big\|\lambda^{-\frac{1}{\alpha}}\eta\big(t,\lambda^{-\frac{2}{\alpha}}.\big)\big\|_{L^\infty(|x|>a_0)}^{2\alpha}\big) \int u^2 \varphi_A'(\tilde{x}) \\
&\leqslant C\lambda^{-2}(t)\big(\|Q\|_{L^\infty(|y|\geqslant 2^{-\frac{2}{\alpha}}a_0)}^{2\alpha} + \|\eta\|_{L^\infty}^{2\alpha}\big) \int u^2 \varphi_A'(\tilde{x}) \\
&\leqslant C\rho_t\big(\|Q\|_{L^\infty(|y|\geqslant 2^{-\frac{2}{\alpha}}a_0)}^{2\alpha} + \|\eta\|_{H^{\frac{\alpha}{2}}}^{2\alpha}\big) \int u^2 \varphi_A'(\tilde{x}) \leqslant \frac{1-\mu}{8}\rho_t \int u^2 \varphi_A'(\tilde{x}),
\end{aligned}$$

for a_0 large enough and ϵ_0 small enough (recall that $\lambda(t) \leqslant 2$, $1 \leqslant \alpha \leqslant 2$).

On the other hand, a_0 being now fixed, by (44) and (47),

$$\big\|u(t)\big\|_{L^\infty}^{2\alpha} \leqslant \frac{C}{\lambda^2(t)} \leqslant C'\rho_t.$$

Thus, by the definition of φ_A

$$\mathbf{II} \leqslant \big\|u(t)\big\|_{L^2}^2 \big\|u(t)\big\|_{L^\infty}^{2\alpha} \big\|\varphi_A'(\tilde{x})\big\|_{L^\infty(|x-\rho(t)|<a_0)} \leqslant C\rho_t\big\langle x_0 + \mu\big(\rho(t_2) - \rho(t)\big)\big\rangle^{-2r}.$$

Estimate (49) is thus obtained by integration on $[t_1, t_2]$. □

Now, we prove Lemmas 6–7.

Proof of Lemma 6. We use commutator arguments and pseudo-differential operators tools. We recall here some well-known results which can be found for instance in Hörmander [9, Chapter 18]. For simplicity we denote by $(u|v) = \int u(x)\overline{v(x)}\,dx$ and $\|u\|^2 = (u|u)$.

We denote by $S^{m,q}$ the symbolic class of symbol defined by

$$a(x,\xi) \in S^{m,q} \quad \Leftrightarrow \quad \begin{cases} a \in \mathcal{C}^\infty(\mathbb{R}^2), \\ \forall k,\beta \in \mathbb{N},\ \exists C_{k,\beta} > 0 \text{ such that } \left|\partial_x^k \partial_\xi^\beta a(x,\xi)\right| \leqslant C_{k,\beta} \langle x\rangle^{q-k} \langle \xi\rangle^{m-\beta}. \end{cases} \tag{55}$$

Following Hörmander's notation, we have $S^{m,q} = S(\langle x\rangle^q \langle\xi\rangle^m, g)$ where $g = \frac{dx^2}{\langle x\rangle^2} + \frac{d\xi^2}{\langle\xi\rangle^2}$. We define the operator associated to a by the following formula for $u \in \mathcal{S}$,

$$a(x,D)u = \frac{1}{2\pi} \int e^{ix\xi} a(x,\xi)\hat{u}(\xi)\,d\xi \tag{56}$$

where the Fourier transform is defined by $\hat{u}(\xi) = \int e^{-ix\xi} u(x)\,dx$. We recall here some results about the pseudo-differential calculus.

$$\text{Let } a(x,\xi) \in S^{m,q},\ \exists C > 0,\ \forall u \in \mathcal{S} \text{ then } \left\|a(x,D)u\right\| \leqslant C\left\|\langle x\rangle^q \langle D\rangle^m u\right\|. \tag{57}$$

$$\begin{aligned} &\text{Let } a(x,\xi) \in S^{m,q},\ \text{there exists } b(x,\xi) \in S^{m,q} \text{ such that } a(x,D)^* = b(x,D) \\ &\text{moreover, there exists } r_0(x,\xi) \in S^{m-3,q-3} \text{ such that} \\ &b(x,\xi) = \overline{a(x,\xi)} + \frac{1}{i}\partial_x\partial_\xi \overline{a(x,\xi)} - \frac{1}{2}\partial_x^2\partial_\xi^2 \overline{a(x,\xi)} + r_0(x,\xi). \end{aligned} \tag{58}$$

We recall that A^* is the unique operator satisfying for all u and v in $\mathcal{S}$, $(Au|v) = (u|A^*v)$. We remark that $\partial_x\partial_\xi \overline{a(x,\xi)} \in S^{m-1,q-1}$ and $\partial_x^2\partial_\xi^2 \overline{a(x,\xi)} \in S^{m-2,q-2}$.

$$\begin{aligned} &\text{Let } a(x,\xi) \in S^{m,q} \text{ and } b(x,\xi) \in S^{m',q'} \text{ then there exists } c(x,\xi) \in S^{m+m',q+q'} \\ &\text{such that } a(x,D)b(x,D) = c(x,D). \end{aligned} \tag{59}$$

Remark that following (56), we have $a(x,D)D = c(x,D)$ where $c(x,\xi) = a(x,\xi)\xi$.

$$\begin{aligned} &\text{Let } a(x,\xi) \in S^{m,q} \text{ and } b(x,\xi) \in S^{m',q'} \text{ then there exists } c(x,\xi) \in S^{m+m'-1,q+q'-1} \\ &\text{such that } \left[a(x,D), b(x,D)\right] = c(x,D) \text{ moreover} \\ &\text{there exists } r_0(x,\xi) \in S^{m+m'-2,q+q'-2} \text{ such that } c(x,\xi) = \frac{1}{i}\{a,b\}(x,\xi) + r_0(x,\xi). \end{aligned} \tag{60}$$

We recall for operators A and B we have $[A,B] = AB - BA$ and $\{a,b\} = \partial_\xi a \partial_x b - \partial_x a \partial_\xi b$. In some cases we have exact formula, for instance $[D, a(x,D)] = \frac{1}{i}(\partial_x a)(x,D)$.

In Lemma 6 u is real-valued but it is convenient to write the integral in the following form

$$\int \left(-|D|^\alpha u\right) u_x \varphi(x) = \operatorname{Im}\left(\varphi(x)Du \,\middle|\, |D|^\alpha u\right) = -\frac{i}{2}\left(\left(|D|^\alpha \varphi D - D\varphi |D|^\alpha\right)u \,\middle|\, u\right). \tag{61}$$

Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $0 \leqslant \chi \leqslant 1$, $\chi(\xi) = 1$ if $|\xi| \leqslant 1$ and $\chi(\xi) = 0$ if $|\xi| \geqslant 2$. We set

$$\begin{aligned} &T = |D|^\alpha \varphi D - D\varphi |D|^\alpha = T_1 + T_2 \quad \text{where} \\ &T_1 = |D|^\alpha \left(1 - \chi(D)\right)\varphi D - D\varphi\left(1 - \chi(D)\right)|D|^\alpha, \\ &T_2 = |D|^\alpha \chi(D)\varphi D - D\varphi\chi(D)|D|^\alpha. \end{aligned} \tag{62}$$

The proof of Lemma 6 follows from (61), (62) and the two following claims.

Claim 1. *There exists $C > 0$ such that for all $u \in \mathcal{S}$ we have*

$$i(T_1 u|u) = (\alpha - 1)\left(\phi|D|^\alpha\left(1 - \chi(D)\right)\phi u \,\middle|\, u\right) + R \tag{63}$$

where R satisfies $|R| \leqslant C\|\phi u\|^2$.

Claim 2. *There exists $C > 0$ such that for all $u \in \mathcal{S}$ we have*

$$i(T_2u|u) = (\alpha - 1)\big(\phi|D|^\alpha \chi(D)\phi u|u\big) + R \tag{64}$$

where R satisfies $|R| \leqslant C\|\phi u\|^2$.

Proof of Claim 1. In the following we set $a(x,\xi) = \varphi(x)|\xi|^\alpha(1-\chi(\xi))$ and we have $a(x,\xi) \in S^{\alpha,0}$. With this notation we have $T_1 = a(x,D)^*D - Da(x,D)$. Following (58), the symbol of $a(x,D)^*$ is $a(x,\xi) + \frac{1}{i}\partial_x\partial_\xi a(x,\xi) - \frac{1}{2}\partial_x^2\partial_\xi^2 a(x,\xi) + r_0(x,\xi)$ where $r_0(x,\xi) \in S^{\alpha-3,-3}$. We obtain, following (60) and remark below,

$$\begin{aligned} T_1 &= \big[a(x,D), D\big] + \frac{1}{i}(\partial_x\partial_\xi a)(x,D)D - \frac{1}{2}\big(\partial_x^2\partial_\xi^2\big)a(x,D)D + r_1(x,D) \\ &= i(\partial_x a)(x,D) + \frac{1}{i}(\partial_x\partial_\xi a)(x,D)D - \frac{1}{2}\big(\partial_x^2\partial_\xi^2 a\big)(x,D)D + r_1(x,D) \end{aligned} \tag{65}$$

where $r_1(x,\xi) = r_0(x,\xi)\xi \in S^{\alpha-2,-3} \subset S^{0,-2r}$. We have, by (57)

$$\big|\big(r_1(x,D)u|u\big)\big| = \big|\big(\langle x\rangle^r r_1(x,D)u|\langle x\rangle^{-r}u\big)\big| \leqslant C\left\|\frac{u}{\langle x\rangle^r}\right\|^2 \tag{66}$$

because $\langle x\rangle^r r_1(x,D) = r_2(x,D)$ where $r_2(x,\xi) \in S^{0,-r}$.

We remark that the symbol of $(\partial_x^2\partial_\xi^2 a)(x,D)D$ is real-valued, we can apply the following claim.

