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Expanding measures ☆

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Abstract

We prove that any $C^{1+\alpha}$ transformation, possibly with a (non-flat) critical or singular region, admits an invariant probability measure absolutely continuous with respect to any expanding measure whose Jacobian satisfies a mild distortion condition. This is an extension to arbitrary dimension of a famous theorem of Keller (1990) [33] for maps of the interval with negative Schwarzian derivative.

Given a non-uniformly expanding set, we also show how to construct a Markov structure such that any invariant measure defined on this set can be lifted. We used these structure to study decay of correlations and others statistical properties for general expanding measures.

1. Introduction

In this work we propose a general construction of Markov structures for non-uniformly expanding transformations. A distinctive feature is that these Markov structures capture all trajectories with expanding behavior.

In particular, we are able to use them to prove existence of ergodic invariant measures absolutely continuous with respect to any expanding reference measure with Holder continuous Jacobian. In the special case when Lebesgue measure is the reference, this yields the physical measures of the transformation. Our Markov structures open the way for further development of the ergodic theory of this class of systems. In this direction, we construct Markov transformations induced from the original one, and we prove that any expanding invariant measure of the initial map lifts to invariant measure of these Markov transformations.

Markov partitions were the principal tool for analyzing the qualitative behavior of uniformly hyperbolic (Axiom A) or even uniformly expanding systems (see [51]). For uniformly hyperbolic dynamics, the systematic introduction of these partitions was due to Sinai [52–54] and Bowen [12,13] and became a key technical tool in the ergodic theory of uniformly hyperbolic/expanding systems (see [14]). Sinai, Ruelle, Bowen used Markov partitions to associate these dynamical systems with symbolic ones, prove existence and uniqueness of equilibrium states, and several other properties, in a neighborhood of every transitive hyperbolic set. Recall that a Markov partition for a map $f : \Lambda \to \Lambda$ is a cover $\mathcal{P} = \{P_1, \dots, P_s\}$ of Λ satisfying

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(a) $\operatorname{int} P_i \cap \operatorname{int} P_j = \emptyset$ if $i \neq j$;
(b) if $f(P_j) \cap \operatorname{int} P_i \neq \emptyset$ then $f(P_j) \supset P_i$.

Our setting is much more general than the classical family of uniformly expanding maps. Indeed we assume our systems to be non-uniformly expanding. In this setting one cannot expect, in general, the existence of a classical finite Markov partition as there exist parts of the system that spend arbitrarily large time to present some expanding behavior. Nevertheless, we shall prove the existence of a quite similar partition that will be called an *induced Markov partition*. An *induced Markov partition* is an at most countable cover $\mathcal{P} = \{P_1, P_2, P_3, \dots\}$ of Λ satisfying

(a) $\operatorname{int} P_i \cap \operatorname{int} P_j = \emptyset$ if $i \neq j$;
(b) for each P_j there is an $R_j \geqslant 1$ such that
  (b.1) if $\ell < R_j$ and $f^{\ell}(P_j) \cap \operatorname{int} P_i \neq \emptyset$ then $f^{\ell}(P_j) \subset P_i$;
  (b.2) if $f^{R_j}(P_j) \cap \operatorname{int} P_i \neq \emptyset$ then $f^{R_j}(P_j) \supset P_i$.

Let us be more precise about the kind of systems we will deal with in this paper. Formal statements will appear later. Let $f : M \to M$ be a $C^{1+\alpha}$ transformation outside some critical/singular set $\mathcal{C} \subset M$ (the case $\mathcal{C} = \emptyset$ is a possibility). A positively invariant set $\mathcal{H} \subset M$ is called *expanding* if every point $x \in \mathcal{H}$ satisfies

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \big\| \big(Df\big(f^i(x)\big)\big)^{-1} \big\|^{-1} > 0 \tag{1}$$

and if $\mathcal{H}$ satisfies the condition of slow approximation to the critical set, i.e., for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log \operatorname{dist}_\delta\big(f^j(x), \mathcal{C}\big) \leqslant \varepsilon \tag{2}$$

for every $x \in \mathcal{H}$, where $\operatorname{dist}_\delta(x, \mathcal{C})$ denote the δ-*truncated* distance from x to $\mathcal{C}$ defined as $\operatorname{dist}_\delta(x, \mathcal{C}) = \operatorname{dist}(x, \mathcal{C})$ if $\operatorname{dist}(x, \mathcal{C}) \leqslant \delta$ and $\operatorname{dist}_\delta(x, \mathcal{C}) = 1$ otherwise.

A probability measure is called *expanding* if there is an expanding set $\mathcal{H}$ such that $\mu(\mathcal{H}) = 1$. If f is a $C^{1+\alpha}$ endomorphism then any invariant measure satisfying (1) almost everywhere is automatically an expanding measure (Corollary A.3).

Given any expanding set $\mathcal{H}$, we construct an induced Markov partition $\mathcal{P}$ of $\mathcal{H}$ with respect to f (or an iterate of it). Associated to this partition there is an induced map

$$F : \Lambda \to \Lambda, \qquad F(x) = f^{R(x)}(x),$$

which is Markov, with an appropriate upper bound on the inducing time.

Given any reference measure ν which gives positive weight to $\mathcal{H}$, we can use the induced Markov map to construct f-invariant probabilities absolutely continuous with respect to ν, and study decay of correlations and others statistical properties.

A crucial point to be noted is that every f-invariant measure μ that gives positive weight to $\mathcal{H}$ can be lifted to the level of the induced map (the induced map does not depend on the measure μ).

We also give several examples of expanding measures and applications of these results.

1.1. Statement of main results

Let M be a compact Riemannian manifold of dimension $d \geqslant 1$ and $f : M \to M$ a map defined on M.

The map f is called *non-flat* if it is a local C^{1+} (i.e., $C^{1+\alpha}$ with $\alpha > 0$) diffeomorphism in the whole manifold except in a *non-degenerate critical/singular set* $\mathcal{C} \subset M$. We say that $\mathcal{C} \subset M$ is a *non-degenerate critical/singular set* if $\exists \beta, B > 0$ such that the following two conditions hold.

(C.1) $$\frac{1}{B} \operatorname{dist}(x, \mathcal{C})^{\beta} \leqslant \frac{\|Df(x)v\|}{\|v\|} \leqslant B \operatorname{dist}(x, \mathcal{C})^{-\beta} \quad \text{for all } v \in T_x M.$$

For every $x, y \in M \setminus \mathcal{C}$ with $\operatorname{dist}(x, y) < \operatorname{dist}(x, \mathcal{C})/2$ we have

$$\text{(C.2)} \qquad \bigl|\log\bigl\|Df(x)^{-1}\bigr\| - \log\bigl\|Df(y)^{-1}\bigr\|\bigr| \leqslant \frac{B}{\operatorname{dist}(x, \mathcal{C})^{\beta}} \operatorname{dist}(x, y).$$

If $\dim(M) = 1$ and f satisfies the usual one dimensional definition of non-flatness (see [36]), then it also satisfies the definition given above.

In the whole paper, a measure will be a countable additive measure defined on the Borel sets. A measure μ is called *f-non-singular* if $f_*\mu \ll \mu$, where $f_*\mu$ $(= \mu \circ f^{-1})$ is the push-forward of μ by f.

Let f be a non-flat map with critical/singular set $\mathcal{C} \subset M$. A finite measure μ is called *f-non-flat* if it is f-non-singular, $\mu(\mathcal{C}) = 0$, $J_\mu f(x)$ is well defined and positive for μ-almost every $x \in M$, and for μ-almost every $x, y \in M \setminus \mathcal{C}$ with $\operatorname{dist}(x, y) < \operatorname{dist}(x, \mathcal{C})/2$ we have

$$\left|\log \frac{J_\mu f(x)}{J_\mu f(y)}\right| \leqslant \frac{B}{\operatorname{dist}(x, \mathcal{C})^{\beta}} \operatorname{dist}(x, y).$$

1.1.1. Expanding sets and measures

Definition 1.1. A positively invariant set $\mathcal{H} \subset M$ (i.e., $f(\mathcal{H}) \subset \mathcal{H}$) is called *$\lambda$-expanding*, $\lambda \geqslant 0$, if

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log\bigl\|\bigl(Df\bigl(f^i(x)\bigr)\bigr)^{-1}\bigr\|^{-1} > \lambda, \tag{3}$$

for every $x \in \mathcal{H}$, and $\mathcal{H}$ satisfies the slow approximation condition, i.e., for each $\varepsilon > 0$ there is a $\delta > 0$ such that (2) holds for every $x \in \mathcal{H}$.

An expanding set is a positively invariant set but, in general, it is not a compact one. In the one-dimensional case (3) reduces to the Lyapunov exponent of f on x to be bigger than λ, i.e.,

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log\bigl|f'\bigl(f^n(x)\bigr)\bigr| = \limsup_{n\to\infty}\bigl|\bigl(f^n\bigr)'(x)\bigr| > \lambda.$$

Definition 1.2 *(Expanding measures).* We call a measure μ (non-necessarily invariant) a *λ-expanding* measure (with respect to f) if μ is f-non-singular and there exists a λ-expanding set $\mathcal{H}$ such that $\mu(M \setminus \mathcal{H}) = 0$.

Theorem A *(Existence of absolutely continuous invariant measures). Let $f : M \to M$ be a non-flat map. If μ is an f-non-flat λ-expanding measure, $\lambda > 0$, then there exists a finite collection of μ absolutely continuous ergodic f-invariant probabilities such that μ-almost every point in M belongs to the basin of one of these probabilities.*

Recall that the *basin of measure* η is the set $\mathcal{B}(\eta)$ of the points $x \in M$ such that

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi \circ f^j(x) = \int \varphi\, d\eta,$$

for every continuous function $\varphi : M \to \mathbb{R}$.

1.1.2. Markov partitions

Let $f : U \to U$ a measurable map defined on a Borel set U of a compact, separable metric space X. A countable collection $\mathcal{P} = \{P_1, P_2, P_3, \dots\}$ of Borel subsets of U is called a *Markov partition* if

(1) $\operatorname{int}(P_i) \cap \operatorname{int}(P_j) = \emptyset$ if $i \neq j$;
(2) if $f(P_i) \cap \operatorname{int}(P_j) \neq \emptyset$ then $f(P_i) \supset \operatorname{int}(P_j)$;
(3) $\#\{f(P_i);\ i \in \mathbb{N}\} < \infty$;

(4) $f|_{P_i}$ is a homeomorphism and it can be extended to a homeomorphism sending $\overline{P_i}$ onto $\overline{f(P_i)}$;
(5) $\lim_n \text{diameter}(\mathcal{P}_n(x)) = 0$ $\forall x \in \bigcap_{n\geqslant 0} f^{-n}(\bigcup_i P_i)$,

where $\mathcal{P}_n(x) = \{y;\ \mathcal{P}(f^j(y)) = \mathcal{P}(f^j(x))\ \forall 0 \leqslant j \leqslant n\}$ and $\mathcal{P}(x)$ denotes the element of $\mathcal{P}$ that contains x.

Definition 1.3 *(Induced Markov partition).* A countable collection $\mathcal{P} = \{P_1, P_2, P_3, \dots\}$ of Borel subsets of U is called a *induced Markov partition* if it satisfies all conditions of a Markov partition except the second one which has to be replaced by the following

(2) for each $P_i \in \mathcal{P}$ there is an $R_i \geqslant 1$ such that
  (2.1) if $\ell < R_i$ and $\text{int}(f^\ell(P_i)) \cap \text{int}(P_j) \neq \emptyset$ then $\text{int}(f^\ell(P_i)) \subset \text{int}(P_j)$ or $\text{int}(f^\ell(P_i)) \supset \text{int}(P_j)$;
  (2.2) if $f^{R_i}(P_i) \cap \text{int}(P_j) \neq \emptyset$ then $f^{R_i}(P_i) \supset \text{int}(P_j)$.

Definition 1.4 *(Markov map).* The pair $(F, \mathcal{P})$, where $\mathcal{P}$ is a Markov partition of $F : U \to U$, is called a *Markov map* defined on U. If $F(P) = U$ $\forall P \in \mathcal{P}$, $(F, \mathcal{P})$ is called a *full Markov map*.

Note that if $(F, \mathcal{P})$ is a full Markov map defined on an open set U then the elements of $\mathcal{P}$ are open sets (because $F(P) = U$ and $F|_P$ is a homeomorphism $\forall P \in \mathcal{P}$).

Consider a measurable map $f : M \to M$ from M to M (or, more in general, from the metric space X to X).

Definition 1.5 *(Induced Markov map).* A Markov map $(F, \mathcal{P})$ defined on U is called a *induced Markov map* for f on U if is there is a function $R : U \to \mathbb{N} = \{0, 1, 2, 3, \dots\}$ (called *inducing time*) such that $\{R \geqslant 1\} = \bigcup_{P\in\mathcal{P}} P$, $R|_P$ is constant $\forall P \in \mathcal{P}$ and $F(x) = f^{R(x)}(x)$ $\forall x \in U$.

If an induced Markov map $(F, \mathcal{P})$ is a full Markov map, we call $(F, \mathcal{P})$ an *induced full Markov map*.

Given an induced Markov map $(F, \mathcal{P})$, an ergodic f-invariant probability μ is said *liftable* to F if there exists F-invariant finite measure $\nu \ll \mu$ such that

$$\mu = \sum_{P\in\mathcal{P}} \sum_{j=0}^{R(P)-1} f_*^j(\nu|_P),$$

where R is the inducing time of F, $\nu|_P$ denotes the measure given by $\nu|_P(A) = \nu(A \cap P)$ and f_*^j is the push-forward by f^j.

Definition 1.6 *(Markov structure).* A *Markov structure* for a set $U \subset M$ (or X) is an at most countable collection $\mathfrak{F} = \{(F_i, \mathcal{P}_i)\}_i$ of induced Markov maps such that if μ is an ergodic f-invariant probability with $\mu(U) = 1$ then $\exists (F_i, \mathcal{P}_i) \in \mathfrak{F}$ such that μ is liftable to F_i.

Theorem B *(Markov structure for an expanding set). Let M be a compact Riemannian manifold and $f : M \to M$ a non-flat map. Let $\lambda \geqslant 0$ and $\mathcal{H}$ be a λ-expanding set. Then there is a Markov structure $\mathfrak{F} = \{(F_i, \mathcal{P}_i)\}_i$ for $\mathcal{H}$. Furthermore, denoting the domain of F_i by U_i and its inducing time by R_i, we have the following additional properties.*

(1) *If $\lambda > 0$ then $\mathfrak{F} = \{(F_1, \mathcal{P}_1), \dots, (F_s, \mathcal{P}_s)\}$, that is, $\mathfrak{F}$ is a finite collection of Markov induced maps.*
(2) *U_i is a connected open set $\forall i$.*
(3) *Each $(F_i, \mathcal{P}_i) \in \mathfrak{F}$ is a full Markov map, i.e., is a Markov map with $F_i(P) = U_i$ $\forall P \in \mathcal{P}_i$. In particular, every $P \in \mathcal{P}_i$ is a connected open set $\forall i$.*
(4) *For each $(F_i, \mathcal{P}_i) \in \mathfrak{F}$ there is $\lambda_i > 0$ such that*

$$\log\left\|\left(DF_i(x)\right)^{-1}\right\|^{-1} > \lambda_i, \quad \forall x \in \bigcup_{P\in\mathcal{P}_i} P.$$

As an expanding set $\mathcal{H}$ is positively invariant, it follows from Theorem B that every ergodic f-invariant probability having $\mu(\mathcal{H}) > 0$ is liftable to one of the full induced Markov map $F_1, \dots, F_s$ given by Theorem B.

It is important to observe that in Section 5 we introduce the zooming sets and the theorems above are corollaries of Theorems C, D and E for zooming sets. The zooming sets (or measures) generalize the expanding ones and allows us to deal with non-exponential expansions.

1.2. *Overview of the paper*

In Section 2 we introduced the notion of *nested sets* adapted to the kind of pre-images we want to deal with (for example, pre-images with some contraction).

In Section 3 we study the ergodic components for non-(necessarily) invariant measures for maps on metric spaces.

In Section 4 we obtain a statistical characterization of the liftable measures for a given induced map.

Although we are basically interested in expanding measures (Section 1.1), we weakened the expansion condition to permit more flexibility in the applications. For this we introduce the *zooming measures* in Section 5.

In Sections 6 and 7 we show most of the results for zooming sets and measures. In particular the existence of induced Markovian maps for zooming sets (Theorems D and E) and the existence of an invariant measure $\nu \ll \mu$ that is absolutely continuous with respect to a given zooming measure with some distortion control (Theorem C).

Section 8 is dedicated to the definition and properties of expanding measures, as well as to establish the connection between these measures with the zooming ones. The existence of an absolutely continuous invariant measure for a given expanding measure, the induced Markovian maps for expanding sets and so on are consequences of the analogous result for zooming measures and in this section we use the zooming results to get the expanding ones.

In Section 9 we give many examples of expanding and zooming sets and measures. We give also some applications of the results of the previous sections. In particular, we study the decay of correlations for general expanding measures.

2. Nested sets

The notion of nice interval, introduced by Martens in [35], is a useful tool in the theory of real and complex one-dimensional dynamical systems (see, for instance, [36,46]). A *nice interval* is an open interval I such that the forward orbit $\mathcal{O}^+(\partial I)$ of the boundary of I does not return to I, i.e., $\mathcal{O}^+(\partial I) \cap I = \emptyset$. Note that nice intervals are natural and easy to construct for interval maps. For instance, two consecutive points of a periodic orbit define a nice interval. Its main property is that there are no linked pre-images of a nice interval, that is, if I_1 and I_2 are sent homeomorphically onto an open nice interval I by f^{n_1} and f^{n_2} respectively then either $I_1 \cap I_2 = \emptyset$, $I_1 \subset I_2$ or $I_2 \subset I_1$.

In the multidimensional case, the boundary of topological open balls are connect topological manifolds and if a chaotic transitive dynamic is not much symmetric, it is natural to expect that this dynamic will spread these boundaries to the whole manifold, forbidding any "nice ball". In general, the same seems true for sets whose the boundary is not totally disconnected.

In this section we present the abstract construction of nested sets. This reformulates and generalizes the concept of *nice interval*. In Section 5 we show their abundance in the presence of some expansion (see Lemma 5.12).

Let $f : X \to X$ be a map defined on a complete, separable metric space X. Fixed some $K \subset X$, a set $P \subset X$ is called a *regular pre-image of order* $n \in \mathbb{N}$ of K if f^n sends P homeomorphically onto K. Denote the order of P (with respect to K) by $\mathrm{ord}(P)$.

Let us fix in all Section 2 a collection $\mathcal{E}_0$ of connected open subsets of X (for instance, $\mathcal{E}_0$ can be the collection $\{f^n(V_n(x));\ x \in \mathrm{H}_n \text{ and } n \in \mathbb{N}\}$ of all hyperbolic balls of X, see Proposition 8.2). For each $n \in \mathbb{N}$ and $V \in \mathcal{E}_0$ consider some collection $\mathcal{E}_n(V)$ of regular pre-images of order n of K. Set $\mathcal{E}_n = (\mathcal{E}_n(V))_{V \in \mathcal{E}_0}$. We call the sequence $\mathcal{E} = (\mathcal{E}_n)_n$ a *dynamically closed family of* (*regular*) *pre-images* if $f^\ell(E) \in \mathcal{E}_{n-\ell}$ $\forall E \in \mathcal{E}_n$ and $\forall 0 \leqslant \ell \leqslant n$. Given $Q \in \mathcal{E}_n$ we denote $f^n|_Q$ by f^Q and we denote the $\mathcal{E}$-inverse branch of associated to Q, $(f^n|_Q)^{-1}$, by f^{-Q}.

Let $\mathcal{E} = (\mathcal{E}_n)_n$ be a dynamically closed family of pre-images. A set P is called an $\mathcal{E}$-pre-image of a set $W \subset X$ if there is $n \in \mathbb{N}$ and $Q \in \mathcal{E}_n$ such that $\overline{W} \subset f^n(Q)$ and $P = f^{-Q}(W)$, where $\overline{W}$ is the closure of W.

Remark 2.1. Two distinct $\mathcal{E}$-pre-images $\mathcal{X}_1$ and $\mathcal{X}_2$ of some set $\mathcal{X} \subset X$ having the same order cannot intersect. Indeed, write $n = \mathrm{ord}(\mathcal{X}_1) = \mathrm{ord}(\mathcal{X}_2)$ and for each $i \in \{1, 2\}$ write $\mathcal{X}_i = f^{-Q_i}(\mathcal{X})$, with $Q_i \in \mathcal{E}_n$. Let $P_j = f^{-Q_j}(Q_1 \cap Q_2)$, for $j = 1, 2$. It follows that $P_1 \cap P_2 \supset \mathcal{X}_1 \cap \mathcal{X}_2 \neq \emptyset$. Of course $P_1 \neq P_2$, otherwise $\mathcal{X}_2 = f^{-Q_2}(\mathcal{X}) = (f^n|_{P_2})^{-1}(\mathcal{X}) = (f^n|_{P_1})^{-1}(\mathcal{X}) = f^{-Q_1}(\mathcal{X}) = \mathcal{X}_1$. Thus $P_1 \cap \partial P_2 \neq \emptyset$ or $P_2 \cap \partial P_1 \neq \emptyset$. Assume that $P_1 \cap \partial P_2 \neq \emptyset$. So, $\emptyset \neq f^n(P_1 \cap \partial P_2) \subset f^n(P_1) \cap \partial(f^n(P_2)) \subset (Q_1 \cap Q_2) \cap \partial(Q_1 \cap Q_2) = \emptyset$. An absurd.

Definition 2.2 *(Linked sets).* We say that two open sets U_1 and U_2 are *linked* if both $U_1 \setminus U_2$ and $U_2 \setminus U_1$ are not empty sets.

Note that two connected open sets U_1 and U_2 are linked if and only if $\partial U_1 \cap U_2$ and $U_1 \cap \partial U_2$ are not empty sets.

Definition 2.3 *($\mathcal{E}$-nested set).* A set V is called *$\mathcal{E}$-nested* if V is an open set and V is not linked with any $\mathcal{E}$-pre-image of V.

The fundamental property of a nested set is that any $\mathcal{E}$-pre-images P_1 and P_2 of it are not linked (see Corollary 2.6). We can extend the concept of $\mathcal{E}$-nested set to a collection of sets in the following way.

Definition 2.4 *($\mathcal{E}$-nested collection of sets).* A collection $\mathcal{A}$ of open sets is called an *$\mathcal{E}$-nested collection of sets* if every $A \in \mathcal{A}$ is not linked with any $\mathcal{E}$-pre-image of an element of $\mathcal{A}$ with order bigger than zero. Precisely, if $A_1 \in \mathcal{A}$ and P is an $\mathcal{E}$-pre-image of some $A_2 \in \mathcal{A}$, then either A_1 and P are not linked or $P = A_2$.

It follows from the definition of an $\mathcal{E}$-nested collection of sets that every sub-collection of an $\mathcal{E}$-nested collection is also an $\mathcal{E}$-nested collection. In particular, each element of an $\mathcal{E}$-nested collection is an $\mathcal{E}$-nested set.

Lemma 2.5 *(Main property of a nested collection). If $\mathcal{A}$ is an $\mathcal{E}$-nested collection of open sets and P_1 and P_2 are $\mathcal{E}$-pre-images of two elements of $\mathcal{A}$ with $\operatorname{ord}(P_1) \neq \operatorname{ord}(P_2)$ then P_1 and P_2 are not linked.*

Proof. Let $\ell_j = \operatorname{ord}(P_j)$ for $j = 1, 2$. We may assume that $\ell_1 < \ell_2$ and, by contradiction, assume that P_1 and P_2 are linked. Let, for $i = 1, 2$, $p_j \in P_j \cap \partial P_{3-j}$, $Q_i \in \mathcal{E}_{\ell_i}$ and $A_i \in \mathcal{A}$ be such that $P_i = f^{-Q_i}(A_i)$. As $\mathcal{E}$ is a dynamically closed family of pre-images of elements of $\mathcal{E}_0$, $Q = f^{\ell_1}(Q_2) \in \mathcal{E}_{\ell_2 - \ell_1}$ and $P = f^{\ell_1}(P_2) = f^{-Q}(A_2)$ is an $\mathcal{E}$-pre-image of A_2. On the other hand $f^{\ell_1}(P_1) = A_1 \in \mathcal{A}$. As $f^{\ell_1}(p_1) \in f^{\ell_1}(P_1) \cap \partial(f^{\ell_1}(P_2)) = A_1 \cap \partial P$ and $f^{\ell_1}(p_2) \in f^{\ell_1}(P_2) \cap \partial(f^{\ell_1}(P_1)) = P \cap \partial A_1$, it follows that P and A_1 are linked, but this is impossible because $\mathcal{A}$ is $\mathcal{E}$-nested. $\square$

Corollary 2.6 *(Main property of a nested set). If V is an $\mathcal{E}$-nested set and P_1 and P_2 are $\mathcal{E}$-pre-images of V then P_1 and P_2 are not linked. Furthermore,*

(1) *if $P_1 \cap P_2 \neq \emptyset$ then* $\operatorname{ord}(P_1) \neq \operatorname{ord}(P_2)$;
(2) *if $P_1 \subsetneq P_2$ with* $\operatorname{ord}(P_1) < \operatorname{ord}(P_2)$ *then V is contained in an $\mathcal{E}$-pre-image of itself with order bigger than zero,*

$$f^{\operatorname{ord}(P_2) - \operatorname{ord}(P_1)}(V) \subset V.$$

Proof. Lets suppose that $P_1 \neq P_2$ are $\mathcal{E}$-pre-images of V and set $\ell_j = \operatorname{ord}(P_j)$ for $j = 1, 2$. By Remark 2.1, $\ell_1 \neq \ell_2$. Thus, we may assume that $\ell_1 < \ell_2$. By Lemma 2.5 it follows that P_1 and P_2 are not linked.

Now, suppose in addition that $P_1 \subset P_2$. Then $V = f^{\ell_1}(P_1) \subset f^{\ell_1}(P_2)$ ($f^{\ell_1}(P_2)$ is an $\mathcal{E}$-pre-image of V) and this will imply that $f^{\ell_2 - \ell_1}(V) \subset f^{\ell_2}(P_2) = V$. $\square$

2.1. Constructing nested sets

In this section (Section 2.1) let $\mathcal{A}$ be a collection of connected open subsets of X such that the elements of $\mathcal{A}$ are not contained in any $\mathcal{E}$-pre-image of order bigger than zero of an element of $\mathcal{A}$.

A finite sequence $\mathcal{K} = (P_0, P_1, \dots, P_n)$ of $\mathcal{E}$-pre-images of elements of $\mathcal{A}$ is called a *chain of $\mathcal{E}$-pre-images of $\mathcal{A}$* beginning in $A \in \mathcal{A}$ (Fig. 1) if

(1) $0 < \operatorname{ord}(P_0) \leqslant \cdots \leqslant \operatorname{ord}(P_{n-1}) \leqslant \operatorname{ord}(P_n)$;
(2) A and P_0 are linked;

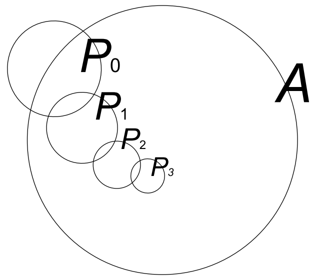

Fig. 1. A chain (P_0, P_1, P_2, P_3) of pre-images beginning in A.

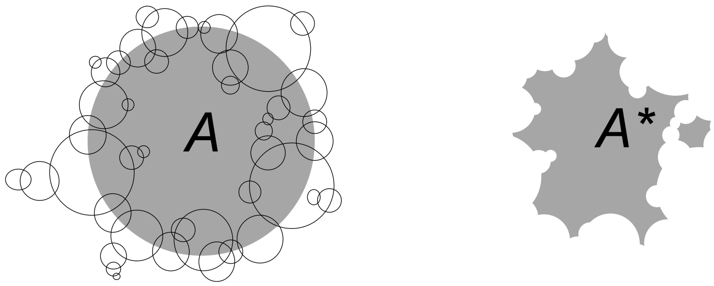

Fig. 2. On the left side it is shown a ball A (in grey) and the boundaries of the pre-images of A that belong to the chains. On the right side $A^\star$ is shown.

(3) P_{j-1} and P_j are linked $\forall 1 \leqslant j \leqslant n$;
(4) $P_i \neq P_j$ $\forall i \neq j$.

Denote by $ch_{\mathcal{E}}(A)$ the collection of all chain of pre-images of $\mathcal{A}$ beginning in $A \in \mathcal{A}$. As the elements of $\mathcal{A}$ are connected and open, it is easy to check the following remark.

Remark 2.7. If $(P_0, P_1, \dots, P_n) \in ch_{\mathcal{E}}(A)$, with $A \in \mathcal{A}$, then $\bigcup_{j=n_0}^{n_1} P_j$ is a connected open set $\forall 0 \leqslant n_0 \leqslant n_1 \leqslant n$.

For each $A \in \mathcal{A}$ define the open set

$$A^\star = A \setminus \overline{\bigcup_{(P_j)_j \in ch_{\mathcal{E}}(A)} \bigcup_j P_j}. \tag{4}$$

Proposition 2.8 *(An abstract construction of a nested collection). For each $A \in \mathcal{A}$ such that $A^\star \neq \emptyset$ choose a connected component A' of $A^\star$. If $\mathcal{A}' = \{A';\ A \in \mathcal{A} \text{ and } A^\star \neq \emptyset\}$ is not an empty collection then $\mathcal{A}'$ is an $\mathcal{E}$-nested collection of sets.* (*See Fig.* 2.)

Proof. Suppose that $\mathcal{A}' \neq \emptyset$. By contradiction, assume that there exist $A_1, A_2 \in \mathcal{A}$ and an $\mathcal{E}$-pre-image P of A_2', with $\operatorname{ord}(P) > 0$, such that A_1' and P are linked. So, as A_1' and P are connected sets, $\exists p \in P \cap \partial A_1'$. Let $\wp = \operatorname{ord}(P)$ and let $E \in \mathcal{E}_\wp$ be such that $P = f^{-E}(A_2')$. Setting $Q = f^{-E}(A_2)$, we get $P \subset Q$.

Claim. $Q \subset A_1$.

Proof. First note that $Q \cap A_1 \supset Q \cap A_1' \supset P \cap A_1' \neq \emptyset$. On the other hand, if $Q \cap \partial A_1 \neq \emptyset$, the unitary sequence (Q) will be a chain of $\mathcal{E}$-pre-images beginning in A_1, i.e., $(Q) \in ch_{\mathcal{E}}(A_1)$. But this is a contradiction to the definition of $A_1^\star$ because $Q \cap A_1^\star \supset Q \cap A_1' \neq \emptyset$. Thus, $Q \cap \partial A_1 = \emptyset$. As Q and A_1 are connected sets and $Q \cap A_1 \supset Q \neq \emptyset$, we get $Q \supset A_1$ or $Q \subset A_1$. The first option is not possible because (by hypothesis) the elements of $\mathcal{A}$ are not contained in any $\mathcal{E}$-pre-image of order bigger than zero of an element of $\mathcal{A}$. Therefore, $Q \subset A_1$. $\square$

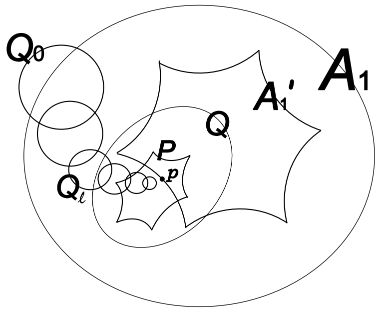

Fig. 3.

As $p \in \partial A_1'$, for a given $\varepsilon > 0$ there exists a chain $(Q_0, \dots, Q_n) \in ch_{\mathcal{E}}(A_1)$ such that $\mathrm{dist}(p, \bigcup_{j=0}^n Q_j) < \varepsilon$. (See Fig. 3.) On the other hand, as P and Q are open sets and $p \in P \subset Q$, taking ε small enough, $P \cap (\bigcup_{j=0}^n Q_j) \neq \emptyset$ and so,

$$Q_m \cap Q \supset Q_m \cap P \neq \emptyset, \tag{5}$$

for some $1 \leqslant m \leqslant n$. As $Q_0 \cup \dots \cup Q_m$ is a connected set (Remark 2.7) and $Q_0 \cap (X \setminus Q) \supset Q_0 \cap (X \setminus A_1) \neq \emptyset$ (because Q_0 and A_1 are linked), there exists $0 \leqslant j \leqslant m$ such that $Q_j \cap \partial Q \neq \emptyset$. So, setting $\ell = \min\{0 \leqslant j \leqslant m;\ Q_j \cap \partial Q \neq \emptyset\}$, it follows that Q_ℓ and Q are linked. Indeed, Q_ℓ cannot contains Q, otherwise $A' \cap (Q_0 \cup \dots \cup Q_n) \neq \emptyset$ and this contradicts $A_1' \subset A_1^\star$.

We have two cases, either $\mathrm{ord}(Q_\ell) \leqslant \mathrm{ord}(Q)$ or $\mathrm{ord}(Q_\ell) > \mathrm{ord}(Q)$. Suppose first that $\mathrm{ord}(Q_\ell) \leqslant \mathrm{ord}(Q)$. By the minimality of ℓ, $Q \neq Q_j$ $\forall 0 \leqslant j \leqslant \ell$. Thus, it is easy to check that $\mathcal{K} = (Q_0, \dots, Q_\ell, Q) \in ch_{\mathcal{E}}(A_1)$. As $Q \cap A_1^\star \supset Q \cap A_1' \neq \emptyset$, the existence of the chain $\mathcal{K}$ is a contradiction to (4) and so, this case cannot occur. For the second case ($\mathrm{ord}(Q_\ell) > \mathrm{ord}(Q)$), consider the sequence $\mathcal{K} = (f^{\wp}(Q_\ell), \dots, f^{\wp}(Q_m))$. It is also easy to check that $\mathcal{K} \in ch_{\mathcal{E}}(A_2)$ (note that, as $f^{\wp}(Q) = A_2$, $f^{\wp}(Q_\ell) \cap \partial A_2 = f^{\wp}(Q_\ell \cap \partial Q) \neq \emptyset$). But, as $f^{\wp}(P) = A_2' \subset A_2^\star$, it follows from (5) that $f^{\wp}(Q_m) \cap A_2^\star \supset f^{\wp}(Q_m \cap P) \neq \emptyset$, contradicting (4) again and concluding the proof. □

An easy way to assure the existence of nested sets (or collections) is to show that the chains have small diameter, where the diameter of a chain $(P_j)_j$ is defined as the diameter of $\bigcup_j P_j$.

