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Finding new families of rank-one convex polynomials

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Abstract

We introduce a method to find, in a systematic way, rank-one convex polynomials. We show how it works in several examples. It can also be applied to convexity along general cones.

Keywords: Rank-one convexity; Laminate

1. Introduction

It is well known that quasiconvexity is a fundamental concept for vector problems in the Calculus of Variations [7,3]. Two important related convexity conditions are polyconvexity (a sufficient condition [2]), and rank-one convexity (a necessary condition [7]). Even these two types of convexity, though more manageable, are not easy to check on explicit examples [4,6]. In particular, rank-one convexity is an appealing property as it is like the usual convexity. Namely, we say that $\varphi : \mathbb{M} \to \mathbb{R}$ is rank-one convex provided that

$$\varphi(t_1\xi_1 + t_2\xi_2) \leqslant t_1\varphi(\xi_1) + t_2\varphi(\xi_2)$$

whenever $t_i \geqslant 0$, $t_1 + t_2 = 1$, and $\xi_2 - \xi_1$ is a rank-one matrix. $\mathbb{M}$ stands for the space of $m \times n$ matrices. If φ is smooth, the rank-one convexity is equivalent to the Legendre–Hadamard condition

$$A^T \nabla^2\varphi(\xi) A \geqslant 0,$$

for every $A \in \mathcal{A}$, $\xi \in \mathbb{M}$, where $\mathcal{A}$ is the rank-one cone.

Deciding when a given function is or is not rank-one convex is not an easy task. Our aim is to provide a way to determine (at least in some specific situations) the rank-one convexity of functions of a particular structure.

Our method can be applied to the following situation. Let

$$\varphi_i : \mathbb{M} \to \mathbb{R}, \quad i = 1, 2,$$

be two polynomials such that

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1. the combination
$$\varphi(\xi) = \varphi_1(\xi) - c\varphi_2(\xi) \tag{1}$$
for any constant $c \in \mathbb{R}$ is coercive with superlinear growth;
2. φ_1 is strictly convex.

The basic important problem we would like to address is

Problem 1. Determine the range of the constant c so that $\varphi(\xi)$ is rank-one convex.

For a general parameter c, it is possible to determine the range of this constants for which the corresponding family of functions are rank-one convex. In fact, the rank-one convexity of (1) is then equivalent to
$$A^T \nabla^2 \varphi_1(\xi) A - c A^T \nabla^2 \varphi_2(\xi) A \geqslant 0,$$
or to
$$\frac{c A^T \nabla^2 \varphi_2(\xi) A}{A^T \nabla^2 \varphi_1(\xi) A} \leqslant 1, \quad A \in \mathcal{A},\ \xi \in \mathbb{M}.$$
If
$$\frac{1}{c_-}\left(\text{resp. } \frac{1}{c_+}\right) = \inf_{A \in \mathcal{A},\, \xi \in \mathbb{M}} (\text{resp. } \sup) \frac{A^T \nabla^2 \varphi_2(\xi) A}{A^T \nabla^2 \varphi_1(\xi) A},$$
then it is easy to derive

Theorem 1. *Let*
$$\varphi = \varphi_1 - c\varphi_2,$$
where φ_i are smooth and φ_1 is strictly convex. Then φ is rank-one convex if and only if

1. $c \in [c_-, c_+]$, *if φ_2 is neither rank-one convex nor rank-one concave* (*alternatively, we can write*: $A^T \nabla^2 \varphi_2(\xi) A$ *attains both positive and negative values*);
2. $c \in (-\infty, c_+]$, *if*
$$\inf_{A \in \mathcal{A},\, \xi \in \mathbb{M}} \frac{A^T \nabla^2 \varphi_2(\xi) A}{A^T \nabla^2 \varphi_1(\xi) A} = 0;$$
3. $c \in [c_-, +\infty)$, *if*
$$\sup_{A \in \mathcal{A},\, \xi \in \mathbb{M}} \frac{A^T \nabla^2 \varphi_2(\xi) A}{A^T \nabla^2 \varphi_1(\xi) A} = 0.$$

Remark 1. We will make the assumption that if $\frac{1}{c_-} = -\infty$ (resp. $\frac{1}{c_+} = +\infty$) then $c_- = 0$ (resp. $c_+ = 0$).

Though the proof of this result is straightforward in these terms, it is quite remarkable that these optimal constants can be computed explicitly in specific examples, as we show in Section 3.

Before that, we also provide an appropriate description of this theorem in terms of laminates. This seems interesting as this strategy looks more promising for other situations like polyconvexity and, even, quasiconvexity. The proof of this theorem from this point of view can be found in Section 4.

2. Alternative route: Laminates

We know that laminates are the class of probability measures which play a fundamental role with respect to rank-one convexity through duality with Jensen's inequality [8]. In this section it is presented the result of the previous one, from the point of view of laminates. We think that this gives further insight into the problem, specially because it is more easily visualized. To state the main result in terms of laminates requires some notation.

Let $\Lambda(\xi_0)$ denote the set of laminates with barycenter ξ_0. Consider the linear mapping

$$T:\Lambda(\xi_0)\mapsto\mathbb{R}^2,\quad T(\mu)=\left(\int\varphi_1(\xi)\,d\mu(\xi),\int\varphi_2(\xi)\,d\mu(\xi)\right).$$

It is clear that $T(\Lambda(\xi_0))$ is a convex set in $\mathbb{R}^2$. If (x,y) designate usual coordinates in $\mathbb{R}^2$, and we put

$$x_0=\varphi_1(\xi_0),\qquad y_0=\varphi_2(\xi_0),$$

we know, due to convexity of φ_1, that

$$T\big(\Lambda(\xi_0)\big)\subset\big\{(x,y)\in\mathbb{R}^2:\ x\geqslant x_0\big\}.$$