Claim 3. *Let $b(x,\xi) \in S^{m,q}$, real-valued then there exists $C > 0$ such that for all $u \in \mathcal{S}$, we have*

$$\big|\mathrm{Im}\big(b(x,D)u|u\big)\big| \leqslant C\big\|\langle x\rangle^{\frac{q-1}{2}}\langle D\rangle^{\frac{m-1}{2}}u\big\|^2. \tag{67}$$

By definition $(T_1u|u) = 2i\,\mathrm{Im}(Du|a(x,D)u)$, it is sufficient to consider the imaginary part of the term of (65). In particular $\mathrm{Im}((\partial_x^2\partial_\xi^2 a)(x,D)Du|u)$ and we have $(\partial_x^2\partial_\xi^2 a)(x,\xi)\xi \in S^{\alpha-1,-2}$. Claim 3 gives

$$\big|\mathrm{Im}\big(\big(\partial_x^2\partial_\xi^2 a\big)(x,D)Du|u\big)\big| \leqslant C\big\|\langle x\rangle^{-\frac{3}{2}}\langle D\rangle^{\frac{\alpha-2}{2}}u\big\|^2 \leqslant C\left\|\frac{u}{\langle x\rangle^r}\right\|^2 \tag{68}$$

following (57) and $\langle x\rangle^{-\frac{3}{2}}\langle\xi\rangle^{\frac{\alpha-2}{2}} \in S^{\frac{\alpha-2}{2},-\frac{3}{2}} \subset S^{0,-2r}$.

Proof of Claim 3. We have $2i\,\mathrm{Im}(b(x,D)u|u) = ((b(x,D) - b(x,D)^*)u|u)$. By (58) we have $b(x,D)^* = b(x,D) + r_0(x,D)$ where $r_0(x,\xi) \in S^{m-1,q-1}$. We have $2i\,\mathrm{Im}(b(x,D)u|u) = (\langle x\rangle^{-\frac{q-1}{2}}\langle D\rangle^{-\frac{m-1}{2}}r_0(x,D)u|\langle x\rangle^{\frac{q-1}{2}}\langle D\rangle^{\frac{m-1}{2}}u)$ and following (59) $\langle x\rangle^{-\frac{q-1}{2}}\langle D\rangle^{-\frac{m-1}{2}}r_0(x,D) = c(x,D)$ where $c(x,\xi) \in S^{\frac{m-1}{2},\frac{q-1}{2}}$. We conclude by (57). □

Following (65), (66), (68) and notation of Claim 1, we have

$$(T_1u|u) = \big(i\big((\partial_x a)(x,D) - (\partial_x\partial_\xi a)(x,D)D\big)u|u\big) + R. \tag{69}$$

We have

$$\begin{aligned} (\partial_x a)(x,\xi) - (\partial_x\partial_\xi a)(x,\xi)\xi &= \varphi'(x)|\xi|^\alpha\big(1-\chi(\xi)\big) \\ &\quad - \alpha\varphi'(x)|\xi|^{\alpha-2}|\xi|^2\big(1-\chi(\xi)\big) + \varphi'(x)|\xi|^\alpha\chi'(\xi) \\ &= (1-\alpha)\varphi'(x)|\xi|^\alpha\big(1-\chi(\xi)\big) + \varphi'(x)|\xi|^\alpha\chi'(\xi). \end{aligned} \tag{70}$$

We have $\varphi'(x)|\xi|^\alpha\chi'(\xi) \in S^{0,-2r}$ because χ' is compact supported in $\mathbb{R}\setminus 0$. We have

$$\big|\big(\varphi'(x)\big)|D|^\alpha\chi'(D)u|u\big)\big| = \big|\big(\langle x\rangle^r\varphi'(x)\big)|D|^\alpha\chi'(D)u|\langle x\rangle^{-r}u\big)\big| \leqslant C\left\|\frac{u}{\langle x\rangle^r}\right\|^2 \tag{71}$$

following (59) and (57). By (69), (70) and (71), we obtain

$$(T_1u|u) = (1-\alpha)\big(i\phi^2|D|^\alpha\big(1-\chi(D)\big)u|u\big) + R$$
$$= (1-\alpha)\big(\big(i\phi|D|^\alpha\big(1-\chi(D)\big)\phi u|u\big) + \big(i\phi\big[\phi, |D|^\alpha\big(1-\chi(D)\big)\big]u|u\big)\big) + R. \tag{72}$$

Following (60), we have $i[\phi, |D|^\alpha(1-\chi(D))] = c(x,D) + r_0(x,D)$ where $c(x,\xi) = \{\phi, |\xi|^\alpha(1-\chi(\xi))\} = -\phi'(x)\partial_\xi(|\xi|^\alpha(1-\chi(\xi)))$ and $r_0(x,\xi) \in S^{\alpha-2,-r-2} \subset S^{0,-r}$ then $|(r_0(x,D)u|\phi u)| \leqslant C\|\langle x\rangle^{-r}u\|^2$. We have $\phi(x)c(x,\xi) \in S^{\alpha-1,-2r-1} \subset S^{1,-2r+1}$ and real-valued, we can apply Claim 3 to obtain $|\mathrm{Im}(\phi(x)c(x,D)u|u)| \leqslant C\|\langle x\rangle^{-r}u\|^2$. With (72), this proves Claim 1. □

Proof of Claim 2. Since $[D, a(x,D)] = \frac{1}{i}(\partial_x a)(x,D)$ for any $a(x,D)$,

$$T_2 = |D|^\alpha D\chi(D)\varphi(x) - \varphi(x)|D|^\alpha D\chi(D) + i|D|^\alpha\chi(D)\varphi'(x) + i\varphi'(x)|D|^\alpha\chi(D)$$
$$= \big[|D|^\alpha D\chi(D), \varphi(x)\big] + 2i\phi|D|^\alpha\chi(D)\phi + i\big[\big[|D|^\alpha\chi(D),\phi\big],\phi\big] = A_1 + A_2 + A_3. \tag{73}$$

We remark that $D|D|^\alpha\chi(D)u = g * u$ where $\hat{g}(\xi) = |\xi|^\alpha\xi\chi(\xi)$.

Claim 4. *Let $A_1 = [|D|^\alpha D\chi(D), \varphi]$, then there exists $C > 0$ such that for all $u \in \mathcal{S}$,*

$$i(A_1u|u) = (\alpha+1)\big(\phi|D|^\alpha\chi(D)\phi u|u\big) + R \tag{74}$$

where $|R| \leqslant C\|\langle x\rangle^{-r}u\|^2$. In particular,

$$i\big((A_1+A_2)u|u\big) = (\alpha-1)\big(\phi|D|^\alpha\chi(D)\phi u|u\big) + R. \tag{75}$$

Proof. We have, by a direct computation $[|D|^\alpha D\chi(D),\varphi]u(x) = \int g(x-y)(\varphi(y)-\varphi(x))u(y)\,dy$. To prove Claim 4 we need the following two claims, proved below.

Claim 5. *There exists $C > 0$ such that we have*

$$\varphi(y) - \varphi(x) = \frac{y-x}{\langle x\rangle^r\langle y\rangle^r} + Q(x,y) \tag{76}$$

where $Q(x,y)$ satisfies

$$\big|Q(x,y)\big| \leqslant C\frac{|x-y|^2}{(\langle x\rangle + \langle y\rangle)^{2r+1}} \quad \text{if } |x-y| \leqslant \frac{1}{2}\big(\langle x\rangle + \langle y\rangle\big), \tag{77}$$

$$\big|Q(x,y)\big| \leqslant C + C\frac{|x-y|}{\langle x\rangle^r\langle y\rangle^r} \quad \text{if } |x-y| \geqslant \frac{1}{2}\big(\langle x\rangle + \langle y\rangle\big). \tag{78}$$

We remark that if $|x-y| \leqslant \frac{1}{2}(\langle x\rangle + \langle y\rangle)$ then $\langle x\rangle \sim \langle y\rangle$ and if $|x-y| \geqslant \frac{1}{2}(\langle x\rangle + \langle y\rangle)$ then $\langle x-y\rangle \sim |x-y| \sim \langle x\rangle + \langle y\rangle$.

Claim 6. *Let $Ku(x) = \int Q(x,y)g(x-y)u(y)\,dy$, there exists $C > 0$ such that for all $u \in \mathcal{S}$ we have*

$$\big|(Ku|u)\big| \leqslant C\big\|\langle x\rangle^{-r}u\big\|^2. \tag{79}$$

Following Claims 5 and 6, we have $A_1u = \phi(h * (\phi u)) + Ru$ where $|(Ru|u)| \leqslant C\|\langle x\rangle^{-r}u\|^2$ and $h(x) = -xg(x)$. By definition of g we have

$$h(x) = \frac{1}{2\pi}\int -xe^{ix\xi}\xi|\xi|^\alpha\chi(\xi)\,d\xi$$
$$= \frac{i}{2\pi}\int \partial_\xi\big(e^{ix\xi}\big)\xi|\xi|^\alpha\chi(\xi)\,d\xi$$
$$= \frac{-i}{2\pi}\int e^{ix\xi}\partial_\xi\big(\xi|\xi|^\alpha\chi(\xi)\big)\,d\xi. \tag{80}$$

In the last equality we use that $\xi|\xi|^\alpha$ is a C^1 function, and we have $\partial_\xi(\xi|\xi|^\alpha\chi(\xi)) = (\alpha+1)|\xi|^\alpha\chi(\xi) + \xi|\xi|^\alpha\chi'(\xi)$. Then we have $h(x) = h_1(x) + h_2(x)$ where $\hat{h}_1(\xi) = -i(\alpha+1)|\xi|^\alpha\chi(\xi)$ and $\hat{h}_2(\xi) = -i\xi|\xi|^\alpha\chi'(\xi)$. We have $\phi(h_1 *$

$(\phi u)) = -i(\alpha+1)(\phi|D|^\alpha\chi(D)\phi u)(x)$. This term gives the first term of the right-hand side of (74). We have $\phi(h_2 * (\phi u)) = (\phi D|D|^\alpha\chi'(D)\phi u)(x)$ and by (59), $D|D|^\alpha\chi'(D)\phi$ is an operator with symbol in $S^{0,-r}$ (we recall χ' is supported in $1 \leqslant |\xi| \leqslant 2$), we have by (57), $|(D|D|^\alpha\chi'(D)\phi u|\phi u)| \leqslant C\|\langle x\rangle^{-r}u\|^2$. This proves Claim 4. □