Corollary 2.9. *Let $\varepsilon \in (0, 1/2)$ and let $A = B_r(p)$ be a connected open ball with radius r centered in $p \in X$ such that $f^n(A) \not\subset A$ $\forall n > 0$. If every chain of $\mathcal{E}$-pre-images of A has diameter smaller than $2\varepsilon r$ then the set $A^\star$, given by* (4) *contains the ball $B_{r(1-2\varepsilon)}(p)$. Moreover, the connected component A' of $A^\star$ that contains p is an $\mathcal{E}$-nested set containing $B_{r(1-2\varepsilon)}(p)$.*

Proof. Set $\mathcal{A} = \{A\}$. As $f^n(A) \not\subset A$ $\forall n > 0$, it follows that A is not contained in any $\mathcal{E}$-pre-image of itself (with order bigger than zero). Let Γ be the collection of all chains of $\mathcal{E}$-pre-images of A. If $(P_j)_j \in \Gamma$ then $\bigcup_j P_j$ is a connected open set intersecting ∂A with diameter smaller than $2\varepsilon r$. Thus, $\bigcup_j P_j \subset B_{2\varepsilon r}(\partial A)$, $\forall (P_j)_j \in \Gamma$. As a consequence, $A^\star = A \setminus \overline{\bigcup_{(P_j)_j \in \Gamma} \bigcup_j P_j} \supset A \setminus \overline{B_\varepsilon(\partial A)} \supset B_{r(1-2\varepsilon)}(p)$ is a non-empty open set. Taking A' as the connected component of $A^\star$ that contains p (and so, contains $B_{r(1-2\varepsilon)}(p)$), it follows from Proposition 2.8 that A' is an $\mathcal{E}$-nested set. □

3. Ergodic components

Before constructing the Markov partition using the adapted *nested sets*, we need also some preliminary knowledge of the so called ergodic components for non-(necessarily) invariant measures. This knowledge is important to assure good statistical properties for these nested sets with respect to the class of measures that we are working on.

Let μ be a finite measure defined on the Borel sets of the compact, separable metric space X and let $f : X \to X$ be a measurable map. A subset $U \subset X$ is called an invariant set (with respect to f) if $f^{-1}(U) = U$, and it is called a positively invariant set if $f(U) \subset U$.

Definition 3.1 *(Ergodic components).* An invariant set U with $\mu(U) > 0$ is called an *ergodic component* (indeed, a μ *ergodic component* with respect to f), if it does not admit any smaller invariant subset with positive measure, that is, if $V \subset U$ is invariant, $f^{-1}(V) = V$, then either $\mu(V)$ or $\mu(U \setminus V)$ is zero. The measure μ is called *ergodic* if X is an ergodic component.

We stress that in the definition of ergodic measure and ergodic components we are not assuming the invariance of the measure μ with respect to f. Let us give some examples of non-invariant ergodic measures.

Example 3.2. Given any $p \in X \setminus \mathrm{Fix}(f)$, the Dirac measure δ_p is ergodic and non-invariant ($\mathrm{Fix}(f)$ is the set of fixed points of f). More in general, given a finite subset $\mathcal{U} \subset \mathcal{O}_f^-(p)$ of the pre-orbit of a point $p \in X$, let $\mu = \frac{1}{\#\mathcal{U}} \sum_{q \in \mathcal{U}} \delta_q$. If $f^{-1}(\mathcal{U}) \neq \mathcal{U}$ then μ is an ergodic probability but not invariant.

Example 3.3. Given an ergodic (not necessarily invariant) measure μ, let $Y \subset X$ be such that $\mu(Y)\mu(f^{-1}(Y) \setminus Y) > 0$. Then $\mu|_Y$, the restriction of μ to Y, is non-invariant and ergodic.

Example 3.4. By Martens [35], the Lebesgue measure is ergodic and non-invariant for every non-flat S-unimodal map f without a periodic attractor. In particular when f is an infinitely renormalizable map the Lebesgue measure is ergodic but there is no absolutely continuous invariant measure (for multimodal maps, see Blokh and Lyubich [9,10] and van Strien and Vargas [55]).

Following Milnor's definition of attractor (indeed, minimal attractor [37]), a compact positively invariant set A will be called a μ-attractor, or for short, an attractor, if its basin of attraction $\mathcal{B}_f(A) = \{x \in X;\ \omega_f(x) \subset A\}$ has positive measure and, in contrast, the basin of every positively invariant compact subset $A' \subsetneqq A$ has zero measure. Here, $\omega_f(x)$ denotes the ω-limit set of $x \in X$.

A collection $\mathcal{P}$ of sets with positive measure is called a *partition mod* μ of $U \subset X$ if this collection covers U almost everywhere ($\mu(U \setminus \bigcup_{P \in \mathcal{P}} P) = 0$) and $\mu(P \cap Q) = 0$ for every $P, Q \in \mathcal{P}$ with $P \neq Q$. The *diameter* of a partition $\mathcal{P}$ is defined by $\mathrm{diameter}(\mathcal{P}) = \sup\{\mathrm{diameter}(P);\ P \in \mathcal{P}\}$.

Proposition 3.5 *(Ergodic attractors). Given an ergodic component $U \subset X$, there exists a unique attractor $A \subset X$ that attracts almost every point of U. Moreover, $\omega_f(x) = A$ for almost every point of U.*

Proof. Let $\mathcal{P}_1$ be any finite partition (mod $\mu|_U$) of X formed by open subsets and with $\mathrm{diameter}(\mathcal{P}_1) < 1$ and such that $\bigcup_{P \in \mathcal{P}_1} \overline{P} \supset X$ (see Lemma A.1). We will construct by induction a sequence of partitions $\mathcal{P}_1 < \mathcal{P}_2 < \cdots$ of X in the measure-theoretical sense. Thus, suppose that the collection $\mathcal{P}_{n-1}$ has already been constructed. Set, for each $P \in \mathcal{P}_{n-1}$, $U_P = \{x \in X;\ \omega_f(x) \cap \overline{P} \neq \emptyset\}$. As $\mu|_U$ is ergodic and $f^{-1}(U_P) = U_P$ (because $\omega_f(x) = \omega(f(x))\ \forall x$), either U_P or $X \setminus U_P$ is a zero measure set.

Given $P \in \mathcal{P}_{n-1}$, we define a partition $\mathbb{P}_P$ (mod $\mu|_U$) of P as follows. If $\mu(U_P) = 0$, we set $\mathbb{P}_P$ as the trivial refinement, i.e., $\mathbb{P}_P = \{P\}$. On the other hand, if $\mu(U_P) > 0$, we choose any $\mathbb{P}_P$ in the collection of finite partitions (mod $\mu|_U$) of $P \in \mathcal{P}_{n-1}$ formed by open subset of P with diameter smaller that $\frac{1}{2}\,\mathrm{diameter}(P)$ and $\bigcup_{Q \in \mathbb{P}_P} \overline{Q} = \overline{P}$. Now, define

$$\mathcal{P}_n = \{Q \in \mathbb{P}_P;\ P \in \mathcal{P}_{n-1}\}.$$

For each $n \in \mathbb{N}$, set $\mathcal{P}_n^* = \{P \in \mathcal{P}_n;\ U \setminus U_P \text{ is a zero measure set}\}$ and $K_n = \bigcup_{P \in \mathcal{P}_n^*} \overline{P}$. As $K_1 \supset K_2 \supset \cdots \supset K_n \supset \cdots$ is a nested sequence of non-empty compact sets, $A = \bigcap_n K_n$ is also a non-empty compact set. By construction, for almost every point $x \in U$ and $\forall n \in \mathbb{N}$, $\omega_f(x) \subset K_n$ and $\omega_f(x) \cap \overline{P}\ \forall P \in \mathcal{P}_n^*$. Moreover, as $\mathrm{diameter}(\overline{P}) < 2^{-n}\ \forall P \in \mathcal{P}_n^*$, it follows that $\sup\{\mathrm{dist}(y, \mathcal{O}_f(x));\ y \in A\} \leqslant 2^{-n}$ and $\omega_f(x) \subset K_n \subset \overline{B_{2^{-n}}(A)} = \{p \in X;\ \mathrm{dist}(p, A) \leqslant 2^{-n}\}$ for every $n \in \mathbb{N}$ and $\mu|_U$ almost every point x. Thus, $\omega_f(x) = A$ for μ-almost every point $x \in U$. $\square$

Consider for each point x of a positively invariant set $U \subset X$, a subset $\mathcal{U}(x) \subset \mathcal{O}^+(x)$ of the positive orbit of x.

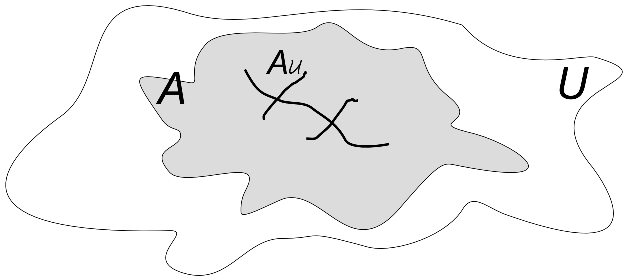

Fig. 4. U is an ergodic component with its attractor A and its omega-$\mathcal{U}$ set $A_{\mathcal{U}}$.

Definition 3.6. The collection $\mathcal{U} = (\mathcal{U}(x))_{x\in U}$ is called *asymptotically invariant* if for every $x \in U$,

(1) $\#\{j \in \mathbb{N};\ f^j(x) \in \mathcal{U}(x)\} = \infty$, and
(2) $\mathcal{U}(x) \cap \mathcal{O}^+(f^n(x)) = \mathcal{U}(f(x)) \cap \mathcal{O}^+(f^n(x))$ for every big $n \in \mathbb{N}$.

Definition 3.7 *($\omega_{f,\mathcal{U}}$).* Given an asymptotically invariant collection $\mathcal{U} = (\mathcal{U}(x))_{x\in U}$, define for each x the *omega-$\mathcal{U}$ limit set* of x (*omega-$\mathcal{U}$* of x, for short), denoted by $\omega_{f,\mathcal{U}}(x)$, as the set of accumulation points of $\mathcal{U}(x)$ and, that is, the set of points $p \in X$ such that there is a sequence $n_j \to +\infty$ satisfying $\mathcal{U}(x) \ni f^{n_j}(x) \to p$.

It is easy to check that $\omega_{\mathcal{U}}(x)$ is a non-empty compact set but not necessarily invariant.

We say that the asymptotically invariant collection $\mathcal{U} = (\mathcal{U}(x))_{x\in U}$ has *positive frequency* if $\limsup \frac{1}{n}\#\{1 \leqslant j \leqslant n;\ f^j(x) \in \mathcal{U}(x)\} > 0$, for every $x \in U$.

Definition 3.8 *($\omega_{+,f,\mathcal{U}}$).* If $\mathcal{U}$ is an asymptotically invariant collection with positive frequency, define $\omega_{+,f,\mathcal{U}}(x)$, the *set of $\mathcal{U}$-frequently visited points* of x orbit, as the set of points $p \in X$ such that $\limsup \frac{1}{n}\#\{1 \leqslant j \leqslant n;\ f^j(x) \in \mathcal{U}(x) \cap V\} > 0$ for every neighborhood V of p.

Lemma 3.9. *Let $\mathcal{U} = (\mathcal{U}(x))_{x\in U}$ be an asymptotically invariant collection defined in an ergodic component U and let $A \subset X$ be the attractor associated to U. There is a compact set $A_{\mathcal{U}} \subset A$ such that $\omega_{f,\mathcal{U}}(x) = A_{\mathcal{U}}$ for μ-almost every $x \in U$. Furthermore, if $\mathcal{U}$ has positive frequency then there is also a compact set $A_{+,\mathcal{U}} \subset A_{\mathcal{U}}$ such that $\omega_{+,f,\mathcal{U}}(x) = A_{+,\mathcal{U}}$ for μ-almost every $x \in U$.*

Proof. We construct the compact sets $A_{\mathcal{U}}$ and $A_{+,\mathcal{U}}$ in the same way we did for A in the proof of Proposition 3.5. For $A_{\mathcal{U}}$ the only difference is that we have to change $\omega_f(x)$ by $\omega_{f,\mathcal{U}}(x)$ in the proof. Note that the key property of $\omega_f(x)$ used there is that $\omega_f(x) = \omega_f(f(x))$ and we also have the same property for $\omega_{f,\mathcal{U}}$, i.e., $\omega_{f,\mathcal{U}}(x) = \omega_{f,\mathcal{U}}(f(x))$. (See Fig. 4.)

For $A_{+,\mathcal{U}}$ we have, of course, to change in the proof ω_f by $\omega_{+,f,\mathcal{U}}$ (again $\omega_{+,f,\mathcal{U}}(x) = \omega_{+,f,\mathcal{U}}(f(x))\ \forall x$) and we have also to change the definition of the set U_P. For this we proceed as follows. Given a point $x \in U$ and a set $K \subset X$ denote the $\mathcal{U}$-visit frequency of x to K by $\phi_K(x) = \limsup \frac{1}{n}\#\{0 \leqslant j < n;\ f^j(x) \in K \cap \mathcal{U}(x)\}$. Set, for each $P \in \mathcal{P}_{n-1}$, $U_P = \{x \in U;\ \phi_{\overline{P}}(x) > 0\}$. As we are using lim sup in the definition of ϕ_K, we get $\phi_K(x) > 0$ or $\phi_{X\setminus K}(x) > 0$. This is important to ensure that $K_n \neq \emptyset\ \forall n$ (see proof of Proposition 3.5).

To finish the proof, we remark that every point of $A_{+,\mathcal{U}} = \bigcap_n K_n$ is accumulated by the sequence $\{f^n(x);\ n \in \mathbb{N}$ and $f^n(x) \in \mathcal{U}(x)\}$ for almost every point $x \in U$ and so, $A_{+,\mathcal{U}}$ is contained in $A_{\mathcal{U}}$ which is contained in A. Moreover, if B is an open set with $B \cap A_{+,\mathcal{U}} \neq \emptyset$ then for any big n there will be some element P of $\mathcal{P}_n^*$ such that $\overline{P} \subset B$. Therefore, by construction, $\limsup \frac{1}{n}\#\{0 \leqslant j < n;\ f^j(x) \in B \cap \mathcal{U}(x)\} \geqslant \limsup \frac{1}{n}\#\{0 \leqslant j < n;\ f^j(x) \in \overline{P} \cap \mathcal{U}(x)\} > 0$. $\square$

As defined in Section 5, a measure μ is f-non-singular if the pre-image by f of any set with zero measure has also zero measure ($f_*\mu \ll \mu$). The ergodic measures that appears in Examples 3.2 and 3.3 are not in general f-non-singular. The lemma below gives a way to construct new f-non-singular ergodic measures.

Lemma 3.10. *If μ is an f-non-singular ergodic measure (not necessarily invariant) then $\frac{1}{\mu(E)}\mu|_E$ is an f-non-singular ergodic probability whenever $E \subset X$ is a positively invariant Borel set with positive measure (i.e., $f(E) \subset E$ and $\mu(E) > 0$).*

Proof. As in Example 3.3, $\frac{1}{\mu(E)}\mu|_E$ is an ergodic probability. We need only to show that this probability is f-non-singular. Given $Y \subset X$, we have $\mu(f^{-1}(Y) \cap E) \leqslant \mu(f^{-1}(Y) \cap f^{-1}(f(E))) = \mu(f^{-1}(Y \cap f(E)))$. Thus, if $\mu|_E(Y) = \mu(Y \cap E) = 0$ then $0 \leqslant \mu|_E(f^{-1}(Y)) \leqslant \mu(f^{-1}(Y \cap f(E))) = 0$ (because μ is f-non-singular). As a consequence, $\frac{1}{\mu(E)}\mu|_E$ is f-non-singular. □

Lemma 3.11. *Let μ be a finite measure. If there exists some $\delta > 0$ such that every invariant set has μ measure either zero or bigger than δ, then X can be decomposed into a finite number of μ ergodic components.*

Proof. Let $W_1 \subset X$ be any invariant subset of X (for example, $W_1 = X$) with non-zero μ measure and let $\mathcal{F}(W_1)$ be the collection of all invariant subsets $U \subset W_1$ with μ measure bigger than zero. Note that $\mathcal{F}(W_1)$ is non-empty, because $W_1 \in \mathcal{F}(W_1)$. Let us consider the inclusion (mod μ) as a partial order on $\mathcal{F}(W_1)$.

Claim. *Every totally ordered subset $\Gamma \subset \mathcal{F}(W_1)$ is finite. In particular, it has an upper bound.*

Proof. Otherwise there is an infinite sequence $\gamma_0 \supset \gamma_1 \supset \gamma_3 \supset \cdots$ with $\mu(\gamma_k \setminus \gamma_{k+1}) > 0$ $\forall k$. But as $\sum_k \mu(\gamma_k \setminus \gamma_{k+1}) = \mu(\gamma_0) < \infty$, it follows that $\mu(\gamma_k \setminus \gamma_{k+1}) < \delta$ for k big and this contradicts our hypothesis as every $\gamma_k \setminus \gamma_{k+1}$ is an invariant set. □

From Zorn's lemma, there exists a maximal element $U_1 \in \mathcal{F}(W_1)$ and this is necessarily an ergodic component.

As $W_2 = X \setminus U_1$ is an invariant set, either it has zero μ measure or we can use the argument above to W_2 and obtain a new ergodic component U_2 inside $X \setminus U_1$. Inductively, we can construct a collection of ergodic components $U_1, \dots, U_r$ while $\mu(X \setminus U_1 \cup \cdots \cup U_r) > 0$. But, as $\mu(U_j) > \delta$, this processes will stop and we will get the decomposition of X into μ ergodic components as desired. □

Proposition 3.12 *(A criterion for ergodicity). Let μ be an f-non-singular finite measure. If there exists some $\delta > 0$ such that every positively invariant set has μ measure either zero or bigger than δ, then X can be decomposed into a finite number of μ ergodic components. Moreover, the attractor associated to each ergodic component has positive μ measure.*

Proof. As every invariant set is positively invariant, it follows from Lemma 3.11 that X can be decomposed into a finite number of μ ergodic components.

From Proposition 3.5 each ergodic component U of X is the basin of some attractor A. Let us, for instance, suppose that $\mu(A) = 0$. In this case, one can choose an open neighborhood V of A such that $\mu(V) < \delta$ and an integer n_0 such that $\mu(U') > 0$, where $U' = \{x \in U\ ;\ f^n(x) \in V\ \forall n \geqslant n_0\}$. Note that $\mu(f^{n_0}(U')) > 0$ because μ is f-non-singular. As U' is positively invariant, $f^{n_0}(U')$ is a positively invariant set with $0 < \mu(f^{n_0}(U')) < \mu(V) < \delta$, but this is impossible by ours hypothesis. So, $\mu(A) > 0$ (indeed, $\mu(A) > \delta$). □

We end this section relating the number of μ ergodic components with respect to f to the number of μ ergodic components with respect to f^k.

Lemma 3.13. *Let μ be an f-non-singular finite measure. If U is an ergodic component with respect to f then U can be partitioned in at most k ergodic components with respect to f^k. Furthermore, if $U_1, U_2 \subset U$ are ergodic components with respect to f^k then $U_2 = f^{-j}(U_1)$ (mod μ) for some $0 \leqslant j < k$.*

Proof. First we will prove by induction that U can be partitioned (mod μ) in a finite number of ergodic components with respect to f^k. Of course this claim is true for $k = 1$. Thus, suppose by induction that for every $1 \leqslant j \leqslant k - 1$ we can decompose U (mod μ) in a finite number of ergodic components with respect to f^j.

If U is ergodic with respect to f^k there is nothing to prove. Thus, we may assume that there is an invariant set $Y \subset U$ (that is, $f^{-k}(Y) = Y$) with $0 < \mu(Y) < \mu(U)$.

Let $\{j_1, \dots, j_s\}$ be a maximal subset of $\{1, \dots, k\}$ (with respect to the inclusion) such that $\mu(Y \cap f^{-j_1}(Y) \cap \dots \cap f^{-j_s}(Y)) > 0$. Set $Y_1 = Y \cap f^{-j_1}(Y) \cap \dots \cap f^{-j_s}(Y)$. Note that $f^{-k}(Y_1) = Y_1$. Furthermore, by maximality, if $\mu(f^{-\ell}(Y_1) \cap Y_1) > 0$ then $f^{-\ell}(Y_1) = Y_1$ (mod μ). Let $a_1 = \min\{1 \leqslant \ell \leqslant k;\ f^{-\ell}(Y_1) = Y_1\}$. Of course, $f^{-1}(\bigcup_{j=0}^{a_1-1} f^{-j}(Y_1)) = \bigcup_{j=0}^{a_1-1} f^{-j}(Y_1)$ (mod μ). As U is ergodic component for f, we get

$$U = \bigcup_{j=0}^{a_1-1} f^{-j}(Y_1) \text{ (mod } \mu).$$

Claim. *Y_1 is an ergodic component for f^{a_1}.*

Proof of Claim. Suppose that $Y_1' \subset Y_1$ is f^{a_1} invariant and $\mu(Y_1 \setminus Y_1') > 0$. As $f^{-a_1}(Y_1 \setminus Y_1') = Y_1 \setminus Y_1'$, we get $f^{-1}(\bigcup_{j=0}^{a_1-1} f^{-j}(Y_1 \setminus Y_1')) = \bigcup_{j=0}^{a_1-1} f^{-j}(Y_1 \setminus Y_1')$ and, as $\widetilde{U}$ is ergodic component for f, $\widetilde{U} = \bigcup_{j=0}^{a_1-1} f^{-j}(Y_1 \setminus Y_1')$ (mod μ). Thus

$$\sum_{j=0}^{a_1-1} \mu\big(f^{-j}(Y_1)\big) = \mu(\widetilde{U}) = \sum_{j=0}^{a_1-1} \mu\big(f^{-j}\big(Y_1 \setminus Y_1'\big)\big), \tag{6}$$

because $\mu(f^{-i}(Y_1) \cap f^{-j}(Y_1)) = 0$ $\forall 0 \leqslant i < j \leqslant a_1 - 1$ (here we are using that μ is f-non-singular). As $\mu(f^{-j}(Y_1)) \geqslant \mu(f^{-j}(Y_1 \setminus Y_1'))$ $\forall j$, it follows from (6) that $\mu(f^{-j}(Y_1)) = \mu(f^{-j}(Y_1 \setminus Y_1'))$ $\forall j$ and so, $\mu(Y_1') = 0$. $\square$

Denote by $\mathcal{U}$ the collection of all ergodic component $\widetilde{U} \subset U$ with respect to some iterate f^j, $j = 1, \dots, k-1$. By induction $\mathcal{U}$ is finite and so, $\delta = \min\{\mu(\widetilde{U});\ U \in \mathcal{U}\} > 0$.

From the claim above follows that if U is not an ergodic component with respect to f^k then every f^k-invariant set $Y \subset U$ with $0 < \mu(Y) < \mu(U)$ contains some element of $\mathcal{U}$. Thus, every positively invariant subset of U has μ measure either zero or bigger than δ. Applying Lemma 3.11 to μ (indeed to $\widetilde{\mu} = \mu|_U$), it follows that U can be decomposed into a finite number of μ ergodic components with respect to f^k.

To finish the proof of the lemma, let $W \subset U$ be an ergodic component with respect to f^k. As $f^{-k}(W) = W$, $f^{-1}(\bigcup_{j=0}^{k-1} f^{-j}(W)) = \bigcup_{j=0}^{k-1} f^{-j}(W)$. Thus, by the ergodicity of U, $U = \bigcup_{j=0}^{k-1} f^{-j}(W)$ (mod μ). Note that, if $\widetilde{W} \subset U$ is an ergodic component with respect to f^k and $\mu(\widetilde{W} \cap f^{-j}(W)) > 0$, then $\widetilde{W} = f^{-j}(W)$ (mod μ), because $f^{-k}(\widetilde{W} \cap f^{-j}(W)) = \widetilde{W} \cap f^{-j}(W)$ and $\widetilde{W}$ is ergodic with respect to f^k. As $U = \bigcup_{j=0}^{k-1} f^{-j}(W)$ (mod μ), we can conclude that any ergodic component $\widetilde{W} \subset U$ with respect to f^k is (mod μ) an element of $\{W, f^{-1}(W), \dots, f^{-(k-1)}(W)\}$. $\square$

4. Characterizing the liftable measures

In this section we obtain a statistical characterization of the liftable measures for a given induced map (see Corollary 4.6). Differently of Zweimüller's results [66], this characterization is given by a statistical condition, condition (7), not by the integrability of the induced time with respect to the reference measure (the one that we want to lift). This is important to avoid an additional condition of integrability of the induced time with respect to the reference measure (in our context this is not a natural condition).

Let X be a compact separable metric space and $f : X \to X$ a measurable map defined on X.

Definition 4.1 *(Markov map compatible with a measure).* We say that a Markov map $(F, \mathcal{P})$ defined on an open set $Y \subset X$ is *compatible* with a measure μ if

(1) $\mu(Y) > 0$;
(2) μ is F-non-singular;
(3) $\mu(\bigcup_{P \in \mathcal{P}} P) = \mu(Y)$ (in particular, $\mu(\partial P) = 0$ $\forall P \in \mathcal{P}$).

We say that a measure μ has a *Jacobian with respect to the map* $f : X \to X$ if there is a function $J_\mu f \in L^1(\mu)$ such that

$$\mu\big(f(A)\big) = \int_A J_\mu f \, d\mu$$

for every measurable set A such that $f|_A$ is injective. When the Jacobian exists, it is essentially unique. In general, the Jacobian may not exist, but if, for instance, μ is an f-invariant measure and f is a countable to one map then the Jacobian of μ with respect to f is well defined (see [40]).

Definition 4.2 *(Markov map with μ-bounded distortion).* We say that a Markov map $(F, \mathcal{P})$ defined on an open set $Y \subset X$ has *bounded distortion with respect to a measure* μ (for short, has *μ-bounded distortion*) if $(F, \mathcal{P})$ *compatible* with μ, μ has a Jacobian with respect to F and $\exists K > 0$ such that

$$\left|\log \frac{J_\mu F(x)}{J_\mu F(y)}\right| \leqslant K \operatorname{dist}\big(F(x), F(y)\big),$$

for μ almost every $x, y \in P$ and for all $P \in \mathcal{P}$.

The remark below is a well-known fact about projections of invariant measures of induced maps, see for instance Lemma 3.1 in Chapter V of [36].

Remark 4.3. Let $(F, \mathcal{P})$ be an induced Markov map for f defined on some $Y \subset X$ and let R be its induced time. If ν is an F-invariant finite measure such that $\int R \, d\nu < \infty$ then

$$\eta = \sum_{P \in \mathcal{P}} \sum_{j=0}^{R(P)-1} f_*^j(\nu|_P) \quad \left(= \sum_{j=0}^{+\infty} f_*^j(\nu|_{\{R>j\}})\right)$$

is an f-invariant finite measure.

Note that, if $(F, \mathcal{P})$ is compatible with a measure μ, the σ-algebra generated by $\{F^{-n}(P);\ P \in \mathcal{P} \text{ and } n \geqslant 0\}$ is equal to the Borel sets of U (mod μ). Thus, using for example Lemma 4.4.1 of [1], it is easy to obtain the following result.

Proposition 4.4 *(Folklore Theorem). Let μ be f-non-singular measure. If $(F, \mathcal{P})$ is an induced full Markov map for f with μ-bounded distortion then there exists an ergodic F invariant probability $\nu \ll \mu$ whose density belongs to $L^\infty(\mu)$. Indeed, $\log \frac{d\nu}{d\mu} \in L^\infty(\mu|_{\{\frac{d\nu}{d\mu}>0\}})$.*

Moreover, if the inducing time R of F is ν-integrable, then $\eta = \sum_{P \in \mathcal{P}} \sum_{j=0}^{R(P)-1} f_*^j(\nu|_P)$ is a μ absolutely continuous ergodic f-invariant finite measure.

In Theorem 1 we obtain an absolutely continuous F-invariant measure ν replacing the condition of bounded distortion (that appears in Proposition 4.4) by μ being f-invariant and the statistical condition (7). Furthermore, this statistical condition assures that projecting ν by the dynamics of f we recover μ. That is, every invariant measure satisfying (7) can be lifted (indeed this is necessary and sufficient condition, see Corollary 4.6).

Theorem 1. *Let $(F, \mathcal{P})$ be an induced full Markov map for f defined on an open set $B \subset X$. Let R be the inducing time of F and μ be an ergodic f-invariant probability such that $\mu(\{R = 0\}) = 0$ and $\mathcal{O}_f^+(x) \cap \mathcal{O}_f^+(y) \neq \emptyset \Rightarrow \mathcal{O}_F^+(x) \cap \mathcal{O}_F^+(y) \neq \emptyset$ for μ almost every $x, y \in B$. If there exists $\Theta > 0$ such that*

$$\limsup \frac{1}{n} \#\big\{0 \leqslant j < n;\ f^j(x) \in \mathcal{O}_F^+(x)\big\} \geqslant \Theta \tag{7}$$

for μ almost every $x \in B$ then there is a non-trivial ($\not\equiv 0$) finite F-invariant measure ν such that $\nu(Y) \leqslant \mu(Y)$ for all Borel set $Y \subset B$ and such that $\int R \, d\nu \leqslant \Theta^{-1}$.

Proof. Let $\mathfrak{B} = \{x \in B;\ F^j(x) \in \bigcup_{P\in\mathcal{P}} P\ \forall j \geqslant 0\}$. Of course, $\mathfrak{B}$ is a metric space with the distance of X and $\mathfrak{B} = B$ (μ mod).

Let $\mathcal{W}$ be a collection of subsets of $\mathfrak{B}$ formed by the empty set $\emptyset$ and all $Y \subset \mathfrak{B}$ such that $Y = (F|_{P_1})^{-1} \circ \cdots \circ (F|_{P_s})^{-1}(\mathfrak{B})$ for some sequence of $P_1, \dots, P_s \in \mathcal{P}$. That is, the elements of $\mathcal{W}$ are the empty set and all homeomorphic F pre-image of $\mathfrak{B}$. Note that $\mathcal{W}$ is a collection of open sets of $\mathfrak{B}$. Given $Y \subset \mathfrak{B}$ and $r > 0$, let $\mathcal{W}(r, Y)$ be the set of all countable covers $\{I_i\}$ of Y by elements of $\mathcal{W}$ with diameter$(I_i) \leqslant r\ \forall i$. It is clear that $\mathcal{W}(r, Y) \neq \emptyset\ \forall Y \subset \mathfrak{B}$ and $\forall r > 0$.

Given a Borel set $Y \subset \mathfrak{B}$, let $\tau(Y) \in [0, 1]$ be such that

$$\tau(Y) = \limsup_{n\to+\infty} \frac{1}{n}\bigl\{0 \leqslant j < n;\ f^j(x) \in Y \cap \mathcal{O}_F^+\bigl(f^{\pi(x)}(x)\bigr)\bigr\},$$

for μ almost every $x \in X$, where $\pi(x) = \min\{i \geqslant 0;\ f^i(x) \in B\}$. As μ is ergodic and $\mathcal{O}_f^+(x) \cap \mathcal{O}_f^+(y) \neq \emptyset \Rightarrow \mathcal{O}_F^+(x) \cap \mathcal{O}_F^+(y) \neq \emptyset$, for μ almost every $x, y \in B$, it follows that $\tau(Y)$ is well defined.

Claim 1. *The function τ has the following properties.*

(1) $\tau(\emptyset) = 0$;
(2) $\tau(\mathfrak{B}) \geqslant \Theta > 0$;
(3) $\tau(Y_1) \leqslant \tau(Y_2)$ *whenever $Y_1 \subset Y_2$ are Borel subsets of* $\mathfrak{B}$;
(4) $\tau(\bigcup_{i=1}^{\infty} Y_i) \leqslant \sum_{i=1}^{\infty} \tau(Y_i)\ \forall Y_1, Y_2, Y_3, \dots$ *Borel subsets of* $\mathfrak{B}$;
(5) $\tau(Y) \leqslant \mu(Y)$ *for all Borel set* $Y \subset \mathfrak{B}$;
(6) $\tau(F^{-1}(Y)) = \tau(Y)$ *for all Borel set* $Y \subset \mathfrak{B}$.

Proof of Claim 1. The first four items follows from (7) and the definition of τ. From Birkhoff Theorem follows the fifth item. Indeed, $\mu(Y) = \lim \frac{1}{n}\#\{0 \leqslant j < n;\ f^j(x) \in Y\}$ for every Borel set $Y \subset \mathfrak{B}$ and μ almost every x. Thus, $\tau(Y) \leqslant \mu(Y)$ for every Borel set $Y \subset \mathfrak{B}$. To check the last item considers a Borel set $Y \subset \mathfrak{B}$. As $F^j(x) \in Y \Leftrightarrow F^{j-1}(x) \in F^{-1}(Y)\ \forall j \geqslant 1$ and $\forall x \in \mathfrak{B}$, we get $\tau(F^{-1}(Y)) = \tau(Y)$. $\square$

Following the definition of pre-measure of Rogers [47], τ restricted to $\mathcal{W}$ is a pre-measure (Definition 5 of [47]). Given $Y \subset \mathfrak{B}$, define

$$\nu(Y) = \sup_{r>0} \nu_r(Y) \quad \Bigl(= \lim_{r\searrow 0} \nu_r(Y)\Bigr),$$

where $\nu_r(Y) = \inf_{\mathcal{I}\in\mathcal{W}(r,Y)} \sum_{I\in\mathcal{I}} \tau(I)$ and $\mathcal{W}(r, Y)$ is the set of all countable covers $\mathcal{I} = \{I_i\}$ of Y by elements of $\mathcal{W}$ with diameter$(I_i) \leqslant r\ \forall i$. The function ν, defined on the class of all subset of $\mathfrak{B}$, is called in [47] the *metric measure constructed from the pre-measured τ by Method II* (Theorem 15 of [47]).

As $(F, \mathcal{P})$ is a Markov map,

$$\text{either } I_1 \subset I_2 \text{ or } I_2 \subset I_1 \quad \text{or} \quad I_1 \cap I_2 = \emptyset, \quad \forall I_1, I_2 \in \mathcal{W}. \tag{8}$$

Thus,

$$\nu_r(Y) = \inf_{\mathcal{I}\in\widetilde{\mathcal{W}}(r,Y)} \sum_{I\in\mathcal{I}} \tau(I),$$

where $\widetilde{\mathcal{W}}(r, Y) = \{\{I_i\} \in \mathcal{W}(r, Y);\ I_i \cap I_j = \emptyset\ \forall i \neq j\}$.

Claim 2. $\nu(Y) \leqslant \mu(Y)$ *for every Borel set* $Y \subset \mathfrak{B}$.

Proof of Claim 2. Let $Y \subset \mathfrak{B}$. As we are working only with countable additive measures defined on the Borel sets (see Section 1.1), μ is a regular measure. So, $\mu(Y) = \inf_{\mathcal{I}\in\widetilde{\mathcal{W}}(r,Y)} \mu(\bigcup_{I\in\mathcal{I}} I) = \inf_{\mathcal{I}\in\widetilde{\mathcal{W}}(r,Y)} \sum_{I\in\mathcal{I}} \mu(I) \geqslant \inf_{\mathcal{I}\in\widetilde{\mathcal{W}}(r,Y)} \sum_{I\in\mathcal{I}} \tau(I) = \nu_r(Y)$ for every $r > 0$. Thus $\nu(Y) \leqslant \mu(Y)$. $\square$

It follows from Claims 1 and 2 that ν restricted to the Borel subsets $\mathfrak{B}$ is finite and non-trivial, i.e., $\nu \not\equiv 0$. Indeed, $\nu(\emptyset) = 0 < \theta \leqslant \tau(\mathfrak{B}) \leqslant \nu(\mathfrak{B}) \leqslant \mu(B)$. Therefore, Theorems 19 and 3 of [47] assures that ν restrict to the Borel subsets of $\mathfrak{B}$ is a countable additive measure.

