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Some controllability results for the 2D Kolmogorov equation

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Abstract

In this article, we prove the null controllability of the 2D Kolmogorov equation both in the whole space and in the square. The control is a source term in the right-hand side of the equation, located on a subdomain, that acts linearly on the state. In the first case, it is the complementary of a strip with axis x and in the second one, it is a strip with axis x.

The proof relies on two ingredients. The first one is an explicit decay rate for the Fourier components of the solution in the free system. The second one is an explicit bound for the cost of the null controllability of the heat equation with potential that the Fourier components solve. This bound is derived by means of a new Carleman inequality.

Keywords: Kolmogorov equation; Controllability; Carleman inequalities

1. Introduction

1.1. Main result

We consider the Kolmogorov equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{\partial^2 f}{\partial v^2} = u(t,x,v)1_\omega(x,v), \quad (x,v) \in \Omega,\ t \in (0,+\infty), \tag{1}$$

where Ω is an open subset of $\mathbb{R}^2$, $\omega \subset \Omega$, 1_ω is the characteristic function of this set and $u(t,x,v)$ is a source term located on the subdomain ω. It is a linear control system in which

- the state is f,
- the control is u and it is supported in the subset ω.

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We investigate the null controllability of Eq. (1) in two different geometric configurations,

$$\Omega_1 = \mathbb{R}_x \times \mathbb{R}_v, \qquad \omega_1 = \mathbb{R}_x \times \big[\mathbb{R} - (a_1, b_1)\big]_v$$

where $-\infty < a_1 < b_1 < +\infty$ and

$$\Omega_2 = (0, 2\pi)_x \times (0, 2\pi)_v, \qquad \omega_2 = (0, 2\pi)_x \times (a_2, b_2)_v,$$

where $0 \leqslant a_2 < b_2 \leqslant 2\pi$. More precisely, we study the Cauchy problems

$$\begin{cases} \dfrac{\partial f}{\partial t} + v\dfrac{\partial f}{\partial x} - \dfrac{\partial^2 f}{\partial v^2} = u(t,x,v)1_{\omega_1}(v), \quad (x,v) \in \Omega_1,\ t \in (0,+\infty), \\ f(0,x,v) = f_0(x,v), \end{cases} \tag{2}$$

and

$$\begin{cases} \dfrac{\partial f}{\partial t} + v\dfrac{\partial f}{\partial x} - \dfrac{\partial^2 f}{\partial v^2} = u(t,x,v)1_{\omega_2}(x,v), \quad (x,v) \in \Omega_2,\ t \in (0,+\infty), \\ f(t,0,v) = f(t,2\pi,v), \\ f(t,x,0) = f(t,x+2\pi t,2\pi), \\ \partial_v f(t,x,0) = \partial_v f(t,x+2\pi t,2\pi), \\ f(0,x,v) = f_0(x,v). \end{cases} \tag{3}$$

The boundary conditions in (3) may seem strange. We chose them to ensure that the function $h(t,x,v) := f(t,x+vt,v)$ is 2π-periodic with respect to x and v, which facilitates the Fourier analysis of solutions. Notice that, thanks to the second line of (3), one can identify the function f and the function from $(0,+\infty)_t \times \mathbb{R}_x \times (0,2\pi)_v$ to $\mathbb{R}$, which is 2π-periodic with respect to the variable x and coincides with f on $(0,+\infty)_t \times (0,2\pi)_x \times (0,2\pi)_v$. This gives sense to the third and fourth lines of (3).

The main result of this article guarantees the null controllability of systems (2) and (3):

Theorem 1. *For every $T > 0$ and $f_0 \in L^2(\Omega_1,\mathbb{R})$ (resp. $f_0 \in L^2(\Omega_2,\mathbb{R})$), there exists $u \in L^2((0,T)\times\Omega_1,\mathbb{R})$ (resp. $u \in L^2((0,T)\times\Omega_2,\mathbb{R})$) such that the solution of* (2) (*resp.* (3)) *satisfies $f(T) = 0$.*

By duality, this result is equivalent to the following observability inequalities for the corresponding adjoint systems (see for instance [2, Lemma 2.48]).