Proof of Claim 5. By definition φ is bounded then (78) is obvious. We have $\varphi(y)-\varphi(x) = \int_x^y \frac{1}{\langle s\rangle^{2r}}\,ds$ then $Q(x,y) = \int_x^y (\frac{1}{\langle s\rangle^{2r}} - \frac{1}{\langle x\rangle^r\langle y\rangle^r})\,ds$. We have

$$\frac{1}{\langle s\rangle^{2r}} - \frac{1}{\langle x\rangle^r\langle y\rangle^r} = \frac{1}{\langle s\rangle^r}\left(\frac{1}{\langle s\rangle^r} - \frac{1}{\langle x\rangle^r}\right) + \frac{1}{\langle x\rangle^r}\left(\frac{1}{\langle s\rangle^r} - \frac{1}{\langle y\rangle^r}\right). \tag{81}$$

We have $\langle s\rangle \leqslant \langle x\rangle + \langle y\rangle$ because $s \in [x,y]$, and $\langle s\rangle \geqslant \inf(\langle x\rangle, \langle y\rangle) \sim \langle x\rangle \sim \langle y\rangle \sim \langle x\rangle + \langle y\rangle$ if $|x-y| \leqslant \frac{1}{2}(\langle x\rangle + \langle y\rangle)$. To prove (77), it is sufficient to prove

$$\left|\frac{1}{\langle s\rangle^r} - \frac{1}{\langle x\rangle^r}\right| \leqslant C\frac{|s-x|}{\langle x\rangle^{r+1}}. \tag{82}$$

Writing $\frac{1}{\langle s\rangle^r} - \frac{1}{\langle x\rangle^r} = \int_x^s \psi(t)\,dt$ where $\psi(t) = \partial_t(\frac{1}{\langle t\rangle^r})$, we have $|\psi(t)| \leqslant C\frac{1}{\langle t\rangle^{r+1}}$, this gives (82). □

Proof of Claim 6. Writing $((Ku)(x)|u(x)) = (\langle x\rangle^r K(\langle y\rangle^r\langle y\rangle^{-r}u)(x)|\langle x\rangle^{-r}u(x))$, it is sufficient to prove that $\langle x\rangle^r K(\langle y\rangle^r v)(x)$ defines a bounded operator on L^2. The kernel of this operator is $H(x,y) = \langle x\rangle^r\langle y\rangle^r Q(x,y)g(x-y) = H_1(x,y) + H_2(x,y)$, where H_1 and H_2 are H restricted respectively to the regions $|x-y| \leqslant \frac{1}{2}(\langle x\rangle + \langle y\rangle)$ and $|x-y| \geqslant \frac{1}{2}(\langle x\rangle + \langle y\rangle)$. Following Lemma A.2, we have $|g(x-y)| \leqslant \frac{C}{\langle x-y\rangle^{\alpha+2}}$.

From Claim 5 we have

$$\begin{aligned}
\left|H_1(x,y)\right| &\leqslant C\frac{\langle x\rangle^r\langle y\rangle^r|x-y|^2}{\langle x-y\rangle^{\alpha+2}(\langle x\rangle + \langle y\rangle)^{2r+1}} \\
&\leqslant \frac{C}{\langle x-y\rangle^{\alpha}(\langle x\rangle + \langle y\rangle)} \\
&\leqslant \frac{C}{\langle x-y\rangle^{\alpha+1}}
\end{aligned} \tag{83}$$

and

$$\begin{aligned}
\left|H_2(x,y)\right| &\leqslant C\frac{\langle x\rangle^r\langle y\rangle^r}{\langle x-y\rangle^{\alpha+2}}\left(C + \frac{C|x-y|}{\langle x\rangle^r\langle y\rangle^r}\right) \\
&\leqslant C\frac{\langle x\rangle^r\langle y\rangle^r}{\langle x-y\rangle^{\alpha+2}} + \frac{C}{\langle x-y\rangle^{\alpha+1}} = H_3(x,y) + H_4(x,y).
\end{aligned} \tag{84}$$

We claim $\int H_3(x,y)\,dy \leqslant C$ (and by symmetry $\int H_3(x,y)\,dx \leqslant C$). Indeed,

$$\begin{aligned}
\int H_3(x,y)\,dy &\leqslant \int\limits_{|y|<|x|} H_3(x,y)\,dy + \int\limits_{|y|>|x|} H_3(x,y)\,dy \\
&\leqslant C\langle x\rangle^{r-(\alpha+2)} \int\limits_{|y|<|x|} \langle y\rangle^r\,dy + C\langle x\rangle^r \int\limits_{|y|>|x|} \langle y\rangle^{r-(\alpha+2)}\,dy \leqslant C\langle x\rangle^{2r-(\alpha+1)} \leqslant C.
\end{aligned}$$

The same estimate is trivially true for H_1 and H_4. Thus, by Schur's lemma, the operator with kernel H is bounded on L^2. □

Claim 7. *Let $A_3 = i[[|D|^\alpha\chi(D), \phi], \phi]$, there exists $C > 0$ such that for all $u \in \mathcal{S}$ we have*

$$\left|(A_3u|u)\right| \leqslant C\left\|\langle x\rangle^{-r}u\right\|^2. \tag{85}$$

Proof. We set $h(x) = \frac{1}{2\pi}\int e^{ix\xi}|\xi|^\alpha\chi(\xi)\,d\xi$. Following Lemma A.2, there exists $C > 0$ such that $|h(x)| \leqslant \frac{C}{\langle x\rangle^{\alpha+1}}$. We have $[[|D|^\alpha\chi(D), \phi], \phi]u = \int h(x-y)(\phi(x) - \phi(y))^2u(y)\,dy$. We need the following claim to continue.

Claim 8. *There exists $C > 0$ such that*

$$\left|\phi(x)-\phi(y)\right| \leqslant C\frac{|x-y|}{(\langle x\rangle+\langle y\rangle)^{r+1}} \quad \text{if } |x-y| \leqslant \frac{1}{2}\left(\langle x\rangle+\langle y\rangle\right),$$
$$\left|\phi(x)-\phi(y)\right| \leqslant \frac{1}{\langle x\rangle^r}+\frac{1}{\langle y\rangle^r} \quad \text{if } |x-y| \geqslant \frac{1}{2}\left(\langle x\rangle+\langle y\rangle\right).$$

Proof. We have $\phi(x)-\phi(y)=\int_y^x \zeta(s)\,ds$ where $\zeta=\partial_s(\frac{1}{\langle s\rangle^r})$ and we have $|\zeta(s)| \leqslant \frac{C}{\langle s\rangle^{r+1}}$. If $|x-y| \leqslant \frac{1}{2}(\langle x\rangle+\langle y\rangle)$, we have $\langle s\rangle \sim \langle x\rangle \sim \langle y\rangle \sim \langle x\rangle+\langle y\rangle$, this gives the first inequality. The second one is obvious. □

We argue as in the proof of Claim 6. Let $R(x,y)=\langle x\rangle^r h(x-y)(\phi(x)-\phi(y))^2\langle y\rangle^r = R_1(x,y)+R_2(x,y)$ where R_1 and R_2 are R restricted respectively to the regions $|x-y| \leqslant \frac{1}{2}(\langle x\rangle+\langle y\rangle)$ and $|x-y| \geqslant \frac{1}{2}(\langle x\rangle+\langle y\rangle)$. It is sufficient to prove that R defines a bounded operator on L^2.

We have

$$\begin{aligned}\left|R_1(x,y)\right| &\leqslant C\frac{\langle x\rangle^r\langle y\rangle^r|x-y|^2}{\langle x-y\rangle^{\alpha+1}(\langle x\rangle+\langle y\rangle)^{2r+2}}\\ &\leqslant \frac{C}{\langle x-y\rangle^{\alpha+1}}\end{aligned} \tag{86}$$

and

$$\begin{aligned}\left|R_2(x,y)\right| &\leqslant C\frac{\langle x\rangle^r\langle y\rangle^r}{\langle x-y\rangle^{\alpha+1}}\left(\frac{1}{\langle x\rangle^{2r}}+\frac{1}{\langle y\rangle^{2r}}\right)\\ &\leqslant \frac{C\langle x\rangle^r}{\langle x-y\rangle^{\alpha+1}\langle y\rangle^r}+\frac{C\langle y\rangle^r}{\langle x-y\rangle^{\alpha+1}\langle x\rangle^r}=R_3(x,y)+R_4(x,y).\end{aligned} \tag{87}$$

By symmetry, it is now sufficient to prove that R_3 defines a bounded operator on L^2. We have

$$\int R_3(x,y)\langle x\rangle^{-\frac{1}{2}}\,dx \leqslant C\langle y\rangle^{-r} \quad \text{and} \quad \int R_3(x,y)\langle y\rangle^{-r}\,dy \leqslant C\langle x\rangle^{-\frac{1}{2}} \tag{88}$$

if $r \leqslant \frac{\alpha+1}{2}$. Using a variant of Schur's lemma (see e.g. [8, Theorem 5.2]), the operator with kernel $R(x,y)$ is bounded on L^2. □

By (73), Claims 4 and 7 we obtain $i(T_2u|u)=(\alpha-1)(\phi|D|^\alpha\chi(D)\phi u|u)+R$ where R satisfies the required estimates to prove Claim 2. □

Lemma 6 follows from Claims 1 and 2. □

Proof of Lemma 7. We have

$$\int\left(-|D|^\alpha u\right)u\varphi'\,dx=\left(-\phi^2|D|^\alpha u|u\right)=\left(-\phi|D|^\alpha\phi u|u\right)-\left(\phi\left[\phi,|D|^\alpha\right]u|u\right). \tag{89}$$

As the left-hand side is real, we can take the real part of the last term and we have

$$\begin{aligned}2\operatorname{Re}\left(\phi\left[\phi,|D|^\alpha\right]u|u\right)&=\left(\phi\left[\phi,|D|^\alpha\right]u|u\right)+\left(u|\phi\left[\phi,|D|^\alpha\right]u\right)\\ &=\left(\phi\left[\phi,|D|^\alpha\right]u|u\right)-\left(\left[\phi,|D|^\alpha\right]\phi u|u\right)=\left(\left[\phi,\left[\phi,|D|^\alpha\right]\right]u|u\right).\end{aligned} \tag{90}$$

By pseudo-differential calculus (60), the symbol of $[\phi,[\phi,|D|^\alpha(1-\chi(D))]]$ is in $S^{\alpha-2,-2r-2}\subset S^{0,-2r}$ and then it satisfies

$$\left|\left(\left[\phi,\left[\phi,|D|^\alpha\left(1-\chi(D)\right)\right]\right]u|u\right)\right| \leqslant C\left\|\langle x\rangle^{-r}u\right\|^2. \tag{91}$$

The term $([\phi,[\phi,|D|^\alpha\chi(D)]]u|u) \leqslant C\|\langle x\rangle^{-r}u\|^2$ by Claim 7. This proves that

$$\int\left(-|D|^\alpha u\right)u\varphi'\,dx \leqslant -\left\||D|^{\frac{\alpha}{2}}(\phi u)\right\|^2+C\left\|\langle x\rangle^{-r}u\right\|^2, \tag{92}$$

and completes the proof of Lemma 7. □

4.3. Monotonicity result on $\eta(t)$

For future use, we also state a monotonicity result for $\eta(t)$, restricted to the regular regime, i.e. the situation where the solution stays close to a fixed soliton.