Before we show that ν is F-invariant (Claim 3) let us introduce some notation.

Notation 4.5. Let Y being Borel subset of $\mathfrak{B}$ and $r, r' > 0$.

- Given $\mathcal{I} \in \widetilde{\mathcal{W}}(r, Y)$, set $F^*\mathcal{I} = \{(F|_P)^{-1}(I)\}_{I \in \mathcal{I}, P \in \mathcal{P}}$.
- Given $\mathcal{I} \in \widetilde{\mathcal{W}}(r, Y)$ and $x \in \mathfrak{B}$, let $\mathcal{I}(x)$ be the element of $\mathcal{I}$ that contains x, if $x \in \sum_{I \in \mathcal{I}} I$. Otherwise, $\mathcal{I}(x) = \emptyset$.
- Given $\mathcal{I} \in \widetilde{\mathcal{W}}(r, Y)$ and $\mathcal{I}' \in \widetilde{\mathcal{W}}(r', Y)$, define $\mathcal{I} \cap_Y \mathcal{I}' = \{\mathcal{I}(x) \cap \mathcal{I}'(x);\ x \in Y\}$.

Given $\mathcal{I} \in \widetilde{\mathcal{W}}(r, Y)$ and $\mathcal{I}' \in \widetilde{\mathcal{W}}(r', Y)$, follows easily from (8) that

$$\mathcal{I} \cap_Y \mathcal{I}' \in \widetilde{\mathcal{W}}\big(\min\{r, r'\}, Y\big). \tag{9}$$

Furthermore, (8) and Claim 1 give

$$\sum_{I \in \mathcal{I} \cap_Y \mathcal{I}'} \tau(I) \leqslant \min\left\{\sum_{I \in \mathcal{I}} \tau(I), \sum_{I \in \mathcal{I}'} \tau(I)\right\}. \tag{10}$$

Claim 3. *ν is F-invariant.*

Proof of Claim 3. Let Y be a Borel subset of $\mathfrak{B}$. Let be a sequence of $a_1 > a_2 > \cdots > a_j \searrow 0$ a sequence of positive real numbers and $\mathcal{I}_1, \mathcal{I}_2, \ldots$ a sequence of covers of Y by elements of $\mathcal{W}$ satisfying the following properties.

(P1) $\nu_{a_j}(Y) \leqslant \nu_{a_{j+1}}(Y)\ \forall j \geqslant 1$;
(P2) $\nu_{a_j}(Y) \leqslant \nu(Y) < \nu_{a_j}(Y) + (1/j)\ \forall j \geqslant 1$;
(P3) $\mathcal{I}_j \in \widetilde{\mathcal{W}}(a_j, Y)\ \forall j \geqslant 1$.

Given $\varepsilon > 0$, let $\delta > 0$ be such that

$$\mu\left(\bigcup_{P \in \mathcal{P}_1} P\right) < \frac{\varepsilon}{6}, \tag{11}$$

where $\mathcal{P}_1 = \{P \in \mathcal{P};\ \mu(P) < \delta\}$.

Set $\mathcal{P}_0 = \{P \in \mathcal{P};\ \mu(P) \geqslant \delta\}$. Of course, $n_0 := \#\mathcal{P}_0 < \infty$. For each $P \in \mathcal{P}_0$, let $0 < b_p <$ diameter(P) be such that

$$\nu_{b_P}\big((F|_P)^{-1}(Y)\big) \leqslant \nu\big((F|_P)^{-1}(Y)\big) \leqslant \nu_{b_P}\big((F|_P)^{-1}(Y)\big) + \frac{\varepsilon}{6n_0}, \tag{12}$$

and let $\mathcal{J}_P \in \widetilde{\mathcal{W}}(b_P, (F|_P)^{-1}(Y))$ such that

$$\nu_{b_p}\big((F|_P)^{-1}(Y)\big) \leqslant \sum_{J \in \mathcal{J}_P} \tau(J) \leqslant \nu_{b_P}\big((F|_P)^{-1}(Y)\big) + \frac{\varepsilon}{6n_0}. \tag{13}$$

As $b_P <$ diameter(P), it follows from (8) that $J \subset (F|_P)^{-1}(\mathfrak{B}) = P \cap \mathfrak{B}\ \forall J \in \mathcal{J}_P$. Thus, for every $P \in \mathcal{P}_0$ we have

$$(F|_P)^{-1}(Y) \subset \bigcup_{J \in \mathcal{J}_P} J \subset P.$$

As $\bigcup_{J \in \mathcal{J}_P} J \subset P$ and $F|_P$ is a homeomorphism, it follows that $\{F(J)\}_{J \in \mathcal{J}_P} \in \widetilde{\mathcal{W}}(r_B, Y)\ \forall P \in \mathcal{P}_0$, where $r_B =$ diameter(B). So, by (9),

$$\mathcal{J}_0 := \bigcap_{P \in \mathcal{P}_0}^{Y} \{F(J)\}_{J \in \mathcal{J}_P} \in \widetilde{\mathcal{W}}(r_B, Y)$$

and

$$\mathcal{I}_j \cap_Y \mathcal{J}_0 \in \widetilde{\mathcal{W}}(a_j, Y) \quad \text{for every } j \geqslant 1.$$

Given $P \in \mathcal{P}_0$, note that

$$\big\{(F|_P)^{-1}(I)\big\}_{I \in \mathcal{I}_j \cap_Y \mathcal{J}_0} = \mathcal{K}_1 \cap_{Y_P} \mathcal{J}_P \cap_{Y_P} \mathcal{K}_2, \tag{14}$$

where $Y_P = (F|_P)^{-1}(Y)$, $\mathcal{K}_1 = \{(F|_P)^{-1}(I)\}_{I \in \mathcal{I}_j}$ and

$$\mathcal{K}_2 = \bigcap_{\substack{Y_P \\ P \neq Q \in \mathcal{P}_0}} \big\{(F|_P)^{-1}\big(F(I)\big)\big\}_{I \in \mathcal{J}_Q}.$$

It follows from (9) and (14) that

$$\big\{(F|_P)^{-1}(I)\big\}_{I \in \mathcal{I}_j \cap_Y \mathcal{J}_0} \in \widetilde{\mathcal{W}}\big(b_P, (F|_P)^{-1}(Y)\big), \quad \forall P \in \mathcal{P}_0. \tag{15}$$

Furthermore, by (10) and (14) we get

$$\sum_{I \in \{(F|_P)^{-1}(I)\}_{I \in \mathcal{I}_j \cap_Y \mathcal{J}_0}} \tau(I) \leqslant \sum_{I \in \mathcal{J}_P} \tau(I), \quad \forall P \in \mathcal{P}_0. \tag{16}$$

Using the definition of ν_{b_p}, (15), (16) and (13), we obtain for all $P \in \mathcal{P}_0$ that

$$\begin{aligned}\nu_{b_P}\big((F|_P)^{-1}(Y)\big) &\leqslant \sum_{I \in \{(F|_P)^{-1}(I)\}_{I \in \mathcal{I}_j \cap_Y \mathcal{J}_0}} \tau(I) \\ &= \sum_{I \in \mathcal{I}_j \cap_Y \mathcal{J}_0} \tau\big((F|_P)^{-1}(I)\big) \leqslant \sum_{I \in \mathcal{J}_P} \tau(I) \leqslant \nu_{b_P}\big((F|_P)^{-1}(Y)\big) + \frac{\varepsilon}{6n_0}.\end{aligned} \tag{17}$$

Therefore

$$\begin{aligned}&\Big|\nu\big(F^{-1}(Y)\big) - \sum_{I \in F^*(\mathcal{I}_j \bigcap_Y \mathcal{J}_0)} \tau(I)\Big| \\ &\quad = \Big|\sum_{P \in \mathcal{P}} \nu\big((F|_P)^{-1}(Y)\big) - \sum_{P \in \mathcal{P}} \sum_{I \in \mathcal{I}_j \cap_Y \mathcal{J}_0} \tau\big((F|_P)^{-1}(I)\big)\Big| \\ &\quad \leqslant \sum_{P \in \mathcal{P}_0} \Big|\nu\big((F|_P)^{-1}(Y)\big) - \sum_{I \in \mathcal{I}_j \cap_Y \mathcal{J}_0} \tau\big((F|_P)^{-1}(I)\big)\Big| \\ &\qquad + \sum_{P \in \mathcal{P}_1} \nu\big((F|_P)^{-1}(Y)\big) + \sum_{P \in \mathcal{P}_1} \sum_{I \in \mathcal{I}_j \cap_Y \mathcal{J}_0} \tau\big((F|_P)^{-1}(I)\big) \\ &\quad < \sum_{P \in \mathcal{P}_0} \Big|\nu\big((F|_P)^{-1}(Y)\big) - \sum_{I \in \mathcal{I}_j \cap_Y \mathcal{J}_0} \tau\big((F|_P)^{-1}(I)\big)\Big| \\ &\qquad + \underbrace{\sum_{P \in \mathcal{P}_1} \mu\big((F|_P)^{-1}(Y)\big)}_{*} + \underbrace{\sum_{P \in \mathcal{P}_1} \sum_{I \in \mathcal{I}_j \cap_Y \mathcal{J}_0} \mu\big((F|_P)^{-1}(I)\big)}_{**}.\end{aligned}$$

As $* \leqslant \mu(\bigcup_{P \in \mathcal{P}_1} P)$ and also $** \leqslant \sum_{P \in \mathcal{P}_1} \mu((F|_P)^{-1}(\bigcup_{I \in \mathcal{I}_j \cap_Y \mathcal{J}_0} I)) \leqslant \sum_{P \in \mathcal{P}_1} \mu(P) = \mu(\bigcup_{P \in \mathcal{P}_1} P)$, it follows from (11) that

$$\Big|\nu\big(F^{-1}(Y)\big) - \sum_{I \in F^*(\mathcal{I}_j \cap_Y \mathcal{J}_0)} \tau(I)\Big| < \sum_{P \in \mathcal{P}_0} \underbrace{|\nu\big((F|_P)^{-1}(Y)\big) - \sum_{I \in \mathcal{I}_j \cap_Y \mathcal{J}_0} \tau\big((F|_P)^{-1}(I)\big)|}_{***} + \varepsilon/3.$$

By (12) and (17),

$$***\leqslant \left|\nu\big((F|_P)^{-1}(Y)\big)-\nu_{b_P}\big((F|_P)^{-1}(Y)\big)\right|+\left|\nu_{b_P}\big((F|_P)^{-1}(Y)\big)-\sum_{I\in\mathcal{I}_j\cap_Y\mathcal{J}_0}\tau\big((F|_P)^{-1}(I)\big)\right|<\frac{\varepsilon}{3n_0}.$$

Therefore,

$$\left|\nu\big(F^{-1}(Y)\big)-\sum_{I\in F^*(\mathcal{I}_j\cap_Y\mathcal{J}_0)}\tau(I)\right|<\frac{2}{3}\varepsilon. \tag{18}$$

Let $j>3/\varepsilon$. Using the properties of τ (Claim 1), the fact that $\mathcal{I}_j\cap_Y\mathcal{J}_0\in\widetilde{\mathcal{W}}(a_j,Y)$, (P2) and (18), we get

$$\begin{aligned}\nu(Y)&<\frac{1}{j}+\nu_{a_j}(Y)\leqslant\frac{1}{j}+\sum_{I\in\mathcal{I}_j\cap_Y\mathcal{J}_0}\tau(I)\\&=\frac{1}{j}+\sum_{I\in\mathcal{I}_j\cap_Y\mathcal{J}_0}\tau\big(F^{-1}(I)\big)=\frac{1}{j}+\sum_{I\in\mathcal{I}_j\cap_Y\mathcal{J}_0}\tau\Big(\sum_{P\in\mathcal{P}}(F|_P)^{-1}(I)\Big)\\&\leqslant\frac{1}{j}+\sum_{I\in\mathcal{I}_j\cap_Y\mathcal{J}_0}\sum_{P\in\mathcal{P}}\tau\big((F|_P)^{-1}(I)\big)=\frac{1}{j}+\sum_{I\in F^*(\mathcal{I}_j\cap_Y\mathcal{J}_0)}\tau(I)\\&\leqslant\frac{1}{j}+\nu\big(F^{-1}(Y)\big)+\frac{2}{3}\varepsilon<\nu\big(F^{-1}(Y)\big)+\varepsilon.\end{aligned}$$

Thus, given a Borel set $Y\subset\mathfrak{B}$, we can conclude that $\nu(Y)<\nu(F^{-1}(Y))+\varepsilon$ for every $\varepsilon>0$. That is,

$$\nu(Y)\leqslant\nu\big(F^{-1}(Y)\big)\quad\text{for all Borel set } Y\subset\mathfrak{B}.$$

To conclude the proof of Claim 3, let us assume the existence of a Borel set $L\subset\mathfrak{B}$ such that $\nu(L)<\nu(F^{-1}(L))$. As $\nu(\mathfrak{B}\setminus L)\leqslant\nu(F^{-1}(\mathfrak{B}\setminus L))$, we obtain $\nu(\mathfrak{B})=\nu(L)+\nu(\mathfrak{B}\setminus L)<\nu(F^{-1}(L))+\nu(F^{-1}(\mathfrak{B}\setminus L))=\nu(\mathfrak{B})$, which is an absurd. $\square$

Now, suppose that $\int R\,d\nu\in(\gamma,+\infty]$, for some $\frac{1}{\Theta}<\gamma\in\mathbb{R}$. As ν is F invariant and $R\geqslant 0$, it follows from Birkhoff Theorem that $\exists\widetilde{\mathfrak{B}}\subset\mathfrak{B}$ with $\nu(\widetilde{\mathfrak{B}})>0$ such that for every $x\in\widetilde{\mathfrak{B}}$ there is some $n_x\in\mathbb{N}$ satisfying $\sum_{k=0}^{n}R\circ F^k(x)>\gamma n$ $\forall n\geqslant n_x$. In this case, for every $n>\gamma n_x$ $(\geqslant n_x)$ and every $\frac{1}{\gamma}n\leqslant j<n$ we get $\sum_{k=0}^{j}R\circ F^k(x)>\gamma j=\gamma\frac{j}{n}n\geqslant n$. Thus,

$$\sup\left\{j\geqslant 0;\ \sum_{k=0}^{j}R\circ F^k(x)<n\right\}\leqslant\frac{1}{\gamma}n<\Theta n,$$

for all $n\geqslant\gamma n_x$ and all $x\in\widetilde{\mathfrak{B}}$.

Because $\{j\geqslant 0;\ \sum_{k=0}^{j}R\circ F^k(x)<n\}=\{0\leqslant j<n;\ f^j(x)\in\mathcal{O}_F^+(x)\}$ and $\sup\{j\geqslant 0;\ \sum_{k=0}^{j}R\circ F^k(x)<n\}=\#\{j\geqslant 0;\ \sum_{k=0}^{j}R\circ F^k(x)<n\}$, it follows that

$$\#\big\{0\leqslant j<n;\ f^j(x)\in\mathcal{O}_F^+(x)\big\}=\sup\left\{j\geqslant 0;\ \sum_{k=0}^{j}R\circ F^k(x)<n\right\}. \tag{19}$$

So, for every $x\in\widetilde{\mathfrak{B}}$, we get

$$\limsup\frac{1}{n}\#\big\{0\leqslant j<n;\ f^j(x)\in\mathcal{O}_F^+(x)\big\}<\Theta. \tag{20}$$

But this is a contradiction. Indeed, as $\nu\ll\mu$, we have by hypothesis that $\nu(\{x\in\mathfrak{B};\ (20)\text{ holds}\})=\nu(\mathfrak{B}\setminus\{x\in\mathfrak{B};\ (7)\text{ holds}\})=0$. This proves that $\int R\,d\nu\leqslant\Theta^{-1}$. To finish the proof of the theorem, we extend ν to B by setting $\nu(B\setminus\mathfrak{B})=0$. $\square$

Using Theorem 1 we obtain the following characterization of the liftable measures.

Corollary 4.6. *Let $(F, \mathcal{P})$ be an induced full Markov map for f defined on an open set $B \subset X$. Let R be the inducing time of F and μ be an ergodic f-invariant probability such that $\mu(\{R = 0\}) = 0$ and $\mathcal{O}_f^+(x) \cap \mathcal{O}_f^+(y) \neq \emptyset \Rightarrow \mathcal{O}_F^+(x) \cap \mathcal{O}_F^+(y) \neq \emptyset$ for μ almost every $x, y \in B$. The following statements are equivalent.*

(i) *There is an F-invariant finite measure $\nu \ll \mu$ such that $\mu = \sum_{j=0}^{+\infty} f_*^j(\nu|_{\{R>j\}})$.*
(ii) *For μ almost every $x \in B$, $\limsup_n \frac{1}{n}\#\{0 \leqslant j < n;\ f^j(x) \in \mathcal{O}_F^+(x)\} > 0$.*
(iii) *For μ almost every $x \in B$, $\limsup_n \frac{1}{n}\sup_j\{j \geqslant 0;\ \sum_{k=0}^j R \circ F^k(x) < n\} > 0$.*
(iv) *There is an F-invariant finite measure $\nu \ll \mu$ such that $0 < \int R\,d\nu < \infty$.*

Proof. By (19) follows that (ii) $\Leftrightarrow$ (iii). As $\mu = \sum_{j=0}^{+\infty} f_*^j(\nu|_{\{R>j\}})$ implies that $\int R\,d\nu = \sum_{j=0}^{+\infty} f_*^j(\nu|_{\{R>j\}})(X) = \mu(X)$, it follows that (i) $\Rightarrow$ (iv). We get (i) $\Leftarrow$ (iv) from Proposition 4.4. As (ii) $\Rightarrow$ (i) follows from Theorem 1, only (iv) $\Rightarrow$ (iii) remains to be proved.

Suppose that (iv) holds. For every $n \in \mathbb{N}$ and each $x \in \mathfrak{B} := \{x \in B;\ F^j(x) \in \bigcup_{P\in\mathcal{P}} P\ \forall j \geqslant 0\}$, let $i_x(n) = \sup_j\{j \geqslant 0;\ \sum_{k=0}^j R \circ F^k(x) < n\}$. Thus, for every $x \in B$,

$$\frac{1}{i_x(n)+1}\sum_{k=0}^{i_x(n)+1} R \circ F^k(x) \geqslant \frac{1}{i_x(n)+1}n = \frac{n}{i_x(n)}\left(\frac{i_x(n)}{i_x(n)+1}\right). \tag{21}$$

If, by contradiction, $\limsup_n \frac{1}{n}\sup_j\{j \geqslant 0;\ \sum_{k=0}^j R \circ F^k(x) < n\} = 0$ for μ almost every $x \in B$, then $\lim_n n/i_x(n) = \infty$ for μ almost every $x \in B$. Using (21) it follows that $\limsup_k \frac{1}{k}\sum_{j=0}^{k-1} R \circ F^j(x) = \infty$ for μ almost every $x \in B$. This contradicts (iv) as, by Birkhoff Theorem, $\limsup_k \frac{1}{k}\sum_{j=0}^{k-1} R \circ F^j(x) = \int R\,d\nu < \infty$ for ν for a.e. $x \in B$ and so, $\mu(\{x \in B;\ \limsup_k \frac{1}{k}\sum_{j=0}^{k-1} R \circ F^j(x) < \infty\}) > 0$. $\square$

Lemma 4.7 just below will be useful to bound the space average of the induced time by having some information about the time average of the induced time. This will be necessary for projecting an invariant measure of the induced map onto an f-invariant measure.

Lemma 4.7. *Let $\{G_j\}_{j\in\mathbb{N}}$ be a collection of subsets of X such that $f^j(x) \in G_{n-j}\ \forall 0 \leqslant j < n\ \forall x \in G_n$. Let $B \subset X$ and let $x \in B$ be a point such that $\#\{j \geqslant 0;\ x \in G_j$ and $f^j(x) \in B\} = \infty$. Let $T : \mathcal{O}^+(x) \cap B \to \mathcal{O}^+(x) \cap B$ be a map given by $T(y) = f^{g(y)}(y)$, with $1 \leqslant g(y) \leqslant \min\{j \in \mathbb{N};\ y \in G_j$ and $f^j(y) \in B\}$. Then*

$$\#\{1 \leqslant j \leqslant n;\ x \in G_j \text{ and } f^j(x) \in B\} \leqslant \#\left\{j \geqslant 0;\ \sum_{k=0}^{j} g(T^k(x)) \leqslant n\right\}.$$

Moreover, if $\limsup_n \frac{1}{n}\#\{1 \leqslant j \leqslant n;\ x \in G_j$ and $f^j(x) \in B\} > \Theta > 0$ then

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} g \circ T^j(x) \leqslant \Theta^{-1}.$$

Proof. Given $n \in \mathbb{N}$, set $\Gamma_n = \{1 \leqslant j \leqslant n \mid x \in G_j$ and $f^j(x) \in B\}$ and $\Sigma_n = \{j \geqslant 0;\ \sum_{k=0}^j g(T^k(x)) \leqslant n\}$.

As $\Gamma_0 = \emptyset = \Sigma_0$, we have $\#\Gamma_0 \leqslant \#\Sigma_0$. By induction, assume that $\#\Gamma_j \leqslant \#\Sigma_j\ \forall 0 \leqslant j < n$. To prove that $\#\Gamma_n \leqslant \#\Sigma_n$ we may assume that $n \in \Gamma_n$, otherwise $\#\Gamma_{n-1} = \#\Gamma_n$ and so, $\#\Gamma_n = \#\Gamma_{n-1} \leqslant \#\Sigma_{n-1} \leqslant \#\Sigma_n$. Let $\ell = \max\{j;\ j \in \Sigma_{n-1}\}$ and $s = \sum_{k=0}^{\ell} g(T^k(x))$. As $s \leqslant n-1$ and $x \in G_n$, we have $T^{\ell+1}(x) = f^s(x) \in G_{n-s}$. Moreover, we also know that $f^s(x) \in B$, $f^{n-s}(f^s(x)) = f^n(x) \in B$ and so, $g(f^s(x)) \leqslant n-s$ and, as a consequence, $\sum_{k=0}^{\ell+1} g(T^k(x)) = \sum_{k=0}^{\ell} g(T^k(x)) + g(T^{\ell+1}(x)) \leqslant s + (n-s) \leqslant n$. Therefore, $\ell+1 \in \Sigma_n \setminus \Sigma_{n-1}$ and so, $\#\Gamma_n = \#\Gamma_{n-1} + 1 \leqslant \#\Sigma_{n-1} + 1 \leqslant \#\Sigma_n$ (as $n \in \Gamma_n$, $\Gamma_n = \{n\} \cup \Gamma_{n-1}$), completing the induction.

Assume now that $\limsup_n \frac{1}{n}\#\{1 \leqslant j \leqslant n;\ x \in G_j$ and $f^j(x) \in B\} > \Theta > 0$. If $\liminf_n \frac{1}{n}\sum_{k=0}^{n-1} g \circ T^k(x) > \Theta^{-1}$, there is some n_0 such that $\sum_{k=0}^{n} g \circ T^k(x) > \Theta^{-1}n\ \forall n \geqslant n_0$. In this case, if $n_0 \leqslant \Theta n \leqslant j \leqslant n$ then $\sum_{k=0}^{j} g \circ T^k(x) > \Theta^{-1}j = \Theta^{-1}\frac{j}{n}n \geqslant n$. So, $\#\Sigma_n(x) \leqslant \Theta n\ \forall n \geqslant n_0$ and, as a consequence of $\#\Gamma_n \leqslant \#\Sigma_n\ \forall n$,

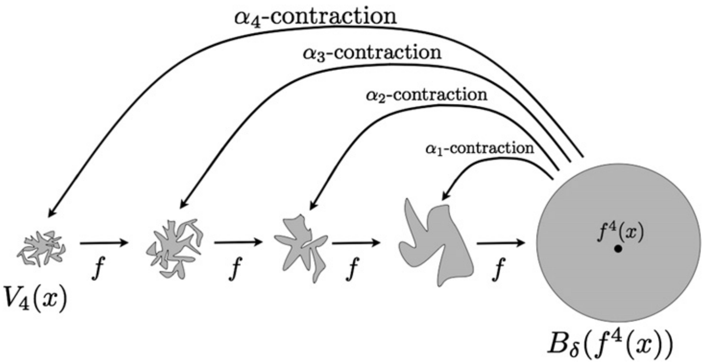

Fig. 5. A zooming time for $x \in \mathrm{Z}_4(\alpha, \delta, f)$.

$$\limsup_{n\to+\infty} \frac{1}{n}\#\big\{1 \leqslant j \leqslant n;\ x \in G_j \text{ and } f^j(x) \in B\big\} = \limsup_{n\to+\infty} \frac{1}{n}\#\Gamma_n \leqslant \Theta,$$

contradicting our hypotheses. □

5. Zooming sets and measures

In this section we introduce the notion of *zooming times*. This notion captures and weakens the geometric aspects of the *hyperbolic times* (Section 8), allowing more flexibility in the applications and examples.

Let $f : X \to X$ be a measurable map defined on a connected, compact, separable metric space.

Definition 5.1 *(Zooming contraction).* A sequence $\alpha = \{\alpha_n\}_{1\leqslant n\in\mathbb{N}}$ of functions $\alpha_n : [0, +\infty) \to [0, +\infty)$ is called a *zooming contraction* if it satisfies the following conditions

- $\alpha_n(r) < r$ $\forall r > 0$ and $\forall n \geqslant 1$;
- $\alpha_n(r) \leqslant \alpha_n(\widetilde{r})$ $\forall$ $0 \leqslant r \leqslant \widetilde{r}$ and $\forall n \geqslant 1$;
- $\alpha_n \circ \alpha_m(r) \leqslant \alpha_{n+m}(r)$ $\forall r > 0$ and $\forall n, m \geqslant 1$;
- $\sup_{0\leqslant r\leqslant 1}(\sum_{n=1}^{\infty} \alpha_n(r)) < \infty$.

For instance, an exponential contraction corresponds to a zooming contraction $\alpha_n(r) = \lambda^n r$ with $0 < \lambda < 1$. We note that we can deal with polynomial contractions ($\alpha_n(r) = n^{-a} r$, $a > 1$) and also with contractions that becomes in small scales as weak as we want ($\alpha_n(r) := (\frac{1}{1+n\sqrt{r}})^2 r$ defines a zooming contraction and $\lim_{r\to 0} \frac{a_n(r)}{r} = 1$, see Example 9.14).

Let $\alpha = \{\alpha_n\}_n$ be a zooming contraction and $\delta > 0$ be a positive constant. (See Fig. 5.)

Definition 5.2 *(Zooming times).* We say that $n \geqslant 1$ is a (α, δ)-*zooming time* for $p \in X$ (with respect to f) if there is a neighborhood $V_n(p)$ of p satisfying

(1) f^n sends $\overline{V_n(p)}$ homeomorphically onto $\overline{B_\delta(f^n(p))}$;
(2) $\mathrm{dist}(f^j(x), f^j(y)) \leqslant \alpha_{n-j}(\mathrm{dist}(f^n(x), f^n(y)))$, for every $x, y \in V_n(p)$ and every $0 \leqslant j < n$.

The ball $B_\delta(f^n(p))$ is called a *zooming ball* and the set $V_n(p)$ is called a *zooming pre-ball*. Denote by $\mathrm{Z}_n(\alpha, \delta, f)$ the set of points of X for which n is an (α, δ)-zooming time.

Definition 5.3 *(Zooming sets).* A positively invariant set $\Lambda \subset X$ is called a *zooming set* if (22) holds for every $x \in \Lambda$.

Definition 5.4 *(Zooming measures).* A f-non-singular finite measure μ defined on the Borel set of X is called a *weak zooming measure* if μ almost every point has infinitely many (α, δ)-zooming times. A weak zooming measure is called a *zooming measure* if

$$\limsup \frac{1}{n}\#\{1 \leqslant j \leqslant n;\ x \in \mathrm{Z}_j(\alpha, \delta, f)\} > 0, \tag{22}$$

for μ almost every $x \in X$.

Definition 5.5 *(Bounded distortion).* We say that a weak zooming measure μ has bounded distortion if $\exists\rho > 0$ such that, $\forall n \in \mathbb{N}$ and μ almost every $p \in \mathrm{Z}_n(\alpha, \delta, f)$, the Jacobian of f^n with respect to μ, $J_\mu f^n$, is well defined on $V_n(p)$ and

$$\left|\log \frac{J_\mu f^n(x)}{J_\mu f^n(y)}\right| \leqslant \rho \operatorname{dist}\bigl(f^n(x), f^n(y)\bigr),$$

for μ-almost every x and $y \in V_n(p)$.

Remark 5.6. We use the connectivity (indeed, local connectivity is enough) only in the proof of Lemma 5.12 (the local connectivity is necessary to apply Proposition 2.8, where $\mathcal{A} = \{B_r(x)\}$). This lemma assures the existence of *nested sets* containing a given point $x \in X$. Thus, to obtain all the results of Sections 5, 6 and 7 we can remove the additional hypotheses above if the existence of sets like $(B_r(x))^\star$ can be ensured in another way.

Lemma 5.7. *The zooming times have the following properties.*

(1) *If $p \in \mathrm{Z}_j(\alpha, \delta, f)$ then $f^\ell(p) \in \mathrm{Z}_{j-\ell}(\alpha, \delta, f)$ for all $0 \leqslant \ell < j$.*
(2) *If $p \in \mathrm{Z}_j(\alpha, \delta, f)$ and $f^j(p) \in \mathrm{Z}_\ell(\alpha, \delta, f)$ then $p \in \mathrm{Z}_{j+\ell}(\alpha, \delta, f)$.*
(3) *If $p \in \mathrm{Z}_{j\ell}(\{\alpha_n\}_n, \delta, f)$ then $p \in \mathrm{Z}_j(\{\alpha_{\ell n}\}_n, \delta, f^\ell)$.*

Proof. Follows easily from the properties of zooming times. $\square$

It follows from Lemma 5.7 that if $x \in \mathrm{Z}_{km+j}(\{\alpha_n\}_n, \delta, f)$, with $0 \leqslant j < k$, then $f^j(x) \in \mathrm{Z}_{km}(\{\alpha_n\}_n, \delta, f) \subset \mathrm{Z}_m(\{\alpha_{kn}\}_n, \delta, f^k)$. Thus,

$$\begin{aligned}\limsup_m \mathrm{Z}_m\bigl(\{\alpha_n\}_n, \delta, f\bigr) &\subset \bigcup_{j=0}^{k-1} f^{-j}\Bigl(\limsup_m \mathrm{Z}_{km}\bigl(\{\alpha_n\}_n, \delta, f\bigr)\Bigr)\\ &\subset \bigcup_{j=0}^{k-1} f^{-j}\Bigl(\limsup_m \mathrm{Z}_m\bigl(\{\alpha_{kn}\}_n, \delta, f^k\bigr)\Bigr).\end{aligned} \tag{23}$$

Let $\mathcal{Z}$ be the set of all points of X with positive frequency of $(\{\alpha_n\}_n, \delta)$-zooming times, that is, (22) holds.

Notation 5.8. Denote by $\mathcal{E}_{\mathcal{Z}} = (\mathcal{E}_{\mathcal{Z},n})_n$ as the collection of all (α, δ)-zooming pre-balls, where $\mathcal{E}_{\mathcal{Z},n} = \{V_n(x);\ x \in \mathrm{Z}_n(\alpha, \delta, f)\}$ is the collection of all (α, δ)-zooming pre-balls of order n. One can check easily that the collection of all (α, δ)-zooming pre-balls is a *dynamically closed family of pre-images* as defined in Section 2.

Given $x \in X$ and $0 < r < \delta$ let $(B_r(x))^\star$ be the set defined by (4). If $x \in (B_r(x))^\star$, it follows from Proposition 2.8 (taking $\mathcal{A} = \{B_r(x)\}$) that the connected component of $(B_r(x))^\star$ which contains x is an $\mathcal{E}_{\mathcal{Z}}$-nested set.

Definition 5.9 *(Zooming nested balls).* If $x \in (B_r(x))^\star$, define *the* (α, δ)*-zooming nested ball* (*with respect to* f) *of radius r and center on* x, denoted by $B_r^\star(x)$, as the connected component of $(B_r(x))^\star$ which contains x.

Note that, as we have contraction in any zooming time, $B_r(x)$ cannot be contained in any zooming pre-image (with order bigger than zero) of itself. So $\mathcal{A} = \{B_r(x)\}$, in the definition above, is indeed a collection of open sets as desired on Section 2.1.

Remark 5.10. As two distinct $\mathcal{E}_{\mathcal{Z}}$-pre-images of the same set cannot intersect (Remark 2.1), the order of the elements of a chain are strictly increasing. That is, if $(P_0, \ldots, P_n)$ is a chain of $\mathcal{E}_{\mathcal{Z}}$-pre-images of $B_r(x)$ then $0 < \operatorname{ord}(P_0) < \operatorname{ord}(P_1) < \cdots < \operatorname{ord}(P_n)$.

Definition 5.11 *(Backward separated map).* We say that f is backward separated if for every $x \in X$ we have

$$\operatorname{dist}\left(x, \bigcup_{j=1}^{n} f^{-j}(x) \setminus \{x\}\right) > 0 \quad \forall n \geqslant 1. \tag{24}$$

For instance, every continuous map f with bounded number of pre-images ($\sup\{\# f^{-1}(x);\ x \in X\} < +\infty$) is backward separated.

Lemma 5.12 *(Existence of zooming nested balls). If for some $0 < r < \delta/2$ we have $\sum_{n \geqslant 1} \alpha_n(r) < r/4$ then the zooming nested ball $B_r^\star(x)$ is well defined and $B_r^\star(x) \supset B_{r/2}(x)$, $\forall x \in X$. Furthermore, if f is backward separated and $\sup_{r>0}(\sum_{n \geqslant 1} \alpha_n(r)/r) < +\infty$ then for each $x \in X$ there exists $0 < r_0 < \delta/2$ such that $B_r^\star(x)$ is well defined $\forall 0 < r \leqslant r_0$ and, given any $0 < \gamma < 1$, one can find $0 < r_\gamma \leqslant r_0$ such that $B_r^\star(x) \supset B_{\gamma r}(x)$ and $\forall 0 < r \leqslant r_\gamma$.*

Proof. If $\sum_{n \geqslant 1} \alpha_n(r) < r/4$, $0 < r < \delta/2$, as the order of the elements of a chain of $\mathcal{A} = \{B_r(x)\}$, $0 < r < \delta$, are strictly increasing (Remark 5.10), the diameter of any chain is smaller than $\sum_{n \geqslant 1} \alpha_n(\operatorname{diameter}(B_r(x))) < r/2$. Thus, using Corollary 2.9, we get $B_r^\star(x) \supset B_{r/2}(x)$.