Theorem 2. *For every $T > 0$, there exists $C > 0$ such that, for every $g_0 \in L^2(\Omega_1,\mathbb{R})$, the solution of*

$$\begin{cases} \dfrac{\partial g}{\partial t} - v\dfrac{\partial g}{\partial x} - \dfrac{\partial^2 g}{\partial v^2} = 0, \quad (x,v) \in \Omega_1,\ t \in (0,T), \\ g(0,x,v) = g_0(x,v), \quad (x,v) \in \Omega_1 \end{cases} \tag{4}$$

satisfies

$$\int_{\Omega_1} \big|g(T,x,v)\big|^2 dx\,dv \leqslant C \int_0^T\!\!\int_{\omega_1} \big|g(t,x,v)\big|^2 dx\,dv\,dt.$$

Theorem 3. *For every $T > 0$, there exists $C > 0$ such that, for every $g_0 \in L^2(\Omega_2,\mathbb{R})$, the solution of*

$$\begin{cases} \dfrac{\partial g}{\partial t} - v\dfrac{\partial g}{\partial x} - \dfrac{\partial^2 g}{\partial v^2} = 0, \quad (x,v) \in \Omega_2,\ t \in (0,T), \\ g(t,0,v) = g(t,2\pi,v), \\ g(t,x,0) = g\big(t,x+2\pi(T-t),2\pi\big), \\ \dfrac{\partial g}{\partial v}(t,x,0) = \dfrac{\partial g}{\partial v}\big(t,x+2\pi(T-t),2\pi\big), \\ g(0,x,v) = g_0(x,v), \end{cases} \tag{5}$$

satisfies

$$\int_{\Omega_2} \left|g(T,x,v)\right|^2 dx\,dv \leqslant C \int_0^T \int_{\omega_2} \left|g(t,x,v)\right|^2 dx\,dv\,dt.$$

1.2. Some bibliographical comments

The null controllability property of the Kolmogorov equation has not been much explored in the literature. On the contrary, the null and approximate controllability of the heat equation are essentially well understood subjects both for linear equations and for semilinear ones, both for bounded and unbounded domains (see, for instance, [4,6,8–10, 12,16–20,23,24]).

Let us summarize some of the existing main results. We consider the linear heat equation

$$\begin{cases} y_t(t,x) - \Delta y(t,x) = u(t,x)1_\omega(x), \quad x \in \Omega,\ t \in (0,T), \\ y = 0 \quad \text{on } (0,T) \times \partial\Omega, \\ y(0) = y^0, \end{cases} \tag{6}$$

where Ω is an open subset of $\mathbb{R}^l$, $l \in \mathbb{N}^*$ and ω a subset of Ω. One has the following theorem, which is due to H. Fattorini and D. Russell [7, Theorem 3.3] if $l = 1$, to O. Imanuvilov [14,15] (see also the book [11] by A. Fursikov and O. Imanuvilov), and to G. Lebeau and L. Robbiano [17] for $l \geqslant 2$. We also refer to the book [2, Theorem 2.66] by J.-M. Coron for a pedagogical presentation.

Theorem 4. *Let us assume that Ω is bounded, of class C^2 and connected, $T > 0$, and ω is a nonempty open subset of Ω. Then the control system* (6) *is null controllable in time T: for every $y^0 \in L^2(\Omega, \mathbb{R})$, there exists $u \in L^2((0,T) \times \Omega, \mathbb{R})$ such that the solution of* (6) *satisfies $y(T) = 0$.*

In particular, the heat equation on a bounded domain is null controllable

- in arbitrarily small time,
- with an arbitrarily small control support ω.

As a consequence of Theorem 4, we also have the following result [1].

Theorem 5. *Let us assume that $\Omega = \mathbb{R}^l$, $T > 0$, and ω is the complementary in $\mathbb{R}^l$ of a compact set. Then the control system* (6) *is null controllable in time T: for every $y^0 \in L^2(\mathbb{R}^l, \mathbb{R})$, there exists $u \in L^2((0,T) \times \mathbb{R}^l, \mathbb{R})$ such that the solution of* (6) *satisfies $y(T) = 0$.*

In particular, the heat equation on the whole space is null controllable

- in arbitrarily small time,
- when the control support is the complementary of a compact subset of $\mathbb{R}^l$.

The Kolmogorov equation (1) diffuses both in space and velocity variables: it diffuses in v thanks to $\partial_v^2 f$ and also in x, in a hidden way, thanks to an interplay between the transport term $v\partial_x f$ and the diffusive term $\partial_v^2 f$ (see, for instance, [13] where the hypoellipticity of this operator and more general systems is proved and characterized and [22] for the study of the asymptotic behavior, see also Lemmas 1 and 2 of this article). Thus, it is natural to ask if the null controllability results known for the heat equation also hold for the Kolmogorov equation.

The results proved in this article constitute a first step in this direction. Indeed, Theorem 1 shows that one can generalize, for the 2D Kolmogorov equation (1), the results known for the 1D heat equation in the velocity variable

$$\frac{\partial f}{\partial t} - \frac{\partial^2 f}{\partial v^2} = u(t,v)1_\omega(v), \quad v \in \Omega,\ t \in (0,+\infty).$$

In particular, our results show that the transport term $v\partial_x f$ does not destroy the zero controllability produced by the diffusive term $\partial_v^2 f$, but they have the drawback of needing the support of the control to be independent of the x variable. This is due to the fact that our proof uses the Fourier transform in the variable x to reduce the problem to a one-parameter family of heat equations in the variable v.