Even more, because of strict convexity of φ_1, the intersection of $T(\Lambda(\xi_0))$ with the vertical line $x=x_0$ is the unique point (x_0,y_0). Then solving Problem 1 is equivalent to determining the best constants c_-, c_+ so that

$$T\big(\Lambda(\xi_0)\big)\subset C\big((x_0,y_0),c_-,c_+\big),$$

for every $\xi_0\in\mathbb{M}$, where $C((\bar{x},\bar{y}),c_1,c_2)$ is the cone in $\mathbb{R}^2$ defined by

$$C\big((\bar{x},\bar{y}),c_1,c_2\big)=\big\{(x,y)\in\mathbb{R}^2:\ c_1(x-\bar{x})+\bar{y}\leqslant y\leqslant c_2(x-\bar{x})+\bar{y},\,x\geqslant\bar{x}\big\}.$$

For $s\in[0,1]$, we consider our basic first-order laminates

$$\mu_s=\frac{1}{2}\delta_{\xi_0+sA}+\frac{1}{2}\delta_{\xi_0-sA},$$

for A of rank one. Finally, consider the plane curve

$$\sigma^{(A,\xi_0)}(s)=T(\mu_s)=\left(\frac{1}{2}\varphi_1(\xi_0+sA)+\frac{1}{2}\varphi_1(\xi_0-sA),\frac{1}{2}\varphi_2(\xi_0+sA)+\frac{1}{2}\varphi_2(\xi_0-sA)\right).$$

$\mathcal{A}$ stands for the cone of rank-one matrices.

Theorem 2. *Let φ be as in Theorem* 1 *and*

$$\frac{1}{c_-}\left(\textit{resp. }\frac{1}{c_+}\right)=\inf_{A\in\mathcal{A},\,\xi_0\in\mathbb{M}}(\textit{resp. }\sup)\frac{\ddot{\sigma}_2^{(A,\xi_0)}(0)}{\ddot{\sigma}_1^{(A,\xi_0)}(0)}.$$

Then φ is rank-one convex if and only if

1. $c\in[c_-,c_+]$, *if $\ddot{\sigma}_2$ attains both positive and negative values*;
2. $c\in(-\infty,c_+]$, *if*
$$\inf_{A\in\mathcal{A},\,\xi_0\in\mathbb{M}}\frac{\ddot{\sigma}_2^{(A,\xi_0)}(0)}{\ddot{\sigma}_1^{(A,\xi_0)}(0)}=0;$$
3. $c\in[c_-,+\infty)$, *if*
$$\sup_{A\in\mathcal{A},\,\xi_0\in\mathbb{M}}\frac{\ddot{\sigma}_2^{(A,\xi_0)}(0)}{\ddot{\sigma}_1^{(A,\xi_0)}(0)}=0.$$

Remark 2. Obviously, we have that

$$\ddot{\sigma}_i^{A,\xi_0}(0)=A^T\nabla^2\varphi_i(\xi_0)A,$$

where

$$\operatorname{rank}(A)\leqslant 1.$$

3. Examples

We now want to solve problem

$$\inf_{A,\xi_0\in\mathbb{M}} (\text{resp. } \sup) \frac{A^T\nabla^2\varphi_2(\xi_0)A}{A^T\nabla^2\varphi_1(\xi_0)A}$$

subject to the restriction

$$\operatorname{rank}(A) \leqslant 1.$$

To fix ideas, consider the minimization problem as a partial double minimization problem. If we minimize first in $A \in \mathbb{M}$, the above quotient is always a quotient of two expressions which are homogeneous of degree two in A, where

$$A^T\nabla^2\varphi_1(\xi_0)A > 0.$$

So, we can consider the equivalent problem

$$\min_{A\in\mathbb{M}} A^T\nabla^2\varphi_2(\xi_0)A$$

subject to the restrictions

$$\begin{cases} A^T\nabla^2\varphi_1(\xi_0)A = 1, \\ A, \quad \text{rank-one.} \end{cases}$$

In the particular case of 2×2 matrices, we can replace the rank-one condition on A by the more quantitative condition $A^T DA = \det A$. Anyhow, this minimum is attained since the function to minimize is continuous, and the domain is the intersection between a compact set and a closed set.

Let us stick to the 2×2 situation for the sake of this short discussion. If α, β are Lagrange multipliers, we put

$$L(A,\alpha,\beta) = A^T\nabla^2\varphi_2(\xi_0)A - \alpha\big(A^T\nabla^2\varphi_1(\xi_0)A - 1\big) - \beta A^T DA.$$

From first-order optimality conditions, if A is a critical point of the objective function, one obtains

$$A^T\nabla^2\varphi_2(\xi_0)A = \alpha,$$

where α can be recovered from solving the following system

$$\begin{cases} \big(\nabla^2\varphi_2(\xi_0) - \alpha\nabla^2\varphi_1(\xi_0) - \beta D\big)A = 0, \\ A^T\nabla^2\varphi_1(\xi_0)A = 1, \\ A^T DA = 0. \end{cases}$$

α will be a function of ξ_0, and to finish, we would have to compute the infimum with respect to the variable $\xi_0 \in \mathbb{M}^{2\times 2}$. In the case where the φ_i's are polynomials, the above system of equations is indeed a parametric system of polynomial equations, where ξ_0 is the parameter, and A, α, β are the variables to solve for. There exist several algorithms which deal with the problem of describing the solutions of these systems in terms of the parameters, such as comprehensive Gröbner bases [11], triangular sets decomposition [10] and rational parametrizations [9]. The description of the generic solutions of this systems is in general difficult and is beyond the scope of this work. Here we will deal with a simple example, whose system can be solved with several recent symbolic mathematical softwares.

For a more general situation, we can replace the matrix A by $a \otimes n$ even under the constraints $|a| = |n| = 1$. In this case, we would have to solve the problem

$$\inf_{\xi_0} \min_{a,n} n \otimes a\nabla^2\varphi_2(\xi_0)a \otimes n$$

subject to the constraint

$$n \otimes a\nabla^2\varphi_2(\xi_0)a \otimes n = 1.$$

We can then use optimality conditions to make some progress in the calculations. However, one has to keep track of the dependence on a and ξ_0 when solving the minimization problem for n. In general, it is not so easy to compute the range for the constant c through this approach.