Let suppose now that f is backward separated and $\sup_{r>0}(\sum_{n \geqslant 1} \alpha_n(r)/r) < +\infty$. Given $0 < \gamma < 1$, let $n_0 \in \mathbb{N}$ be such that $\sum_{n > n_0} \alpha_n(r) < (1-\gamma)r/2$. Let $x \in X$ and $\varepsilon > 0$ be such that $\inf_x \operatorname{dist}(x, \bigcup_{j=1}^{n_0} f^{-j}(x) \setminus \{x\}) > \varepsilon$, $r_\gamma = \frac{1}{3}\min\{\varepsilon, \delta\}$ and $0 < r \leqslant r_\gamma$. Note that if $j < n_0$ then $B_r(x) \cap P = \emptyset\ \forall P \in \mathcal{E}_{\mathcal{Z},j}$ (because $P \cap (\bigcup_{j=1}^{n_0} f^{-j}(x)) \neq \emptyset$ and $\operatorname{diameter}(P) < r < \varepsilon/2$). Thus, every chain of $\mathcal{E}_{\mathcal{Z}}$-pre-images of $B_r(x)$ begins with a pre-image of order bigger than n_0. By Remark 5.10, the diameter of any chain is smaller than $\sum_{n > n_0} \alpha_n(\operatorname{diameter}(B_r(x))) < (1-\gamma)r$ and, as a chain intersects the boundary of $B_r(x)$, we can conclude that a chain cannot intersect $B_{\gamma r}(x)$. So, $(B_r(x))^\star$, and also $B_r^\star(x)$, contains $B_{\gamma r}(x)$. $\square$

Notation 5.13. Given any sequence of sets $\{U_n\}_n$, denote by $\limsup_n U_n$ the set of points that belong to infinitely many elements of this sequence, i.e.,

$$\limsup_n U_n = \bigcap_{n \geqslant 1} \bigcup_{j \geqslant n} U_j.$$

Using the notation above, f-non-singular finite measure μ is weak zooming if $\mu(X \setminus \limsup \mathrm{Z}_m(\alpha, \delta, f)) = 0$ (see Definition 5.4).

If $x \in X$ has a zooming time, we can define the *first zooming time* of x as $\min\{n;\ x \in \mathrm{Z}_n(\alpha, \delta, f)\}$. It is easy to show that, if μ is a finite f-non-singular measure and the first zooming time is well defined for μ-almost everywhere, then μ is a weak zooming measure. That is,

$$\mu\left(X \setminus \bigcup_{j=1}^{\infty} \mathrm{Z}_j(\alpha, \delta, f)\right) = 0 \quad \Rightarrow \quad \mu\big(X \setminus \limsup \mathrm{Z}_m(\alpha, \delta, f)\big) = 0.$$

Notation 5.14 *(The zooming images set).* Denote the collection of zooming images of f by $\mathfrak{z} = (\mathfrak{z}(x))_{x \in \limsup \mathrm{Z}_m(\alpha,\delta,f)}$, where $\mathfrak{z}(x) = \{f^m(x);\ m \in \mathbb{N} \text{ and } x \in \mathrm{Z}_m(\alpha, \delta, f)\}$ is the set of zooming images of x by f.

It is easy to see that if $x \in \mathrm{Z}_m(\alpha, \delta, f)$ then $f^{m-j}(x) \in \mathrm{Z}_{m-j}(\alpha, \delta, f)$, $\forall 0 \leqslant j < m$. Thus, using this information and Lemma 5.7, one can prove that $\mathfrak{z}$ is an asymptotically invariant collection. Indeed, if $x \in \mathrm{Z}$ and m_0 is the first zooming time for x then $\{f^m(x);\ m \geqslant 2 \text{ and } x \in \mathrm{Z}_m(\alpha, \delta, f)\} = \{f^m(f(x));\ m \geqslant \max\{m_0 - 1, 1\} \text{ and } f(x) \in \mathrm{Z}_m(\alpha, \delta, f)\}$.

In the lemma below, let μ be a weak (α, δ)-zooming measure with bounded distortion (see Definition 5.5), where $\alpha = \{\alpha_n\}_n$.

Lemma 5.15. *Suppose that for some $0 < r_0 < \delta/2$ and $p \in X$ the (α, δ)-zooming nested ball $B_{r_0}^\star(p)$ is well defined connected open set and contains $B_{r_0/2}(p)$. If $U \subset X$ is positively invariant, $\mu(U) > 0$ and $\mu(\{x \in U;\ B_{r_0/2}(p) \cap \omega_{\mathfrak{z}}(x) \neq \emptyset\}) > 0$ then $\mu(B_{r_0/2}(p) \cap U) = \mu(B_{r_0/2}(p)) > 0$.*

Proof. Let $\rho > 0$ be the distortion constant that appear in Definition 5.5 and let $K \subset \{x \in U;\ B_{r_0}(p) \cap \omega_{\mathfrak{z}}(x) \neq \emptyset\}$ be a compact set with positive μ measure.

Given $\ell > 0$ choose an open neighborhood $V \supset K$ of K such that $\mu(V \setminus K) < \mu(K)/\ell$. Choose for each $x \in K$ a zooming time $n(x)$ such that $V_{n(x)}(x) \subset V$ and $f^{n(x)}(x) \in B_{r_0/2}(p)$. As $V_{n(x)}(x)$ is mapped diffeomorphically by $f^{n(x)}$ onto $B_\delta(f^{n(x)}(x))$ and $B_\delta(f^{n(x)}(x)) \supset B^\star_{r_0}(p)$ (because $r_0 < \delta/2$), set, for each $x \in K$,

$$W(x) = \left(f^{n(x)}|_{V_{n(x)}(x)}\right)^{-1}\left(B^\star_{r_0}(p)\right).$$

By compactness $K \subset W(x_1) \cup \cdots \cup W(x_s)$ for some $x_1, \ldots, x_m \in K$. As $B^\star_{r_0}(p)$ is a nested set, we can assume that $W(x_j) \cap W(x_i) = \emptyset$ whenever $j \neq i$. Thus, at least for one j we have $\mu(W(x_j) \setminus K) < \mu(W(x_j))/\ell$. Otherwise, $\mu(V \setminus K) \geqslant \mu((\bigcup_j W(x_j)) \setminus K) = \sum_j \mu(W(x_j) \setminus K) \geqslant \sum_j \mu(W(x_j))/\ell = \mu(\bigcup_j W(x_j))/\ell \geqslant \mu(K)/\ell$. Therefore, for each $\ell \in \mathbb{N}$ we can find some pre-ball W_ℓ which is sent by some iterate f^{n_ℓ} of f diffeomorphically onto $B^\star_{r_0}(p)$ and with the distortion bounded by ρ. Furthermore, $\mu(W_\ell \setminus K) < \mu(W_\ell)/\ell\ \forall \ell$. By the bounded distortion we get

$$\frac{\mu(B^\star_{r_0}(p) \setminus U)}{\mu(B^\star_{r_0}(p))} \leqslant \frac{\mu(B^\star_{r_0}(p) \setminus f^{n_\ell}(K))}{\mu(B^\star_{r_0}(p))} \leqslant \rho \frac{\mu(W_\ell \setminus K)}{\mu(W_\ell)} < \frac{\rho}{\ell} \to 0.$$

As $B^\star_{r_0}(p) \supset B_{r_0/2}(p)$ and $\mu(B_{r_{0/2}}(p)) > 0$ (because μ is f-non-singular), we conclude the proof. $\square$

Corollary 5.16. *If μ is a weak zooming measure with compact support and bounded distortion, then there is $\varepsilon > 0$ such that every positively invariant set has either μ-measure bigger than ε or equal to zero. Furthermore, if $f|_{\operatorname{supp}\mu}$ is transitive, continuous and $K \subset \operatorname{supp}\mu$ is a compact positive invariant set with $\mu(K) > 0$ then $K = \operatorname{supp}\mu$.*

Proof. Let μ be a weak (α, δ)-zooming measure with compact support and bounded distortion, where $\alpha = \{\alpha_n\}_n$. Let U be a positively invariant set with $\mu(U) > 0$.

First, assume that $\sum_{n \geqslant 1} \alpha_n(r_0) < r_0/4$ for some $0 < r_0 < \delta/2$. It follows from Lemma 5.12 that for every $p \in X$ the (α, δ)-zooming nested ball $B^\star_{r_0}(r)$ is well defined and contains $B_{r_0/2}(p)$. Let p be any point on the support of μ such that $\mu(\{x \in U;\ \omega_{f,\mathfrak{z}}(x) \cap B_{r_0/2}(p) \neq \emptyset\}) > 0$ (of course at least one of such point exist). It follows from Lemma 5.15 that

$$\mu(U) \geqslant \mu\left(B_{r_0/2}(p) \cap U\right) = \mu\left(B_{r_0/2}(p)\right) > 0. \tag{25}$$

Let $\varepsilon := \inf\{\mu(B_{r_0/2}(x));\ x \in \operatorname{supp}\mu\}$. It is easy to see that $\varepsilon > 0$, ε does not depend on U and $\mu(U) \geqslant \varepsilon > 0$.

Assuming that U is compact and $f|_{\operatorname{supp}\mu}$ is transitive, we claim that $B_{r_0/2}(p) \cap \operatorname{supp}\mu \subset U$. Otherwise, $(B_{r_0/2}(p) \setminus U)$ is an open set with $(B_{r_0/2}(p) \setminus U) \cap \operatorname{supp}\mu \neq \emptyset$. Thus, by the definition of the support of a measure, $\mu(B_{r_0/2}(p) \setminus U) > 0$, contradicting (25). Now, it is easy to see that $U = \operatorname{supp}\mu$. Indeed, let $q \in \operatorname{supp}\mu$ be such that $\omega(q) = \operatorname{supp}\mu$. Then there is $n > 0$ such that $f^n(q) \in (B_{r_0/2}(p) \cap \operatorname{supp}\mu) \subset U$. As U is positive invariant and compact, we get $\operatorname{supp}\mu = \omega(q) \subset U \subset \operatorname{supp}\mu$

In the general case (that is, when we do not know if $\sum_{n \geqslant 1} \alpha_n(r_0) < r_0/4$), let $r_0 = \frac{\delta}{3}$ and $\widetilde{f} = f^k$, where $k \geqslant 1$ is such that $\sum_{n \geqslant 1} \alpha_{kn}(r_0) < \frac{r_0}{4}$.

By (23), there is $0 \leqslant j < k$ such that $\mu(f^{-j}(\limsup_m \mathrm{Z}_m(\{\alpha_{kn}\}_n, \delta, f^k)) \cap U) > 0$ and, as $\mu \circ f^{-1} \ll \mu$ and $f(U) \subset U$, we get

$$\mu\Big(\limsup_m \mathrm{Z}_m\left(\{\alpha_{kn}\}_n, \delta, f^k\right) \cap U\Big) > 0. \tag{26}$$

Taking $\widetilde{\mu} = \mu|_{\limsup_m \mathrm{Z}_m(\{\alpha_{kn}\}_n, \delta, f^k)}$, it is easy to see that $\widetilde{\mu}$ is a weak $(\{\widetilde{\alpha}_n\}_n, \delta)$-zooming measure with respect to $\widetilde{f}$, where $\widetilde{\alpha}_n = \alpha_{kn}$. Moreover $\widetilde{\mu}$ has compact support, bounded distortion and $\sum_{n \geqslant 1} \widetilde{\alpha}_n(r_0)/r_0 < 1/4$. As $\widetilde{f}(U) \subset U$ and, by (26), $\widetilde{\mu}(U) > 0$, we can apply the particular case and get $\varepsilon > 0$, not depending on U, such that $\mu(U) \geqslant \widetilde{\mu}(U) > \varepsilon$.

When U is compact and $f|_{\operatorname{supp}\mu}$ is transitive and continuous, there is a finite number of transitive components for $\widetilde{f}|_{\operatorname{supp}\mu}$, that is, $\operatorname{supp}\mu = M_1 \cup \cdots \cup M_s$, where M_j is compact, positive invariant, and $\widetilde{f}|_{M_j}$ is transitive $\forall j$. If we consider $\widetilde{\mu}$ restrict to one of those components, say M_j, such that $\widetilde{\mu}|_{M_j}(U) = \widetilde{\mu}(M_j \cap U) > 0$, we can apply the particular case to the $\widetilde{f}$-positive invariant set $\widetilde{U} = U \cap M_j$ and get $\widetilde{U} = \operatorname{supp}\widetilde{\mu}|_{M_j} = \operatorname{supp}\mu \cap M_j$. As $U = \widetilde{U} \cup \cdots \cup f^s(\widetilde{U}) = (\operatorname{supp}\mu \cap M_j) \cup \cdots \cup f^s(\operatorname{supp}\mu \cap M_j) = \operatorname{supp}\mu \cap (M_j \cap \cdots \cup f^s(M_j)) = \operatorname{supp}\mu$, we end the proof. $\square$

As a consequence of Proposition 3.5, Proposition 3.12, Corollary 5.16 and Lemma 3.9 we have the following result.

Theorem 2. *If μ is a weak zooming measure with bounded distortion then X can be partitioned into a finite collection of μ-ergodic components. Inside each μ-ergodic component U there exists a fat attractor A (i.e., $\mu(A) > 0$) such that $\omega_f(x) = A$ for μ-almost every point $x \in U$.*

Furthermore, there is a compact set $A_{\mathfrak{z}} \subset A$ such that $\omega_{f,\mathfrak{z}}(x) = A_{\mathfrak{z}}$ for μ-almost every point $x \in U$ and, if μ is a zooming measure, there is a compact set $A_{+,\mathfrak{z}} \subset A_{\mathfrak{z}}$ such that $\omega_{+,f,\mathfrak{z}}(x) = A_{+,\mathfrak{z}}$ for μ-almost every point $x \in U$.

Corollary 5.17. *If μ is a weak zooming measure with bounded distortion and $f|_{\operatorname{supp}\mu}$ is transitive and continuous then $\omega_f(x) = \operatorname{supp}\mu$ for μ-almost every x.*

6. Constructing a local inducing Markov map

Sections 6 and 7 are the kernel of this paper. Most of the results for zooming sets and measures are proved in these sections, and from them we will obtain their analogues for expanding sets and measures. The existence of an invariant measure $\nu \ll \mu$ that is absolutely continuous with respect to a given zooming measure with some distortion control is given by Theorem C. In Theorem D we prove the existence of Markov structures for zooming sets. The existence of global induced Markov maps for zooming sets is given in Section 7 by Theorem E. Note that our approach to construct induced Markov map for dynamics with some hyperbolic behavior has to be very different from the one of Alves, Luzzatto, Pinheiro [5,6], Gouëzel [31], Pinheiro [45] and Young [61]. That is because this construction in those papers depends in an essential way on the good relation between the diameter and the volume (Lebesgue measure) of balls and this is not true for general zooming (or expanding) measures.

Let X, f, δ, $\alpha = \{\alpha_n\}_n$ and $\mathfrak{z} = (\mathfrak{z}(x))_{x \in \limsup Z_n(\alpha,\delta,f)}$ be as in Section 5. Let

$$\Lambda \subset \limsup_{n\to\infty} Z_n(\alpha, \delta, f) \subset X$$

be a positively invariant set.

Let Δ be an (α, δ)-zooming nested open set. Assume also that diameter$(\Delta) < \delta/2$. For example, if $\sum_{n\geqslant 1} \alpha_n(r) < r/4$ for some $0 < r < \delta/4$ (or if f is backward separated and $\sup_{r>0} \sum_{n\geqslant 1} \alpha_n(r)/r < +\infty$) we can take Δ as any zooming nested ball $B^\star_r(q)$ given by Lemma 5.12.

It is sometimes useful not to use all the zooming times but a sub-collection of them in the construction of the induced Markov map (for instance, this is necessary in the proof of Theorem E). This motivates the definitions below.

For each $x \in \Lambda$ consider a set $\widetilde{\mathfrak{z}}(x) \subset \mathfrak{z}(x)$. We say that n is a $\widetilde{\mathfrak{z}}$-time for x if $f^n(x) \in \widetilde{\mathfrak{z}}(x)$. A zooming pre-ball $V_n(x)$ is called a $\widetilde{\mathfrak{z}}$-pre-ball if n is a $\widetilde{\mathfrak{z}}$-time for x. Let $\widetilde{\mathcal{E}}_{\mathcal{Z}} \subset \mathcal{E}_{\mathcal{Z}}$ be the collection of all $\widetilde{\mathfrak{z}}$-pre-balls $V_n(x)$ for all $x \in \Lambda$ and all $\widetilde{\mathfrak{z}}$-time for x.

Definition 6.1. We say that $\widetilde{\mathfrak{z}} = (\widetilde{\mathfrak{z}}(x))_{x\in\Lambda}$ is a *proper zooming sub-collection* if

(1) $\widetilde{\mathfrak{z}}$ is asymptotically invariant;
(2) $\widetilde{\mathfrak{z}}(x) \subset \mathfrak{z}(x)$ for all $x \in \Lambda$;
(3) $\widetilde{\mathfrak{z}}$ has positive frequency whenever $\mathfrak{z}$ has positive frequency;
(4) $\widetilde{\mathcal{E}}_{\mathcal{Z}}$ is a dynamically closed family of pre-images.

The zooming collection itself is an example of a proper zooming sub-collection. Another example of proper zooming sub-collection that we are interested in is the following. Fixed $\ell \geqslant 1$, set $\widetilde{f} = f^\ell$ and $\widetilde{\alpha} = \{\widetilde{\alpha}_n\}_n$, where $\widetilde{\alpha}_n = \alpha_{\ell n}$. For instance, denote the collection of (α, δ)-zooming images of f by $\mathfrak{z}_f = (\mathfrak{z}_f(x))_{x\in\limsup_n Z_n(\alpha,\delta,f)}$ and the collection of $(\widetilde{\alpha}, \delta)$-zooming images of $\widetilde{f}$ by $\mathfrak{z}_{\widetilde{f}} = (\mathfrak{z}_{\widetilde{f}}(x))_{x\in\limsup_n Z_n(\widetilde{\alpha}\delta,\widetilde{f})}$. It follows from Lemma 5.7 that $\limsup_n Z_{\ell n}(\alpha\delta, f) \subset \limsup_n Z_n(\widetilde{\alpha}, \delta, \widetilde{f})$. Thus, taking $\widetilde{\mathfrak{z}}_{\widetilde{f}}(x) = \{\widetilde{f}^n(x);\ f^{\ell n}(x) \in \mathfrak{z}_f(x)\}$, the collection $\widetilde{\mathfrak{z}} = \widetilde{\mathfrak{z}}_{\widetilde{f}} = (\widetilde{\mathfrak{z}}_{\widetilde{f}}(x))_{x\in\limsup_n Z_{\ell n}(\alpha,\delta,f)}$ is a proper $(\widetilde{\alpha}, \delta)$-zooming sub-collection for the map $\widetilde{f}$.

Essentially, sub-collections will be necessary only in the proof of item (2) of Theorem D (when $m_0 > 1$) (and also in Remark 8.8) to acquire more contraction on the pre-balls (changing α for $\widetilde{\alpha}$ and f for $\widetilde{f}$) maintaining the

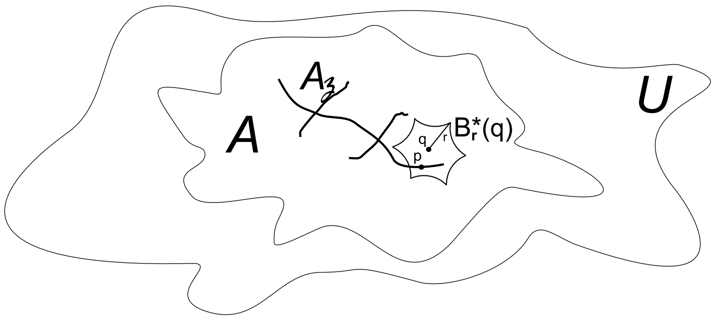

Fig. 6. $\Delta = B_r^\star(x)$.

contraction (and distortion control) for each iterate of the original map. We emphasize that for all the other results we do not really need to work with sub-collections.

Now, let $\widetilde{\mathfrak{z}} = (\widetilde{\mathfrak{z}}(x))_{x \in \Lambda}$ be a proper zooming sub-collection and let $\widetilde{\mathcal{E}}_{\mathcal{Z}} \subset \mathcal{E}_{\mathcal{Z}}$ be the collection of all $\widetilde{\mathfrak{z}}$-pre-balls. Given $x \in \Delta$, let $\Omega(x)$ be the collection of $\widetilde{\mathcal{E}}_{\mathcal{Z}}$-pre-images V of Δ such that $x \in V$.

The set $\Omega(x)$ is not empty for every $x \in \Delta$ that has a $\widetilde{\mathfrak{z}}$-return to Δ. Indeed, if $x \in \Delta$ and $f^n(x) \in \Delta \cap \widetilde{\mathfrak{z}}(x)$ then $B_\delta(f^n(x)) = f^n(V_n(x)) \supset \Delta$ (because diameter(Δ) $< \delta/2$). Thus, for each $\widetilde{\mathfrak{z}}$-return time of a point $x \in \Delta$ we can associated the $\widetilde{\mathcal{E}}_{\mathcal{Z}}$-pre-image $P = (f^n|_{V_n(x)})^{-1}(\Delta)$ of Δ with $x \in P$.

Definition 6.2. The *inducing time* on Δ associated to "*the first $\widetilde{\mathcal{E}}_{\mathcal{Z}}$-return time to* Δ" is the function $R : \Delta \to \mathbb{N}$ given by

$$R(x) = \begin{cases} \min\{\operatorname{ord}(V);\ V \in \Omega(x)\} & \text{if } \Omega(x) \neq \emptyset, \\ 0 & \text{if } \Omega(x) = \emptyset. \end{cases} \tag{27}$$

Note that $R(x)$ is smaller than or equal to the first $\widetilde{\mathfrak{z}}$-return time to Δ, i.e., $R(x) \leqslant \min\{n \geqslant 1;\ f^n(x) \in \widetilde{\mathfrak{z}}(x) \cap \Delta\}$.

Definition 6.3. The induced map F on Δ associated to "*the first $\widetilde{\mathcal{E}}_{\mathcal{Z}}$-return time to* Δ" is the map $F : \Delta \to \Delta$ given by

$$F(x) = f^{R(x)}(x), \quad \forall x \in \Delta. \tag{28}$$

As the collection of sets $\Omega(x)$ is totally ordered by inclusion, it follows from Corollary 2.6 that there is a unique $I(x) \in \Omega(x)$ such that $\operatorname{ord}(I(x)) = R(x)$, whenever $\Omega(x) \neq \emptyset$.

Lemma 6.4. *If $\Omega(x) \neq \emptyset \neq \Omega(y)$ then either $I(x) \cap I(y) = \emptyset$ or $I(x) = I(y)$.*

Proof. We claim that, if $\Omega(x) \neq \emptyset$, $I(x) \supset V\ \forall V \in \Omega(x)$. Indeed, if $I(x) \subsetneq V$ with $V \in \Omega(x)$, as $\operatorname{ord}(I(x)) < \operatorname{ord}(V)$, it follows from Corollary 2.6 that Δ is contained in an $\widetilde{\mathcal{E}}_{\mathcal{Z}}$-pre-image of itself of order bigger than zero. But this is impossible because we have contraction in the zooming times, i.e., the diameter of an $\widetilde{\mathcal{E}}_{\mathcal{Z}}$-pre-image of Δ has diameter smaller than the diameter of Δ. (See Fig. 6.)

Let $x, y \in X$ with $\Omega(x) \neq \emptyset \neq \Omega(y)$. As $I(x)$ and $I(y)$ are $\widetilde{\mathcal{E}}_{\mathcal{Z}}$-pre-images of Δ, if $I(x) \cap I(y) \neq \emptyset$ then $I(x) \supset I(y)$ or $I(x) \subset I(y)$. Thus, $I(x) \cap I(y) \neq \emptyset$ implies that $I(x) \in \Omega(y)$ or $I(y) \in \Omega(x)$. In any case, by uniqueness, $I(x) = I(y)$. $\square$

Definition 6.5. The Markov partition associated to "*the first $\widetilde{\mathcal{E}}_{\mathcal{Z}}$-return time to* Δ" is the collection of open sets $\mathcal{P}$ given by

$$\mathcal{P} = \{I(x);\ x \in \Delta \text{ and } \Omega(x) \neq \emptyset\}. \tag{29}$$

The corollary below shows that $\mathcal{P}$ is indeed a Markov partition of open sets.

Corollary 6.6 *(Existence of a full induced Markov map for a zooming set). Let F be given by* (28)*, R given by* (27) *and $\mathcal{P}$ by* (29)*. If $\mathcal{P} \neq \emptyset$ then $(F, \mathcal{P})$ is an induced full Markov map for f on Δ.*

Proof. By construction the elements of $\mathcal{P}$ are open sets. By Lemma 6.4, $\mathcal{P}$ satisfies the first condition of a Markov partition for F. As $F(P) = \Delta \supset Q$ $\forall P, Q \in \mathcal{P}$, $\mathcal{P}$ also satisfies the second and third conditions of a Markov partition. On the other hand, as $F|_P = f^{\mathrm{ord}(P)}|_P$ and P is an $\widetilde{\mathcal{E}}_{\mathcal{Z}}$-pre-image of order $n = \mathrm{ord}(P)$, there is a zooming pre ball $V_n(x)$, $x \in \mathrm{Z}_n(\alpha, \delta, f)$, containing P and $F|_P$ can be extended to a homeomorphism between $\overline{P}$ and $\overline{\Delta}$ (because $f^n|_{\overline{V_n(x)}}$ is a homeomorphism). Given $x \in \bigcap_{n \geqslant 0} F^{-n}(\bigcup_{P \in \mathcal{P}} P)$, set $P_j = \mathcal{P}(F^j(x))$. As $\mathrm{diameter}(\mathcal{P}_n(x)) = \mathrm{diameter}(F|_{P_1}^{-1} \circ F|_{P_2}^{-1} \circ \cdots \circ F|_{P_n}^{-1}(\Delta)) < \prod_{j=1}^n \alpha_{\mathrm{ord}(P_j)}(\mathrm{diameter}(\Delta)) \leqslant \alpha_{\sum_{j=1}^n \mathrm{ord}(P_j)}(\mathrm{diameter}(\Delta)) \to 0$, we conclude that $\mathcal{P}$ is a Markov partition for F. Finally, as $\{R > 0\} = \bigcup_{P \in \mathcal{P}} P$ and $F(P) = \Delta$ $\forall P \in \mathcal{P}$, it follows that the Markov map $(F, \mathcal{P})$ is indeed an induced full Markov map. □

Let μ be an (α, δ)-weak zooming measure with $\mu(X \setminus \Lambda) = 0$ and let $U \subset X$ be a μ ergodic component. Let A be the attractor associated to U and $A_{\widetilde{\mathfrak{Z}}} \subset A$ the compact set such that $\omega_{f,\widetilde{\mathfrak{Z}}}(x) = A_{\widetilde{\mathfrak{Z}}}$ for μ-almost every point $x \in U$ (given by Proposition 3.5 and by Lemma 3.9 applied to $\mathcal{U} = \widetilde{\mathfrak{Z}}$).

Lemma 6.7. *Let $(F, \mathcal{P})$ be as in Corollary* 6.6 *and suppose that $\Delta \cap A_{\widetilde{\mathfrak{Z}}} \neq \emptyset$. Then $(F, \mathcal{P})$ is an induced full Markov map defined on Δ and it is compatible with $\mu|_U$.*

Proof. Let $p \in \Delta \cap A_{\widetilde{\mathfrak{Z}}}$. As $p \in \omega_{f,\widetilde{\mathfrak{Z}}}(x)$ for μ almost every $x \in U$, we get $\mu|_U(U \setminus \bigcup_{n \geqslant 0} f^{-n}(\Delta)) = 0$. Thus, as $\mu|_U \circ f^{-1} \ll \mu|_U$, $\mu|_U(\Delta) > 0$.

By Corollary 6.6, we only need to show that $\mu|_U(\Delta \setminus \bigcup_{P \in \mathcal{P}} P) = \mu((\Delta \setminus \bigcup_{P \in \mathcal{P}} P) \cap U) = 0$. As $p \in \omega_{f,\widetilde{\mathfrak{Z}}}(x)$ for μ almost every $x \in U$, it follows that $\Omega(x) \neq \emptyset$ for μ almost every $x \in \Delta$. Thus, $\mu|_U(\{R = 0\}) = \mu|_U(\Delta \setminus \bigcup_{P \in \mathcal{P}} P) = 0$. □

Theorem 3. *Suppose that for some $0 < r_0 < \delta/2$ and every x the (α, δ)-zooming nested ball $B^{\star}_{r_0}(x)$ is well defined and contains $B_{r_0/2}(x)$. Let $\Lambda \subset \limsup_n \mathrm{Z}(\alpha, \delta, f)$ be a positively invariant set and $\widetilde{\mathfrak{Z}} = (\widetilde{\mathfrak{Z}}(x))_{x \in \Lambda}$ a proper (α, δ)-zooming sub-collection. Let μ be an (α, δ)-weak zooming probability with bounded distortion and $\mu(\Lambda) = 1$. Let $U \subset X$ an ergodic component for μ and $A_{\widetilde{\mathfrak{Z}}}$ be the compact set such that $\omega_{f,\widetilde{\mathfrak{Z}}}(x) = A_{\widetilde{\mathfrak{Z}}}$ for μ-almost every point $x \in U$ (given by Theorem* 2*). Let Δ be an (α, δ)-zooming nested open set with* $\mathrm{diameter}(\Delta) \leqslant r_0/2$ *and such that $\Delta \cap A_{\widetilde{\mathfrak{Z}}} \neq \emptyset$.*

If $(F, \mathcal{P})$ is the induced Markov map associated to "the first $\widetilde{\mathcal{E}}_{\mathcal{Z}}$-return time to Δ" *(as in Corollary* 6.6*) then $(F, \mathcal{P})$ is an induced full Markov map with μ-bounded distortion. Furthermore, there exists $\nu \ll \mu$ an ergodic F-invariant probability with* $\log \frac{d\nu}{d\mu} \in L^{\infty}(\mu|_{\{\frac{d\nu}{d\mu} > 0\}})$ *and $\nu(\Delta) = 1$.*

Proof. Let us show that, as μ has bounded distortion, $\mu|_U(\Delta) = \mu(\Delta)$. To prove this, let $p \in \Delta \cap A_{\widetilde{\mathfrak{Z}}}$. By Lemma 5.15, $\mu(B_{r_0/2}(p) \cap U) = \mu(B_{r_0/2}(p))$. As $\mathrm{diameter}(\Delta) \leqslant r_0/2$, $\Delta \subset B_{r_0/2}(p)$. So, $\mu|_U(\Delta) = \mu(\Delta)$.

As $\mu|_U(\Delta) = \mu(\Delta)$, Lemma 6.7 implies that $(F, \mathcal{P})$ is an induced full Markov map defined on Δ compatible with μ.

Finally, as $|\log \frac{J_\mu F(x)}{J_\mu F(y)}| \leqslant \rho \operatorname{dist}(F(x), F(y))$, $\forall x, y \in P$ and $\forall P \in \mathcal{P}$ (because P is contained in a zooming pre-ball of order $R(P)$ and μ has bounded distortion at the zooming times), we obtain that $(F, \mathcal{P})$ has μ-bounded distortion.

Applying Proposition 4.4, we obtain an F-invariant ergodic probability $\nu \ll \mu$ with $\log \frac{d\nu}{d\mu} \in L^{\infty}(\mu|_{\{\frac{d\nu}{d\mu} > 0\}})$ and, of course, $\nu(\Delta) = 1$. □

Given $\theta > 0$ and $n \in \mathbb{N}$, let $\mathcal{Z}_n(\alpha, \delta, \theta, f)$ be the set of points $x \in X$ such that $\#\{1 \leqslant j \leqslant n;\ x \in \mathrm{Z}_j(\alpha, \delta, f)\} \geqslant \theta n$. Thus, the set of points of X with infinitely many moments with θ-frequency of (α, δ)-zooming times (with respect to f) is

$$\limsup_n \mathcal{Z}_n(\alpha, \delta, \theta, f) = \bigcap_{j=1}^{+\infty} \bigcup_{n \geqslant j} \mathcal{Z}_n(\alpha, \delta, \theta, f).$$

If μ is an (α,δ)-zooming measure with bounded distortion, X can be decomposed in a finite collection of $\{U_1,\dots,U_s\}$ of μ-ergodic components (Theorem 2). By ergodicity, $\exists\theta_i>0$ such that

$$\limsup \frac{1}{n}\#\big\{1\leqslant j\leqslant n;\ x\in \mathrm{Z}_j(\alpha,\delta,f)\big\}\geqslant\theta_i$$

for μ almost every $x\in U_i$ $\forall i$. Furthermore, if $\widetilde{\mathfrak{z}}=(\widetilde{\mathfrak{z}}(x))_{x\in\Lambda}$ is a proper zooming sub-collection and $\mu(X\setminus\Lambda)=0$, there are also $\widetilde{\theta}_1,\dots,\widetilde{\theta}_s>0$ such that

$$\limsup \frac{1}{n}\#\{1\leqslant j\leqslant n;\ j \text{ is a } \widetilde{\mathfrak{z}}\text{-time to } x\}\geqslant\widetilde{\theta}_i$$

for μ almost every $x\in U_i$ and all $1\leqslant i\leqslant s$. Thus, we get the following remark.

Remark 6.8. Let Λ be a zooming set and $\widetilde{\mathfrak{z}}=(\widetilde{\mathfrak{z}}(x))_{x\in\Lambda}$ a proper zooming sub-collection. Let μ be a zooming measure with $\mu(X\setminus\Lambda)=0$. If μ has bounded distortion or, more in general, has a finite number of ergodic components then $\exists\widetilde{\theta}>0$ such that

$$\limsup \frac{1}{n}\#\{j\leqslant n;\ \text{is a } \widetilde{\mathfrak{z}}\text{-time to } x\}\geqslant\widetilde{\theta}$$

for μ almost every $x\in X$. In particular, for every zooming measure μ with bounded distortion (or having a finite number of ergodic components) there is $\theta>0$ such that

$$\mu\Big(X\setminus\limsup_m \mathcal{Z}_m(\alpha,\delta,\theta,f)\Big)=0.$$

Theorem 4. *Suppose that for some $0<r_0<\delta/2$ and every x the (α,δ)-zooming nested ball $B^{\star}_{r_0}(x)$ is well defined and contains $B_{r_0/2}(x)$. Let $\Lambda\subset X$ be an (α,δ)-zooming set and μ an ergodic f-invariant zooming probability with $\mu(\Lambda)=1$. Let $\widetilde{\mathfrak{z}}=(\widetilde{\mathfrak{z}}(x))_{x\in\Lambda}$ be a proper (α,δ)-zooming sub-collection and $A_{+,\widetilde{\mathfrak{z}}}$ the compact set such that $\omega_{+,f,\widetilde{\mathfrak{z}}}(x)=A_{+,\widetilde{\mathfrak{z}}}$ for μ-almost every point $x\in X$ (given by Lemma 3.9 applied to $\mathcal{U}=\widetilde{\mathfrak{z}}$). Let Δ be an (α,δ)-zooming nested open set with* diameter$(\Delta)\leqslant r_0/2$ *and such that $\Delta\cap A_{+,\widetilde{\mathfrak{z}}}\neq\emptyset$.*

If R is "the first $\widetilde{\mathcal{E}}_{\mathcal{Z}}$-return time to Δ" *and $(F,\mathcal{P})$ is the induced Markov map associated to R (as in Corollary 6.6) then $(F,\mathcal{P})$ is a full induced Markov map compatible with μ and there exists an F-invariant finite measure $\nu\ll\mu$ (indeed, $\nu(Y)\leqslant\mu(Y)$ for every Borel set $Y\subset\Delta$) such that $\int R\,d\nu<+\infty$ and*

$$\mu=\frac{1}{\gamma}\sum_{j=0}^{+\infty}f_*^j(\nu|_{\{R>j\}}),$$

where $\gamma=\sum_{j=0}^{+\infty}f_*^j(\nu|_{\{R>j\}})(X)$.