Proposition 5. *Let $r \in (\frac{1}{2}, \frac{1}{2}(\alpha+1)]$ and $0 < \mu < 1$. Under the assumptions of Lemma 5, with the restriction $\lambda_0(t) = 1$, for $\epsilon_0 = \epsilon_0(\mu, r)$ small enough and $A = A(\mu, r)$ large enough, there exists $C = C(\mu, r, A) > 0$ such that for all $x_0 > 1$,*

$$\int \eta^2(s_2, y)\big[\varphi_A\big(\lambda^{\frac{2}{\alpha}}(s_2)y - x_0\big) - \varphi_A(-x_0)\big]\,dy$$
$$\leqslant \int \eta^2(s_1, y)\big[\varphi_A\big(\lambda^{\frac{2}{\alpha}}(s_1)y - x_0 - \mu(s_2 - s_1)\big) - \varphi_A\big(-x_0 - \mu(s_2 - s_1)\big)\big]\,dx$$
$$+ C\int_{s_1}^{s_2} \frac{\|\eta(s)\|_{L^2}^2}{(x_0 + \mu(s_2 - s))^{2r}}\,ds. \tag{93}$$

Sketch of proof. Using Lemmas 6–7, the proof is similar to the one of Proposition 2 in [11], the only difference being the additional scaling parameter $\lambda(s)$ (close to 1) in the present situation. Let

$$\tilde{y} = \lambda^{\frac{2}{\alpha}}(s)y - x_0 - \mu(s_2 - s), \qquad M_\eta(s) = \frac{1}{2}\int \eta^2(s)\big[\varphi_A(\tilde{y}) - \varphi_A\big(-x_0 - \mu(s_2 - s)\big)\big].$$

Using the equation of $\eta(s)$ (see Lemma 5), Lemmas 6–7 and estimates on φ_A, as in [11], one finds

$$M'_\eta(s) \leqslant \frac{C\|\eta(s)\|_{L^2}^2}{(x_0 + \mu(s_2 - s))^{2r}},$$

and the result follows by integration on $[s_1, s_2]$. □

5. Nonlinear Liouville property and asymptotic stability

This section is devoted to the regular regime: we study rigidity properties of the nonlinear equation (1) in a neighborhood of a soliton. In this section, $\alpha_0 < \alpha < 2$, where α_0 is given by Proposition 3 and Q denotes the ground state solution of (9). Note that we could also work with a general $1 \leqslant \alpha < 2$, assuming the linear Liouville property.

5.1. Nonlinear Liouville property

Proposition 6 *(Nonlinear Liouville property). Let $\alpha_0 < \alpha < 2$. There exists $\epsilon > 0$ such that if $u(t)$ is a global ($t \in \mathbb{R}$) solution of (1) satisfying for some $x_0(t)$,*

$$\forall t \in \mathbb{R}, \quad \big\|u(t) - Q\big(. - x_0(t)\big)\big\|_{H^{\frac{\alpha}{2}}} \leqslant \epsilon, \tag{94}$$

$$\forall \delta > 0,\ \exists B > 0,\ \forall t \in \mathbb{R}, \quad \int_{|x|>B} \big|u\big(t, x - x_0(t)\big)\big|^2\,dx \leqslant \delta, \tag{95}$$

then $u(t,x) \equiv Q_{\lambda_0}(x - x_0 - \lambda_0^{-2}t)$ for some $x_0 \in \mathbb{R}$ and some λ_0 close to 1.

Proof. The proof of Proposition 6 is by contradiction. Assume that there exists a sequence $u_n(t)$ of global $H^{\frac{\alpha}{2}}$ solutions of (1) close to a translation of Q for all time and such that their decomposition parameters $\eta_n(t)$, $\lambda_n(t)$, $\rho_n(t)$ given by Lemma 5 satisfy

$$\sup_{s\in\mathbb{R}}\big(\big|\lambda_n(s) - 1\big| + \big\|\eta_n(s)\big\|_{H^{\frac{\alpha}{2}}}\big) \to 0 \quad \text{as } n \to +\infty, \tag{96}$$

$$\eta_n \not\equiv 0, \tag{97}$$

$$\forall n,\ \forall \delta,\ \exists B_{n,\delta}>0,\ \forall t\in\mathbb{R},\qquad \int\limits_{|x|>B_{n,\delta}} \left|u_n\big(t,x+\rho_n(t)\big)\right|^2 dx \leqslant \delta. \tag{98}$$

We follow the strategy of [11], proof of Theorem 2. Define $0\not\equiv b_n=\sup_{s\in\mathbb{R}}\|\eta_n(s)\|_{L^2}$, $b_n\to 0$ as $n\to+\infty$. Then, there exists s_n such that $\|\eta_n(s_n)\|_{L^2}\geqslant \frac{1}{2}b_n$. We set

$$w_n(s,y)=\frac{\eta_n(s_n+s,y)}{b_n},$$

and we claim the following convergence result for the sequence (w_n).

Lemma 8. *There exists a subsequence of (w_n), denoted $(w_{n'})$ and $w\in C(\mathbb{R},L^2(\mathbb{R}))\cap L^\infty(\mathbb{R},L^2(\mathbb{R}))$ such that*

$$\forall s\in\mathbb{R},\quad w_{n'}(s)\rightharpoonup w(s)\quad \text{in } L^2(\mathbb{R}) \text{ weak as } n\to+\infty.$$

Moreover, $w(s)$ satisfies for some continuous functions $\beta(s)$, $\gamma(s)$,

$$w_s=(Lw)_y+\beta(s)Q'+\gamma(s)\Lambda Q\quad \text{on } \mathbb{R}\times\mathbb{R},$$
$$w\neq 0,\quad \int \chi_0 w=\int Q'w=0,$$
$$\forall s\in\mathbb{R},\ \forall y_0>1,\quad \int\limits_{|y|>y_0} w^2(s,y)\,dy\leqslant \frac{C}{y_0^\alpha}.$$

Sketch of the proof of Lemma 8. We proceed as in [11, proof of Proposition 5].

Decay estimate. From Proposition 5 (with $r=\frac{\alpha+1}{2}$, $s_2=s$ and $s_1\to-\infty$) and (98), it follows that

$$\forall y_0>1,\ \forall s\in\mathbb{R},\quad \int\limits_{|y|>y_0}\eta_n^2(s,y)\,dy\leqslant\frac{Cb_n^2}{y_0^\alpha},\qquad \int\limits_{|y|>y_0} w_n^2(s,y)\,dy\leqslant\frac{C}{y_0^\alpha}. \tag{99}$$

Local smoothing estimate. As in [11], we obtain using the equation of $w_n(s)$

$$\int_0^1\!\!\int \left|D^{\frac{\alpha}{2}}\big(w_n(s,y)\sqrt{\varphi'(y)}\big)\right|^2 dy\,ds\leqslant C. \tag{100}$$

Compactness in L^2. Following (99) and (100), there exists $\tau_n\in[0,1]$ and a subsequence of (w_n) still denoted by (w_n), $s_0\in[0,1]$ and $w_{s_0}\in L^2$ such that

$$w_n(\tau_n)\to w_{s_0}\quad \text{in } L^2,\qquad \tau_n\to s_0\quad \text{as } n\to+\infty.$$

Moreover, $\int w_{s_0}Q'=\int w_{s_0}\chi_0=0$.

Next, note that

$$\begin{aligned} w_{ns}&=\partial_y(Lw_n)-\partial_y\left(\frac{1}{b_n}\mathcal{R}(b_nw_n)\right)+\frac{1}{b_n}\frac{\lambda_{ns}}{\lambda_n}(\Lambda Q+b_n\Lambda w_n)+\frac{1}{b_n}\left(\frac{\rho_{ns}}{\lambda_n^{\frac{2}{\alpha}}}-1\right)\partial_y(Q+b_nw_n)\\ &=\partial_y(Lw_n)-\partial_y\left(\frac{1}{b_n}\mathcal{R}(b_nw_n)\right)+\beta_nQ'+\gamma_n\Lambda Q+b_nF_n'+b_nG_n+b_n\tilde\beta_nw_{ny}+b_n\tilde\gamma_n\Lambda w_n,\end{aligned}$$

where

$$\beta_n=\frac{1}{\int (Q')^2}\int w_nL(Q''),\qquad \tilde\beta_n=\frac{1}{b_n}\left(\frac{\rho_{ns}}{\lambda_n^{\frac{2}{\alpha}}}-1\right),\qquad F_n=\frac{1}{b_n}(\tilde\beta_n-\beta_n)Q,$$
$$\gamma_n=\frac{1}{\int\Lambda Q\chi_0}\int w_nL(\chi_0'),\qquad \tilde\gamma_n=\frac{1}{b_n}\frac{\lambda_{ns}}{\lambda_n},\qquad G_n=\frac{1}{b_n}(\tilde\gamma_n-\gamma_n)\Lambda Q.$$