In the case of 4th degree homogeneous polynomials, we can easily overcome this difficulties. For this special situation, we can take advantage of the fact that $A^T\nabla^2\varphi_i(\xi_0)A$ is also quadratic in ξ_0. In other words, we can write

$$A^T\nabla^2\varphi_i(\xi_0)A=\xi_0^T M_i(A)\xi_0,$$

where $M_i(A)$, for $i=1,2$ is a matrix whose entries only depend on $A\in\mathcal{A}$. This is a huge advantage, as in this case we can perform first the minimization in ξ_0, and then in A, avoiding in this way to include the additional rank-one restriction, but still dealing with quadratic problems. We want hence to compute

$$\min_{A\in\mathcal{A}}\left(\min_{\xi_0\in\mathbb{M}}\frac{\xi_0^T M_2(A)\xi_0}{\xi_0^T M_1(A)\xi_0}\right).$$

To evaluate the first minimum, we can now fix

$$\xi_0^T M_1(A)\xi_0=1,$$

and calculate

$$\min_{\xi_0}\xi_0^T M_2(A)\xi_0$$

subject to this restriction. Notice that this minimum is attained, as the smallest eigenvalue of $\nabla^2\varphi_1(\xi_0)$ is strictly positive. If α is a Lagrange multiplier, we put

$$L(A,\alpha)=\xi_0^T M_2(A)\xi_0-\alpha\big(\xi_0^T M_1(A)\xi_0-1\big),$$

and from first-order optimality conditions, if ξ_0 is a critical point, one obtains

$$\xi_0^T M_2(A)\xi_0=\alpha,$$

where α are the solutions of

$$\det\big(M_2(A)-\alpha M_1(A)\big)=0.$$

Notice that in this case this condition is a necessary and sufficient condition for the existence of minimizers.

α will be a function of A, and to finish we have to compute the minimum with respect to this variable $A\in\mathbb{M}$ with $\operatorname{rank}(A)\leqslant 1$.

3.1. Classical examples

We deal first with some classical examples [1,3,5].

Example 1.

$$\varphi:\mathbb{M}^{2\times 2}\to\mathbb{R},$$

given by

$$\varphi(\xi)=|\xi|^4-c|\xi|^2\det\xi.$$

If $A\in\mathbb{M}^{2\times 2}$ is such that $|A|=1$, by putting

$$A=\begin{pmatrix}a & b\\ c & d\end{pmatrix},$$

we get here that

$$M_2(A)=\begin{pmatrix}2ad & bd-ac & cd-ab & \frac{1}{2}+a^2+d^2\\ bd-ac & -2bc & -\frac{1}{2}-b^2-c^2 & ab-cd\\ cd-ab & -\frac{1}{2}-b^2-c^2 & -2bc & ac-bd\\ \frac{1}{2}+a^2+d^2 & ab-cd & ac-bd & 2ad\end{pmatrix}$$

and

$$M_1(A) = \begin{pmatrix} 2+4a^2 & 4ab & 4ac & 4ad \\ 4ab & 2+4b^2 & 4bc & 4bd \\ 4ac & 4bc & 2+4c^2 & 4cd \\ 4ad & 4bd & 4cd & 2+4d^2 \end{pmatrix}.$$

To obtain the values of α we have to solve the equation

$$\det\bigl(M_2(A) - \alpha M_1(A)\bigr) = 0.$$

But if we now perform the substitution

$$A = (\cos\theta_1, \sin\theta_1) \otimes (\cos\theta_2, \sin\theta_2),$$

with $\theta_1, \theta_2 \in [0, 2\pi]$, the above equation becomes

$$\frac{9}{16} - 12\alpha^2 + 48\alpha^4 = 0,$$

and the maximum and the minimum values are, respectively, $\alpha = \frac{\sqrt{3}}{4}$ and $\alpha = -\frac{\sqrt{3}}{4}$. So, φ is rank-one convex if and only if

$$c \in \left[-\frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right].$$

In the case of convexity, it is known [1] that φ is convex if and only if

$$c \in \left[-\frac{4\sqrt{2}}{3}, \frac{4\sqrt{2}}{3}\right].$$

Example 2.

$$\varphi : \mathbb{M}^{2\times 2} \to \mathbb{R},$$

given by

$$\varphi(\xi) = |\xi|^4 - c(\det\xi)^2.$$

If we proceed as in the previous example, and put

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

for $A \in \mathbb{M}^{2\times 2}$ with $|A| = 1$, $M_1(A)$ will be the same as before, and

$$M_2(A) = \begin{pmatrix} 2d^2 & -2cd & -2bd & 2ad \\ -2cd & 2c^2 & 2bc & -2ac \\ -2bd & 2bc & 2b^2 & -2ab \\ 2ad & -2ac & -2ab & 2a^2 \end{pmatrix}.$$

For

$$A = (\cos\theta_1, \sin\theta_1) \otimes (\cos\theta_2, \sin\theta_2),$$

with $\theta_1, \theta_2 \in [0, 2\pi]$, we have

$$\det\bigl(M_2(A) - \alpha M_1(A)\bigr) = 384\alpha^3(-1 + 2\alpha) = 0,$$

and so, the maximum value of α is $\frac{1}{2}$ and the minimum is 0. In this case, it is clear that φ is rank-one convex if and only if

$$c \in (-\infty, 2].$$

The range for the constant c for which the corresponding φ is convex is given by

$$c \in [-4, 1].$$

3.2. New examples

We now present some other examples to stress our main result.