Proof. Let $A_{\widetilde{\mathfrak{z}}}$ be the compact set (given by Lemma 3.9) such that $\omega_{f,\widetilde{\mathfrak{z}}}(x)=A_{\widetilde{\mathfrak{z}}}$ for μ-almost every $x\in X$. As $A_{+,\widetilde{\mathfrak{z}}}\subset A_{\widetilde{\mathfrak{z}}}$, we have $\Delta\cap A_{\widetilde{\mathfrak{z}}}\neq\emptyset$. Thus, it follows from Lemma 6.7 that $(F,\mathcal{P})$ is an induced full Markov map defined on Δ and compatible with μ (see Definition 4.1).

Let $\mathfrak{B}=\{x\in\Delta;\ F^j(x)\in\bigcup_{P\in\mathcal{P}}P,\ \forall j\geqslant 0\}$. Because μ is f-invariant (in particular, f-non-singular), we get $\Delta=\mathfrak{B}$ (mod μ). As $\Delta\cap A_{+,\widetilde{\mathfrak{z}}}\neq\emptyset$ and μ is f-ergodic, there is $\Theta>0$ such that

$$\limsup_{n\to\infty}\frac{1}{n}\#\big\{1\leqslant j\leqslant n;\ x\in G_j \text{ and } f^j(x)\in\Delta\big\}\geqslant\Theta$$

for μ almost every $x\in\Delta$, where $G_j=\{x\in\Lambda;\ j \text{ is a } \widetilde{\mathfrak{z}}\text{-time to } x\}$. Thus, taking $B=\Delta$, $g=R$ and applying the first part of Lemma 4.7 to f we get

$$\limsup_{n\to\infty}\frac{1}{n}\#\left\{j\geqslant 0;\ \sum_{k=0}^{j}R\circ F^k(x)\leqslant n\right\}\geqslant\Theta \tag{30}$$

for μ almost every $x\in\Delta$. Because

$$\left\{j \geqslant 0;\ \sum_{k=0}^{j} R \circ F^k(x) < n\right\} = \left\{0 \leqslant j < n;\ f^j(x) \in \mathcal{O}_F^+(x)\right\},$$

it follows form (30) and Theorem 1 that there exists a non-trivial F-invariant measure such that $\nu(Y) \leqslant \mu(Y)$ for every Borel set $Y \subset \Delta$ (in particular, $\nu \ll \mu$) with $\int R d\nu < +\infty$. Thus, $\eta = \sum_{j=0}^{+\infty} f_*^j(\nu|_{\{R>j\}})$ is an f-invariant finite measure (see Remark 4.3). Note that, if $\eta(Y) > 0$ for some Borel set $Y \subset X$ then $\nu(f^{-j}(Y)) > 0$ for some $j \geqslant 0$ and, as $\nu \ll \mu$, $\mu(Y) = \mu(f^{-j}(Y)) > 0$. Thus, $\eta \ll \mu$. As μ is f-ergodic probability, we get

$$\mu = \frac{1}{\eta(X)}\eta = \frac{1}{\eta(X)} \sum_{j=0}^{+\infty} f_*^j(\nu|_{\{R>j\}}). \qquad \square$$

Lemma 6.9. *For every $k \geqslant 1$,*

$$\begin{aligned}\limsup_m \mathcal{Z}_m\big(\{\alpha_n\}_n, \delta, \theta, f\big) &\subset \bigcup_{j=0}^{k-1} f^{-j}\Big(\limsup_m \mathcal{Z}_{km}\big(\{\alpha_n\}_n, \delta, \theta/k, f\big)\Big)\\ &\subset \bigcup_{j=0}^{k-1} f^{-j}\Big(\limsup_m \mathcal{Z}_m\big(\{\alpha_{kn}\}_n, \delta, \theta/k, f^k\big)\Big).\end{aligned}$$

Proof. Let $k \geqslant 1$. For each $x \in \limsup_m \mathrm{Z}_m(\{\alpha_n\}_n, \delta, f)$ and $0 \leqslant i < k$, set $\mathbb{N}_x(i) = \{kj + i;\ j \in \mathbb{N}$ and $x \in \mathrm{Z}_{kj+i}(\{\alpha_n\}_n, \delta, f)\}$. So, $x \in \mathrm{Z}_m(\{\alpha_n\}_n, \delta, f) \Leftrightarrow m \in \bigcup_{i=0}^{k-1} \mathbb{N}_x(i)$. Note also that $\mathbb{N}_x(j) \cap \mathbb{N}_x(i) = \emptyset$, whenever $i \neq j$.

So, for each $x \in \limsup \mathcal{Z}_m(\{\alpha_n\}_n, \delta, \theta, f) \subset \limsup \mathrm{Z}_m(\{\alpha_n\}_n, \delta, f)$ one can choose $\ell(x) \in \{0, \dots, k-1\}$ such that $\limsup_m \frac{1}{m}\#\{1 \leqslant j \leqslant m;\ j \in \mathbb{N}_x(\ell(x))\} \geqslant \theta/k$. Otherwise, $\limsup_m \frac{1}{m}\#\{1 \leqslant j \leqslant m;\ x \in \mathrm{Z}_j(\{\alpha_n\}_n, \delta, f)\} < \theta$, contradicting $x \in \limsup \mathcal{Z}_m(\{\alpha_n\}_n, \delta, \theta, f)$. As $j \in \mathbb{N}_x(\ell(x)) \Leftrightarrow x \in \mathrm{Z}_{jk+\ell(x)}(\{\alpha_n\}_n, \delta, f) \Leftrightarrow f^{\ell(x)}(x) \in \mathrm{Z}_{jk}(\{\alpha_n\}_n, \delta, f)$, it follows from Lemma 5.7 that

$$j \in \mathbb{N}_x\big(\ell(x)\big) \quad \Rightarrow \quad f^{\ell(x)}(x) \in \mathrm{Z}_{kj}\big(\{\alpha_n\}_n, \delta, f\big) \subset \mathrm{Z}_j\big(\{\alpha_{kn}\}_n, \delta, f^k\big).$$

Therefore,

$$\begin{aligned}&\limsup_m \frac{1}{m}\#\big\{1 \leqslant j \leqslant m;\ f^{\ell(x)}(x) \in \mathrm{Z}_{kj}\big(\{\alpha_n\}_n, \delta, f\big)\big\}\\ &\quad \geqslant \limsup_m \frac{1}{m}\#\big\{1 \leqslant j \leqslant m;\ f^{\ell(x)}(x) \in \mathrm{Z}_j\big(\{\alpha_{kn}\}_n, \delta, f^k\big)\big\} \geqslant \theta/k.\end{aligned}$$

As a consequence, if $x \in \limsup_m \mathrm{Z}_m(\{\alpha_n\}_n, \delta, f)$ then $f^{\ell(x)}(x) \in \limsup_m \mathrm{Z}_m(\{\alpha_{kn}\}_n, \delta, \theta/k, f^k)$, with $0 \leqslant \ell(x) < k$. $\square$

Corollary 6.10. *Let μ be an f-ergodic (α, δ)-zooming measure (not necessarily invariant). For each $k > 0$ there is a positively invariant set $E \subset X$ with μ positive measure and such that $\frac{1}{\mu(E)}\mu|_E$ is an $(\{\alpha_{kn}\}, \delta)$-zooming ergodic probability with respect to f^k. Furthermore, if μ is f-invariant then E is a μ-ergodic component with respect to f^k, $\frac{1}{\mu(E)}\mu|_E$ is f^k-invariant and $\mu = (\sum_{j=0}^{k-1} f^j{}_*)\mu|_E$.*

Proof. It follows from Remark 6.8 that $\exists\theta > 0$ such that $\mu(X \setminus \limsup_n \mathcal{Z}_n(\alpha, \delta, \theta, f)) = 0$. By Lemma 6.9, there is $0 \leqslant j < k$ such that $\mu(f^{-j}(\widetilde{\mathcal{Z}})) > 0$, where $\widetilde{\mathcal{Z}} = \limsup_n \mathcal{Z}_n(\{\alpha_{kn}\}, \delta, \theta/k, f^k)$. As μ is f-non-singular (by the definition of zooming measure), we get $\mu(\widetilde{\mathcal{Z}}) > 0$.

Because μ has at most k ergodic components with respect to f^k (Lemma 3.13), there is one of these ergodic components $E_0 \subset X$ such that $\mu(E_0 \cap \widetilde{\mathcal{Z}}) > 0$. Set $E = E_0 \cap \widetilde{\mathcal{Z}}$. As $\widetilde{\mathcal{Z}} \supset f^k(\widetilde{\mathcal{Z}})$ and $f^{k^{-1}}(E_0) = E_0$, we have $f^k(E) \subset E \subset E_0$. Because $\mu|_{E_0}$ is f^k-ergodic and f^k-non-singular, it follows from Lemma 3.10 that $\frac{1}{\mu(E)}\mu|_E = \frac{1}{\mu(E)}(\mu|_{E_0})|_E$ is a probability f^k-non-singular and f^k-ergodic. Of course $\frac{1}{\mu(E)}\mu|_E(\widetilde{\mathcal{Z}}) = 1$ and so, $\frac{1}{\mu(E)}\mu|_E$ is an $(\{\alpha_{kn}\}, \delta)$-zooming ergodic probability with respect to f^k.

Suppose now that μ is f-invariant. In this case, as $E \subset f^{-k}(E)$, it follows that $E = f^{-k}(E)$ (mod μ). Thus, changing E by $\bigcap_{j\geqslant 0} f^{-jk}(E)$, it follows that E is a μ-ergodic component with respect to f^k. So, it is easy to conclude that $\mu|_E$ is f^k-invariant and $\mu = (\sum_{j=0}^{k-1} f^j{}_*)\mu|_E$. □

Theorem C *(Existence of invariant zooming measures). If μ is a zooming measure with bounded distortion then there exists a finite collection of ergodic f-invariant probabilities absolutely continuous with respect to μ such that μ-almost every point in X belongs to the basin of one of these probabilities.*

Proof of Theorem C. By Theorem 2, X can be partitioned in a finite collection of μ-ergodic components with respect to f. Let U be one of these ergodic components. Choose any $0 < r_0 < \delta/2$ and let $k \geqslant 1$ be such that $\sum_{n=1}^{+\infty} \alpha_{kn}(\widetilde{r}) < \widetilde{r}/4$ for $\widetilde{r} = r_0/4$ and for $\widetilde{r} = r_0$.

By Lemma 3.13, U can be decomposed into a finite collection of disjoint ergodic components with respect to f^k. As U is invariant (in particular, $f(U) \subset U$), it follows from Lemma 3.10 that $\mu|_U$ is f-non-singular. Thus, $\mu|_U$ is an ergodic (α, δ)-zooming measure. From Corollary 6.10, there is $E \subset U \subset X$, with $f(E) \subset E$ and $\mu|_U(E) > 0$, such that $\widetilde{\mu} = \frac{1}{\mu(E)}\mu|_E = \frac{1}{\mu|_U(E)}(\mu|_U)|_E$ is an $(\widetilde{\alpha}, \delta)$-zooming ergodic probability with respect to $\widetilde{f} = f^k$, where $\widetilde{\alpha} = \{\alpha_{kn}\}_n$.

Denote the set of $(\widetilde{\alpha}, \delta)$-zooming images of $\widetilde{f}$ by $\widetilde{\mathfrak{z}} = (\widetilde{\mathfrak{z}}(x))_{x\in\Lambda}$, where $\widetilde{\mathfrak{z}}(x) = \{\widetilde{f}^n(x);\ n \in \mathbb{N}$ and $x \in \mathrm{Z}_n(\widetilde{\alpha}, \delta, \widetilde{f})\}$ is the set of $(\widetilde{\alpha}, \delta)$-zooming images of x by $\widetilde{f}$ and $\Lambda = \limsup \mathrm{Z}_n(\widetilde{\alpha}, \delta, \widetilde{f})$.

By Theorem 2, there exists a fat attractor A (with respect to $\widetilde{f}$) such that $\omega_{\widetilde{f}}(x) = A$ for $\widetilde{\mu}$-almost every point $x \in X$. Moreover, there are compact sets $A_{+,\widetilde{\mathfrak{z}}}, A_{\widetilde{\mathfrak{z}}} \subset A$, with $A_{+,\widetilde{\mathfrak{z}}} \subset A_{\widetilde{\mathfrak{z}}}$, such that $\omega_{\widetilde{f},\widetilde{\mathfrak{z}}}(x) = A_{\widetilde{\mathfrak{z}}}$ and $\omega_{+,\widetilde{f},\widetilde{\mathfrak{z}}}(x) = A_{+,\widetilde{\mathfrak{z}}}$ for $\widetilde{\mu}$-almost every point $x \in X$.

Let $r = r_0/4$ and choose any point $q \in A_{+,\widetilde{\mathfrak{z}}}$. As $A_{+,\widetilde{\mathfrak{z}}} \subset A_{\widetilde{\mathfrak{z}}}$, we get $B_r^\star(q) \cap A_{\widetilde{\mathfrak{z}}} \supset B_r^\star(q) \cap A_{+,\widetilde{\mathfrak{z}}} \neq \emptyset$, where $B_r^\star(q)$ is the $(\widetilde{\alpha}, \delta)$-zooming nested ball with respect to $\widetilde{f}$, radius r and center on q (see Definition 5.9 and Lemma 5.12).

Taking $\Delta = B_r^\star(q)$ and $\widetilde{\mathcal{E}}$ as the collection of all $(\widetilde{\alpha}, \delta)$-zooming pre-balls with respect to $\widetilde{f}$ (see Notation 5.8), let R be the first $\widetilde{\mathcal{E}}$-return time to Δ (with respect to $\widetilde{f}$) given by (27), let $F = \widetilde{f}^R$ be induced map associated to R and let $\mathcal{P}$ be the Markov partition given by (29).

Applying Theorem 3 to $\widetilde{f}, \widetilde{\alpha}, \Delta = B_r^\star(q)$ (note that diameter$(\Delta) = r_0/2$), $(F, \mathcal{P})$ and $\widetilde{\mu}$, we obtain an F-invariant measure $\nu \ll \widetilde{\mu}$. To prove the existence of an $\widetilde{f}$-invariant ergodic probability $\widetilde{\eta} \ll \widetilde{\mu}$ we need only to show that the induced time R is ν integrable (see Proposition 4.4).

Let $\mathfrak{B} = \{x \in B_r^\star(q);\ F^j(x) \in \bigcup_{P\in\mathcal{P}} P\ \forall j \geqslant 0\}$. Note that $\widetilde{\mu}(B_r^\star(q) \setminus \mathfrak{B}) = \nu(B_r^\star(q) \setminus \mathfrak{B}) = 0$. Let B be the set of $x \in \mathfrak{B}$ such that $\limsup_n \frac{1}{n}\#\{1 \leqslant j \leqslant n;\ x \in \mathrm{Z}_j(\widetilde{\alpha}, \delta, \widetilde{f})$ and $\widetilde{f}^j(x) \in B\} > 0$. As $B_r^\star(q) \cap A_{+,\widetilde{\mathfrak{z}}} \neq \emptyset$, $\nu(B \setminus \mathfrak{B}) = \widetilde{\mu}(B \setminus \mathfrak{B}) = 0$.

Taking $G_j = \mathrm{Z}_j(\widetilde{\alpha}, \delta, \widetilde{f})$, $g = R$ and $T = F$, it follows from Lemma 4.7 that

$$\liminf_n \frac{1}{n}\sum_{j=0}^{n-1} R \circ F^j(x) < +\infty$$

for every $x \in B$. By Birkhoff's Ergodic Theorem, $\int R\, d\nu = \lim_n \frac{1}{n}\sum_{j=0}^{n-1} R \circ F^j(x)$ for ν-almost every $x \in B$. Thus, $\int R\, d\nu < +\infty$. As a consequence, the projection $\widetilde{\eta} = \sum_{P\in\mathcal{P}} \sum_{j=0}^{R(P)-1} \widetilde{f}_*^j(\nu|_P)$ is a $\widetilde{\mu}$ absolutely continuous $\widetilde{f}$-invariant finite measure.

Taking $\eta = \frac{1}{k}\sum_{j=0}^{k-1} f_*^j \widetilde{\eta}$, it is easy to see that η is f-invariant finite measure and $\eta \ll \mu$. So, to finish the proof of the theorem we only need to verify that U belongs to the basin of η.

By Birkhoff's Theorem, $\eta(\mathcal{B}(\eta) \cap U) = \eta(\mathcal{B}(\eta)) > 0$ and, as $\eta \ll \mu$, we get $\mu(\mathcal{B}(\eta) \cap U) > 0$. As $\mathcal{B}(\eta)$ is an f invariant set and U is a μ ergodic component with respect to f, we conclude that $U = \mathcal{B}(\eta)$ (mod μ). □

Before we begin the proof of Theorem D which gives the existence of a Markov structure for a zooming set, we want to emphasize a difference between the proof of Theorem C and proof of Theorem D.

In both proofs we begin with a reference measure μ and we need to show the existence of an induced invariant measure $\nu \ll \mu$ and also the ν integrability of the inducing time R. In the hypothesis of Theorem C, we have a zooming measure μ with bounded distortion, but we do not know if μ is invariant. On the other hand, in Theorem D, we want to study zooming measures for which we do not know anything about distortion, but we know that they are invariant measures. In the proof of Theorem C the existence of ν is given by Proposition 4.4 (this proposition is used to prove Theorem 3) and in the proof of Theorem D the existence of ν is assured by Theorem 1 (this theorem is central in the proof of Theorem 4). In both case, the estimate to get the integrability of the inducing time is given by Lemma 4.7 (this lemma appears in the proof of Theorems 3 and 4).

Theorem D *(Markov Structure for the zooming set). Every zooming set Λ admits a finite Markov structure $\mathfrak{F} = \{(F_1, \mathcal{P}_1), \dots, (F_s, \mathcal{P}_s)\}$. Furthermore, each $(F_i, \mathcal{P}_i) \in \mathfrak{F}$ is a full Markov map defined on some connected open set U_i (also the elements of $\mathcal{P}_i$ are connected open sets). Furthermore, denoting the induced time of F_j by R_j, $\forall j$, $\mathfrak{F}$ has the following additional properties.*

(1) *There is some $m_0 \geqslant 1$ such that each F_j is defined on an $(\{\alpha_{m_0 n}\}_n, \delta)$-zooming nested ball $B^\star_j$ with respect to f^{m_0} ($m_0 = 1$, if f is backward separated). Moreover,*

$$R_j(x) \leqslant \min\big\{n \geqslant 1;\ x \in \mathrm{Z}_{m_0 n}(\alpha, \delta, f) \text{ and } f^{m_0 n}(x) \in B^\star_j\big\}.$$

(2) *Each $P \in \mathcal{P}_j$ is a connected open set, $\forall 1 \leqslant j \leqslant s$. Furthermore, there is an (α, δ)-pre-ball V_n with respect to f, where $n = R_j|_P$, such that $P = (f^n|_{V_n})^{-1}(\mathcal{P}_j) \subset V_n$. In particular,*

$$\operatorname{dist}\big(f^\ell(x), f^\ell(y)\big) \leqslant \bigg(\sum_{k>n-\ell} \alpha_k\bigg) \operatorname{dist}\big(F_j(x), F_j(y)\big) \quad \forall x, y \in P.$$

(3) *If μ is an ergodic (not necessarily invariant) zooming measure with bounded distortion and $\mu(B^*_j) > 0$ then $\exists K > 0$ such that*

$$\left|\log \frac{J_\mu F_j(x)}{J_\mu F_j(y)}\right| \leqslant K \operatorname{dist}\big(F_j(x), F_j(y)\big),$$

for μ almost every $x, y \in P$, all $P \in \mathcal{P}_j$ and $1 \leqslant j \leqslant s$.

Proof of Theorem D. If f is backward separated and $\sup_{r>0} \sum_{n\geqslant 1} \alpha_n(r)/r < \infty$, choose any $0 < r \leqslant \frac{1}{4} r_0$, where r_0 is given by Lemma 5.12, and set $m_0 = 1$. Otherwise, choose $0 < r_0 < \delta/2$, set $r = r_0/4$ and let m_0 be an integer big enough such that $\sum_{n\geqslant 1} \alpha_{m_0 n}(\widetilde{r}) < \widetilde{r}/4$ for $\widetilde{r} = r_0$ and $\widetilde{r} = r$. Set also $\widetilde{f} = f^{m_0}$, $\widetilde{\theta} = \theta/m_0$, $\widetilde{\alpha}_j = \alpha_{m_0 j}$ and $\widetilde{\alpha} = \{\widetilde{\alpha}_j\}_j$.

Let $\widetilde{\mathfrak{z}}$ be the proper zooming sub-collection for $\widetilde{f}$ given by $\widetilde{\mathfrak{z}} = (\widetilde{\mathfrak{z}}(x))_{x \in \widetilde{\Lambda}}$, where

$$\widetilde{\mathfrak{z}}(x) = \big\{\widetilde{f}^n(x);\ n \in \mathbb{N} \text{ and } x \in \mathrm{Z}_{m_0 n}(\alpha, \delta, f)\big\}$$

and

$$\widetilde{\Lambda} = \Lambda \cap \limsup_{n\to\infty} \mathrm{Z}_{m_0 n}(\alpha, \delta, f) \subset \Lambda \cap \limsup_{n\to\infty} \mathrm{Z}_n(\widetilde{\alpha}, \delta, \widetilde{f}).$$

One can easily check that $\widetilde{\mathfrak{z}}$ is indeed a proper $(\widetilde{\alpha}, \delta)$-zooming collection with respect to $\widetilde{f}$ (see comments just below Definition 6.1).

As X is compact, one can find $q_1, q_2, \dots, q_s \in X$ such that $\{B^\star_r(q_1), \dots, B^\star_r(q_s)\}$ is a finite cover of X by $(\widetilde{\alpha}, \delta)$-zooming nested balls, with respect to $\widetilde{f}$.

For each $1 \leqslant j \leqslant s$, let $\widetilde{F}_j : B^\star_r(q_j) \to B^\star_r(q_j)$ be the induced map (with respect to $\widetilde{f}$) on $B^\star_r(q_j)$ associated to "*the first $\widetilde{\mathcal{E}}$-return time to $B^\star_r(q_j)$*" given by (28), where $\widetilde{\mathcal{E}}$ is the collection of all $\widetilde{\mathfrak{z}}$-pre-balls $V_n(x, \widetilde{f})$ (with respect to $\widetilde{f}$) such that $x \in \Lambda$ and $n \geqslant 1$ is a $\widetilde{\mathfrak{z}}$-time (with respect to $\widetilde{f}$) for x. Note that $V_n(x, \widetilde{f})$ is an $(\widetilde{\alpha}, \delta)$-zooming pre-ball of order n with respect to $\widetilde{f}$ and also an (α, δ)-zooming pre-ball of order $m_0 n$ with respect to f

Let $\widetilde{R}_j$ be the inducing time of $\widetilde{F}_j$ (with respect to $\widetilde{f}$) and let $\mathcal{P}_j$ be the Markov partition associated to "*the first $\widetilde{\mathcal{E}}$-return time to $B^\star_r(q_j)$*" (see (29)).

Set, for every $x \in B^\star_r(q_j)$, $F_j(x) = f^{m_0 \widetilde{R}_j(x)}(x)$. As $\widetilde{f} = f^{m_0}$, it is easy to see that $(F_j, \mathcal{P}_j)$ is also an induced full Markov map with respect to f with inducing time $R_j = m_0 \widetilde{R}_j$. Of course, as it is a Markov map, $(F_j, \mathcal{P}_j) = (\widetilde{F}_j, \mathcal{P}_j)$.

Remark 6.11. Note that $\mathfrak{F} = \{(F_1, \mathcal{P}_1), \dots, (F_s, \mathcal{P}_s)\}$ is a finite collection of induced full Markov maps for f and satisfies the following additional properties.

(1) Each F_j is defined on a connected open sets. Indeed, each F_j is defined on an $(\widetilde{\alpha}, \delta)$-zooming nested ball with respect to $\widetilde{f}$.
(2) For every $1 \leqslant j \leqslant s$, each $P \in \mathcal{P}_j$ is a connected open set. Furthermore, setting $n = \widetilde{R_j}|_P$, we get $P \subset V_n(x, \widetilde{f})$ for some $x \in P \cap \mathrm{Z}_{m_0 n}(\alpha, \delta, f) \cap \Lambda$, where $V_n(x, \widetilde{f})$ is an (α, δ)-zooming pre-ball of order $m_0 n$ with respect to f (and also a $(\widetilde{\alpha}, \delta)$-zooming pre-ball of order n with respect to $\widetilde{f}$).

Thus, to finish the proof we only need to show that $\mathfrak{F}$ is a Markov structure for Λ.

Let μ be an ergodic f-invariant probability such that $\mu(\Lambda) > 0$. By ergodicity, $\mu(\Lambda) = 1$ (we are also using that Λ is positively invariant and μ is invariant). As μ is f-non-singular (because μ is f-invariant) and Λ is an (α, δ)-zooming set, it follows that μ is an (α, δ)-zooming measure.

It follows from Corollary 6.10 that there is a μ-ergodic component $U \subset X$ with respect to $\widetilde{f}$ such that $\widetilde{\mu} = \frac{1}{\mu(U)}\mu|_U$ is an $(\widetilde{\alpha}, \delta)$-zooming ergodic invariant probability with respect to $\widetilde{f}$ and $\mu = (\sum_{j=0}^{m_0-1} f^j{}_*)\mu|_U$.

By Proposition 3.5, there exists an $\widetilde{f}$-attractor $A \subset X$ which attracts $\widetilde{\mu}$-almost every point of X and such that $\omega_{\widetilde{f}}(x) = A$ for $\widetilde{\mu}$-almost every $x \in X$ (indeed, as $\widetilde{\mu}$ is $\widetilde{f}$-invariant, $A = \operatorname{supp}\widetilde{\mu}$). By Lemma 3.9, there are compact sets $A_{+,\widetilde{\mathfrak{z}}}$ and $A_{\widetilde{\mathfrak{z}}}$, with $A_{+,\widetilde{\mathfrak{z}}} \subset A_{\widetilde{\mathfrak{z}}} \subset A$, such that $\omega_{\widetilde{f},\widetilde{\mathfrak{z}}}(x) = A_{\widetilde{\mathfrak{z}}}$ and $\omega_{+,\widetilde{f},\widetilde{\mathfrak{z}}}(x) = A_{+,\widetilde{\mathfrak{z}}}$ for $\widetilde{\mu}$-almost every $x \in X$ (see Definitions 3.7 and 3.8).

Let $1 \leqslant j_0 \leqslant s$ be such that $B_r^\star(q_{j_0}) \cap A_{+,\widetilde{\mathfrak{z}}} \neq \emptyset$. It follows from Theorem 4 that $(\widetilde{F}_{j_0}, \mathcal{P}_{j_0})$ is an induced full Markov map (with respect to $\widetilde{f}$) defined on $B_r^\star(q_{j_0})$ and compatible with $\widetilde{\mu}$. As a consequence, $(F_{j_0}, \mathcal{P}_{j_0})$ is an induced full Markov map with respect to f (defined on $B_r^\star(q_{j_0})$ and compatible with $\widetilde{\mu}$). Also by Theorem 4, there exists an $\widetilde{F}_{j_0}$-invariant measure $\nu \ll \widetilde{\mu}$ such that

$$\widetilde{\mu} = \frac{1}{\widetilde{\gamma}} \sum_{j=0}^{+\infty} \widetilde{f}_*^j (\nu|_{\{\widetilde{R}_{j_0} > j\}}),$$

where $\widetilde{\gamma} = \sum_{j=0}^{+\infty} \widetilde{f}_*^j (\nu|_{\{\widetilde{R}_{j_0} > j\}})(X)$.

It follows from the relation $R_{j_0} = m_0 \widetilde{R}_{j_0}$ that

$$\{R_{j_0} > m_0 j + k\} = \{R_{j_0} > m_0 j\}, \tag{31}$$

$\forall\, 0 \leqslant k < m_0$ and $\forall j \geqslant 0$.

Setting $\gamma = \widetilde{\gamma}/\mu(U)$, we get

$$\begin{aligned}
\mu &= \left(\sum_{k=0}^{m_0-1} f^k{}_*\right)\mu|_U = \mu(U)\left(\sum_{k=0}^{m_0-1} f^k{}_*\right)\widetilde{\mu} \\
&= \frac{\mu(U)}{\widetilde{\gamma}}\left(\sum_{k=0}^{m_0-1} f^k{}_*\right)\sum_{j=0}^{+\infty} \widetilde{f}_*^j (\nu|_{\{\widetilde{R}_{j_0} > j\}}) = \frac{1}{\gamma}\left(\sum_{k=0}^{m_0-1} f^k{}_*\right)\sum_{j=0}^{+\infty} f_*^{m_0 j} (\nu|_{\{R_{j_0}/m_0 > j\}}) \\
&= \frac{1}{\gamma}\sum_{k=0}^{m_0-1}\sum_{j=0}^{+\infty} f_*^{m_0 j + k} (\nu|_{\{R_{j_0} > m_0 j\}}) \underbrace{=}_{(31)} \frac{1}{\gamma}\sum_{j=0}^{+\infty}\sum_{k=0}^{m_0-1} f_*^{m_0 j + k} (\nu|_{\{R_{j_0} > m_0 j + k\}}) \\
&= \frac{1}{\gamma}\sum_{n=0}^{+\infty} f_*^n (\nu|_{\{R_{j_0} > n\}}),
\end{aligned}$$

finishing the proof of Theorem D. □

7. A global induced Markov map

In [5], Alves, Luzzatto and Pinheiro study the decay of correlations of non-uniformly expanding maps using a local induced Markov map. Using a global induced map, Gouëzel [31] could improve the results of [5] to deal with decay faster then super-polynomially. The advantage of a global induced map is the possibility of dominating the induced time by the first hyperbolic time. In this section we construct a global induced map (adapted to any zooming measure) with the induced time smaller or equal to the first zooming time with respect f or, when we do not have enough backward contraction, with respect to a fixed iterate of the original map.

Consider a connected, compact, separable metric space X. It was introduced in Section 2 the notion of an open set being nested and this notion can be extended straightforwardly to *essentially open sets*. A Borel set A is called *essentially open* if $\overline{\mathrm{int}(A)} \supset A$, that is, the closure of the interior of A contains A.

Definition 7.1 *(Essentially open linked sets).* We say that two essentially open sets U_1 and U_2 are *linked* if their interior are linked.

Let $f : X \to X$ be a measurable map and $\mathcal{E} = (\mathcal{E}_n)_n$ be a dynamically closed family of pre-images.

Exactly as we have done to open sets, we say that a collection of essentially open sets $\mathcal{A}$ is a *$\mathcal{E}$-nested collection* if every $A \in \mathcal{A}$ is not linked with any $\mathcal{E}$-pre-image of an element of $\mathcal{A}$ with order bigger than zero.

Note that a collection $\mathcal{A} = \{A_1, \dots, A_s\}$ of essentially open sets is $\mathcal{E}$-nested if and only if the collection $\mathrm{int}(\mathcal{A}) := \{\mathrm{int}(A_1), \dots, \mathrm{int}(A_s)\}$ is an $\mathcal{E}$-nested collection of sets as in Definition 2.4. As a consequence, we get the following remark.

Remark 7.2. Lemma 2.5 is also valid for collections of essentially open sets. That is, if $\mathcal{A}$ is an $\mathcal{E}$-nested collection of essentially open sets and P_1 and P_2 are $\mathcal{E}$-pre-images of two elements of $\mathcal{A}$ with $\mathrm{ord}(P_1) \neq \mathrm{ord}(P_2)$ then P_1 and P_2 are not linked.

Let $\delta > 0$ and let $\alpha = \{\alpha_n\}_{1 \leqslant n \in \mathbb{N}}$ be a zooming contraction (Definition 5.1).

Assume that there exists a set $\mathcal{C} \subset X$, called critical set, such that f is injective on each connected component of $X \setminus \mathcal{C}$ and such that $\mathcal{C}$ does not intersect any (α, δ)-zooming pre-ball, i.e., $V_n(p) \cap \mathcal{C} = \emptyset$ $\forall p \in \mathrm{Z}_n(\alpha, \delta, f)$ and $\forall n \geqslant 1$.

Remark 7.3. Instead to assume the condition above that $\mathcal{C}$ does not intersect any (α, δ)-zooming pre-ball, one may consider in Theorem E a set $\Lambda \subset \limsup_n \mathcal{Z}_n^{\mathcal{C}}(\alpha, \theta, \delta, f)$, where $\mathcal{Z}_n^{\mathcal{C}}(\alpha, \delta, \theta, f)$ is the set of points $x \in X$ such that $\#\{1 \leqslant j \leqslant n;\ x \in \mathrm{Z}_j^{\mathcal{C}}(\alpha, \delta, f)\} \geqslant \theta n, \theta > 0$ and $\mathrm{Z}_n^{\mathcal{C}}(\alpha, \delta, f) \subset \mathrm{Z}_n(\alpha, \delta, f)$ is the set of all (α, δ)-zooming points $p \in X$ such that $\mathcal{C}$ does not intersect $f^j(V_n(p))$ for every $0 \leqslant j < n$. In this case we have replace the sets $\mathrm{Z}_{\ell j}(\alpha, \delta, f)$ and $\mathrm{Z}_j(\widetilde{\alpha}, \delta, \widetilde{f})$ in item (6) by respectively $\mathrm{Z}_{\ell j}^{\mathcal{C}}(\alpha, \delta, f)$ and $\mathrm{Z}_j^{\mathcal{C}}(\widetilde{\alpha}, \delta, \widetilde{f})$.