Set

$$\tilde{w}_n(s) = w_n(s) - \Lambda Q \int_{\tau_n}^{s} \gamma_n(s')\,ds' - Q' \int_{\tau_n}^{s} \left(\beta_n(s') + 2 \int_{\tau_n}^{s'} \gamma_n(s'')\,ds'' \right) ds',$$

then

$$\tilde{w}_{ns} = \partial_y(L\tilde{w}_n) - \partial_y\left(\frac{1}{b_n}\mathcal{R}(b_n w_n)\right) + b_n F_n' + b_n G_n + b_n \tilde{\beta}_n w_{ny} + b_n \tilde{\gamma}_n \Lambda w_n.$$

Consider $\tilde{w}(s, y)$ the unique global solution of

$$\tilde{w}_s = \partial_y(L\tilde{w}) \quad \text{on } \mathbb{R}\times\mathbb{R}, \qquad \tilde{w}(s_0) = w_{s_0} \quad \text{on } \mathbb{R}.$$

Then (see proof of Lemma 9 in [11]), we have

$$\forall s \in \mathbb{R}, \quad \tilde{w}_n(s) \rightharpoonup \tilde{w}(s) \quad \text{in } L^2 \text{ weak}.$$

Finally, Lemma 8 is proved with

$$w(s, y) = \tilde{w}(s, y) + \Lambda Q \int_{s_0}^{s} \gamma(s')\,ds' + Q' \int_{s_0}^{s} \left(\beta(s') + 2 \int_{s_0}^{s'} \gamma(s'')\,ds'' \right) ds'$$

where

$$\gamma(s) = \frac{1}{\int \Lambda Q \chi_0} \int \tilde{w} L(\chi_0'), \qquad \beta(s) = \frac{1}{\int (Q')^2} \int \left(\tilde{w} + \Lambda Q \int_{s_0}^{s} \gamma(s')\,ds' \right) L(Q''). \qquad \square$$

We finish the proof of Proposition 6 by observing that the function $w(s, y)$ constructed in Lemma 8 contradicts the linear Liouville property, thus reaching the desired contradiction. Indeed, using the strategy of the proof of Corollary 1 in [17], we obtain

$$w(s, y) = a(t)\Lambda Q + b(t)Q'.$$

But since $\int w\chi_0 = \int wQ' = 0$, we obtain $a(t) = b(t) \equiv 0$ and thus $w \equiv 0$, which is a contradiction. $\square$

5.2. Asymptotic stability in the bounded regime

The next proposition is not used in the proof of Theorem 2 but it is stated as a consequence of Proposition 6 and the monotonicity arguments of Section 4.

Proposition 7 *(Asymptotic stability). Assume* $\alpha_0 < \alpha \leqslant 2$. *There exists* $\epsilon > 0$ *such that if* $u(t)$ *is a global* ($t \in \mathbb{R}$) *solution of* (1) *satisfying*

$$\forall t \in \mathbb{R}, \quad \inf_{x_0 \in \mathbb{R}} \left\| u(t) - Q(. - x_0) \right\|_{H^{\frac{\alpha}{2}}} \leqslant \epsilon, \tag{101}$$

then there exist $\lambda(t) > 0$, $\rho(t) \in \mathbb{R}$ *such that*

$$\eta(t, y) = \lambda^{\frac{1}{\alpha}}(t) u\left(t, \lambda^{\frac{2}{\alpha}}(t) y + \rho(t)\right) - Q(y)$$

satisfies

$$\eta(t) \rightharpoonup 0 \quad \text{in } H^{\frac{\alpha}{2}} \text{ as } t \to +\infty.$$

Except for the presence of the scaling parameter, it is similar to the proof of Theorem 2 from Theorem 1 in [11]. It is also close to the original proof for the gKdV equation in [19]. We thus omit the proof.

6. Finite or infinite time blow-up in the energy space

In this section, we prove Theorem 2 following the strategy of [22] and using the classification result given by Proposition 6.

Let $\alpha \in (\alpha_0, 2]$ where α_0 be given by Proposition 3. Consider an initial data $u(0) \in H^{\frac{\alpha}{2}}(\mathbb{R})$ such that

$$E\big(u(0)\big) < 0 \quad \text{and} \quad 0 < \beta\big(u(0)\big) = \int u^2(0) - \int Q^2 < \beta_0,$$

where β_0 is small enough (to be chosen) and $u(t)$ the corresponding solution of (1). Let $[0, T)$, $0 < T \leqslant +\infty$ be the maximal interval of existence of $u(t)$ as a solution of (1) in $H^{\frac{\alpha}{2}}$ (for $t \geqslant 0$).

We need the following variational result concerning negative energy $H^{\frac{\alpha}{2}}$ functions, with L^2 norm close to the L^2 norm of Q.

Lemma 9. *There exists $\beta_0 > 0$ such that for all $v \in H^{\frac{\alpha}{2}}$, if $E(v) < 0$ and $\beta(v) < \beta_0$ then there exist $x_0 \in \mathbb{R}$, $\lambda_0 > 0$, $\epsilon = \pm 1$ such that*

$$\big\| Q - \epsilon \lambda_0^{\frac{1}{\alpha}} v\big(\lambda_0^{\frac{2}{\alpha}}(x + x_0)\big)\big\|_{H^{\frac{\alpha}{2}}} \leqslant \delta(\beta),$$

where $\delta(\beta) \to 0$ as $\beta \to 0$.

We omit the proof since it is similar to the one of Lemma 1 in [22], using (41).

By conservation of mass, of energy and under the assumptions on $u(0)$, for β_0 small enough, it follows from Lemma 9 applied to $u(t)$ for all $t \in [0, T)$, that $u(t)$ is close to $\pm Q_{\lambda_0(t)}(x - \rho_0(t))$ for some $\lambda_0(t)$, $\rho_0(t)$. Without loss of generality, and by continuity in $H^{\frac{\alpha}{2}}$, we assume that u is close to $+Q$ (up to scaling and translation), by possibly considering $-u$ instead of u and using the invariance of the equation.

Now, from Lemma 5, possibly taking β_0 smaller, there exist $\lambda(t)$, $\rho(t)$ on $[0, T)$ such that, for all $t \in [0, T)$,

$$\eta(t, y) = \lambda^{\frac{1}{\alpha}}(t) u\big(t, \lambda^{\frac{2}{\alpha}}(t) y + \rho(t)\big) - Q(y)$$

satisfies

$$\int Q'(y)\eta(t, y)\,dy = \int \chi_0(y)\eta(t, y)\,dy = 0, \tag{102}$$

$$\big\|\eta(t)\big\|_{H^{\frac{\alpha}{2}}} \leqslant C\sqrt{\beta\big(u(0)\big)}, \tag{103}$$

$$\left|\frac{\lambda_s}{\lambda}\right| + \left|\left(\frac{\rho_s}{\lambda^{\frac{2}{\alpha}}} - 1\right)\right| \leqslant C\sqrt{\beta\big(u(0)\big)}. \tag{104}$$

Note that Lemmas 9 and 5 only give $\|\eta\|_{H^{\frac{\alpha}{2}}} \leqslant C\delta(\beta(0))$, where $\delta(\beta)$ is defined in Lemma 9, but not explicit. Actually, in this context, this estimate can be refined to get (103) by using energy arguments, exactly as in the proof of Lemma 3 in [22].

Now, we prove that

- either the solution $u(t)$ ceases to exist in finite time $0 < T < +\infty$ and consequently by Theorem 1, $\lim_{t \to T} \|u(t)\|_{H^{\frac{\alpha}{2}}} = +\infty$;
- or it exists for all time and then $\lim_{t \to +\infty} \|u(t)\|_{H^{\frac{\alpha}{2}}} = +\infty$.

The proof is by contradiction. Assume on the contrary that the solution $u(t)$ is globally defined in $H^{\frac{\alpha}{2}}$ for $t \geqslant 0$ and that there exists an increasing sequence $\bar{t}_m \to +\infty$ and $c_0 > 0$ such that

$$\big\|u(\bar{t}_m)\big\|_{H^{\frac{\alpha}{2}}} \leqslant c_0. \tag{105}$$

We proceed in four steps to reach a contradiction.

Step 1. Renormalisation and reduction of the problem. We recall that $\|u(t)\|_{L^2}$ is bounded and we define

$$\ell = \liminf_{t\to+\infty}\big\||D|^{\frac{\alpha}{2}}u_n(t)\big\|_{L^2} < \infty.$$

Note first that $\ell > 0$. Indeed, for all time t, $\int |u(t)|^{2\alpha+2} > -(\alpha+1)(2\alpha+1)E(u(0)) > 0$ and by the Gagliardo–Nirenberg inequality (8), we obtain $\ell > 0$. From the definition of ℓ, there exists t_0 such that

$$\big\||D|^{\frac{\alpha}{2}}u(t_0)\big\|_{L^2} \leqslant \ell(1+\beta_0) \quad \text{and} \quad \forall t \geqslant t_0, \quad \big\||D|^{\frac{\alpha}{2}}u(t)\big\|_{L^2} \geqslant \ell(1-\beta_0).$$

We consider the following rescaled version of $u(t,x)$: let $\bar\lambda = \frac{\||D|^{\frac{\alpha}{2}}Q\|_{L^2}}{\ell}$ and

$$\bar u(t,x) = \bar\lambda^{\frac{1}{\alpha}}u\big(\bar\lambda^{2+\frac{2}{\alpha}}t + t_0, \bar\lambda^{\frac{2}{\alpha}}x\big).$$

Note that $\|Q\|_{L^2}^2 < \|\bar u(0)\|_{L^2}^2 < \|Q\|_{L^2}^2 + \beta_0$, $E(\bar u(0)) < 0$, $\beta(\bar u(0)) < \beta_0$, $\bar u(t)$ is still a solution of (1) in $H^{\frac{\alpha}{2}}$ defined for all $t \geqslant 0$, and for all $t \geqslant 0$, $\||D|^{\frac{\alpha}{2}}\bar u(t)\|_{L^2} \geqslant (1-\beta_0)\||D|^{\frac{\alpha}{2}}Q\|_{L^2}$. Moreover, there exists a sequence $t_m \to +\infty$, such that

$$\lim_{m\to+\infty}\big\||D|^{\frac{\alpha}{2}}\bar u(t_m)\big\|_{L^2} = \big\||D|^{\frac{\alpha}{2}}Q\big\|_{L^2} \quad \text{and} \quad \lim_{m\to+\infty} t_{m+1} - t_m = +\infty.$$