Example 3. For

$$\varphi : \mathbb{M}^{2\times 2} \to \mathbb{R},$$

put

$$\varphi(\xi) = |\xi|^4 - c(\operatorname{tr}\xi)^4,$$

where $\operatorname{tr}\xi$ represents the trace of the matrix ξ. For

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with $A \in \mathbb{M}^{2\times 2}$, $|A| = 1$, $M_1(A)$ is given above, and

$$M_2(A) = \begin{pmatrix} 12(a+d)^2 & 0 & 0 & 12(a+d)^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 12(a+d)^2 & 0 & 0 & 12(a+d)^2 \end{pmatrix}.$$

In the rank-one directions

$$A = (\cos\theta_1, \sin\theta_1) \otimes (\cos\theta_2, \sin\theta_2),$$

where $\theta_1, \theta_2 \in [0, 2\pi]$, we have

$$\begin{aligned} &\det\bigl(M_2(A) - \alpha M_1(A)\bigr) = 0 \\ &\quad\Leftrightarrow 768\alpha^3\bigl(-4 + 2\cos(\theta_2)^2 - 16\cos(\theta_1)^2\cos(\theta_2)^4 + 2\cos(\theta_2)^4 + 2\cos(\theta_1)^2 + 2\cos(\theta_1)^4 \\ &\qquad + 8\cos(\theta_2)^2\cos(\theta_1)^2 - 4\cos(\theta_1)\cos(\theta_2)\sin(\theta_1)\sin(\theta_2) + 16\cos(\theta_1)^4\cos(\theta_2)^4 \\ &\qquad - 8\cos(\theta_1)^3\cos(\theta_2)\sin(\theta_1)\sin(\theta_2) + 16\cos(\theta_1)^3\cos(\theta_2)^3\sin(\theta_1)\sin(\theta_2) \\ &\qquad - 8\cos(\theta_1)\cos(\theta_2)^3\sin(\theta_1)\sin(\theta_2) - 16\cos(\theta_2)^2\cos(\theta_1)^4 + \alpha\bigr) = 0. \end{aligned}$$

Consequently the maximum value for α is 4. Regarding the minimum value of α, notice that φ_2 is convex and so φ is rank-one convex if and only if

$$c \in \left(-\infty, \frac{1}{4}\right].$$

φ is convex if and only if

$$c \in \left(-\infty, \frac{2}{9}\right].$$

Example 4. An example with a non-homogeneous polynomial

$$\varphi : \mathbb{M}^{2\times 2} \to \mathbb{R},$$

defined by

$$\varphi(\xi) = (\operatorname{tr}\xi)^4 + |\xi|^2 - c(\operatorname{tr}\xi)^3.$$

For

$$\xi = \begin{pmatrix} x & y \\ z & w \end{pmatrix},$$

we have

$$\nabla^2\varphi_1(\xi)=\begin{pmatrix}12(x+w)^2+2 & 0 & 0 & 12(x+w)^2\\ 0 & 2 & 0 & 0\\ 0 & 0 & 2 & 0\\ 12(x+w)^2 & 0 & 0 & 12(x+w)^2+2\end{pmatrix}$$

and

$$\nabla^2\varphi_2(\xi)=\begin{pmatrix}6(x+w) & 0 & 0 & 6(x+w)\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 6(x+w) & 0 & 0 & 6(x+w)\end{pmatrix}.$$

In addition, for

$$A=\begin{pmatrix}a\\ b\\ c\\ d\end{pmatrix}$$

and

$$D=\begin{pmatrix}0 & 0 & 0 & \frac{1}{2}\\ 0 & 0 & -\frac{1}{2} & 0\\ 0 & -\frac{1}{2} & 0 & 0\\ \frac{1}{2} & 0 & 0 & 0\end{pmatrix},$$

the first-order necessary conditions will be the parametric system of polynomial equations

$$\begin{cases}\big(6x+6w-12\alpha(x+w)^2-\beta\big)a+\big(6x+6w-\alpha\big(12(x+w)^2+2\big)\big)d=0,\\ -2\alpha b+\beta c=0,\\ \beta b-2\alpha c=0,\\ \big(6x+6w-\alpha\big(12(x+w)^2+2\big)\big)a+\big(6x+6w-12\alpha(x+w)^2-\beta\big)d=0,\\ \big(a\big(12(x+w)^2+2\big)+12d(x+w)^2\big)a+2b^2+2c^2+\big(12a(x+w)^2+d\big(12(x+w)^2+2\big)\big)d=1,\\ ad-bc=0\end{cases}$$

which give us the real solutions

$$\alpha=0,\qquad \alpha=\frac{3(x+w)}{6(x+w)^2+1},$$

which by its turn provide the range of the constant c to be

$$c\in\left[-\frac{2\sqrt{6}}{3},\frac{2\sqrt{6}}{3}\right].$$

For convexity, we have

$$c\in\left[-\frac{2\sqrt{3}}{3},\frac{2\sqrt{3}}{3}\right].$$

Example 5. An example for 2×3 matrices

$$\varphi:\mathbb{M}^{2\times 3}\to\mathbb{R},$$

given by

$$\varphi(\xi)=|\xi|^4-c|\xi|^2\big(\xi^1_{2\times 2}+\xi^2_{2\times 2}+\xi^3_{2\times 2}\big),$$

where $\xi^j_{2\times 2}$, $j=1,2,3$, represents the 2×2 minor that is obtained from ξ, by removing the j column. If $A\in\mathbb{M}^{2\times 3}$ with $|A|=1$ we set

$$A=\begin{pmatrix}a & c & e\\ b & d & f\end{pmatrix}.$$

We have

$$\frac{M_1(A)}{2}=\begin{pmatrix} 4a^2+2 & 4ba & 4ca & 4da & 4ea & 4fa\\ 4ba & 4b^2+2 & 4cb & 4db & 4eb & 4fb\\ 4ca & 4cb & 4c^2+2 & 4dc & 4ec & 4fc\\ 4da & 4db & 4dc & 4d^2+2 & 4ed & 4fd\\ 4ea & 4eb & 4ec & 4ed & 4e^2+2 & 4fe\\ 4fa & 4fb & 4fc & 4fd & 4fe & 4f^2+2 \end{pmatrix}$$