Theorem E *(Global zooming induced Markov map). Given an (α, δ)-zooming set $\Lambda \subset X$ there are an induced Markov map $(F, \mathcal{P})$ defined on X with induced time R, a finite partition $\mathcal{P}_0$ of X by essentially open sets and an integer $\ell \geqslant 1$ satisfying the following properties.*

(1) *For each $Q \in \mathcal{P}$ there exists $P \in \mathcal{P}_0$ such that* $\mathrm{int}(Q) \subset P$.
(2) *$F(P) \in \mathcal{P}_0$ $\forall P \in \mathcal{P}$ (in particular, the elements of P are essentially open sets).*
(3) *Given $P \in \mathcal{P}$ there is a zooming pre-ball $V_{R(P)}(x)$, $x \in \mathrm{Z}_{R(P)}(\alpha, \delta, f) \cap \Lambda$, such that $F|_P = (f^{R(P)}|_{V_{R(P)}(x)})|_P$. In particular,*
    (3.1) $\mathrm{dist}(F(x), F(y)) \geqslant 8\, \mathrm{dist}(x, y)$ $\forall x, y \in P$ *and* $\forall P \in \mathcal{P}$;
    (3.2) *for all $x, y \in P$, $P \in \mathcal{P}$ and $0 \leqslant n < R(P)$,*
    $$\mathrm{dist}\big(f^n(x), f^n(y)\big) < \alpha_{R(P)-n}\, \mathrm{dist}\big(F(x), F(y)\big);$$
    (3.3) *if μ is (α, δ)-zooming measure (not necessarily invariant) with bounded distortion (with respect to f) and $\mu(X \setminus \Lambda) = 0$ then $\exists \rho > 0$ such that*
    $$\left|\log \frac{J_\mu F(x)}{J_\mu F(y)}\right| \leqslant \rho\, \mathrm{dist}\big(F(x), F(y)\big),$$
    for μ almost every $x, y \in P$ and $\forall P \in \mathcal{P}$.
(4) *If μ is an f-invariant measure with $\mu(\Lambda) > 0$ then $\mu|_{\bigcap_{j \geqslant 0} f^{-\ell j}(\{R > 0\})}$ is an invariant measure with respect to f^ℓ.*
(5) *Every ergodic f-invariant zooming probability μ with $\mu(\Lambda) > 0$ is liftable to F.*

(6) *There is a good relationship between the tail of the partition and the tail of zooming times, i.e.,*

$$\{R > n\} \cap \Lambda \subset \Lambda \setminus \bigcup_{j=1}^{n} \mathrm{Z}_{\ell j}(\alpha, \delta, f).$$

Furthermore, if we do not need F satisfying item 3.2 *then we can get*

$$\{R > n\} \cap \Lambda \subset \Lambda \setminus \bigcup_{j=1}^{n} \mathrm{Z}_j\big(\widetilde{\alpha}, \delta, f^{\ell}\big),$$

where $\widetilde{\alpha} = \{\alpha_{\ell n}\}_n$.

Proof of Theorem E. Choose $0 < r_0 < \delta/2$ and set $r = r_0/4$. Let $s_0 = \operatorname{diameter}(X)/r$ and let ℓ be an integer big enough such that $\sum_{j=1}^{\infty} \alpha_{\ell j}(r_0) \leqslant \frac{r_0}{64 s_0}$. Thus, $\sum_{j=1}^{\infty} \alpha_{\ell j}(2r) \leqslant \sum_{j=1}^{\infty} \alpha_{\ell j}(r_0) \leqslant r_0/(64 s_0) = r/(16 s_0)$.

Set $\widetilde{f} = f^{\ell}$, $\widetilde{\alpha}_j = \alpha_{\ell j}$ and $\widetilde{\alpha} = \{\widetilde{\alpha}_j\}_j$. Denote by $\mathcal{E}_{\mathcal{Z},\widetilde{f}} = (\mathcal{E}_{\mathcal{Z},\widetilde{f},n})_n$, where $\mathcal{E}_{\mathcal{Z},\widetilde{f},n} = \{V_n(x, \widetilde{f});\ x \in \mathrm{Z}_n(\widetilde{\alpha}, \delta, \widetilde{f})\}$ is the collection of all $(\widetilde{\alpha}, \delta)$-zooming pre-balls with respect $\widetilde{f}$ of order n. Let us denote the order with respect to $\widetilde{f}$ by $\operatorname{ord}_{\widetilde{f}}$.

Let $\{q_1, \dots, q_s\} \subset X$ be a maximal $r/2$-separated set and consider the collection $\mathcal{A} = \{B_r(q_1), \dots, B_r(q_s)\}$ of open balls. As we have contraction in zooming times, the elements of $\mathcal{A}$ are not contained in any $\mathcal{E}_{\mathcal{Z},\widetilde{f}}$-pre-image of an element of $\mathcal{A}$ with order bigger than zero (see Notation 5.8).

As f is injective on each connected component of $X \setminus \mathcal{C}$ and $\mathcal{C}$ does not intersect any (α, δ)-zooming pre-ball, a chain $\mathcal{K}$ of $\mathcal{E}_{\mathcal{Z},\widetilde{f}}$-pre-images of $\mathcal{A}$ cannot have two pre-images the same element of $\mathcal{A}$ with the same order, that is, if Q and P belongs to $\mathcal{K}$ and $\widetilde{f}^{\operatorname{ord}(Q)}(Q) = \widetilde{f}^{\operatorname{ord}(P)}(P)$ then $\operatorname{ord}_{\widetilde{f}}(P) \neq \operatorname{ord}_{\widetilde{f}}(Q)$. This implies that in each chain the number of pre-images of a given order $j \in \mathbb{N}$ is bounded by the number of elements of $\mathcal{A}$, i.e., $s = \#\mathcal{A} \leqslant \operatorname{diameter}(X)/(r/2) + 1 = 2s_0 + 1 < 3s_0$.

As $\sum_n \widetilde{\alpha}_n(2r) \leqslant r/(16 s_0)$, every chain of $\mathcal{E}_{\mathcal{Z},\widetilde{f}}$-pre-images of $\mathcal{A}$ has diameter smaller that $r/4$. Indeed, if $(P_0, \dots, P_n) \in ch_{\mathcal{E}_{\mathcal{Z},\widetilde{f}}}(B_r(q_i))$ then

$$\operatorname{diameter}\Big(\bigcup_j P_j\Big) < \sum_{i=1}^{\operatorname{ord}(P_n)} \sum_{\operatorname{ord}(P_j)=i} \operatorname{diameter}(P_j) \leqslant \sum_{i=1}^{\operatorname{ord}(P_n)} s\widetilde{\alpha}_j(2r) < \frac{r}{4}.$$

Thus,

$$\big(B_r(q_i)\big)^{\star} = B_r(q_i) \setminus \Big(\overline{\bigcup_{(P_j)_j \in ch_{\mathcal{E}_{\mathcal{Z},\widetilde{f}}}(B_r(q_i))} \bigcup_j P_j}\Big) \supset B_{(3/4)r}(q_i),$$

for all $1 \leqslant i \leqslant s$.

Let $\mathcal{A}' = \{\Delta_1, \dots, \Delta_s\}$, where Δ_i is the connected component of $(B_r(q_i))^{\star}$ containing $B_{(3/4)r}(q_i)$. It follows from Proposition 2.8 that $\mathcal{A}'$ is an $\mathcal{E}_{\mathcal{Z},\widetilde{f}}$-nested collection of sets. Moreover, as $\{q_1, \dots, q_s\} \subset X$ is maximal $r/2$-separated, $\mathcal{A}'$ is a cover of X by opens sets.

Setting $\mathcal{P}^s = \{\Delta_1 \cap \dots \cap \Delta_s\}$ and, for $1 \leqslant \wp < s$,

$$\mathcal{P}^{\wp} = \Big\{\Delta_{i_1} \cap \dots \cap \Delta_{i_{\wp}} \setminus \Big(\bigcup_{1 \leqslant k_1 < \dots < k_{\wp+1} \leqslant s} \Delta_{k_1} \cap \dots \cap \Delta_{k_{\wp+1}}\Big);\ 1 \leqslant i_1 < \dots < i_{\wp} \leqslant s\Big\},$$

it follows that $\mathcal{P}_0 := \mathcal{P}^1 \cup \dots \cup \mathcal{P}^s$ is a partition of X by essentially open sets. Note that, $\bigcup_{P \in \mathcal{P}_0} \partial P \subset \bigcup_{j=1}^{s} \partial \Delta_j$.

Claim 4. *Let Q be an $\mathcal{E}_{\mathcal{Z},\widetilde{f}}$-pre-image* (*with respect to $\widetilde{f}$*) *of some $\Delta_i \in \mathcal{A}'$ with $\operatorname{ord}_{\widetilde{f}}(Q) > 0$ and let $P \in \mathcal{P}_0$. If $Q \cap P \neq \emptyset$ then $Q \subset \operatorname{int}(P)$.*

Proof of Claim 4. Suppose that $Q \cap P \neq \emptyset$ and $Q \not\subset \operatorname{int}(P)$. As Q is a connected open set, $Q \cap \partial P \neq \emptyset$. Thus, there exists $\Delta_k \in \mathcal{A}'$ such that $Q \cap \partial \Delta_k \neq \emptyset$. As $\mathcal{A}'$ is $\mathcal{E}_{\mathcal{Z},\widetilde{f}}$-nested collection of open sets and $\operatorname{ord}_{\widetilde{f}}(Q) > 0$, $\Delta_k \subset Q$. As $\operatorname{diameter}(Q) < (\sum_{j=1}^{\operatorname{ord}_{\widetilde{f}}(Q)} \widetilde{\alpha}_j) \operatorname{diameter}(\Delta_i) < r/8$. But this leads to a contradiction because $B_{(3/4)r}(p_k) \subset \Delta_k$. $\square$

Claim 5. *If Q_1 and Q_2 are $\mathcal{E}_{\mathcal{Z},\widetilde{f}}$-pre-images (with respect to $\widetilde{f}$) of respectively $P_1, P_2 \in \mathcal{P}_0$ then Q_1 and Q_2 are not linked. In particular, $\mathcal{P}_0$ is an $\mathcal{E}_{\mathcal{Z},\widetilde{f}}$-nested collection (with respect to $\widetilde{f}$) of essentially open sets. Furthermore,*

(1) *if* $Q_1 \cap Q_2 \neq \emptyset$ *then* $\operatorname{ord}_{\widetilde{f}}(Q_1) \neq \operatorname{ord}_{\widetilde{f}}(Q_2)$;
(2) *if* $Q_1 \subsetneqq Q_2$ *then* $\operatorname{ord}_{\widetilde{f}}(Q_1) > \operatorname{ord}_{\widetilde{f}}(Q_2)$.

Proof of Claim 5. Note that if $Q_1 \cap Q_2 \neq \emptyset$ then $\operatorname{ord}_{\widetilde{f}}(Q_1) \neq \operatorname{ord}_{\widetilde{f}}(Q_2)$. Otherwise $P_1 \cap P_2 = f^{\operatorname{ord}_{\widetilde{f}}(Q_1)}(Q_1 \cap Q_2) \neq \emptyset$. This shows the first item.

Suppose that $\operatorname{int}(Q_1 \cap Q_2) \neq \emptyset$, with $Q_1 \neq Q_2$. We may assume that $P_1 \neq P_2$ (if $P_1 = P_2$, the claim follows from Corollary 2.6). Set $\wp_j = \operatorname{ord}_{\widetilde{f}}(P_j)$ (with respect to $\widetilde{f}$) for $j = 1, 2$. By the first item, assume for instance that $\wp_1 < \wp_2$.

Write $Q_2 = (f^{\wp_2}|V_{\wp_2}(x, \widetilde{f}))^{-1}(P_2)$, with $x \in \mathrm{Z}_{\wp_2}(\widetilde{\alpha}, \delta, \widetilde{f})$. Let $\Delta_{j_2} \in \mathcal{A}'$ be such that $\operatorname{int}(P_2) \subset \Delta_{j_2}$ and set $\widetilde{\Delta}_{j_2} = (f^{\wp_2}|V_{\wp_2}(x, \widetilde{f}))^{-1}(\Delta_{j_2})$. Thus,

$$\operatorname{int}\big(P_1 \cap \widetilde{f}^{\wp_1}(\widetilde{\Delta}_{j_2})\big) \supset \operatorname{int}\big(P_1 \cap \widetilde{f}^{\wp_1}(Q_2)\big) = \widetilde{f}^{\wp_1}\big(\operatorname{int}(Q_1 \cap Q_2)\big) \neq \emptyset. \tag{32}$$

As $\widetilde{f}^{\wp_1}(\widetilde{\Delta}_{j_2})$ is an $\mathcal{E}_{\mathcal{Z},\widetilde{f}}$-pre-image of Δ_{j_2} with order $\wp_2 - \wp_1 > 0$, it follows from Claim 4 that $\operatorname{int}(P_1) \supset \widetilde{f}^{\wp_1}(\widetilde{\Delta}_{j_2})$. So, $\operatorname{int}(P_1) \supset \operatorname{int}(\widetilde{f}^{\wp_1}(Q_2))$. Using (32), we get

$$\widetilde{f}^{\wp_1}\big(\operatorname{int}(Q_1)\big) = \operatorname{int}(P_1) \supset \widetilde{f}^{\wp_1}\big(\operatorname{int}(Q_1 \cap Q_2)\big).$$

As a consequence, $\operatorname{int}(Q_1) \supset \operatorname{int}(Q_2)$.

So, we obtain that $\operatorname{int}(Q_1 \cap Q_2) \neq \emptyset$ and $\wp_1 < \wp_2$ implies that $\operatorname{int}(Q_1) \supset \operatorname{int}(Q_2)$ (or, if $\wp_1 > \wp_2$, $\operatorname{int}(Q_1) \subset \operatorname{int}(Q_2)$). From this we conclude that Q_1 and Q_2 are not linked and also the second item of the claim. □

As in the proof of Theorem D, let $\widetilde{\mathfrak{z}}$ be the proper zooming sub-collection for $\widetilde{f}$ given by $\widetilde{\mathfrak{z}} = (\widetilde{\mathfrak{z}}(x))_{x \in \widetilde{\Lambda}}$, where

$$\widetilde{\mathfrak{z}}(x) = \begin{cases} \{\widetilde{f}^n(x);\ n \in \mathbb{N} \text{ and } x \in \mathrm{Z}_{\ell n}(\alpha, \delta, f)\} & \text{if we need item 3.2,} \\ \{\widetilde{f}^n(x);\ n \in \mathbb{N} \text{ and } x \in \mathrm{Z}_n(\widetilde{\alpha}, \delta, \widetilde{f})\} & \text{otherwise,} \end{cases}$$

and

$$\widetilde{\Lambda} = \begin{cases} \Lambda \cap \limsup_{n\to\infty} \mathrm{Z}_{\ell n}(\alpha, \delta, f) & \text{if we need item 3.2,} \\ \Lambda \cap \limsup_{n\to\infty} \mathrm{Z}_n(\widetilde{\alpha}, \delta, \widetilde{f}) & \text{otherwise.} \end{cases}$$

Remark 7.4. If we are considering the hypothesis of Remark 7.3, i.e., $\Lambda \subset \limsup_n \mathrm{Z}_n^{\mathcal{C}}(\alpha, \delta, f)$, we only have take $\mathrm{Z}_{.}^{\mathcal{C}}(., \delta, .)$ instead of $\mathrm{Z}_{.}(., \delta, .)$ in the proper zooming sub-collection $\widetilde{\mathfrak{z}}$. Note that $\widetilde{\mathfrak{z}}$ will remain a proper zooming sub-collection. Indeed, as $\mathrm{Z}_{.}^{\mathcal{C}}(., \delta, .) \subset \mathrm{Z}_{.}(., \delta, .)$ and $\Lambda \subset \limsup_n \mathcal{Z}_n^{\mathcal{C}}(\alpha, \theta, \delta, f)$, it is easy to check this alternative $\widetilde{\mathfrak{z}}$ satisfies Definition 6.1. Of course, in this case, $\widetilde{\Lambda}$ must be

$$\begin{cases} \Lambda \cap \limsup\limits_{n\to\infty} \mathrm{Z}_{\ell n}^{\mathcal{C}}(\alpha, \delta, f) & \text{if we need item 3.2,} \\ \Lambda \cap \limsup\limits_{n\to\infty} \mathrm{Z}_n^{\mathcal{C}}(\widetilde{\alpha}, \delta, \widetilde{f}) & \text{otherwise.} \end{cases}$$

It is clear that $\widetilde{\Lambda}$ is $\widetilde{f}$-positively invariant. Also note that $\widetilde{\Lambda}$ is a large portion of Λ. Indeed, it follows from (23) that

$$\Lambda \subset \widetilde{\Lambda} \cup f^{-1}(\widetilde{\Lambda}) \cup \cdots \cup f^{-(\ell-1)}(\widetilde{\Lambda}).$$

As a consequence

$$\mu(\Lambda) > 0 \quad \Rightarrow \quad \mu(\widetilde{\Lambda}) > 0$$

for every f-non-singular measure μ.

Let $\widetilde{\mathcal{E}} \subset \mathcal{E}_{\mathcal{Z},\widetilde{f}}$ be the collection of all $\widetilde{\mathfrak{z}}$-pre-ball $V_n(x, \widetilde{f})$ (with respect to $\widetilde{f}$) for all $x \in \Lambda$ and all $\widetilde{\mathfrak{z}}$-time n for x with respect to $\widetilde{f}$.

Define an inducing time $\widetilde{R} : X \to \{0, 1, 2, \ldots\}$ on X as follows. Given $x \in X$, let $\Omega(x)$ be the collection of all $\widetilde{\mathcal{E}}$-pre-images Q of any $P \in \mathcal{P}_0$ such that $x \in Q$. That is, $Q \in \Omega(x)$ if $x \in Q$ and there are $n \in \mathbb{N}$, $y \in \widetilde{\Lambda}$ and $P \in \mathcal{P}_0$

such that $Q = (\widetilde{f}^n|_{V_n(y,\widetilde{f})})^{-1}(P)$, where $n \geqslant 1$ is a $\widetilde{\mathfrak{z}}$-time (with respect to $\widetilde{f}$) for y. Note that $V_n(y, \widetilde{f})$ is both an $(\widetilde{\alpha}, \delta)$-zooming pre-ball of order n with respect to $\widetilde{f}$ and an (α, δ)-zooming pre-ball of order ℓn with respect to f. If $\Omega(x) \neq \emptyset$ let $\widetilde{R}(x) = \min\{\mathrm{ord}_{\widetilde{f}}(V);\ V \in \Omega(x)\}$ and let $\widetilde{R}(x) = 0$ whenever $\Omega(x) = \emptyset$.

Note that if $x \in \widetilde{\Lambda}$ then $\widetilde{R}(x)$ is smaller than or equal to the first $\widetilde{\mathfrak{z}}$-time of x, i.e., $\widetilde{R}(x) \leqslant \min\{n;\ n$ is a $\widetilde{\mathfrak{z}}$-time (with respect to $\widetilde{f}$) to $x\} = \min\{n;\ x \in \mathrm{Z}_{\ell n}(\alpha, \delta, f)\}$. Thus,

$$\{\widetilde{R} > n\} \cap \widetilde{\Lambda} \subset \widetilde{\Lambda} \setminus \bigcup_{j=1}^{n} \mathrm{Z}_j(\widetilde{\alpha}, \delta, \widetilde{f}) \subset \widetilde{\Lambda} \setminus \bigcup_{j=1}^{n} \mathrm{Z}_{\ell j}(\alpha, \delta, f). \tag{33}$$

Define the induced map $\widetilde{F}$ on X associated to *the first* $\mathcal{E}_{\widetilde{\mathcal{Z}}}$ *time* by

$$\widetilde{F}(x) = \widetilde{f}^{\widetilde{R}(x)}(x), \quad \forall x \in X. \tag{34}$$

If $\Omega(x) \neq \emptyset$, it follows from Claim 5 that the collection of sets $\Omega(x)$ is totally ordered by inclusion. Moreover, there is a unique $Q \in \Omega(x)$ such that $\mathrm{ord}_{\widetilde{f}}(Q) = \widetilde{R}(x)$. In this case, set $I(x) = Q$.

Also by Claim 5, $\mathrm{ord}_{\widetilde{f}}(I(x)) < \mathrm{ord}_{\widetilde{f}}(J)$ $\forall I(x) \neq J \in \Omega(x)$ and $\forall x \in X$. Furthermore, if $I(x) \cap I(y) \neq \emptyset$ then $I(x) = I(y)$ (see the proof of Lemma 6.4 which is analogous).

Proceeding as in the proof of Corollary 6.6, one can easily conclude that

$$\mathcal{P} := \{I(x);\ x \in X \text{ and } \Omega(x) \neq \emptyset\}$$

is a Markov partition for $\widetilde{F}$. Besides, defining $R(x) = \ell\widetilde{R}(x)$ and $F(x) = f^{R(x)}(x) = \widetilde{f}^{\widetilde{R}(x)}(x) = \widetilde{F}(x)$, one can see that $\mathcal{P}_0, \mathcal{P}, F$ and R satisfy the first four items of the theorem.

Remark 7.5. For future references we note that $\mathcal{P}|_{\widetilde{\Lambda}} = \{P \cap \widetilde{\Lambda};\ P \in \mathcal{P}\}$ is an induced Markov partition of $\widetilde{\Lambda}$ with respect to $\widetilde{f} = f^\ell$ (this follows from Claim 5).

Let μ be an f-invariant measure with $\mu(\Lambda) > 0$. To check the item (4) set $E = \bigcap_j \widetilde{f}^{-j}(\{R > 0\})$. As $\widetilde{f}^{-1}(E) \supset E \supset \widetilde{\Lambda}$, we get $\widetilde{f}^{-1}(E) = E \pmod{\mu}$ and $\mu(E) \geqslant \mu(\widetilde{\Lambda}) > 0$. Thus $\mu|_E$ is $\widetilde{f}$-invariant.

To prove the last item we will construct a local Markov map induced from $\widetilde{F}$.

Constructing a local induced map from the global one. Let μ be an f-invariant ergodic probability with $\mu(\Lambda) > 0$. By ergodicity, $\mu(\Lambda) = 1$ (we are also using that Λ is positively invariant and μ is invariant). As $\widetilde{\Lambda}$ is $\widetilde{f}$-positively invariant and μ is also $\widetilde{f}$-invariant, $\mu|_{\widetilde{\Lambda}}$ is $\widetilde{f}$-invariant. On the other hand, as X can be decomposed into at most ℓ μ-ergodic components with respect to $\widetilde{f}$, there is a μ-ergodic component $U \subset \widetilde{\Lambda}$.

Thus, $\widetilde{\mu} = \frac{1}{\mu(U)}\mu|_U$ is an $(\widetilde{\alpha}, \delta)$-zooming ergodic invariant probability with respect to $\widetilde{f}$, $\widetilde{\mu}(\widetilde{\Lambda}) = 1$ and $\mu = (\sum_{j=0}^{\ell-1} f^j{}_*)\mu|_U$.

By Proposition 3.5, there exists an $\widetilde{f}$-attractor $A \subset X$ which attracts $\widetilde{\mu}$ almost every point of X (indeed $A = \operatorname{supp}\widetilde{\mu}$ because $\widetilde{\mu}$ is $\widetilde{f}$-invariant). By Lemma 3.9, there are compact sets $A_{+,\widetilde{\mathfrak{z}}}$ and $A_{\widetilde{\mathfrak{z}}}$, with $A_{+,\widetilde{\mathfrak{z}}} \subset A_{\widetilde{\mathfrak{z}}} \subset A$, such that $\omega_{\widetilde{f},\widetilde{\mathfrak{z}}}(x) = A_{\widetilde{\mathfrak{z}}}$ and $\omega_{+,\widetilde{f},\widetilde{\mathfrak{z}}}(x) = A_{+,\widetilde{\mathfrak{z}}}$ for $\widetilde{\mu}$-almost every $x \in X$ (see Definitions 3.7 and 3.8). Let $1 \leqslant j_0 \leqslant s$ be such that $\Delta_{j_0} \cap A_{+,\widetilde{\mathfrak{z}}} \neq \emptyset$.

As in (27), we define the first $\widetilde{\mathcal{E}}$-return time to Δ_{j_0} (with respect to $\widetilde{f}$). Precisely, given $x \in \Delta_{j_0}$, let $\Omega_0(x)$ be the collection of $\widetilde{\mathcal{E}}$-pre-images V of Δ_{j_0} such that $x \in V$ (i.e., $x \in V = (\widetilde{f}^n|_{V_n(y)})^{-1}(\Delta_{j_0})$ for some $y \in \mathrm{Z}_{\ell n}(\alpha, \delta, f) \cap \Lambda$ and $n \in \mathbb{N}$) and define the first $\widetilde{\mathcal{E}}$-return time to Δ_{j_0} as the map $\widetilde{R}_0 : \Delta_{j_0} \to \mathbb{N}$ given by

$$\widetilde{R}_0(x) = \begin{cases} \min\{\mathrm{ord}(V);\ V \in \Omega_0(x)\} & \text{if } \Omega_0(x) \neq \emptyset, \\ 0 & \text{if } \Omega_0(x) = \emptyset. \end{cases} \tag{35}$$

Thus, the induced map $\widetilde{F}_0$ on Δ_{j_0} (with respect to $\widetilde{f}$) associated to "*the first* $\widetilde{\mathcal{E}}$*-return time to* Δ_{j_0}" is given by

$$\widetilde{F}_0(x) = \widetilde{f}^{\widetilde{R}_0(x)}(x), \quad \forall x \in \Delta_{j_0}. \tag{36}$$

We claim that $\widetilde{F}_0$ is also an induced map with respect to $\widetilde{F}$.

Claim 6. *For each $x \in \Delta_{j_0}$ there is $\widetilde{k}(x) \in \mathbb{N}$ such that $\widetilde{F}_0(x) = \widetilde{F}^{\widetilde{k}(x)}(x)$.*

Proof of Claim 6. By definition, $V \in \Omega_0(x)$ if and only if $\exists y \in Z_{\ell n}(\alpha, \delta, f) \cap \Lambda$ such that $x \in V = (\widetilde{f}^n|_{V_n(y,\widetilde{f})})^{-1}(\Delta_{j_0})$, where $n = \operatorname{ord}_{\widetilde{f}}(V)$ and $V_n(y, \widetilde{f})$ is the (α, δ)-zooming pre-ball with respect to f of order ℓn "centered on y" (as noted before, $V_n(y, \widetilde{f})$ is also an $(\widetilde{\alpha}, \delta)$-zooming pre-ball with respect to $\widetilde{f}$ of order n "centered on y"). If P is the element of $\mathcal{P}_0$ that contains $\widetilde{f}^n(x)$, we get $P \subset B_\delta(\widetilde{f}^n(y)) = \widetilde{f}^n(V_n(y, \widetilde{f}))$ (because diameter$(P) \leqslant r_0/2 < \delta/4$). Thus, $V' := (\widetilde{f}^n|_{V_n(y,\widetilde{f})})^{-1}(P) \in \Omega(x)$ and $\operatorname{ord}_{\widetilde{f}}(V') = n$. As a consequence, $0 \leqslant \widetilde{R}(x) \leqslant \widetilde{R}_0(x)$ $\forall x \in \Delta_{j_0}$.

Let $x \in \Delta_{j_0}$ be such that $m := \widetilde{R}_0(x) > 0$. In this case set $s = \sum_{n=0}^{\wp_x - 1} \widetilde{R} \circ \widetilde{F}^n(x)$, where

$$\wp_x = \max\left\{ j \geqslant 1;\ \sum_{n=0}^{j-1} \widetilde{R} \circ \widetilde{F}^n(x) < m \right\}.$$

Let $P \in \mathcal{P}$ be such that $\widetilde{F}_0(x) \in P$ and let $y \in Z_{\ell m}(\alpha, \delta, f) \cap \Lambda$ be such that $I_0(x) = (\widetilde{f}^m|_{V_m(y,\widetilde{f})})^{-1}(\Delta_{j_0})$, where $I_0(x) \in \Omega_0(x)$ is the unique element of $\Omega_0(x)$ such that $\operatorname{ord}_{\widetilde{f}}(I_0(x)) = m = \widetilde{R}_0(x)$ (see the comment just above Lemma 6.4). As $P \subset \overline{\Delta_{j_0}} \subset B_\delta(\widetilde{f}^m(y)) = \widetilde{f}^m(V_m(y, \widetilde{f}))$, $I := (\widetilde{f}^m|_{V_m(y,\widetilde{f})})^{-1}(P) \in \Omega(x)$. Thus, $\widetilde{f}^s(I) \in \Omega(\widetilde{f}^s(x))$ and, as a consequence, $\widetilde{R}(\widetilde{f}^s(x)) \leqslant m - s$ (because $\operatorname{ord}_{\widetilde{f}}(\widetilde{f}^s(I)) = m - s$). On the other hand, as $\widetilde{F}^{\wp_x}(x) = \widetilde{f}^s(x)$ and $\widetilde{R}(\widetilde{F}^{\wp_x}(x)) + s = \sum_{n=0}^{\wp_x} \widetilde{R} \circ \widetilde{F}^n(x) \geqslant m$, we get $\widetilde{R}(\widetilde{f}^s(x)) \geqslant m - s$. Thus, $\widetilde{R}(\widetilde{f}^s(x)) = m - s$. This implies that $\widetilde{R}_0(x) = m = \widetilde{R}(\widetilde{f}^s(x)) + s = \sum_{n=0}^{\wp_x} \widetilde{R} \circ \widetilde{F}^n(x)$, i.e., $\widetilde{F}_0(x) = \widetilde{f}^{\widetilde{R}_0(x)}(x) = \widetilde{F}^{\widetilde{k}(x)}$, where $\widetilde{k}(x) = \wp_x + 1$. $\square$

Now, to finish the proof, we will show the existence of an F-invariant finite measure $\nu \ll \mu$ such that $\mu = \sum_{j=0}^{+\infty} f_*^j(\nu|_{\{R>j\}})$.

Because $\sum_{n \geqslant 1} \widetilde{\alpha}_n(r_0) \leqslant r_0/8$, the $(\widetilde{\alpha}, \delta)$-zooming nested ball $B_{r_0}^\star(x)$ (with respect to $\widetilde{f}$) is well defined and contains $B_{r_0/2}(x)$ for every $x \in X$ (Lemma 5.12). As diameter$(\Delta_{j_0}) \leqslant \frac{1}{2} r_0$, $\Delta_{j_0} \cap A_{+,\widetilde{\mathfrak{z}}} \neq \emptyset$ and $\widetilde{\mu}$ is an ergodic $\widetilde{f}$-invariant $(\widetilde{\alpha}, \delta)$-zooming probability, we can apply Theorem 4 and obtain a finite $\widetilde{F}_0$-invariant finite measure $\nu_0 \ll \widetilde{\mu}$ with $\int \widetilde{R}_0 \, d\nu_0 < +\infty$ and such that

$$\widetilde{\mu} = \frac{1}{\widetilde{\gamma}} \sum_{j=0}^{+\infty} \widetilde{f}_*^j(\nu_0|_{\{\widetilde{R}_0 > j\}}),$$

where $\widetilde{\gamma} = \sum_{j=0}^{+\infty} \widetilde{f}_*^j(\nu_0|_{\{\widetilde{R}_0 > j\}})(X)$. As $\widetilde{F}_0(x) = \widetilde{F}^{\widetilde{k}(x)}(x)$ and $\int \widetilde{k} \, d\nu_0 \leqslant \int \widetilde{R}_0 \, d\nu_0$, it follows from Remark 4.3 that $\nu = \frac{1}{\widetilde{\gamma}} \sum_{j=0}^{+\infty} \widetilde{F}_*^j(\nu_0|_{\{\widetilde{R}_0 > j\}})$ is an $\widetilde{F}$-invariant finite measure (note that ν is not necessarily a probability). Moreover, it is straightforward to check that $\sum_{n=0}^{+\infty} \widetilde{f}_*^n(\nu|_{\{\widetilde{R}>n\}}) = \frac{1}{\widetilde{\gamma}} \sum_{n=0}^{+\infty} \widetilde{f}_*^n((\sum_{j=0}^{+\infty} \widetilde{F}_*^j(\nu_0|_{\{\widetilde{R}_0 > j\}}))|_{\{\widetilde{R}>n\}}) = \frac{1}{\widetilde{\gamma}} \sum_{n=0}^{+\infty} \widetilde{f}_*^n(\nu_0|_{\{\widetilde{R}_0 > n\}}) = \widetilde{\mu}$ (see, for instance, Lemma 4.1 of [66]). That is

$$\widetilde{\mu} = \sum_{n=0}^{+\infty} \widetilde{f}_*^n(\nu|_{\{\widetilde{R}>n\}}).$$

Proceeding as in the end of the proof of Theorem D (or alternatively, using Lemma 4.1 of [66]) we get

$$\mu = \frac{1}{\gamma} \sum_{n=0}^{+\infty} f_*^n(\nu|_{\{R>n\}}),$$

where $\gamma = 1/\mu(U)$. $\square$

8. Expanding measures

Let M be a compact Riemannian manifold of dimension $d \geqslant 1$ and $f : M \to M$ is a non-flat map with a critical/singular set $\mathcal{C} \subset M$.

Hyperbolic times. The idea of hyperbolic times is a key notion on the study of non-uniformly hyperbolic dynamics and was introduced by Alves et al. [2,4]. Let us fix $0 < b = \frac{1}{3}\min\{1, 1/\beta\} < \frac{1}{2}\min\{1, 1/\beta\}$. Given $0 < \sigma < 1$ and $\varepsilon > 0$, we will say that n is a (σ, ε)-*hyperbolic time* for a point $x \in M$ (with respect to the non-flat map f with a β-non-degenerate critical/singular set $\mathcal{C}$) if for all $1 \leqslant k \leqslant n$ we have $\prod_{j=n-k}^{n-1} \|(Df \circ f^j(x))^{-1}\| \leqslant \sigma^k$ and $\mathrm{dist}_\varepsilon(f^{n-k}(x), \mathcal{C}) \geqslant \sigma^{bk}$. We denote the set of points of M such that $n \in \mathbb{N}$ is a (σ, ε)-*hyperbolic time* by $\mathrm{H}_n(\sigma, \varepsilon, f)$.

Proposition 8.1. *(See [4].) Given $\lambda > 0$ there exist $\theta > 0$ and $\varepsilon_0 > 0$ such that, for every $x \in U$ and $\varepsilon \in (0, \varepsilon_0]$,*

$$\#\{1 \leqslant j \leqslant n;\ x \in \mathrm{H}_j(e^{-\lambda/4}, \varepsilon, f)\} \geqslant \theta n,$$

whenever $\frac{1}{n}\sum_{i=0}^{n-1} \log \|(Df(f^i(x)))^{-1}\|^{-1} \geqslant \lambda$ and $\frac{1}{n}\sum_{j=0}^{n-1} -\log \mathrm{dist}_\varepsilon(f^j(x), \mathcal{C}) \leqslant \frac{\lambda}{16\beta}$.

It follows from Proposition 8.1 that the points of an expanding set (recall Definition 1.1) have infinitely many moments with positive frequency of hyperbolic times. In particular, they have infinitely many hyperbolic times.

The proposition below assures that the hyperbolic times are indeed zooming times.

Proposition 8.2. *(See [4].) Given $\sigma \in (0, 1)$ and $\varepsilon > 0$, there is $\delta > 0$, depending only on σ, ε and on the map f, such that if $x \in \mathrm{H}_n(\sigma, \varepsilon, f)$ then there exists a neighborhood $V_n(x)$ of x with the following properties*:

(1) *f^n maps $\overline{V_n(x)}$ diffeomorphically onto the ball $\overline{B_\delta(f^n(x))}$*;
(2) $\mathrm{dist}(f^{n-j}(y), f^{n-j}(z)) \leqslant \sigma^{j/2} \mathrm{dist}(f^n(y), f^n(z))$ $\forall y, z \in V_n(x)$ *and* $1 \leqslant j < n$.

The sets $V_n(x)$ are called *hyperbolic pre-balls* and their images $f^n(V_n(x)) = B_\delta(f^n(x))$, *hyperbolic balls*.

Given $\sigma \in (0, 1)$, $\varepsilon > 0$ and $\theta \in (0, 1]$, define $\mathcal{H}_n(\sigma, \varepsilon, \theta, f)$ as the set $x \in \mathrm{H}_n(\sigma, \varepsilon, f)$ such that $\#\{1 \leqslant j \leqslant n;\ x \in \mathrm{H}_j(\sigma, \varepsilon, f)\} \geqslant \theta n$.