Let $\bar\eta(t)$, $\bar\lambda(t)$ and $\bar\rho(t)$ be the parameters of the decomposition of $\bar u(t)$ given by Lemmas 9 and 5. Then, for $\beta_0 > 0$ small enough,

$$\forall t \geqslant 0, \quad \bar\lambda(t) \leqslant 2.$$

From the bound of $\bar u(t_m)$ in $H^{\frac{\alpha}{2}}$, there exists $\tilde u(0) \in H^{\frac{\alpha}{2}}$ such that after possibly extracting a subsequence (still denoted by (t_m))

$$\bar u\big(t_m, . + \rho(t_m)\big) \rightharpoonup \tilde u(0) \quad \text{in } H^{\frac{\alpha}{2}} \text{ as } m \to +\infty.$$

Taking β_0 small enough, it is clear that $\tilde u(0)$ is close to Q and in particular cannot be zero. Let now $\tilde u(t)$ be the maximal solution of (1) in $H^{\frac{\alpha}{2}}$ corresponding to $\tilde u(0)$ given by Theorem 1. We denote by $(-T_1, T_2)$ the maximal interval of existence of $\tilde u(t)$. Without a further analysis through Steps 2–4 below, we do not know if $\tilde u(t)$ is globally defined for $t > 0$ or $t < 0$.

Step 2. First properties of the limiting problem.

Lemma 10. *The following holds*

$$0 < \beta\big(\tilde u(0)\big) \leqslant \beta_0 \quad \textit{and} \quad E\big(\tilde u(0)\big) < 0. \tag{106}$$

Proof. Let

$$v_m(x) = \bar u\big(t_m, x + \rho(t_m)\big) \rightharpoonup \tilde u(0) \quad \text{in } H^{\frac{\alpha}{2}} \text{ as } m \to +\infty. \tag{107}$$

By weak convergence

$$\beta\big(\tilde u(0)\big) \leqslant \liminf_{m\to+\infty} \beta(v_m) < \beta_0.$$

The positivity $\beta(\tilde u(0)) > 0$ is a consequence of the negativity of the energy of $\tilde u(0)$ and (38), which we prove now.

Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $0 \leqslant \chi \leqslant 1$, $\chi(x) = 1$ if $|x| \leqslant 1$ and $\chi(x) = 0$ if $|x| \geqslant 2$. Let $\chi_A(x) = \chi(x/A)$, for $A > 1$. Then,

$$E(v_m) = \big\|\big(|D|^{\frac{\alpha}{2}}v_m\big)\sqrt{\chi_A}\big\|_{L^2}^2 - \frac{1}{(1+\alpha)(2\alpha+1)}\int |v_m\chi_A|^{2\alpha+2} + E\big(v_m(1-\chi_A)\big) + R_{m,A} + \tilde R_{m,A},$$

where

$$R_{m,A} = \big\||D|^{\frac{\alpha}{2}}v_m\big\|_{L^2}^2 - \big\|\big(|D|^{\frac{\alpha}{2}}v_m\big)\sqrt{\chi_A}\big\|_{L^2}^2 - \big\||D|^{\frac{\alpha}{2}}\big(v_m(1-\chi_A)\big)\big\|_{L^2}^2,$$

$$\tilde R_{m,A} = -\frac{1}{(1+\alpha)(2\alpha+1)}\int |v_m|^{2\alpha+2}\big(1 - \chi_A^{2\alpha+2} - (1-\chi_A)^{2\alpha+2}\big).$$

First, we control the term $R_{m,A}$. Note that from standard arguments, for all u,

$$\big|\big\||D|^{\frac{\alpha}{2}}(1-\chi_A)u\big\|-\big\|(1-\chi_A)|D|^{\frac{\alpha}{2}}u\big\|\big|\leqslant C\big\|\big[|D|^{\frac{\alpha}{2}},(1-\chi_A)\big]u\big\|=C\big\|\big[|D|^{\frac{\alpha}{2}},\chi_A\big]u\big\|\leqslant\frac{C}{A^{\frac{\alpha}{2}}}\|u\|,$$

and so

$$\big\||D|^{\frac{\alpha}{2}}(1-\chi_A)u\big\|^2\leqslant\big(1+A^{-\frac{\alpha}{2}}\big)\big\|(1-\chi_A)|D|^{\frac{\alpha}{2}}u\big\|^2+\frac{C}{A^{\frac{\alpha}{2}}}\|u\|^2.$$

Combining these two estimates, we get

$$R_{m,A}\geqslant -CA^{-\frac{\alpha}{2}}\|v_m\|^2_{H^{\frac{\alpha}{2}}}\geqslant -CA^{-\frac{\alpha}{2}}.$$

Next, we control $\tilde{R}_{m,A}$. By weak convergence in $H^{\frac{\alpha}{2}}$ and the properties of χ_A, we have

$$\lim_{m\to+\infty}\tilde{R}_{m,A}=-\frac{1}{(1+\alpha)(2\alpha+1)}\int\big|\tilde{u}(0)\big|^{2\alpha+2}\big(1-\chi_A^{2\alpha+2}-(1-\chi_A)^{2\alpha+2}\big)=\tilde{R}_A.$$

Moreover, from the definition of χ_A, the following holds $\lim_{A\to+\infty}\tilde{R}_A=0$.

Finally, by (38) (Gagliardo–Nirenberg with best constant), we have $E(v_m(1-\chi_A))\geqslant 0$ since for A large and β_0 small, for all m, $\int v_m^2(1-\chi_A)^2\leqslant\frac{1}{2}\int Q^2$.

Therefore,

$$0>E\big(\bar{u}(0)\big)=E(v_m)\geqslant\big\|\big(|D|^{\frac{\alpha}{2}}v_m\big)\sqrt{\chi_A}\big\|^2_{L^2}-\frac{1}{(1+\alpha)(2\alpha+1)}\int|v_m\chi_A|^{2\alpha+2}$$
$$-CA^{-\frac{\alpha}{2}}\big\|\tilde{u}(0)\big\|^2_{L^2}+\tilde{R}_{m,A}$$

and passing to the limit as $m\to+\infty$, we get

$$0>E\big(\bar{u}(0)\big)\geqslant\big\|\big(|D|^{\frac{\alpha}{2}}\tilde{u}(0)\big)\sqrt{\chi_A}\big\|^2_{L^2}-\frac{1}{(1+\alpha)(2\alpha+1)}\int\big|\tilde{u}(0)\chi_A\big|^{2\alpha+2}-CA^{-\frac{\alpha}{2}}\big\|\tilde{u}(0)\big\|^2_{L^2}+\tilde{R}_A.$$

Finally, passing to the limit as $A\to+\infty$, we obtain $0>E(\bar{u}(0))\geqslant E(\tilde{u}(0))$. □

Lemma 11. *For all $t\in(-T_1,T_2)$,*

$$\bar{u}\big(t_m+t,\bar{\rho}(t_m)+.\big)\rightharpoonup\tilde{u}(t)\quad\textit{in } H^{\frac{\alpha}{2}}(\mathbb{R})\textit{ as } m\to+\infty.\tag{108}$$

Moreover, if $\tilde{\eta}(t,x)$, $\tilde{\lambda}(t)$ and $\tilde{\rho}(t)$ are the parameters of the decomposition of $\tilde{u}(t,x)$, then for all $t\in(-T_1,T_2)$,

$$\bar{\lambda}(t_m+t)\to\tilde{\lambda}(t),\qquad\bar{\rho}(t_m+t)-\bar{\rho}(t_m)\to\tilde{\rho}(t).\tag{109}$$

The first part of Lemma 11 follows from Theorem 3. By Lemmas 10 and 9, $\tilde{u}(t)$ is close to Q (up to scaling and translation) for all $t\in(-T_1,T_2)$, and we can apply Lemma 5 to obtain a refined decomposition of $\tilde{u}(t)$ around Q, denoted by $\tilde{\eta}(t)$, $\tilde{\lambda}(t)$ and $\tilde{\rho}(t)$. Then (109) follows from standard limiting and uniqueness arguments which we omit. See [22], Lemma 8, Corollary 2 and references therein.

Step 3. Decay properties of the limiting problem by monotonicity properties.

Lemma 12. *For all $t\in(-T_1,T_2)$, for all $x_0>1$,*

$$\big\|\tilde{u}(t)\big\|^2_{L^2(|x-\tilde{\rho}(t)|\geqslant x_0)}\leqslant C|x_0|^{-\alpha}.\tag{110}$$

Proof. The main ingredient of the proof is Proposition 4 applied to $\bar{u}(t)$. Fix $\mu=\frac{1}{2}$, $r=\frac{\alpha+1}{2}$, A large enough, and let $C_0=C(\frac{1}{2},r,A)>0$ be the constant given by Proposition 4.