and

$$\frac{M_2(A)}{2}=\left(\begin{matrix} 3ad+3af-bc-be+cf-de & -ca-ea+db+fb\\ -ca-ea+db+fb & ad+af-3bc-3be+cf-de\\ -ba+af+dc+cf & -\frac{1}{2}-b^2+fb-c^2-ec\\ a^2-ea+d^2+fd+\frac{1}{2} & ba-be-dc-de\\ -ba-ad+de+fe & -\frac{1}{2}-b^2-db-ec-e^2\\ a^2+ca+fd+f^2+\frac{1}{2} & ba+bc-cf-fe \end{matrix}\right.$$

$$\begin{matrix} -ba+af+dc+cf & a^2-ea+d^2+fd+\frac{1}{2}\\ -\frac{1}{2}-b^2+fb-c^2-ec & ba-be-dc-de\\ ad+af-3bc-be+3cf-de & ca-db-ec+fd\\ ca-db-ec+fd & 3ad+af-bc-be+cf-3de\\ -bc-be-dc+fe & -\frac{1}{2}+ea-db-d^2-e^2\\ \frac{1}{2}+ca-fb+c^2+f^2 & ad+af+dc-fe \end{matrix}$$

$$\left.\begin{matrix} -ba-ad+de+fe & a^2+ca+fd+f^2+\frac{1}{2}\\ -\frac{1}{2}-b^2-db-ec-e^2 & ba+bc-cf-fe\\ -bc-be-dc+fe & \frac{1}{2}+ca-fb+c^2+f^2\\ -\frac{1}{2}+ea-db-d^2-e^2 & ad+af+dc-fe\\ ad+af-bc-3be+cf-3de & ea-fb+ec-fd\\ ea-fb+ec-fd & ad+3af-bc-be+3cf-de \end{matrix}\right).$$

For

$$A=(\cos\theta_1,\sin\theta_1)\otimes(\cos\theta_2\sin\theta_3,\sin\theta_2\sin\theta_3,\cos\theta_3),$$

$\theta_1,\theta_2\in[0,2\pi]$, $\theta_3\in[0,\pi]$, we have

$$\begin{aligned}&\alpha^2\big(21+12\sin\theta_2\cos\theta_2-160\alpha^2+256\alpha^4+64\alpha^2\sin\theta_2\cos\theta_3^2\cos\theta_2-64\alpha^2\sin\theta_2\sin\theta_3\cos\theta_3\\ &\quad+12\sin\theta_2\sin\theta_3\cos\theta_3-12\sin\theta_2\cos\theta_3^2\cos\theta_2-64\alpha^2\sin\theta_2\cos\theta_2\\ &\quad+64\alpha^2\sin\theta_3\cos\theta_3\cos\theta_2-12\sin\theta_3\cos\theta_3\cos\theta_2\big)=0.\end{aligned}$$

The roots α are

$$\alpha=\pm\sqrt{\frac{7\tan^2\theta_3+7+4\sin\theta_2\tan\theta_3-4\cos\theta_2\tan\theta_3+4\sin\theta_2\tan^2\theta_3\cos\theta_2}{16(\tan^2\theta_3+1)}},$$

$$\alpha=0,\quad \alpha=\pm\frac{\sqrt{3}}{4},$$

and consequently the maximum and minimum values for α are $\alpha=\frac{3}{4}$ and $\alpha=-\frac{3}{4}$ respectively (obtained from maximizing and minimizing, respectively, the above quotients in θ_2,θ_3) so, in this case we have φ rank-one convex if and only if

$$c\in\left[-\frac{4}{3},\frac{4}{3}\right].$$

Remark 3.

1. In this case it is harder to compute the constants for convexity than for rank-one convexity, following this approach. In fact, we were not able to recover those constants.
2. As rank-one convexity is invariant under transposition, that is, $f : \mathbb{M}^{m\times n} \to \mathbb{R}$ is rank-one convex if and only if $f^T : \mathbb{M}^{n\times m} \to \mathbb{R}$ given by

$$f^T(\xi) = f\big(\xi^T\big)$$

is rank-one convex, one can trivially compute the constants for the 3×2 example implicitly given by Example 5.

4. Main proof

This section is devoted to the proof of Theorem 2.

We will use the characterization of rank-one convexity through Jensen's inequality for laminates [8] so that we are interested in determining the exact range for the constant c so that Jensen's inequality holds for every laminate and φ in (1). The key point is that we can control the slope of the secants that pass through the image of the barycenter by the slope of its tangents through zero. In this terminology, secants are related, somehow, to quasiconvexity whereas tangents at the origin reflect rank-one convexity.

We divide the proof in several steps.

Step 1. If μ is a laminate, then by definition [8], there exists a sequence of sets of pairs $\{(\lambda_i^k, A_i^k)\}_{1\leqslant i\leqslant k}$, verifying the (H_k) condition [3] such that

$$\mu_k = \sum_i \lambda_i^k \delta_{A_i^k} \overset{*}{\rightharpoonup} \mu$$

weakly in the sense of measures. So if

$$\varphi\left(\int \xi \, d\mu_k(\xi)\right) \leqslant \int \varphi(\xi)\, d\mu_k(\xi),$$

holds for all k and for some value of c, then by taking weak-$*$ limits on both sides of the above inequality (φ is, in particular, continuous), we have

$$\varphi\left(\int \xi \, d\mu(\xi)\right) \leqslant \int \varphi(\xi)\, d\mu(\xi), \quad \forall \mu \in \Lambda$$

for the same value of c.

Step 2. We will now prove that it suffices to use first-order laminates to determine the range of c. We argue, in particular, that building finite-order laminates recursively from first-order laminates does not reduce the range of the constant c.