Remark 8.3. It follows from Proposition 8.1 that if x is a λ-expanding point then there are $\varepsilon > 0$ and $\theta \in (0, 1)$ such that $x \in \limsup \mathcal{H}_n(e^{-\lambda/4}, \varepsilon, \theta, f)$. That is, every λ-expanding point x belongs not only to $\limsup \mathrm{H}_n(e^{-\lambda/4}, \varepsilon, f)$ but also to $\limsup \mathcal{H}_n(e^{-\lambda/4}, \varepsilon, \theta, f)$. In particular, if μ is a λ-expanding measure then there exists $\varepsilon > 0$ and $\theta \in (0, 1)$ such that

$$\mu(M \setminus \limsup \mathcal{H}_n(e^{-\lambda/4}, \varepsilon, \theta, f)) = 0.$$

The proof of Lemma 8.4 just below is easy and straightforward. For instance, replacing $\det Df$ by $J_\mu f$, the proof proceeds exactly as in the Lebesgue case of Proposition 2.5. of [45].

Lemma 8.4. *If μ is an f-non-flat measure then there is $\rho > 0$ such that*

$$\left|\log \frac{J_\mu f^n(p)}{J_\mu f^n(q)}\right| \leqslant \rho\, \mathrm{dist}(f^n(p), f^n(q))$$

for every $x \in \mathrm{H}_n(\sigma, \varepsilon, f)$ and μ-almost every p and $q \in V_n(x)$.

By Proposition 8.2, given $\sigma \in (0, 1)$ and $\varepsilon > 0$, there is $\delta > 0$ such that if $n \in \mathbb{N}$ is (σ, ε)-hyperbolic time for $x \in M$ then n is an $(\{\alpha_n\}_n, \delta)$-zooming time to x, where $\alpha_n(r) = \sigma^{n/2} r$. Thus, using together Propositions 8.1 and 8.2, we get the following remark.

Remark 8.5. If $\lambda > 0$ and $\mathcal{H}$ is a λ-expanding set and $\varepsilon_0 > 0$ is given by Proposition 8.1 then for every $0 < \varepsilon \leqslant \varepsilon_0$ there is $\delta > 0$ such that $\mathcal{H}$ is an $(\{\alpha_n\}_n, \delta)$-zooming set, where $\alpha_n(r) = e^{-(\lambda/8)n} r$. Furthermore $\mathcal{H} \subset \mathrm{H}_n(e^{-\lambda/4}, \varepsilon, f) \subset \mathrm{Z}_n(\alpha, \delta, f)$ $\forall n \geqslant 1$.

In particular, every λ-expanding measure is an $(\{\alpha_n\}_n, \delta)$-zooming measure. Furthermore, using Lemma 8.4, we obtain the following lemma.

Lemma 8.6. *Given $\lambda > 0$ there is $\delta > 0$ (depending only on λ and f) such that every f-non-flat λ-expanding measure is an $(\{\alpha_n\}_n, \delta)$-zooming measure with bounded distortion at the zooming times, where $\alpha_n(r) = e^{-(\lambda/8)n} r$.*

Proof of Theorem A. The proof of Theorem A follows straightforwardly from Lemma 8.6 and Theorem C. □

Remark 8.7. If in Theorem A we set $\lambda = 0$, then the results will be the same with one difference only: the collection of measures is not finite but countable.

To prove the remark above, let M' be the set of points $y \in M$ such that Eq. (3) holds for every $x \in \bigcup_{k=0}^{+\infty} f^{-k}(y)$. As $\mu \circ f^{-1} \ll \mu$, $\mu(M \setminus M') = 0$. For each $0 < n \in \mathbb{N}$, let M_n be the set of $x \in M'$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log\left\| \left(Df\left(f^i(x)\right)\right)^{-1} \right\|^{-1} \in \left(\frac{1}{n+1}, \frac{1}{n}\right].$$

Note that M_n is an invariant set $\forall n \in \mathbb{N}$, i.e., $f^{-1}(M_n) = M_n$. Let $\mathcal{N} \subset \mathbb{N}$ the set of $n \in \mathbb{N}$ such that $\mu(M_n) > 0$. For each $n \in \mathcal{N}$, we can apply Theorem A to $\mu|_{M_n}$. Thus, we only have to consider the collection of all measure ν such that ν is $\mu|_{M_n}$ absolutely continuous ergodic f-invariant probabilities for some $n \in \mathcal{N}$.

As remarked in the introduction of [45], any multimodal map that satisfies Keller's hypothesis [33] also satisfies the slow approximation condition to the critical set and the expanding condition, that is, satisfies (for Lebesgue measure) the hypothesis of Theorem A.

Proof of Theorem B. This theorem is a direct consequence of Theorem D and the fact that the λ-expanding set $\mathcal{H}$ is an (α, δ)-zooming set, where $\alpha = \{\alpha_n\}_n$, $\alpha_n(r) = e^{-(\lambda/8)n} r$, δ is given by Proposition 8.2 and θ is given by Proposition 8.1. The case $\lambda = 0$ follows directly from the case $\lambda > 0$ by taking any sequence $\lambda_n \searrow 0$ and setting the Markov structure as

$$\mathfrak{F} = \left\{(F, \mathcal{P});\ (F, \mathcal{P}) \in \mathfrak{F}(\lambda_n) \text{ and } n \in \mathbb{N}\right\},$$

where $\mathfrak{F}(\lambda_n)$ is the Markov structure for λ_n. □

The Markov structure in Theorem B satisfies the following additional property.

Remark 8.8 *(Induced time × Hyperbolic times).* In Theorem B, every $P \in \mathcal{P}_i$ is contained in a hyperbolic pre-ball of order $R_i(P)$, i.e., there is $\varepsilon_i > 0$ such that if $P \in \mathcal{P}_i$ then $P \subset V_{R_i(P)}(x)$, where $V_{R_i(P)}(x)$ is an $(e^{-\lambda_i/4}, \varepsilon_i)$-hyperbolic pre-ball for some $x \in \mathcal{H}$. Moreover, there is $m_i \geqslant 1$ ($m_i = 1$ $\forall i$, if f is backward separated) such that

$$R_i(x) \leqslant \min\left\{n \geqslant 1;\ x \in \mathrm{H}_{m_i n} \text{ and } f^{m_i n}(x) \in U_i\right\},$$

where $\mathrm{H}_{m_i n}$ is the set of points having $m_i n$ as an $(e^{-\lambda_i/4}, \varepsilon_i)$-hyperbolic time for f.

9. Examples and applications

The purpose of the current section is to give examples of expanding and zooming measures and also to give some illustrative applications of the theorems and ideas previously developed.

For now, consider a compact Riemannian manifold M of dimension $d \geqslant 1$. Let $f : M \to M$ be a non-flat map and $\mathcal{C} \subset M$ its critical/singular set.

Definition 9.1. We say that a point $x \in M$ has all Lyapunov exponents positive if

$$\limsup \frac{1}{n} \log\left\| \left(Df^n(x)\right)^{-1} \right\|^{-1} > 0.$$

A perionic point p of period n is a repeller if and only if $Df^n(p)$ is well defined and the absolute value of any eigenvalue of $Df^n(p)$ is bigger than one. As

$$\lim_{n_0\to\infty}\left\|\left(\left(Df^n(p)\right)^{-1}\right)^{n_0}\right\|^{\frac{1}{n_0}}=\min\left\{\lambda^{-1};\ \lambda\text{ is an eigenvalue of }Df^n(p)\right\},\tag{37}$$

the periodic point p is a repeller if and only if there is $n_0\geqslant 1$ such that p is a periodic point for $\widetilde{f}=f^{n_0}$ with period n and such that $\log\|(D\widetilde{f}^n(x))^{-1}\|^{-1}>0$ $\forall x\in\mathcal{O}_f^+(p)$ (for this take any prime $n_0\in\mathbb{N}$ big enough).

Lemma 9.2. *If p is a periodic repeller of period $n\geqslant 1$ and $\mathcal{O}_f^-(p)\cap\mathcal{C}=\emptyset$ then given any $\lambda_0>0$ there exists $\ell\geqslant 1$ such that p is a periodic point of period n with respect to $\widetilde{f}=f^\ell$ and $\mathcal{O}_f^-(p)$ is a λ_0-expanding set for $\widetilde{f}$. Furthermore, there are $\delta>0$ and $1>\sigma>0$ such that $\mathcal{O}_f^-(p)\subset\limsup_k \mathrm{Z}_k(\{\sigma^{sj}\}_j,\delta,(\widetilde{f})^s)$ $\forall s\geqslant 1$.*

Proof. Let $m\geqslant 1$ and set $\mathcal{C}_{f^m}=\bigcup_{j=0}^{m-1}f^{-j}(\mathcal{C})$, the critical set of f^m. As $\mathcal{O}_f^-(p)\cap\mathcal{C}=\emptyset$, we get $\mathcal{O}_f^-(p)\cap\mathcal{C}_{f^m}=\emptyset$. From this, it follows that, for every $0<\delta<\operatorname{dist}(\mathcal{O}_f^+(p),\mathcal{C}_{f^m})$ and all $y\in\mathcal{O}_f^-(p)$,

$$\lim_{j\to+\infty}\frac{1}{j}\sum_{i=0}^{j-1}-\log\operatorname{dist}_\delta\left(\left(f^m\right)^i(x),\mathcal{C}_{f^m}\right)=\lim_{j\to+\infty}\frac{1}{j}\sum_{i=0}^{j-1}-\log\operatorname{dist}_\delta\left(\left(f^m\right)^i(p),\mathcal{C}_{f^m}\right)=0.$$

Thus, $\mathcal{O}_f^-(p)$ satisfies the slow approximation condition with respect to f^m (and the critical set of f^m), for every $m\geqslant 1$.

Let n_0 be such that $\log\|(Df^{nn_0}(x))^{-1}\|^{-1}>0$, $\forall x\in\mathcal{O}_f^+(p)$. Set

$$a_0=\min\left\{\log\left\|\left(Df^{nn_0}(x)\right)^{-1}\right\|^{-1};\ x\in\mathcal{O}_f^+(p)\right\}$$

and

$$a_1=\min\left\{\log\left\|Df(x)^{-1}\right\|^{-1};\ x\in\mathcal{O}_f^+(p)\right\}.$$

Let ℓ be a big prime number and write $\ell=mn_0n+r$ with $0\leqslant r<n_0n$. As m goes to infinite with ℓ, one can take ℓ big enough so that $ma_0+n_0na_1>\lambda_0$. By chain rule, we get $\log\|(Df^\ell(p))^{-1}\|^{-1}>\lambda_0$. Of course, p is a periodic point with period n for the map $\widetilde{f}=f^\ell$.

Of course that $\mathcal{O}_f^-(p)$ is a positively invariant set with respect to $\widetilde{f}$. Moreover, as $\mathcal{O}_f^-(p)\cap\mathcal{C}=\emptyset$, it follows that

$$\lim_{j\to\infty}\frac{1}{j}\sum_{k=0}^{j-1}\log\left\|\left(D\widetilde{f}^k(y)\right)^{-1}\right\|=\lim_{j\to\infty}\frac{1}{j}\log\left\|\left(D\widetilde{f}^j(p)\right)^{-1}\right\|$$

for all $y\in\mathcal{O}_f^-(p)$. Thus, $\mathcal{O}_f^-(p)$ is a λ_0-expanding set with respect to $\widetilde{f}$.

Let $\delta>0$ and $0<\sigma<1$ be such that $\mathcal{O}_f^+(p)\subset\liminf_j \mathrm{Z}_j(\{\sigma^{2m}\}_m,\delta,\widetilde{f})$. From Lemma 5.7 follows that

$$\mathcal{O}_f^+(p)\subset\liminf_j \mathrm{Z}_j\left(\left\{\sigma^{2sm}\right\}_m,\delta,\widetilde{f}^s\right),\quad\forall s\in\mathbb{N}.$$

For each $y\in\mathcal{O}_f^-(p)$, let $t_y\in\mathbb{N}$ be such that $\widetilde{f}^{st_y}(y)=q\in\mathcal{O}_f^+(p)$ and let U_y be an open neighborhood of y such that $\widetilde{f}^{st_y}|_{U_y}$ is a diffeomorphism. Let $m_y\in\mathbb{N}$ be such $\overline{V_j(q)}\subset\widetilde{f}^{st_y}(U_y)$ for all $(\{\sigma^{2sm}\}_m,\delta)$-zooming time for q (with respect to $\widetilde{f}^s$) bigger than m_y. Taking m_y big enough we can be assured that $\operatorname{dist}(\widetilde{f}^{sj}(x_1),\widetilde{f}^{sj}(x_2))\leqslant\sigma^{s(k+t_y)-j}\operatorname{dist}(\widetilde{f}^{s(k+t_y)}(x_1),\widetilde{f}^{s(k+t_y)}(x_2))$ for all $x_1,x_2\in(\widetilde{f}^{st_y}|_{U_y})^{-1}(V_k(q))$ and all $0\leqslant j\leqslant k+t_y$, where k is a $(\{\sigma^{2sm}\}_m,\delta)$-zooming time (with respect to $\widetilde{f}^s$) for q and $V_k(y)$ is a zooming pre-ball (also with respect to $\widetilde{f}^s$). From this follows that $k+t_y$ is a $(\{\sigma^{sm}\}_m,\delta)$-zooming time (with respect to $\widetilde{f}^s$) for y. □

Recall that the α-limit set of a point x, denoted by $\alpha_f(x)$, is the set of accumulating points of $\mathcal{O}_f^-(x)=\bigcup_{j=0}^\infty f^{-j}(x)$, the pre-orbit of x. As there is a substantial number of example of dynamics exhibiting periodic repellers whose α-limits have non-empty interior, the proposition below show an easy way to find expanding measures with big support in the topological sense (non-empty interior).

Proposition 9.3. *Let $f : M \to M$ be a non-flat map and $\mathcal{C}$ its critical/singular set. If there is a periodic repeller p contained in the interior of its α-limit set and such that $\mathcal{O}_f^-(p) \cap \mathcal{C} = \emptyset$ then there is an open neighborhood Δ of p and an uncountable collection $\mathcal{M}$ of ergodic invariant probabilities such that if $\mu \in \mathcal{M}$ then all of its Lyapunov exponents are positive and the support of μ contains Δ. Furthermore, $\exists \ell \geqslant 1$ such that every $\mu \in \mathcal{M}$ is an invariant ergodic zooming probability for $\widetilde{f} = f^\ell$ and*

$$\lim \frac{1}{n}\sum_{j=0}^{n-1} \log\big\| D\widetilde{f}\big(\widetilde{f}^j(x)\big)^{-1}\big\|^{-1} \geqslant 8$$

for μ almost every $x \in M$.

Proof. Let ℓ be as in Lemma 9.2 and $\widetilde{f} = f^\ell$. It follows from Proposition 8.1 that there are δ and $\theta_0 > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n}\#\big\{1 \leqslant j \leqslant n;\ x \in \mathrm{H}_j\big(e^{-6\log 2}, \delta, \widetilde{f}\big)\big\} \geqslant \theta_0$$

for every $x \in \mathcal{O}_f^-(p)$.

From Proposition 8.2 it follows that each $(e^{-6\log 2}, \delta)$-hyperbolic time is an (α, δ)-zooming time and every $(e^{-6\log 2}, \delta)$-hyperbolic pre-ball is an (α, δ)-zooming pre-ball (all with respect to $\widetilde{f}$), where $\alpha = \{\alpha_n\}$ and $\alpha_n(r) = (1/8)^n(r)$. In particular, $\mathcal{O}_f^-(p)$ is an (α, δ)-zooming set.

Noting that $\sum_n \alpha_n(r) < r/4\ \forall r > 0$, let $0 < r < \delta/4$ be small such that $\overline{B_r(p)} \subset \mathrm{int}(\alpha_f(p))$. Thus $\Delta := B_r^\star(p)$ is an (α, δ)-zooming nested ball (with respect to $\widetilde{f}$) containing $B_{r/2}(p)$.

Let $\widetilde{\mathfrak{z}} = (\mathfrak{z}(x))_{x\in\limsup \mathrm{H}_n(e^{-6\log 2}, \delta, \widetilde{f})}$ be the collection of zooming images of $\widetilde{f}$ that are $(e^{-6\log 2}, \delta)$-expanding images and let $\widetilde{\mathcal{E}}$ be the collection of all $\widetilde{\mathfrak{z}}$-pre-balls. Let R be the "first $\widetilde{\mathcal{E}}$-return time to Δ", F be the induced map associated to the "first $\widetilde{\mathcal{E}}$-return time to Δ" and $\mathcal{P}$ be the Markov partition associated to the "first $\widetilde{\mathcal{E}}$-return time to Δ" as in Definitions 6.2, 6.3 and 6.5 (all this definitions applied to $\widetilde{f}$ instead to f). By Corollary 6.6, $(F, \mathcal{P})$ is an induced full Markov map for $\widetilde{f}$ on Δ with inducing time R.

As the zooming points are dense on Δ (because $\mathcal{O}_f^-(p)$ is dense), $\{R > 0\} = \bigcup_{P\in\mathcal{P}} P$ is an open and dense subset of Δ. Let $\mathfrak{B} = \bigcap_{j\geqslant 0} F^{-j}(\{R > 0\})$, that is, $\mathfrak{B}$ is the set of points $x \in \Delta$ such that $F^j(x) \in \Delta\ \forall j \geqslant 0$. Of course, $\mathfrak{B}$ is a residual set of Δ. Furthermore, $\mathfrak{B}$ is a metric space with the distance induced by the distance of M and its topology is the induced topology.

Let $\mathcal{W}$ be the collection of subsets of $\mathfrak{B}$ formed by the empty set $\emptyset$ and all $Y \subset \mathfrak{B}$ such that $Y = (F|_{P_1})^{-1} \circ \cdots \circ (F|_{P_s})^{-1}(\mathfrak{B})$ for some sequence of $P_1, \ldots, P_s \in \mathcal{P}$. Note that $\mathcal{W}$ generates all open sets of $\mathfrak{B}$.

Let $\mathcal{A}$ be the collection of all sequence of numbers $\{a_P\}_{P\in\mathcal{P}}$ satisfying $a_P \in (0, 1)$, $\sum_{P\in\mathcal{P}} a_P = 1$ and $\sum_{P\in\mathcal{P}} a_P R(P) < \infty$.

Choose any $\{a_P\}_{P\in\mathcal{P}} \in \mathcal{A}$. Given any $Y \in \mathcal{W} \setminus \{\emptyset, \mathfrak{B}\}$, there is a unique sequence $P_1, \ldots, P_s \in \mathcal{P}$ such that $Y = (F|_{P_1})^{-1} \circ \cdots \circ (F|_{P_s})^{-1}(\mathfrak{B})$. In this case, define $\nu(Y) = \prod_{j=1}^s a_{P_j}$. Set also $\nu(\emptyset) = 0$ and $\nu(\mathfrak{B}) = 1$. It easy to see that $\nu(A \cup B) = \nu(A) + \nu(B)$ for every $A, B \in \mathcal{W}$ with $A \cap B = \emptyset$. Moreover, $\nu(A) \leqslant \nu(B)$ for every $A, B \in \mathcal{W}$ with $A \subset B$. As $\mathcal{W}$ generates the Borel algebra of $\mathfrak{B}$, ν can be extended as a measure on the Borel set of $\mathfrak{B}$. Furthermore, ν is F-invariant. Indeed, given $Y \in \mathcal{W}$, say $Y = (F|_{P_1})^{-1} \circ \cdots \circ (F|_{P_s})^{-1}(\mathfrak{B})$, we get

$$\begin{aligned}
\nu\big(F^{-1}(Y)\big) &= \nu\big(F^{-1}\big((F|_{P_1})^{-1} \circ \cdots \circ (F|_{P_s})^{-1}(\mathfrak{B})\big)\big)\\
&= \sum_{P\in\mathcal{P}} \nu\big((F|_P)^{-1}\big((F|_{P_1})^{-1} \circ \cdots \circ (F|_{P_s})^{-1}(\mathfrak{B})\big)\big)\\
&= \sum_{P\in\mathcal{P}} a_P a_{P_1} \ldots a_{P_s} = a_{P_1} \ldots a_{P_s} \underbrace{\sum_{P\in\mathcal{P}} a_P}_{1} = \nu(Y).
\end{aligned}$$

Note that if $x \in \mathfrak{B}$ and $P_0, P_1, P_2, \ldots$ is the itinerary of x by F (i.e., P_j is the element of $\mathcal{P}$ that contains $F^j(x)$ $\forall j \geqslant 0$) then $x = \bigcap_{j=0}^\infty C_n(x)$, where $C_n(x) = (F|_{P_0})^{-1} \circ \cdots \circ (F|_{P_j})^{-1}(\mathfrak{B})$ is the n-th cylinder containing x. As

$$\frac{\mu(F(C_n(x)))}{\mu(C_n(x))} = \frac{a_{P_1}a_{P_2}\ldots a_{P_n}}{a_{P_0}a_{P_1}a_{P_2}\ldots a_{P_n}} = \frac{1}{a_{P_0}} \quad \forall n > 0,$$

one can prove that the Jacobian of F with respect to μ, $J_\mu F(x)$ is well defined and it is constant on every $P \in \mathcal{P}$ (indeed, $J_\mu F|_P = a_P$). This implies that $\frac{J_\mu F^n(x)}{J_\mu F^n(y)} = 1$ for all $y \in C_n(x)$. As a consequence, one can easily conclude that μ is ergodic with respect to F.

As $\int R\,d\nu = \sum_{P\in\mathcal{P}} R(P)\nu(P) = \sum_{P\in\mathcal{P}} R(P)a_P < \infty$, it follows from Remark 4.3 that

$$\widetilde{\mu} = \sum_{P\in\mathcal{P}} \sum_{j=0}^{R(P)-1} \widetilde{f}_*^j(\nu|_P)$$

is an $\widetilde{f}$-invariant finite measure. Furthermore, as ν is F-ergodic, it follows that $\widetilde{\mu}$ is $\widetilde{f}$-ergodic.

Applying Corollary 4.6, we conclude that

$$\limsup_n \frac{1}{n} \#\{0 \leqslant j < n;\ \widetilde{f}^j(x) \in \mathcal{O}_F^+(x)\} > 0 \tag{38}$$

for $\widetilde{\mu}$ almost every $x \in \Delta$. If for some n and i we have $\widetilde{f}^n(x) = F^i(x)$ then, by construction, x belongs to some $(e^{-6\log 2}, \delta)$-hyperbolic pre-ball $V_n(y, \widetilde{f})$ with respect to $\widetilde{f}$. As a consequence, n is a zooming time for x (with respect to $\widetilde{f}$) and $\frac{1}{n}\sum_{j=0}^{n-1} \log\|D\widetilde{f}(\widetilde{f}^j(x))^{-1}\|^{-1} \geqslant ((e^{-6\log 2})^{1/2})^{-1} = 8$, whenever $\widetilde{f}^n \in \mathcal{O}_F^+(x)$. Thus

$$\limsup_n \frac{1}{n} \#\{0 \leqslant j < n;\ x \in \mathrm{Z}_j(\alpha, \delta, \widetilde{f})\} > 0 \tag{39}$$

and

$$\limsup \frac{1}{n} \sum_{j=0}^{n-1} \log\big\|D\widetilde{f}\big(\widetilde{f}^j(x)\big)^{-1}\big\|^{-1} \geqslant 8 \tag{40}$$

for $\widetilde{\mu}$ almost every $x \in \Delta$. As $\widetilde{\mu}(\Delta) > 0$, it follows by ergodicity that (39) and (40) holds for $\widetilde{\mu}$ almost every point $x \in M$.

One can easy check that $\mu = \sum_{j=0}^{\ell-1} f_*^j \widetilde{\mu}$ is an ergodic f-invariant measure such that (39) and (40) holds for μ almost every point $x \in M$. Therefore, μ is a zooming measure (with respect to $\widetilde{f}$) and it follows from (40) that all Lyapunov exponent are positive for μ almost every point of M.

Of course distinct elements of $\mathcal{A}$ give rise to distinct ergodic F-invariant probabilities. By ergodicity these probabilities are mutually singular and so the f-invariant measures generated from them are also mutually singular. So, as $\mathcal{A}$ are uncountable, this process gives an uncountable collection of f-invariant measures.

To finalize the proof, note that $\mu(U) \geqslant \nu(U) > 0$ for every open subset of Δ, because every open subset of Δ contains a non-empty $Y \in \mathcal{W}$, and $\nu(Y) > 0$ for all $\emptyset \neq Y \in \mathcal{W}$. This implies that $\operatorname{supp}\mu \supset \Delta$. □

Definition 9.4. A map $f : M \to M$ is called *strongly topologically transitive* if we get $\bigcup_{n\geqslant 0} f^n(U) = M$ for all open set $U \subset M$.

Theorem 5. *Let $f : M \to M$ be a C^{1+} map, possible with a critical region $\mathcal{C}$. If f is strongly topologically transitive and it has a periodic repeller $p \notin \mathcal{O}_f^+(\mathcal{C})$ then some iterate of f admits an uncountable number of ergodic invariant expanding probabilities whose supports are the whole manifold.*

Proof. As f is strongly topologically transitive, $\alpha_f(x) = M$ for every point $x \in M$. In particular, if p is a periodic repeller, we get $\alpha_f(p) = M \ni p$. Thus, we can apply Proposition 9.3. Let Δ, ℓ and $\mathcal{M}$ be given by Proposition 9.3.

Let $\mu \in \mathcal{M}$. As f is strongly topologically transitive, given any open set $U \subset M$ there is $n \geqslant 0$ such that $f^{-n}(U) \cap \Delta \neq \emptyset$. Thus $\mu(U) = \mu(f^{-n}(U)) \geqslant \mu(f^{-n}(U) \cap \Delta) > 0$ for every $\mu \in \mathcal{M}$ (because $\Delta \subset \operatorname{supp}\mu$). This implies that $\operatorname{supp}\mu = M$ $\forall \mu \in \mathcal{M}$.

Given $\mu \in \mathcal{M}$, we have that

$$\lim \frac{1}{n} \sum_{j=0}^{n-1} \log\big\|D\widetilde{f}\big(\widetilde{f}^j(x)\big)^{-1}\big\|^{-1} \geqslant 8$$

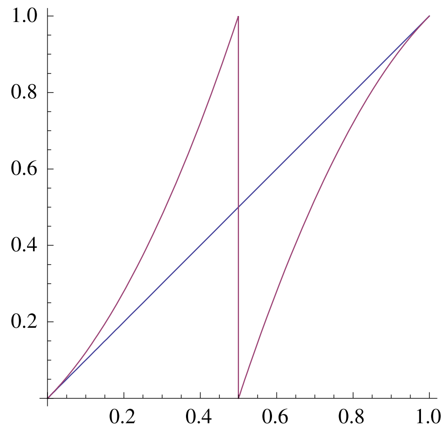

Fig. 7.

for μ almost every $x \in M$. As the equation above implies that there are no negative Lyapunov exponents with respect to any iterate of f and for μ-almost all point, it follows from Lemma A.2 that μ is an expanding measure with respect to $\widetilde{f}$. □

Corollary 9.5. *If a C^{1+} map $f : M \to M$, possibly with a critical region $\mathcal{C}$, is strongly topologically transitive and it has a periodic repeller $p \notin \mathcal{O}_f^+(\mathcal{C})$ then the set of periodic repeller is dense on M.*

Proof. This corollary follows from Theorem 5 and the fact that the support of any expanding invariant measure is contained in the closure of the periodic repellers (see Lemma A.5 of Appendix A). □

Example 9.6. (See Fig. 7.) Let $f : [0, 1] \to [0, 1]$ be given by

$$f(x) = \begin{cases} g(x) & \text{if } x < 1/2, \\ 1 - g(1-x) & \text{if } x \geqslant 1/2, \end{cases}$$

where $g(x) = x + 2x^2$.

The map f can be seen as a C^∞ map of the circle $S^1 = \mathbb{R}/\mathbb{Z}$ and this map is topologically conjugated to the uniformly expanding map $h(x) = 2x \pmod{\mathbb{Z}}$. Thus, f is strongly topologically transitive. Note that f has expanding periodic points. Indeed, f has a periodic point $p \in (0, 1)$ of period two (because h does) and, as $Df^2(x) > 1$ $\forall x \in (0, 1)$, it follows that p is an expanding periodic point. Thus, follows from Theorem 5 that some iterate of f admits an uncountable number of ergodic invariant expanding probabilities whose supports are the whole circle.

In [62] Young shows that for maps like f of Example 9.6 $\frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i(x)}$ converges weakly to the Dirac measure at 0 for Lebesgue almost every x. In particular, f admits no invariant measures that is absolutely continuous with respect to the Lebesgue measure. Furthermore, the Lyapunov exponent is zero for Lebesgue almost every point, contrasting with the existence of an uncountable number of ergodic invariant probabilities whose supports are the whole manifold and whose Lyapunov exponents are positive.

Example 9.7. Let $F : [0, 1]^2 \to [0, 1]^2$ be the skew product given by

$$F(x, y) = \big(f(x), (1 + x)\phi(y)\big)$$

where f is as in Example 9.6 and $\phi(y) = 1/2 - |y - 1/2|$ is the "tent" map of slope one. Taking any periodic point $p \in (0, 1)$ for f, it is easy to see that $\psi(y) = G(p, y)$ is a uniformly expanding map, where $(f^n(x), G(x, y)) = F^n(x, y)$ and n is the period of p. Thus taking any periodic point $q \in (0, 1)$ with respect to ψ, it follows that $(p, q) \in (0, 1)^2$ is an expanding periodic point of F. It is not difficult to check that $F|_{F([0,1]^2)}$ is strongly topologically transitive and so it follows from Theorem 5 that some iterate of F admits an uncountable number of ergodic invariant expanding probabilities whose supports are $F([0, 1]^2)$.

As in Example 9.6, the scenario of the expanding invariant measures of Example 9.7 is much richer than the Lebesgue measure scenario. Indeed, as

$$DF^n(x,y) = \begin{pmatrix} (f^n)'(x) & 0 \\ * & (+/-)\prod_{j=0}^{n-1}(1+f^j(x)) \end{pmatrix},$$

it follows that the Lyapunov exponents of a point (x,y) are $\limsup \frac{1}{n}\log|(f^n)'(x)|$, the Lyapunov exponent of x with respect to f, and $\limsup \frac{1}{n}\log\prod_{j=0}^{n-1}(1+f^j(x))$. As $\limsup \frac{1}{n}\log|(f^n)'(x)| = 0$ and $0 \leqslant \limsup \frac{1}{n}\log\prod_{j=0}^{n-1}(1+f^j(x)) \leqslant \limsup \frac{1}{n}\sum_{j=0}^{n-1} f^j(x) = 0$ for Lebesgue, a.e. $x \in [0,1]$ (Theorem 5 of [62]), we conclude that the *all Lyapunov exponents for Lebesgue almost every point are zero*.

One can find other examples of expanding measures in, for instance, [11] and [29].

Let us apply Theorem 5 to unimodal maps. For this, we note that every non-flat S-unimodal map $f:[0,1]\to[0,1]$ without a periodic attractor has an expanding periodic point $p\in(0,1)$. Moreover, one can show that a non-flat S-unimodal map f has an expanding periodic $p\in(0,1)\setminus\mathcal{O}_f^+(c)$ with dense pre-orbit if and only if f is not an infinitely renormalizable map and f does not have a periodic attractor. Thus, we get from Theorem 5 the following corollary.

Corollary 9.8. *If $f:[0,1]\to[0,1]$ is a non-flat S-unimodal map then one and only one of the following alternatives can occur.*

(1) *f has a periodic attractor.*
(2) *f is an infinitely renormalizable map.*
(3) *f admits an expanding invariant probability whose support has non-empty interior* (*indeed an uncountable number of these probabilities*).

The corollary above shows that the dynamic of any S-unimodal map in the complement of the Axiom A and the infinitely renormalizable maps exhibits uncountable many non-trivial expanding measures, even when there are not SRB measures.

Theorem 6 *(Markov structure for expanding sets of local diffeomorphisms). If $f:M\to M$ a C^{1+} map is a local diffeomorphism then the set of points with all Lyapunov exponents positive admits a Markov structure.*

Proof. Let $\lambda>0$ and, for each $\ell\in\mathbb{N}$, let Λ_ℓ be the set of λ-expanding points of M with respect to f^ℓ (as f is a local diffeomorphism the slow approximation condition is automatically satisfied). It follows from Lemma A.4 that $\bigcup_{\ell\in\mathbb{N}}\Lambda_\ell$ contains the set of points with all Lyapunov exponents positive (indeed, it is equal). As each Λ_ℓ has a Markov structure with respect to f^ℓ (and so, a Markov structure with respect to f), it follows that $\bigcup_{\ell\in\mathbb{N}}\Lambda_\ell$ has a Markov structure with respect to f. □

9.1. Maps with a dense expanding set

Besides the previous examples there are many examples of maps with a dense expanding set. Indeed, most of the results of the so called "non-uniformly expanding maps" was done with the hypothesis of an expanding set of full Lebesgue measure, in particular, a dense expanding set. This is, for instance, the case of Viana maps (see Example 9.11 below).

The crucial property used in the proof of Proposition 9.3 and Theorem 5 is indeed the existence of an expanding (or zooming) set that is dense and also some condition to spread open sets to the whole manifold. In the theorem below the hypotheses are chosen to obtain these properties again.

Theorem 7. *Let $f:M\to M$ be a transitive non-flat map with $\#f^{-1}(x)<\infty$ $\forall x\in M$. If f has a dense λ-expanding set, $\lambda>0$, then there is an uncountable collection of ergodic invariant λ'-expanding probabilities, $\lambda'\geqslant\lambda/8$, whose support are the whole manifold.*

Proof. Given any $x\in \mathrm{H}_n(e^{-\lambda/4},\varepsilon,f)$, let $V_n(x)$ is the $(e^{-\lambda/4},\delta)$-hyperbolic pre-ball of center x and order n, where $\varepsilon,\delta>0$ follows from Propositions 8.1 and 8.2. Note that

$$W := \bigcap_{j=0}^{\infty} \bigcup_{n \geqslant j} \left(\bigcup_{x \in \mathrm{H}_n} V_n(x) \right)$$

is a residual set, where $\mathrm{H}_n = \mathrm{H}_n(e^{-\lambda/4}, \varepsilon, f)$. Thus, the set of points $x \in W$ that are transitive ($\omega(x) = M$) is also a residual set (because the set of transitive points is residual). Choose a transitive point $q \in W$. As $q \in W$, there are sequences $n_k \to \infty$ and $x_k \in \mathrm{H}_{n_k}$ such that $q \in V_{n_k}(x_k)$ $\forall k \in \mathbb{N}$ and $\lim_{k \to \infty} f^{n_k}(x_k) = p$, for some $p \in M$. Of course that $x_k \to q$, indeed, $\mathrm{dist}(x_k, q) \leqslant e^{-(\lambda/8)n_k}\, \delta$, $\forall k$.

Let $\alpha = \{\alpha_n\}_n$, where $\alpha_n(r) = e^{-(\lambda/8)n} r$. As f is backward separated (because $\# f^{-1}(x) < \infty$ $\forall x \in M$) and as $\sup_{r>0} \sum_{n \geqslant 1} \alpha_n(r)/r < +\infty$, we can choose any $0 < r < r_0$ and consider the (α, δ)-zooming nested ball $B_r^\star(p)$, where $0 < r_0 < \delta/2$ is given by Lemma 5.12.