First, we prove the decay estimates on the right. Let $t\in(-T_1,T_2)$ and m be such that $t_m+t>0$. From (49) applied to $\bar{u}$, $t_2=t_m+t$ and $t_1=0$, we have

$$\int \bar{u}^2(t_m+t,x)\varphi_A\big(x-\bar{\rho}(t_m+t)-x_0\big)\,dx$$
$$\leqslant \int \bar{u}^2(0,x)\varphi_A\left(x-\bar{\rho}(0)-\frac{1}{2}\big(\bar{\rho}(t_m+t)-\bar{\rho}(0)\big)-x_0\right)dx+\frac{C_0}{x_0^{2r-1}}.$$

Thus, passing to the limit as $m\to+\infty$, using $\bar{\rho}(t_m+t)\to+\infty$ when $m\to+\infty$, we have

$$\limsup_{m\to+\infty}\int \bar{u}^2(t_m+t,x)\varphi_A\big(x-\bar{\rho}(t_m+t)-x_0\big)\,dx\leqslant\frac{C_0}{x_0^{2r-1}}. \tag{111}$$

It follows from the previous estimate and Lemma 11 that

$$\int \tilde{u}^2(t,x)\varphi_A\big(x-\tilde{\rho}(t)-x_0\big)\,dx\leqslant\frac{C_0}{x_0^{2r-1}}.$$

Second, we prove decay estimate on the left. Let $t\in(-T_1,T_2)$ and let m, m' be such that $t_m>t_{m'}+t$. Using (50), we obtain

$$\int \bar{u}^2(t_m,x)\varphi_A\left(x-\bar{\rho}(t_m)+\frac{1}{2}\big(\bar{\rho}(t_m)-\bar{\rho}(t_{m'}+t)\big)+x_0\right)dx$$
$$\leqslant\int \bar{u}^2(t_{m'}+t,x)\varphi_A\big(x-\bar{\rho}(t_{m'}+t)+x_0\big)\,dx+\frac{C_0}{x_0^{2r-1}}.$$

By Lemma 11, we have on the one hand, for m' fixed,

$$\liminf_{m\to+\infty}\int \bar{u}^2(t_m,x)\varphi_A\left(x-\bar{\rho}(t_m)+\frac{1}{2}\big(\bar{\rho}(t_m)-\bar{\rho}(t_{m'})\big)+x_0\right)dx\geqslant\int\tilde{u}^2(t),$$

and on the other hand, using (111),

$$\limsup_{m'\to+\infty}\int \bar{u}^2(t_{m'}+t,x)\varphi_A\big(x-\bar{\rho}(t_{m'}+t)+x_0\big)\,dx$$
$$\leqslant\int \tilde{u}^2(t,x)\varphi_A\big(x-\tilde{\rho}(t)+x_0\big)\,dx+\frac{C_0}{x_0^{2r-1}}.$$

It follows that

$$\int \tilde{u}^2(t,x)\big(1-\varphi_A\big(x-\tilde{\rho}(t)+x_0\big)\big)\,dx\leqslant\frac{2C_0}{x_0^{2r-1}}.$$

Lemma 12 is now proved. □

Step 4. Conclusion of the proof by rigidity properties. From (106) and Lemma 9, we have

$$\big|\tilde{\lambda}(0)-1\big|\leqslant\delta(\beta_0),\quad \text{where }\lim_{\beta_0\to0}\delta(\beta_0)=0. \tag{112}$$

We claim the following lemma to be used as a bootstrap argument on the behavior of $\tilde{\lambda}(t)$.

Lemma 13. *Assume further that for $-T_1<-t_1<0<t_2<T_2$,*

$$\forall t\in(-t_1,t_2),\quad \big|\tilde{\lambda}(t)-1\big|\leqslant\frac{1}{2}, \tag{113}$$

then for some $\epsilon>0$,

$$\forall t\in(-t_1,t_2),\quad \tilde{\eta}(t)\in L^1(\mathbb{R})\quad \text{and}\quad \int\big|\tilde{\eta}(t,x)\big|\,dx\leqslant C\beta_0^{\epsilon}. \tag{114}$$

Assuming Lemma 13, we finish the proof of Theorem 2. Using the invariant

$$\forall t\in(-T_1,T_2),\qquad \int \tilde u(t)=\int \tilde u(0)$$

and Lemma 13, we prove that the solution $\tilde u(t)$ is global (i.e. $T_1=T_2=\infty$) and

$$\left|\tilde\lambda(0)-1\right|\leqslant \tilde\delta(\beta_0),\qquad \lim_{\beta_0\to 0}\tilde\delta(\beta_0)=0.$$

By (113), (114), for all $t\in(-t_1,t_2)$, we have

$$\left|\int \tilde u(t)-\int Q_{\tilde\lambda(t)}\right|\leqslant C\beta_0^{\epsilon},$$

and so since $\int Q_\lambda=\lambda^{\frac{1}{\alpha}}\int Q$,

$$\left|\tilde\lambda(t)^{\frac{1}{\alpha}}-\tilde\lambda(0)^{\frac{1}{\alpha}}\right|\leqslant\left|\int Q_{\tilde\lambda(0)}-\int Q_{\tilde\lambda(t)}\right|\leqslant C\beta_0^{\epsilon}. \tag{115}$$

Therefore, by a standard continuity argument, (112), (113) and thus (115) are satisfied on $(-T_1,T_2)$. Thus, $\tilde u(t)$ is bounded on $(-T_1,T_2)$ in $H^{\frac{\alpha}{2}}$, which proves that $T_1=T_2=\infty$, and means that $\tilde u(t)$ is global. Moreover, (115) is satisfied for all $t\in\mathbb{R}$. By Proposition 6, $\tilde u$ has to be a soliton but this is a contradiction with $E(\tilde u(0))<0$, since the energy of a soliton is zero. This concludes the proof of Theorem 2 assuming Lemma 13. Thus, we only have to prove Lemma 13.

Proof of Lemma 13. We prove the result for $t\in(0,t_2)$, the proof being the same for negative times. Let $0<\epsilon<\frac{1}{2}(\alpha-1)$ small to be chosen later. As long as (113) is satisfied, we have by Lemma 12,

$$x_0^{\epsilon}\int_{|x|>x_0}\tilde u^2\big(t,x+\tilde\rho(t)\big)\,dx\leqslant C|x_0|^{-\alpha+\epsilon}.$$

Integrating this estimate in x_0 and using the Fubini theorem, we obtain

$$\int |x|^{1+\epsilon}\tilde u^2\big(t,x+\tilde\rho(t)\big)\,dx\leqslant C. \tag{116}$$

By the definition of $\tilde\eta(t)$ and the decay properties of Q, as long as (113) is satisfied, we obtain

$$\int |x|^{1+\epsilon}\tilde\eta^2(t,x)\,dx\leqslant C. \tag{117}$$

In particular, by Hölder inequality,

$$\begin{aligned}\int\left|\tilde\eta(t)\right| &\leqslant \|\tilde\eta\|_{L^\infty}^{\epsilon}\int\left|\tilde\eta(t)\right|^{1-\epsilon}\\ &\leqslant \|\tilde\eta\|_{L^\infty}^{\epsilon}\left(\int\left|\tilde\eta(t)\right|^2\big(1+|x|\big)^{\frac{1}{(1-\epsilon)^2}}\right)^{\frac{1-\epsilon}{2}}\left(\int\big(1+|x|\big)^{-\frac{1}{1-\epsilon^2}}\right)^{\frac{1+\epsilon}{2}}\\ &\leqslant C\|\tilde\eta\|_{L^\infty}^{\epsilon}\end{aligned}$$

and the result follows from $\|\tilde\eta\|_{L^\infty}\leqslant\|\tilde\eta\|_{H^{\frac{\alpha}{2}}}$ and Lemma 5. □

Acknowledgements

Part of this work was done while the second author was visiting the University of Chicago. He would like to thank the Department of Mathematics for its hospitality. The first author was supported in part by NSF grant DMS-0968472. The second author was supported in part by the French Agence Nationale de la Recherche, ANR ONDENONLIN.

At the time when this paper was submitted for publication, Frank and Lenzmann's uniqueness result [5] was not known. The authors would like to thank R.L. Frank and E. Lenzmann for communicating their work.

Appendix A

In the appendix, we gather the proof of standard results for reader's convenience.

Lemma A.1. *Let $r > \frac{1}{2}$ and $\alpha > -1$. Let*

$$g(x) = g_{\alpha,r}(x) = |D|^\alpha \left(\frac{1}{\langle x\rangle^{2r}}\right) \quad \textit{and} \quad h(\xi) = \int e^{-ix\xi} \frac{1}{\langle x\rangle^{2r}}\,dx \quad \textit{so that } \hat{g}(\xi) = |\xi|^\alpha h(\xi).$$

Then

(i) *There exists $C > 0$ such that*

$$\left|g_{\alpha,r}(x)\right| \leqslant \frac{C}{\langle x\rangle^{\alpha+1}}.$$

(ii) *The function h is continuous, and for any $M > 0$, there exists $C_M > 0$ such that $|h(\xi)| \leqslant \frac{C_M}{\langle\xi\rangle^M}$. Moreover, $h \in C^\infty(\mathbb{R}\setminus\{0\})$ and for all $\beta \in \mathbb{N}$, $q > 0$, there exists $C_{\beta,q} > 0$ such that*

$$\left|\partial_\xi^\beta h(\xi)\right| \leqslant \frac{C_{\beta,q}}{|\xi|^\beta \langle\xi\rangle^q}.$$

Proof. The proof is standard. Clearly h is a continuous and bounded function. By integration by part we have

$$\begin{aligned}(i\xi)^N h(\xi) &= \int (-\partial_x)^N \left(e^{-ix\xi}\right) \frac{1}{\langle x\rangle^{2r}}\,dx \\ &= \int e^{-ix\xi} (\partial_x)^N \left(\frac{1}{\langle x\rangle^{2r}}\right) dx. \end{aligned} \tag{118}$$

We have $|(\partial_x)^N(\frac{1}{\langle x\rangle^{2r}})| \leqslant \frac{C}{\langle x\rangle^{2r+N}}$ which is an integrable function and so $\xi^N h(\xi)$ is bounded. This gives the first part of (ii).

Let $\chi \in C_0^\infty(\mathbb{R})$ such that $0 \leqslant \chi \leqslant 1$, $\chi(\xi) = 1$ if $|\xi| \leqslant 1$ and $\chi(\xi) = 0$ if $|\xi| \geqslant 2$, we set $h_N(\xi) = \int e^{-ix\xi} \frac{1}{\langle x\rangle^{2r}} \chi(\frac{x}{N})\,dx$, $h_N \to h$ uniformly and in particular in $\mathcal{D}'$, then $\partial_\xi^\beta h_N \to \partial_\xi^\beta h$ in $\mathcal{D}'$. Let $M > 0$ to be fixed below, we have for some non-important constants C,

$$\begin{aligned}\xi^M \partial_\xi^\beta h_N(\xi) &= C_\beta \int e^{-ix\xi} \frac{\xi^M x^\beta}{\langle x\rangle^{2r}} \chi\left(\frac{x}{N}\right) dx \\ &= C_{\beta,M} \int \partial_x^M \left(e^{-ix\xi}\right) \frac{x^\beta}{\langle x\rangle^{2r}} \chi\left(\frac{x}{N}\right) dx \\ &= \sum_{M_1+M_2=M} C_{\beta,M_1,M_2} \int e^{-ix\xi} \partial_x^{M_1}\left(\frac{x^\beta}{\langle x\rangle^{2r}}\right) \frac{1}{N^{M_2}} \partial_x^{M_2}(\chi)\left(\frac{x}{N}\right) dx. \end{aligned} \tag{119}$$

We have $|\partial_x^{M_1}(\frac{x^\beta}{\langle x\rangle^{2r}})| \leqslant \frac{C}{\langle x\rangle^{2r-\beta+M_1}}$.