Our hypothesis is that c is such that

$$\varphi\left(\int \xi \, d\mu(\xi)\right) \leqslant \int \varphi(\xi)\, d\mu(\xi) \tag{2}$$

for every

$$\mu = \lambda\delta_{A_1} + (1-\lambda)\delta_{A_2} \quad \text{with } \operatorname{rank}(A_1 - A_2) \leqslant 1;$$

and we want to prove that, for the same value of c, we have

$$\varphi\left(\int \xi \, d\mu_N(\xi)\right) \leqslant \int \varphi(\xi)\, d\mu_N(\xi), \tag{3}$$

for every finite-order laminate

$$\mu_N = \sum_{i=1}^N \lambda_i \delta_{A_i}.$$

We proceed by induction (keep in mind that the value c is fixed but arbitrary). For $N = 2$, (3) is just (2). Suppose now that (3) holds for every probability measure associated with (H_{N-1}) conditions. Then, if $\{(\lambda_i^N, A_i^N)\}_{1\leqslant i\leqslant N}$ satisfies the (H_N) condition, we can assume, without loss of generality, that $\operatorname{rank}(A_1 - A_2) \leqslant 1$ (we drop the superindex for simplicity), and by the induction hypothesis, we have

$$\begin{aligned}\int \varphi(\xi)\,d\mu(\xi) &= \sum_{i=1}^N \lambda_i\varphi(A_i)\\ &= (\lambda_1+\lambda_2)\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\varphi(A_1)+\frac{\lambda_2}{\lambda_1+\lambda_2}\varphi(A_2)\right)+\sum_{i=3}^N \lambda_i\varphi(A_i)\\ &\geqslant (\lambda_1+\lambda_2)\varphi\left(\frac{\lambda_1}{\lambda_1+\lambda_2}A_1+\frac{\lambda_2}{\lambda_1+\lambda_2}A_2\right)+\sum_{i=3}^N \lambda_i\varphi(A_i)\\ &\geqslant \varphi\left(\sum_{i=1}^N \lambda_i A_i\right)=\varphi\left(\int \xi\,d\mu(\xi)\right).\end{aligned}$$

In fact, notice that we can further simplify the situation (since φ is continuous), because (2) holds for a value c if and only if

$$\varphi\left(\int \xi\,d\mu(\xi)\right) \leqslant \int \varphi(\xi)\,d\mu(\xi) \tag{4}$$

for every

$$\mu = \frac{1}{2}\delta_{A_1} + \frac{1}{2}\delta_{A_2} \quad \text{with } \operatorname{rank}(A_1 - A_2) \leqslant 1,$$

holds for the same value of c.

After a change of variables, we can write down this measure as

$$\mu = \frac{1}{2}\delta_{\xi_0+A} + \frac{1}{2}\delta_{\xi_0-A},$$

where $\operatorname{rank}(A) \leqslant 1$. For $s \in [0, 1]$, we can take

$$\mu = \mu_s = \frac{1}{2}\delta_{\xi_0+sA} + \frac{1}{2}\delta_{\xi_0-sA}$$

with $\operatorname{rank}(A) = 1$ and $|A| \leqslant 1$ (for $|A| > 1$ just use the fact that $\xi \in \mathbb{M}$ is arbitrary and that φ is continuous). By dealing with this class of measures (which will play the role of “generators”), we can determine the exact range for the constant c that we are interested in.

Step 3. For $s \in [0, 1]$, consider

$$\mu_s = \frac{1}{2}\delta_{\xi_0+sA} + \frac{1}{2}\delta_{\xi_0-sA},$$

and the corresponding plane curve

$$\sigma^{(A,\xi_0)}(s) = T(\mu_s)$$

with end-points

$$\big(\varphi_1(\xi_0), \varphi_2(\xi_0)\big)$$

and

$$\left(\frac{1}{2}\varphi_1(\xi_0+A)+\frac{1}{2}\varphi_1(\xi_0-A), \frac{1}{2}\varphi_2(\xi_0+A)+\frac{1}{2}\varphi_2(\xi_0-A)\right).$$

If σ and μ_s are defined as above, then finding all c’s such that

$$\int \varphi(\xi)\,d\mu_s(\xi) \geqslant \varphi\left(\int \xi\,d\mu_s(\xi)\right),$$

is equivalent to finding all c's for which we have

$$\sigma_1^{(A,\xi_0)}(s) - \sigma_1^{(A,\xi_0)}(0) \geqslant c\big(\sigma_2^{(A,\xi_0)}(s) - \sigma_2^{(A,\xi_0)}(0)\big),$$

for every $\xi_0 \in \mathbb{M}$, $A \in \mathcal{A}$ with $|A| \leqslant 1$, $s \in [0,1]$; or, if we consider $c > 0$ (the other case is similar), that

$$\frac{1}{c} \geqslant \frac{\sigma_2^{(A,\xi_0)}(s) - \sigma_2^{(A,\xi_0)}(0)}{\sigma_1^{(A,\xi_0)}(s) - \sigma_1^{(A,\xi_0)}(0)}$$

for every $\xi_0 \in \mathbb{M}$, $A \in \mathcal{A}$ with $|A| \leqslant 1$, $s \in (0,1]$. If

$$\sigma_2^{(A,\xi_0)}(s) - \sigma_2^{(A,\xi_0)}(0) \leqslant 0,$$

then $c > 0$, and we do not have any additional constraint. Otherwise, we can set

$$\sup_{A,\xi_0,s\in(0,1]} \frac{\sigma_2^{(A,\xi_0)}(s) - \sigma_2^{(A,\xi_0)}(0)}{\sigma_1^{(A,\xi_0)}(s) - \sigma_1^{(A,\xi_0)}(0)} = \frac{1}{c_+} \leqslant \frac{1}{c}.$$