We claim that there is $\Lambda \subset B_r^\star(p)$ dense in $B_r^\star(p)$ and such that every $x \in \Lambda$ has as hyperbolic return to $B_r^\star(p)$, that is, given $x \in \Lambda$ there is $s \geqslant 1$ such that $x \in \mathrm{H}_s$ and $f^s(x) \in B_r^\star(p)$. Indeed, for each $y \in B_r^\star(p)$ and $\gamma > 0$ one can find $\widetilde{y} \in \mathcal{O}^+(q)$, say $\widetilde{y} = f^i(q)$, so that $\mathrm{dist}(\widetilde{y}, y) < \gamma/2$. Taking $k > i$ big enough so that $\mathrm{dist}(f^i(x_k), \widetilde{y}) = \mathrm{dist}(f^i(x_k), f^i(q)) < \gamma/2$, it follows that $\mathrm{dist}(f^i(x_k), y) < \gamma$, $f^i(x_k) \in \mathrm{H}_{n_k - i}$ and $f^{n_k - i}(x_k) \in B_r^\star(p)$.

Now, the proof follows as the proof of Proposition 9.3 with a single difference. Here we do not need to consider an iterate $\widetilde{f} = f^\ell$ of f. Taking $\widetilde{f} = f$ and $\Delta = B_r^\star(p)$, construct the induced map F and everything else as in the proof of Proposition 9.3. □

9.2. Decay of correlation and the Central Limit Theorem

In [5,6] Alves, Luzzatto and Pinheiro study the decay of correlation associated to the decay of the tail of expanding moments. There it was proved that a polynomial decay of the tail of expanding moments, measured by the Lebesgue measure, implies a polynomial decay of correlation for the absolutely continuous invariant measure with respect to the Lebesgue measure (the SRB measure). It was also proved that the Central Limit Theorem holds for the SRB whenever the tail of expanding moments decays more then quadratically. In [31] Gouëzel complemented this study for Lebesgue measure by showing that an exponential (or a stretched exponential) decay of the tail of expanding moments, measured by the Lebesgue measure, implies an exponential (or a stretched exponential) decay of correlation for the SRB measure. Here our construction permits to extend the results of these works for general expanding measures.

For Theorem 8 below, let $f : X \to X$ be a measurable map, backward separated, defined on a compact, connected, separable metric space. Let $\delta > 0$ and let $\alpha = \{\alpha_n\}$ be a zooming contraction with $\sup\{\frac{1}{r} \sum_n \alpha_n(r);\ r > 0\} < \infty$.

Let μ be a reference measure. Assume that μ is an (α, δ)-zooming measure with bounded distortion. Without loss of generality, we may assume that $f|_{\operatorname{supp} \mu}$ is transitive. By Corollary 5.17, $\omega(x) = \operatorname{supp} \mu$ for μ-almost every $x \in X$. In this case, it follows from Theorem C that there is a unique ergodic invariant measure $\nu \ll \mu$. Furthermore, $\operatorname{supp} \nu = \operatorname{supp} \mu$.

By Theorem 2, there is a compact set $A_{+,\mathfrak{z}} \subset \operatorname{supp} \mu$ such that $\omega_{+,f,\mathfrak{z}}(x) = A_{+,\mathfrak{z}}$ for μ-almost every point $x \in X$.

Take any $p \in A_{+,\mathfrak{z}}$ and $0 < r < \delta/2$ small. For μ-almost every $x \in B_r(p)$ let $z_p(x)$ be the first zooming return to $B_r(p)$, that is,

$$z_p(x) = \min\bigl\{n \geqslant 1;\ f^n(x) \in B_r(p) \text{ and } x \in \mathrm{Z}_n(\alpha, \delta, f)\bigr\}.$$

Theorem 8 *(Decay of correlation and Central Limit Theorem for zooming measures with local estimative). For any given functions $\phi, \psi : X \to \mathbb{R}$ with ϕ Hölder and ψ bounded, we have the following estimates for the decay of correlation*

$$\mathrm{Cor}\bigl(\phi, \psi \circ f^n\bigr) = \left| \int \phi \psi \circ f^n \, d\nu - \int \phi \, d\nu \int \psi \, d\nu \right|.$$

(1) *If $\mu\{z_p > n\} = O(n^{-\gamma})$ for some $\gamma > 0$, then* $\mathrm{Cor}(\phi, \psi \circ f^n) = O(n^{-\gamma})$.
(2) *If $\mu\{z_p > n\} = O(\exp(-\rho n^\gamma))$ for some $\rho, \gamma > 0$, then there exist $\widetilde{\rho} > 0$ such that* $\mathrm{Cor}(\phi, \psi \circ f^n) = O(\exp(-\widetilde{\rho} n^\gamma))$.

Furthermore, if $\mu\{z_p > n\} = O(n^{-\gamma})$ for some $\gamma > 1$ then the Central Limit Theorem *holds for any Hölder function $\phi : M \to \mathbb{R}$ such that $\phi \circ f \neq \psi \circ f - \psi$ for any ψ.*

Proof. Of course there is a natural identification of an induced full Markov map with μ-bounded distortion (see Definitions 4.2 and 1.5) with a Young Tower. Thus, this result follows from the existence of a full Markov induced map (F, R) such that μ can be lifted (Theorems 3 and D) and from Theorems 3 and 4 of [62]. $\square$

Remark 9.9. Although there is no explicit reference to the stretched exponential decay in the statement of Theorem 3 of [62], Young proofs can be adapted to the general case (see the comments in the proof Lemma 4.2 of [31] and also the comments in the begging of Section 4 of the same paper).

For Theorem 9 below, let $f : X \to X$ be a measurable map defined on a compact, connected, separable metric space. Let $\delta > 0$ and let $\alpha = \{\alpha_n\}$ be a zooming contraction.

Assume that there exists a set $\mathcal{C} \subset X$, called critical set, such that f is injective on each connected component of $X \setminus \mathcal{C}$ and such that $\mathcal{C}$ does not intersect any (α, δ)-zooming pre-ball, i.e., $V_n(p) \cap \mathcal{C} = \emptyset\ \forall p \in \mathrm{Z}_n(\alpha, \delta, f)$ and $\forall n \geqslant 1$ (alternatively, we may consider Remark 7.3).

Let the reference measure μ be an (α, δ)-zooming measure with bounded distortion. In this case, $\exists \theta > 0$ such that

$$\limsup_n \frac{1}{n} \big\{ j \leqslant n;\ x \in \mathrm{Z}_j(\alpha, \delta, f) \big\} \geqslant \theta$$

for μ-almost every $x \in X$.

As before, we may assume that $f|_{\operatorname{supp}\mu}$ is transitive. So, $\omega(x) = \operatorname{supp}\mu$ for μ-almost every $x \in X$ and there is a unique ergodic invariant measure $\nu \ll \mu$ (also $\operatorname{supp}\nu = \operatorname{supp}\mu$).

For μ-almost every $x \in X$ and any $\ell \geqslant 1$, define

$$z_\ell(x) = \min\big\{ n \geqslant 1;\ x \in \mathrm{Z}_{\ell n}(\alpha, \delta, f) \big\}.$$

Theorem 9 *(Decay of correlation for zooming measures with global estimative). There exists $\ell \geqslant 1$ such that for any given functions $\phi, \psi : X \to \mathbb{R}$, with ϕ Hölder and ψ bounded, we have the following estimates for the decay of correlation*

$$\operatorname{Cor}\big(\phi, \psi \circ f^n\big) = \left| \int \phi \psi \circ f^n \, d\nu - \int \phi \, d\nu \int \psi \, d\nu \right|.$$

(1) *If $\mu\{z_\ell > n\} = O(n^{-\gamma})$ for some $\gamma > 0$, then* $\operatorname{Cor}(\phi, \psi \circ f^n) = O(n^{-\gamma})$.
(2) *If $\mu\{z_\ell > n\} = O(\exp(-\rho n^\gamma))$ for some $\rho, \gamma > 0$, then there exist $\widetilde{\rho} > 0$ such that* $\operatorname{Cor}(\phi, \psi \circ f^n) = O(\exp(-\widetilde{\rho} n^\gamma))$.

Proof. Let $(F, \mathcal{P})$, R and $\mathcal{P}_0$ be the global induced Markov map, the inducing time and the finite partition of M by essentially open sets given by Theorem E. To construct the Young Tower [61,62] (or equivalently, an induced full Markov map with μ-bounded distortion) we can proceed as in the proof of Theorem 4.1 of [31]. In this theorem a global induced Markov map as in Theorem E induces a Young Tower with essentially the same estimates of the tail of the partition. We note that the Lebesgue measure is not important to the proof. Indeed the fundamental ingredient of Theorem 4.1 of [31] is the Lemma 9.10 below.

The construction of the local induced Markov map associated to the global one was already done in the proof of Theorem E, this is precisely the induced map $\widetilde{F}_0$ given by (36) and defined in the connected open set Δ_{j_0}.

To emphasize f instead of its iterate $\widetilde{f} = f^\ell$, set $\tau(x) = \ell \widetilde{R}_0$ and $F_0(x) = f^{\tau(x)}(x) = \widetilde{F}_0(x)$.

Let $k(x) = \ell \widetilde{k}(x)$, where $\widetilde{k}(x)$ is given by Claim 6. Thus, $F_0(x) = F^{k(x)}(x)$. Set $t_j(x) = \sum_{i=0}^{j-1} R(F^i(x))$, for every $j \leqslant k(x)$. Of course that $\tau(x) = t_{k(x)}(x)$.

Let $\mathcal{P}^n$ be the partition of M given by $\mathcal{P}^n = \bigcap_{i=0}^{n-1} F^{-i}(\mathcal{P}_0)$. As Δ_{j_0} is contained in a ν-ergodic component $U \subset M$ with respect to f^ℓ (see the proof of Theorem E), there is some $L > 0$ such that every element of $\mathcal{P}_0$ contains an element of $\mathcal{P}^n$ whose image under F^n is Δ_{j_0}. From the distortion control it follows that there exist constants $C_0, \varepsilon > 0$ such that

$$\mu\big(\{\tau = t_j \text{ or} \ldots \text{or } \tau = t_{j+L-1};\ t_1, \ldots, t_{j-1}, \tau > t_{j-1}\}\big) \geqslant \varepsilon$$

and

$$\mu(\{t_{j+1} - t_j > n;\ t_1, \ldots, t_j\}) \leqslant C_0 \mu(\{R > n\}).$$

Thus, applying Lemma 9.10 and Theorem 3 of [62] the theorem follows (see also Remark 9.9). □

Lemma 9.10. *(See Gouëzel [31].) Let (X, μ) be a space endowed with a finite measure and $k : X \to \mathbb{N}$ and $t_0, t_1, t_2, \ldots : X \to \mathbb{N}$ measurable functions such that $0 = t_0 < t_1 < t_2 < \cdots$ almost everywhere. Set $\tau(x) = t_{k(x)}(x)$, and assume that there exist $L > 0$ and $\epsilon > 0$ such that*

$$\mu\{\tau = t_j \text{ or } \ldots \text{ or } \tau = t_{j+L-1};\ t_1, \ldots, t_{j-1},\ \tau > t_{j-1}\} \geqslant \epsilon. \tag{41}$$

Assume moreover that there exist a positive sequence u_n and a constant C_0 such that

$$\mu\{t_{j+1} - t_j > n;\ t_1, \ldots, t_j\} \leqslant C_0 u_n. \tag{42}$$

Then

(1) *If u_n has polynomial decay, $\mu\{\tau > n\} = O(u_n)$.*
(2) *If $u_n = e^{-cn^\eta}$ with $c > 0$ and $\eta \in (0, 1]$, then there exists $c' > 0$ such that $\mu\{\tau > n\} = O(e^{-c'n^\eta})$.*

Example 9.11 *(Viana maps).* An important class of non-uniformly expanding dynamical systems (with critical sets) in dimension greater than one was introduced by Viana in [59]. This class of maps can be described as follows. Let $a_0 \in (1, 2)$ be such that the critical point $x = 0$ is pre-periodic for the quadratic map $Q(x) = a_0 - x^2$. Let $S^1 = \mathbb{R}/\mathbb{Z}$ and $b : S^1 \to \mathbb{R}$ be a Morse function, for instance, $b(s) = \sin(2\pi s)$. For fixed small $\alpha > 0$, consider the map

$$\begin{aligned} \hat{f}:\quad & S^1 \times \mathbb{R} \longrightarrow S^1 \times \mathbb{R}, \\ & (s, x) \longmapsto (\hat{g}(s), \hat{q}(s, x)) \end{aligned}$$

where $\hat{q}(s, x) = a(s) - x^2$ with $a(s) = a_0 + \alpha b(s)$, and $\hat{g}$ is the uniformly expanding map of the circle defined by $\hat{g}(s) = ds \pmod{\mathbb{Z}}$ for some integer $d \geqslant 16$. It is easy to check that for $\alpha > 0$ small enough there is an interval $I \subset (-2, 2)$ for which $\hat{f}(S^1 \times I)$ is contained in the interior of $S^1 \times I$. Thus, any map f sufficiently close to $\hat{f}$ in the C^0 topology has $S^1 \times I$ as a forward invariant region. We consider from here on these maps restricted to $S^1 \times I$.

Most of the results for $f \in \mathcal{N}$ are summarized below:

(1) $\exists \mathcal{H} \subset S^1 \times I$, with full Lebesgue measure on $S^1 \times I$, and $\lambda > 0$ such that $\mathcal{H}$ is a λ-expanding set with respect to f [59] (indeed, to be coherent with the estimate (43) we may assume that $\mathcal{H}$ is a 2λ-expanding set);
(2) for each $0 < c < 1/4$ and $\varepsilon > 0$ there are constants $C(c, \varepsilon)$ and $\delta(x) > 0$ such that

$$\mathrm{Leb}(\{x;\ h_\varepsilon(x) > n\}) \leqslant C(c, \varepsilon) e^{-c\sqrt{n}}$$

for every $n \geqslant 1$ [3,59], where

$$h_\varepsilon(x) = \inf\left\{ j > 0;\ \frac{1}{j} \sum_{k=0}^{j-1} \log \left\| Df\left(f^k(x)\right)^{-1} \right\|^{-1} \geqslant \lambda \text{ and } \frac{1}{j} \sum_{k=0}^{j-1} -\log \mathrm{dist}_{\delta(\varepsilon)}\left(f^k(x), \mathcal{C}\right) \leqslant \varepsilon \right\}; \tag{43}$$

(3) $f|_{f(S^1 \times I)}$ is strongly topologically transitive and has a unique ergodic absolutely continuous invariant (thus SRB) measure whose support is $f(S^1 \times I)$ [2];
(4) the Central Limit Theorem holds for f [5];
(5) the correlations of Hölder functions decay at least like $e^{-c'\sqrt{n}}$, for some $c' > 0$ [31].

Of course Viana maps satisfies most of hypothesis of the theorems in this section (Section 9). In particular, it follows from Theorem 7 that there is an uncountable collection $\mathcal{M}$ of ergodic invariant probabilities such that all Lyapunov exponents of every $\mu \in \mathcal{M}$ are positive and the support of any $\mu \in \mathcal{M}$ is the whole manifold. Furthermore, every $\mu \in \mathcal{M}$ is an f invariant ergodic zooming probability and

$$\lim \frac{1}{n}\sum_{j=0}^{n-1}\log\left\|Df\left(f^j(x)\right)^{-1}\right\|^{-1} \geqslant \lambda/2$$

for μ almost every $x \in M$ and every $\mu \in \mathcal{M}$.

We can also apply Theorems A and B to the Viana maps. From Theorem A, we conclude that all non-flat expanding measure admits an absolutely continuous invariant measure (in particular one can apply this theorem to obtain the SRB measure). Furthermore, we can apply Theorem 8 (or Theorem 9, if we take into account Remark 7.3) to study the decay of correlation and the Central Limit Theorem of zooming (in particular, expanding) measures with bounded distortion.

Theorem 10. *If f is a Viana map then there exist an uncountable number of ergodic invariant expanding measures with exponential decay of correlation and whose support is the whole $f(S^1 \times I)$.*

Proof. The construction of the collection of expanding measures given by Theorem 7 or Theorem 5 comes from that proof of Proposition 9.3. This measures are associated to an induced full Markov map $(F, \mathcal{P})$ defined on a topological ball Δ and to the collection $\mathcal{A}$ of all sequence $\{a_P\}_{P_\mathcal{P}}$ satisfying $\sum_{P\in\mathcal{P}} a_P = 1$ and $\sum_{P\in\mathcal{P}} a_P R(P) < \infty$, where R is the induced time of F. As one can see in the proof of Proposition 9.3, each $a = \{a_P\}_{P_\mathcal{P}} \in \mathcal{A}$ generates an F-invariant measure ν_a and also an f-invariant measure μ_a, with $\nu_a \ll \mu_a$. Moreover, we have a very good distortion control of $J_{\nu_a}F^n$ in every cylinder C_n. Indeed $\frac{J_{\nu_a}F^n(x)}{J_{\nu_a}F^n(y)} = 1$ $\forall y \in C_n(x)$ (see details in the proof of Proposition 9.3). Let $a = \{a_P\}_{P_\mathcal{P}} \in \mathcal{A}$ be any sequence satisfying $\lim_n \frac{1}{n}\log(\sum_{P\in\mathcal{P}_n} a_P) = \gamma < 0$, where $\mathcal{P}_n = \{P \in \mathcal{P};\ R(P) = n\}$. Thus, $\nu_a(\{R > n\}) = \sum_{j>n} \nu_a(\sum_{P\in\mathcal{P}_j} a_p) = O(e^{-\gamma n})$ and it follows from [62] that μ_a has exponential decay of correlation. $\square$

9.3. Expanding measures on metric spaces

In Sections 1.1 and 8 we deal with expanding sets and measures on Riemannian manifold because the standard way to define these objects is using the derivative of the map. Precisely, using $\|(Df)^{-1}\|^{-1}$. So, to extend the notion of expanding sets or measures we need to rewrite this expression in terms of the distance. For this, note that if $f : M \to M$ is differentiable at a point $p \in M$ then

$$\left\|\left(Df(p)\right)^{-1}\right\|^{-1} = \liminf_{x\to p} \frac{\operatorname{dist}(f(x), f(p))}{\operatorname{dist}(x, p)}.$$

Thus, given a metric spaces X and Y and a map $f : X \to Y$ define

$$\mathbb{D}^- f(p) = \liminf_{x\to p} \frac{\operatorname{dist}(f(x), f(p))}{\operatorname{dist}(x, p)},$$

where we are using the notation dist to assign the distance on both spaces. Define also

$$\mathbb{D}^+ f(p) = \limsup_{x\to p} \frac{\operatorname{dist}(f(x), f(p))}{\operatorname{dist}(x, p)}.$$

Of course one can rewrite the expanding condition (3) in terms of $\mathbb{D}^- f$, that is,

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{\infty} \log\left(\mathbb{D}^- f\right) \circ f^j(x) > 0, \tag{44}$$

and use this condition to define the expanding condition on a metric space. The critical/singular set $\mathcal{C}$ can be defined as the set of points $x \in X$ having $\mathbb{D}^- f(x) = 0$ or $\mathbb{D}^+ f(x) = \infty$. In the condition of non-degenerateness we only need to replace the expressions (C.1) and (C.2) by

(C.1) $$\frac{1}{B}\operatorname{dist}(x, \mathcal{C})^\beta \leqslant \mathbb{D}^- f(x) \leqslant \mathbb{D}^+ f(x) \leqslant B \operatorname{dist}(x, \mathcal{C})^{-\beta}.$$

and

$$\left|\log \mathbb{D}^- f(x) - \log \mathbb{D}^- f(x)\right| \leqslant \frac{B}{\operatorname{dist}(x, \mathcal{C})^\beta} \operatorname{dist}(x, y). \tag{C.2}$$

It is straightforward to check that, Proposition 8.1, Proposition 8.2 et cetera remain true if we change $\|(Df)^{-1}\|^{-1}$ by $\mathbb{D}^- f$. In particular, the expanding sets and measures with this definition are zooming sets and measures. As a consequence, if X is a connected, compact, separable metric space there are results analogues to Theorems A and B for this context (see Remark 5.6 if X is not connected).

Definition 9.12. The map f is called conformal at $p \in X$ if $\mathbb{D}^+ f(p) = \mathbb{D}^- f(p)$. In this case the conformal derivative of f at p is

$$\mathbb{D} f(p) = \lim_{x \to p} \frac{\operatorname{dist}(f(x), f(p))}{\operatorname{dist}(x, p)}.$$

It is easy to check that the *chain rule* holds for the conformal derivative. Moreover, it is obvious that one can rewrite the expanding condition (3) or (44) in terms, if it exists, of the conformal derivative $\mathbb{D} f$.

An example of a conformal in this definition is the shift with the usual metric.

Example 9.13 *(Expanding sets on a metric space).* Consider the *one-side shift* $\sigma : \sum_2^+ \to \sum_2^+$ with its usual metric, that is,

$$\operatorname{dist}(x, y) = \sum_{n=1}^{+\infty} \frac{|x_n - y_n|}{2^n},$$

where $x = \{x_n\}_n$ and $y = \{y_n\}_n$. It is easy to verify that σ is a conformal map and that $\mathbb{D}\sigma(x) = 2$ $\forall x \in \sum_2^+$.

As we could have expected, every positively invariant set (in particular the whole $\sum_2^+$) and all invariant measure for the map σ of the Example 9.13 are expanding.

In this paper we are basically interested in zooming and expanding measures. As we saw, the set of zooming measures contains the expanding measures. Now we will give examples of zooming sets and measures that are not expanding, i.e., examples of sets and invariant measures that are zooming with only a polynomial backward contraction.

Note that if $f : X \to X$ is a conformal map defined on a compact metric space X and $\mathbb{D} f \leqslant 1$ then it follows by compactness that given any $\varepsilon > 0$ there is $\delta > 0$ such that

$$\operatorname{dist}\bigl(f(x), f(y)\bigr) \leqslant (1 + \varepsilon) \operatorname{dist}(x, y)$$

$\forall x, y \in X$ satisfying $\operatorname{dist}(x, y) < \delta$. So, given any $\varepsilon > 0$ there is $\delta > 0$ such that if $x, y \in X$, $n \geqslant 1$ and $\operatorname{dist}(f^j(x), f^j(y)) < \delta$ $\forall 0 \leqslant j < n$ then

$$\operatorname{dist}\bigl(f^n(x), f^n(y)\bigr) \leqslant (1 + \varepsilon)^n \operatorname{dist}(x, y),$$

that is, $\mathbb{D} f \leqslant 1$ prohibits any exponential backward contraction. In particular, it does not admit any expanding set or measure.

Example 9.14 *(Zooming but not expanding).* Consider the *one-side shift* $\sigma : \sum_2^+ \to \sum_2^+$ with its usual topology. Consider the compatible metric given by

$$\operatorname{dist}(x, y) = \begin{cases} 0 & \text{if } x = y, \\ (\phi(x, y))^{-2} & \text{if } x \neq y, \end{cases}$$

where $x = \{x_n\}_n$, $y = \{y_n\}_n$ and $\phi(x, y) = \min\{n \geqslant 1;\ x_n \neq y_n\}$. It is easy to verify that σ is a conformal map and that $\mathbb{D}\sigma(x) = 1$ $\forall x \in \sum_2^+$. In particular,

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbb{D}\sigma^n(x) = \lim_n \frac{1}{n} \sum_{n=0}^{\infty} \log \mathbb{D}\sigma\bigl(\sigma^n(x)\bigr) = 0, \quad \forall x \in \sum_2^+.$$

So, σ does not admit any expanding set or measure. In contrast, given any $p \in \sum_2^+$ and $x, y \in C_n(p) = \{q \in \sum_2^+;\ p_1 = q_1 \dots p_n = q_n\}$, we have $\phi(\sigma^j x, \sigma^j y) = \phi(\sigma^n x, \sigma^n y) + (n-j)$, for $0 \leqslant j \leqslant n$. Thus,

$$\sqrt{\operatorname{dist}(\sigma^j x, \sigma^j y)} = \frac{1}{\phi(\sigma^n x, \sigma^n y) + (n-j)} = \frac{\sqrt{\operatorname{dist}(\sigma^n x, \sigma^n y)}}{1 + (n-j)\sqrt{\operatorname{dist}(\sigma^n x, \sigma^n y)}}$$

and so,

$$\operatorname{dist}(\sigma^j x, \sigma^j y) = \left(\frac{1}{1 + (n-j)\sqrt{\operatorname{dist}(\sigma^n x, \sigma^n y)}}\right)^2 \operatorname{dist}(\sigma^n x, \sigma^n y).$$

As a consequence, the cylinder $C_n(p)$ is an $(\alpha, 1)$-zooming pre-ball for p, where $\alpha = \{\alpha_n\}_n$ and $\alpha_n(r) := (\frac{1}{1+n\sqrt{r}})^2 r$ (one can check that $\alpha_n \circ \alpha_j(r) = \alpha_{n+j}(r)$). This implies that every positively invariant set of $\sum_2^+$ is an $(\alpha, 1)$-zooming set and any σ-invariant measure is $(\alpha, 1)$-zooming.

9.4. Future applications

Recently there was an increasing development of the study of the thermodynamic formalism beyond the uniformly hyperbolic context (including countable Markov shift) by several authors (this list is certainly not complete): Araujo [7], Arbieto, Matheus, Oliveira, Varandas, Viana [8,38,39,57,58], Bruin, Keller, Todd [16,15,17–19], Buzzi, Paccaut, Sarig, Schmitt [22,20,21,48–50], Dobbs [30], Denker, Keller, Nitecki, Przytycki, Rivera-Letelier, Urbański [23,26,24,25,27,28,46,56], Leplaideur, Rios [34], Pesin, Senti, Zhang [41–44], Wang, Young [60], Yuri [63–65]. In many cases, a natural place to look for an equilibrium state is the set of expanding measures. Thus, we believe that the results presented in this paper can be useful to the program of extending the thermodynamic formalism to the general non-uniformly hyperbolic setting.

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Appendix A

The proof of the following fact can be found in, for instance, [32], Lemma A.6.8.

Lemma A.1. *Let X be a connected, compact, separable metric space, μ be a finite measure defined on the Borel sets of X and $U \subset X$ be a measurable set with $\mu(U) > 0$. Given any $\varepsilon > 0$ there exists a finite partition $\mathcal{P}$ (mod μ) of U satisfying the following*:

1. $\mathcal{P} = \{B_1, \dots, B_s\}$, *where $B_1, \dots, B_s$ are open sets of X with* $\operatorname{diameter}(B_j) < \varepsilon\ \forall j$;
2. $\bigcup_j \overline{B_j} \supset U$;
3. $B_j \cap B_k = \emptyset$ *when* $j \neq k$;
4. $\mu(\partial B_j) = 0\ \forall j$.

Now, let M be a compact Riemannian manifold of dimension $d \geqslant 1$.

Lemma A.2. *Let $f : M \to M$ be a C^{1+} map. If μ is an f-invariant ergodic probability with all of its Lyapunov exponent finite* (*i.e.*, $\limsup \frac{1}{n} \log \|(Df^n(x))^{-1}\|^{-1} > -\infty$ *for μ-almost every x*) *then μ satisfies the slow approximation condition, that is, for each $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$\limsup_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}-\log\operatorname{dist}_\delta\big(f^j(x),\mathcal{C}\big)\leqslant\varepsilon,$$

for μ almost every $x\in M$.

Proof. Let $\mathcal{C}$ be the critical region of f (of course we may assume that $\mathcal{C}\neq\emptyset$). As f is C^{1+}, $\mathcal{C}$ is a compact set and also $\det Df$ is Holder. That is, $\exists k_0,k_1>0$ such that $|\det Df(x)-\det Df(y)|\leqslant k_0\operatorname{dist}(x,y)^{k_1}$ $\forall x,y\in M$. Given $x\in M$ there is $y_x\in\mathcal{C}$ such that $\operatorname{dist}(x,y_x)=\operatorname{dist}(x,\mathcal{C})$. Thus, we get $|\det Df(x)|=|\det Df(x)-\det Df(y_x)|\leqslant k_0\operatorname{dist}(x,y_x)^{k_1}=k_0\operatorname{dist}(x,\mathcal{C})^{k_1}$. That is, $\log|\det Df(x)|\leqslant\log k_0+k_1\log\operatorname{dist}(x,\mathcal{C})$. Let $m=\operatorname{dimension}(M)$ and note that $\|A^{-1}\|^{-m}\leqslant|\det A|\leqslant\|A\|^m$ for every $A\in GL(m,\mathbb{R})$. Thus, if $\int\log|\det Df|\,d\mu=-\infty$, it follows from Birkhoff that

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\log\big\|\big(Df^n(x)\big)^{-1}\big\|^{-1}&=\frac{1}{m}\limsup_{n\to\infty}\frac{1}{n}\log\big\|\big(Df^n(x)\big)^{-1}\big\|^{-m}\\&\leqslant\lim\frac{1}{n}\sum_{j=0}^{n-1}\log\big|\det Df\big(f^j(x)\big)\big|=-\infty\end{aligned}$$

for μ-almost every x, contradicting our hypothesis. So, $-\infty<\int\log|\det Df|\,d\mu-\log k_0\leqslant k_1\int\log\operatorname{dist}(x,\mathcal{C})\,d\mu(x)\leqslant k_1\log\operatorname{diameter}(M)$. As the logarithm of the distance to the critical set is integrable, it follows that

$$\int\log\operatorname{dist}_{e^{-n}}(x,\mathcal{C})\,d\mu(x)=\int\limits_{\{x;\ \log\operatorname{dist}(x,\mathcal{C})<-n\}}\log\operatorname{dist}(x,\mathcal{C})\,d\mu\to0$$

when $n\to\infty$. This implies (by Birkhoff) the slow approximation condition. $\square$

From Lemma A.2 follows the Corollary A.3.

Corollary A.3. *Let $f:M\to M$ be a C^{1+} map. An ergodic invariant probability μ is an expanding measure if and only if* (1) *holds for μ almost every $x\in M$.*

The lemma below is a remark that appears in Section 1.1 of [4].

Lemma A.4. *Let $f:M\to M$ be a C^1 local diffeomorphism and let μ be an f-invariant probability. If for μ-almost every $x\in m$ we have*

$$\lim_{n\to\infty}\log\big|Df^n(x)v\big|>0,\quad\forall|v|=1,\tag{45}$$

then there exist an iterate $\tilde f=f^\ell$ of f such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\big\|D\tilde f\big(\tilde f^j(x)\big)^{-1}\big\|^{-1}>0\tag{46}$$

for μ-almost every $x\in M$.

Proof. By the compactness of M, (45) implies that there is $\lambda>0$ such that for each $x\in M$ $\exists n_x\in\mathbb{N}$ satisfying $\log|Df^n(x)v|\geqslant2\lambda$ $\forall|v|=1$ and $\forall n\geqslant n_x$, that is,

$$\log\big\|\big(Df^n(x)\big)^{-1}\big\|^{-1}\geqslant2\lambda\quad\forall n\geqslant n_x.$$

Let $K=|\min_{x\in M}\log\|(Df^n(x))^{-1}\|^{-1}|$ and let $\varepsilon>0$ be such that $\varepsilon(1+K/\lambda)<1$. Let $\ell\geqslant1$ be so that $\mu(U)>1-\varepsilon$, where $U=\{x\in M;\ \log\|(Df^\ell(x))^{-1}\|^{-1}>\lambda\}$. Thus,

$$\int\log\big\|\big(Df^\ell\big)^{-1}\big\|^{-1}\,d\mu>\lambda\mu(U)-K\big(1-\mu(U)\big)=\lambda\big(1-\varepsilon(1+K/\lambda)\big)>0$$

and the proof of the lemma follows from Birkhoff. $\square$

Lemma A.5. *The support of any ergodic invariant expanding measure μ, with respect to a non-flat map $f : M \to M$ (possibly with a critical/singular region $\mathcal{C}$), is contained in the closure of the set of periodic repellers of f. Furthermore, for each $\varepsilon > 0$ there is a periodic repeller whose orbit is ε-dense on the support of μ.*

Proof. Let μ be an ergodic invariant expanding measure and $p \in M$ be a μ-generic point. Thus, $\omega(p) = \operatorname{supp}\mu$. By Proposition 8.2, there is a sequence $n_j \to \infty$ of hyperbolic times for p and a sequence of hyperbolic pre-balls $V_{n_j}(p)$ with f^{n_j} mapping $\overline{V_{n_j}(p)}$ diffeomorphically onto the ball $\overline{B_\delta(f^{n_j}(p))}$.

Let $m \geqslant 1$ be big enough so that $\{p, f(p), \dots, f^m(p)\}$ is $\delta/10$ dense on $\operatorname{supp}\mu$. Given any $\varepsilon > 0$, let k_0 be big enough so that $\{f^m(p),\ f^{m+1}(p),\ \cdots,\ f^{k_0}(p)\}$ is $\varepsilon/2$-dense on $\operatorname{supp}\mu$. Let $0 < r_0$ be small so that $f^m|_{B_{r_0}(p)}$ is a diffeomorphism and diameter$(f^j(B_{r_0}(p))) < \delta/10$ $\forall 0 \leqslant j \leqslant m$ (as x is an expanding point, note that $\{p, f(p), \dots, f^m(p)\} \cap \mathcal{C} = \emptyset$). Choose $0 < r < \varepsilon/3$ so that $\overline{B_r(f^m(p))} \subset f^m(B_{r_0}(p))$ and let $U = (f^m|_{B_{r_0}(p)})^{-1}(B_r(f^m(p)))$. Note that every ball of radius $\delta/2$ and center on a point of $\operatorname{supp}\mu$ contain at least one of the pre-images $U, f(U), \dots, f^m(U) = B_r(f^m(p))$ (because $\{p, f(p), \dots, f^m(p)\}$ be $\delta/10$ is dense).

Let $k \gg k_0$ be a very big hyperbolic time for $f^m(p)$. Thus, the diameter of the associated pre-ball $V_k(f^m(p))$ is smaller then $r/2$ and so, $\overline{V_k(f^m(p))} \subset B_r(f^m(p))$. As noted before, $B_{\delta/2}(f^{k+m}(p))$ contains the closure of some $f^s(U)$. So $f^k(V_k(f^m(p))) = B_\delta(f^{k+m}(p)) \supset \overline{f^s(U)}$. Let $W = (f^k|_{V_k(f^m(p))})^{-1}(f^s(U)) \subset V_k(f^m(p)) \subset B_r(f^m(p))$. Thus, f^{k+m-s} maps $\overline{W} \subset U$ diffeomorphically onto $\overline{U}$. Furthermore, as we can choose k as big as we want, the expansion of $f^k|_W$ is as big as we want. On the other hand, we can loose expansion only on the transport of $f^s(U)$ to $f^m(U) = B_r(f^m(p))$ and this is at most m steps. Therefore, it follows that $g = (f^{k+m-s}|_W)^{-1}$ is a contraction. In particular, f^{k+m-s} has a repeller fixed point $\widetilde{q} \in W$. Of course, $\widetilde{q}$ is a periodic repeller for f and, as the diameter of W is as small as k is big, $\widetilde{q}$ is as close of $f^m(p)$ as we want. From this follows that $\{q, f(q), \dots, f^{k_0}(q)\}$ is ε-dense. $\square$

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