If $M_2 \geqslant 1$, the integral is restricted to $N \leqslant |x| \leqslant 2N$ and we have

$$\left|\int e^{-ix\xi} \partial_x^{M_1}\left(\frac{x^\beta}{\langle x\rangle^{2r}}\right) \frac{1}{N^{M_2}} \partial_x^{M_2}(\chi)\left(\frac{x}{N}\right) dx\right| \leqslant \frac{C}{N^{2r-\beta+M-1}} \tag{120}$$

then these terms go to 0 if $2r - \beta + M - 1 > 0$.

If $M_2 = 0$, $\frac{1}{\langle x\rangle^{2r-\beta+M}}$ is integrable if $2r - \beta + M - 1 > 0$. This implies that

$$\xi^M \partial_\xi^\beta h_N(\xi) \to C_{\beta,M} \int e^{-ix\xi} \partial_x^M \left(\frac{x^\beta}{\langle x\rangle^{2r}}\right) dx$$

uniformly and since $\xi^M \partial_\xi^\beta h_N \to \xi^M \partial_\xi^\beta h$ in $\mathcal{D}'$, we obtain

$$\xi^M \partial_\xi^\beta h(\xi) = C_{\beta,M} \int e^{-ix\xi} \partial_x^M \left(\frac{x^\beta}{\langle x\rangle^{2r}}\right) dx. \tag{121}$$

If we take $M=\beta$ we obtain the second part of the estimate (ii) if $|\xi|\leqslant 1$. If we take $M=\beta+q$ we obtain the second part of the estimate (ii) if $|\xi|\geqslant 1$.

Now, we prove (i). We set $g_{\alpha,r}(x)=\frac{1}{2\pi}(g_1(x)+g_2(x))$ where

$$g_1(x)=\int e^{ix\xi}|\xi|^\alpha h(\xi)\bigl(1-\chi(\xi)\bigr)\,d\xi,$$
$$g_2(x)=\int e^{ix\xi}|\xi|^\alpha h(\xi)\chi(\xi)\,d\xi. \tag{122}$$

Following (ii), $|\xi|^\alpha h(\xi)$ is integrable, thus g_1 is continuous and bounded and for all $M>0$,

$$\bigl|\partial_\xi^\beta\bigl(|\xi|^\alpha h(\xi)\bigl(1-\chi(\xi)\bigr)\bigr)\bigr|\leqslant\frac{C}{\langle\xi\rangle^M} \tag{123}$$

moreover, by integration by part, we have

$$x^\beta g_1(x)=\int i^\beta e^{ix\xi}\partial_\xi^\beta\bigl(|\xi|^\alpha h(\xi)\bigl(1-\chi(\xi)\bigr)\bigr)\,d\xi, \tag{124}$$

(123) and (124) give that $x^\beta g_1(x)$ bounded for all β.

To estimate g_2 we assume $x\geqslant 1$, the case $x\leqslant -1$ follows by the same way. We set $x\xi=\sigma$. We have

$$g_2(x)=x^{-\alpha-1}\int e^{i\sigma}|\sigma|^\alpha h\left(\frac{\sigma}{x}\right)\chi\left(\frac{\sigma}{x}\right)d\sigma=x^{-\alpha-1}\bigl(k_1(x)+k_2(x)\bigr)\quad\text{where}$$
$$k_1(x)=\int e^{i\sigma}\chi(\sigma)|\sigma|^\alpha h\left(\frac{\sigma}{x}\right)\chi\left(\frac{\sigma}{x}\right)d\sigma$$
$$k_2(x)=\int e^{i\sigma}\bigl(1-\chi(\sigma)\bigr)|\sigma|^\alpha h\left(\frac{\sigma}{x}\right)\chi\left(\frac{\sigma}{x}\right)d\sigma. \tag{125}$$

Obviously k_1 is bounded. By integration by part we have

$$\begin{aligned}k_2(x)&=\int(-i\partial_\sigma)^N\bigl(e^{i\sigma}\bigr)\bigl(1-\chi(\sigma)\bigr)|\sigma|^\alpha h\left(\frac{\sigma}{x}\right)\chi\left(\frac{\sigma}{x}\right)d\sigma\\&=\sum_{N_1,N_2,N_3}\int e^{i\sigma}\partial_\sigma^{N_1}\bigl(\bigl(1-\chi(\sigma)\bigr)|\sigma|^\alpha\bigr)\frac{1}{x^{N_2+N_3}}\bigl(\partial_\sigma^{N_2}h\bigr)\left(\frac{\sigma}{x}\right)\bigl(\partial_\sigma^{N_3}\chi\bigr)\left(\frac{\sigma}{x}\right)d\sigma.\end{aligned} \tag{126}$$

We have $|\partial_\sigma^N((1-\chi(\sigma))|\sigma|^\alpha)|\leqslant\frac{C}{\langle\sigma\rangle^{N-\alpha}}$.

If $N_3\geqslant 1$, $x\leqslant|\sigma|\leqslant 2x$ and we obtain

$$\int\left|\partial_\sigma^{N_1}\bigl(\bigl(1-\chi(\sigma)\bigr)|\sigma|^\alpha\bigr)\frac{1}{x^{N_2+N_3}}\bigl(\partial_\sigma^{N_2}h\bigr)\left(\frac{\sigma}{x}\right)\bigl(\partial_\sigma^{N_3}\chi\bigr)\left(\frac{\sigma}{x}\right)\right|d\sigma\leqslant\frac{C}{\langle x\rangle^{N-\alpha-1}} \tag{127}$$

which is bounded if $N\geqslant\alpha+1$.

If $N_3=0$, following (ii), we have

$$\int\left|\partial_\sigma^{N_1}\bigl(\bigl(1-\chi(\sigma)\bigr)|\sigma|^\alpha\bigr)\frac{1}{x^{N_2}}\bigl(\partial_\sigma^{N_2}h\bigr)\left(\frac{\sigma}{x}\right)\chi\left(\frac{\sigma}{x}\right)\right|d\sigma\leqslant\int\frac{C}{\langle\sigma\rangle^{N-\alpha}}\,d\sigma \tag{128}$$

which is bounded for N large enough. This proves (i). □

Lemma A.2. *Let $p(\xi)$ a homogeneous function of degree $\beta>-1$. Let $\chi\in\mathcal{C}_0^\infty(\mathbb{R})$ such that $0\leqslant\chi\leqslant 1$, $\chi(\xi)=1$ if $|\xi|\leqslant 1$ and $\chi(\xi)=0$ if $|\xi|\geqslant 2$. Let*

$$k(x)=\frac{1}{2\pi}\int e^{ix\xi}p(\xi)\chi(\xi)\,d\xi \tag{129}$$

then for all $q\in\mathbb{N}$, there exists $C_q>0$ such that for all $x\in\mathbb{R}$

$$\bigl|\partial_x^q k(x)\bigr|\leqslant\frac{C_q}{\langle x\rangle^{\beta+q+1}}. \tag{130}$$

Proof. The proof is standard. We have $\partial_x^q k(x) = \frac{1}{2\pi}\int e^{ix\xi}(i\xi)^q p(\xi)\chi(\xi)\,d\xi$ and as $(i\xi)^q p(\xi)$ is homogeneous of degree $\beta + q$, it is sufficient to prove Lemma A.2 for $q = 0$. We shall prove the estimate for $x \geqslant 1$, the case $x \leqslant -1$ follows by the same way. We set $y = x\xi$ in integral, we have $\int e^{ix\xi}(i\xi)^q p(\xi)\chi(\xi)\,d\xi = \frac{1}{x^{\beta+1}}\int e^{iy} p(y)\chi(\frac{y}{x})\,dy$. Lemma A.2 will be proved if we prove that $\int e^{iy} p(y)\chi(\frac{y}{x})\,dy$ is bounded. We set $J_1 = \int e^{iy} p(y)\chi(y)\chi(\frac{y}{x})\,dy$ and $J_2 = \int e^{iy} p(y)(1-\chi(y))\chi(\frac{y}{x})\,dy$. We remark that J_1 does not depend of x if x is large enough. We prove that J_2 is bounded by integration by part. For $N > \beta + 1$ we have $\partial_y^N e^{iy} = i^N e^{iy}$ and by integration by part we have

$$J_2 = \sum_{N_1+N_2+N_3=N} C_{N_1,N_2,N_3}\int e^{iy}\partial_y^{N_1} p(y)\partial_y^{N_2}\bigl(1-\chi(y)\bigr)\frac{1}{x^{N_3}}\bigl(\partial_y^{N_3}\chi\bigr)\left(\frac{y}{x}\right)dy. \tag{131}$$

If $N_2 \geqslant 1$ we integrate on compact domain and these integrals are bounded.

If $N_3 \geqslant 1$ in these integrals we have $x \leqslant |y| \leqslant 2x$ and

$$\left|e^{iy}\partial_y^{N_1} p(y)\bigl(1-\chi(y)\bigr)\frac{1}{x^{N_3}}\bigl(\partial_y^{N_3}\chi\bigr)\left(\frac{y}{x}\right)\right| \leqslant C|x|^{\beta-N_1-N_2} \leqslant C|x|^{-1} \tag{132}$$

then these integrals are bounded.

If $N_2 = N_3 = 0$

$$\left|e^{iy}\partial_y^{N} p(y)\bigl(1-\chi(y)\bigr)\chi\left(\frac{y}{x}\right)\right| \leqslant C|y|^{\beta-N}\bigl(1-\chi(y)\bigr) \tag{133}$$

and this function is integrable. This proves Lemma A.2. □

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