Since

$$\sigma_i(s) = \frac{1}{2}\varphi_i(\xi_0 + sA) + \frac{1}{2}\varphi_i(\xi_0 - sA),$$

it follows

$$\dot{\sigma_i}(0) = 0,$$

thus it is obvious that

$$\sup_{A,\xi_0,s\in(0,1]} \frac{\sigma_2^{(A,\xi_0)}(s) - \sigma_2^{(A,\xi_0)}(0)}{\sigma_1^{(A,\xi_0)}(s) - \sigma_1^{(A,\xi_0)}(0)} \geqslant \sup_{A,\xi_0} \frac{\ddot{\sigma_2}^{(A,\xi_0)}(0)}{\ddot{\sigma_1}^{(A,\xi_0)}(0)}. \tag{5}$$

Step 4. To finish the proof, we have to show that the equality holds. First we will suppose that the supremum on the left side of (5) (and where we can suppose $s \geqslant r > 0$, otherwise there is nothing to prove) is indeed a maximum and that a strict inequality holds

$$\frac{1}{c_+} = \max_{A,\xi_0,s\in(0,1]} \frac{\sigma_2^{(A,\xi_0)}(s) - \sigma_2^{(A,\xi_0)}(0)}{\sigma_1^{(A,\xi_0)}(s) - \sigma_1^{(A,\xi_0)}(0)} = \frac{\sigma_2^{(A^*,\xi_0^*)}(s^*) - \sigma_2^{(A^*,\xi_0^*)}(0)}{\sigma_1^{(A^*,\xi_0^*)}(s^*) - \sigma_1^{(A^*,\xi_0^*)}(0)} > \sup_{A,\xi_0} \frac{\ddot{\sigma_2}^{(A,\xi_0)}(0)}{\ddot{\sigma_1}^{(A,\xi_0)}(0)}.$$

Then there has to be a point $s_0 \in (0, s^*)$ such that

$$\begin{aligned} &\frac{\sigma_2^{(A^*,\xi_0^*)}(s^*) - \sigma_2^{(A^*,\xi_0^*)}(s_0)}{\sigma_1^{(A^*,\xi_0^*)}(s^*) - \sigma_1^{(A^*,\xi_0^*)}(s_0)} \\ &\quad = \frac{\frac{1}{2}(\varphi_2(\xi_0^* - s^*A^*) + \varphi_2(\xi_0^* + s^*A^*)) - \frac{1}{2}(\varphi_2(\xi_0^* - s_0A^*) + \varphi_2(\xi_0^* + s_0A^*))}{\frac{1}{2}(\varphi_1(\xi_0^* - s^*A^*) + \varphi_1(\xi_0^* + s^*A^*)) - \frac{1}{2}(\varphi_1(\xi_0^* - s_0A^*) + \varphi_1(\xi_0^* + s_0A^*))} > \frac{1}{c_+}. \end{aligned}$$

But because $\xi_0^* - s_0A^*$ and $\xi_0^* + s_0A^*$ can be regarded as new barycenters of first-order laminates, it is clear, by definition of $\frac{1}{c_+}$, that

$$\frac{\frac{s^*+s_0}{2s^*}\varphi_2((\xi_0^* - s_0A^*) - (s^* - s_0)A^*) + \frac{s^*-s_0}{2s^*}\varphi_2((\xi_0^* - s_0A^*) + (s^* + s_0)A^*) - \varphi_2(\xi_0^* - s_0A^*)}{\frac{s^*+s_0}{2s^*}\varphi_1((\xi_0^* - s_0A^*) - (s^* - s_0)A^*) + \frac{s^*-s_0}{2s^*}\varphi_1((\xi_0^* - s_0A^*) + (s^* + s_0)A^*) - \varphi_1(\xi_0^* - s_0A^*)} \leqslant \frac{1}{c_+}$$

and

$$\frac{\frac{s^*-s_0}{2s^*}\varphi_2((\xi_0^* + s_0A^*) - (s^* + s_0)A^*) + \frac{s^*+s_0}{2s^*}\varphi_2((\xi_0^* + s_0A^*) + (s^* - s_0)A^*) - \varphi_2(\xi_0^* + s_0A^*)}{\frac{s^*-s_0}{2s^*}\varphi_1((\xi_0^* + s_0A^*) - (s^* + s_0)A^*) + \frac{s^*+s_0}{2s^*}\varphi_1((\xi_0^* + s_0A^*) + (s^* - s_0)A^*) - \varphi_1(\xi_0^* + s_0A^*)} \leqslant \frac{1}{c_+}.$$

From here and because φ_1 is strictly convex (and so, in the above fractions both denominators are strictly positive), it is trivial to obtain

$$\frac{\sigma_2^{(A^*,\xi_0^*)}(s^*)-\sigma_2^{(A^*,\xi_0^*)}(s_0)}{\sigma_1^{(A^*,\xi_0^*)}(s^*)-\sigma_1^{(A^*,\xi_0^*)}(s_0)} \leqslant \frac{1}{c_+},$$

which contradicts the above strict inequality, leading to the desired conclusion, that is

$$\max_{A,\xi_0,s\in(0,1]} \frac{\sigma_2^{(A,\xi_0)}(s)-\sigma_2^{(A,\xi_0)}(0)}{\sigma_1^{(A,\xi_0)}(s)-\sigma_1^{(A,\xi_0)}(0)} = \sup_{A,\xi_0} \frac{\ddot{\sigma_2}^{(A,\xi_0)}(0)}{\ddot{\sigma_1}^{(A,\xi_0)}(0)} = \frac{1}{c_+}.$$

Now it remains to prove the case where we have a genuine supremum on the left side of (5). This can only happens if the supremum is obtained by taking $|\xi|\to\infty$. Suppose

$$\frac{1}{c_+} = \sup_{A,\xi,s\in(0,1]} \frac{\sigma_2^{(A,\xi)}(s)-\sigma_2^{(A,\xi)}(0)}{\sigma_1^{(A,\xi)}(s)-\sigma_1^{(A,\xi)}(0)} = \lim_{|\xi|\to\infty} \max_{A,s\in(0,1]} \frac{\sigma_2^{(A,\xi)}(s)-\sigma_2^{(A,\xi)}(0)}{\sigma_1^{(A,\xi)}(s)-\sigma_1^{(A,\xi)}(0)} > \sup_{A,\xi} \frac{\ddot{\sigma_2}^{(A,\xi)}(0)}{\ddot{\sigma_1}^{(A,\xi)}(0)}$$
$$= \lim_{s\to 0} \sup_{A,\xi} \frac{\sigma_2^{(A,\xi)}(s)-\sigma_2^{(A,\xi)}(0)}{\sigma_1^{(A,\xi)}(s)-\sigma_1^{(A,\xi)}(0)}.$$

Then there exists $\delta>0$ such that

$$\sup_{A,\xi} \frac{\ddot{\sigma_2}^{(A,\xi)}(0)}{\ddot{\sigma_1}^{(A,\xi)}(0)} = \frac{1}{c_+} - 3\delta.$$

We also have that for each $\varepsilon>0$, there exists $k=k(\varepsilon)\in\mathbb{R}^+$ such that for $|\xi|\geqslant k(\varepsilon)$,

$$\max_{A,s\in(0,1]} \frac{\sigma_2^{(A,\xi)}(s)-\sigma_2^{(A,\xi)}(0)}{\sigma_1^{(A,\xi)}(s)-\sigma_1^{(A,\xi)}(0)} > \frac{1}{c_+} - \varepsilon.$$

We take $\varepsilon=\delta$, and for ξ such that $|\xi|>k(\delta)$ one has

$$\max_{A,s\in(0,1]} \frac{\sigma_2^{(A,\xi)}(s)-\sigma_2^{(A,\xi)}(0)}{\sigma_1^{(A,\xi)}(s)-\sigma_1^{(A,\xi)}(0)} > \frac{1}{c_+} - \delta > \sup_{A,\xi} \frac{\ddot{\sigma_2}^{(A,\xi)}(0)}{\ddot{\sigma_1}^{(A,\xi)}(0)} = \frac{1}{c_+} - 3\delta.$$

As for such ξ

$$\sup_{A} \frac{\ddot{\sigma_2}^{(A,\xi)}(0)}{\ddot{\sigma_1}^{(A,\xi)}(0)} \leqslant \frac{1}{c_+} - 3\delta,$$

then for each A there must exist a point $s_0\in(0,1)$ for which

$$\lim_{s\to s_0} \frac{\sigma_2^{(A,\xi)}(s)-\sigma_2^{(A,\xi)}(s_0)}{\sigma_1^{(A,\xi)}(s)-\sigma_1^{(A,\xi)}(s_0)} > \frac{1}{c_+} - \delta.$$

Using again the fact that $\xi-s_0A$ and $\xi+s_0A$ can be regarded as new barycenters of first-order laminates, one has

$$\lim_{s\to s_0} \frac{\frac{s+s_0}{2s}\varphi_2((\xi-s_0A)-(s-s_0)A)+\frac{s-s_0}{2s}\varphi_2((\xi-s_0A)+(s+s_0)A)-\varphi_2(\xi-s_0A)}{\frac{s+s_0}{2s}\varphi_1((\xi-s_0A)-(s-s_0)A)+\frac{s-s_0}{2s}\varphi_1((\xi-s_0A)+(s+s_0)A)-\varphi_1(\xi-s_0A)} \leqslant \frac{1}{c_+} - 3\delta$$

and

$$\lim_{s\to s_0} \frac{\frac{s-s_0}{2s}\varphi_2((\xi+s_0A)-(s+s_0)A)+\frac{s+s_0}{2s}\varphi_2((\xi+s_0A)+(s-s_0)A)-\varphi_2(\xi+s_0A)}{\frac{s-s_0}{2s}\varphi_1((\xi+s_0A)-(s+s_0)A)+\frac{s+s_0}{2s}\varphi_1((\xi+s_0A)+(s-s_0)A)-\varphi_1(\xi+s_0A)} \leqslant \frac{1}{c_+} - 3\delta.$$

Consequently, there exists $r_1>0$ such that for each $s\in B(s_0,r_1)$

$$\frac{\frac{s+s_0}{2s}\varphi_2((\xi-s_0A)-(s-s_0)A)+\frac{s-s_0}{2s}\varphi_2((\xi-s_0A)+(s+s_0)A)-\varphi_2(\xi-s_0A)}{\frac{s+s_0}{2s}\varphi_1((\xi-s_0A)-(s-s_0)A)+\frac{s-s_0}{2s}\varphi_1((\xi-s_0A)+(s+s_0)A)-\varphi_1(\xi-s_0A)} \leqslant \frac{1}{c_+} - 2\delta$$

and a $r_2 > 0$ such that for each $s \in B(s_0, r_2)$

$$\frac{\frac{s-s_0}{2s}\varphi_2((\xi + s_0A) - (s+s_0)A) + \frac{s+s_0}{2s}\varphi_2((\xi + s_0A) + (s-s_0)A) - \varphi_2(\xi + s_0A)}{\frac{s-s_0}{2s}\varphi_1((\xi + s_0A) - (s+s_0)A) + \frac{s+s_0}{2s}\varphi_1((\xi + s_0A) + (s-s_0)A) - \varphi_1(\xi + s_0A)} \leqslant \frac{1}{c_+} - 2\delta.$$

For each $s \in B(s_0, r)$, where $r = \min\{r_1, r_2\}$ and noticing that φ_1 is strictly convex, one can get

$$\frac{\sigma_2^{(A,\xi)}(s) - \sigma_2^{(A,\xi)}(s_0)}{\sigma_1^{(A,\xi)}(s) - \sigma_1^{(A,\xi)}(s_0)} \leqslant \frac{1}{c_+} - 2\delta,$$

which is absurd.

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