Finally, let us mention Ref. [21], in which a simplified version of the Kolmogorov equation (the linearized Crocco type equation) is studied. This equation mixes transport in the variable x and diffusion in the variable v but in a simpler way than the Kolmogorov equation, because the transport in variable x is done at constant velocity 1 instead of velocity v,

$$\begin{cases} f_t + f_x - f_{vv} = u(t,x,v)1_\omega(x,v), & (t,x,v) \in (0,T)\times(0,L)\times(0,1), \\ f(t,x,0) = f(t,x,1) = 0, \\ f(t,0,v) = f(t,L,v). \end{cases}$$

Because of this decoupling of the transport and the diffusion phenomena, the linearized Crocco type equation does not diffuse in variable x, thus the question of using an arbitrarily small control domain becomes very different.

For a given open subset ω of $\Omega := (0,L)\times(0,1)$, the authors of [21] prove the property of "regional null controllability", which consists on the control of the solution within the domain of influence of the controls located in ω.

However, for the Kolmogorov equation, the result may be different, because, as we said above, this equation diffuses both in variables v and x, thus the domain of influence of an arbitrarily small subset ω may be the whole domain Ω in any time $T > 0$. This problem is still open.

1.3. Structure of the article

Section 2 is devoted to the case where Ω is the whole space and Section 3 to the case of the square domain.

For each section, in a first subsection, we recall standard results about the existence and uniqueness of solutions. Then, in a second subsection, we present the strategy for the proof of Theorem 1, that relies on two key ingredients:

- an explicit decay rate for the Fourier components of the solution of (1) without control ($u \equiv 0$),
- an explicit bound for the cost of the null controllability of a particular heat equation with potential, that is solved by the Fourier components of the solution of (1).

This cost estimate is new and it is proved in a last subsection, with Carleman inequalities.

2. Control in the whole space

In this section, we prove Theorem 1 on the whole space.

2.1. Well posedness of the Cauchy problem

First, let us define a concept of solution for (2).

Definition 1. Let $T > 0$, $f_0 \in L^2(\Omega_1,\mathbb{R})$ and $u \in L^2((0,T)\times\Omega_1,\mathbb{R})$. A weak solution of the Cauchy problem (2) on $[0,T]$ is a function

$$f \in C^0\big([0,T], L^2(\Omega_1,\mathbb{R})\big)$$

such that $f(0) = f_0$ in $L^2(\Omega_1,\mathbb{R})$ and, for every $\varphi \in C^2([0,T]\times\Omega_1,\mathbb{R}) \cap H^2((0,T)\times\Omega_1,\mathbb{R})$, and $t^* \in (0,T)$,

$$\int_{\Omega_1} \big[f(t^*,x,v)\varphi(t^*,x,v) - f_0(x,v)\varphi(0,x,v)\big]\,dx\,dv = \int_0^{t^*}\int_{\Omega_1} \left\{ f\left(\frac{\partial\varphi}{\partial t} + v\frac{\partial\varphi}{\partial x} + \frac{\partial^2\varphi}{\partial v^2}\right) + u1_\omega\varphi \right\} dx\,dv\,dt.$$

With this definition, one has the following existence and uniqueness result, whose proof is standard.

Proposition 1. *For every $f_0 \in L^2(\Omega_1, \mathbb{R})$, $T > 0$ and $u \in L^2((0,T) \times \Omega_1, \mathbb{R})$, there exists a unique weak solution of the Cauchy problem* (2)*. Moreover, the solutions are continuous with respect to the initial condition for the $C^0([0,T], L^2(\Omega_1))$-topology.*

Proof. First, we get, heuristically, an explicit expression of the solution. Let us consider a solution $f(t,x,v)$. We define the functions h and w by

$$f(t,x,v) := h(t, x - vt, v) \quad \text{and} \quad u(t,x,v)1_{\omega_1}(x,v) = w(t, x - vt, v). \tag{7}$$

Then, formally, h solves

$$\begin{cases} \dfrac{\partial h}{\partial t} - \dfrac{\partial^2 h}{\partial v^2} + 2t\dfrac{\partial^2 h}{\partial x \partial v} - t^2 \dfrac{\partial^2 h}{\partial x^2} = w, \quad (x,v) \in \Omega_1,\ t \in (0, +\infty), \\ h(0,x,v) = f_0(x,v) \end{cases} \tag{8}$$

and its Fourier transform

$$\hat{\hat{h}}(t,\xi,\eta) := \int_{\Omega} h(t,x,v) e^{-i(x\xi + v\eta)}\, dx\, dv$$

solves

$$\begin{cases} \dfrac{\partial \hat{\hat{h}}}{\partial t} + \big(\eta^2 - 2t\eta\xi + t^2\xi^2\big)\hat{\hat{h}} = \hat{\hat{w}}, \quad (\xi,\eta) \in \Omega_1,\ t \in (0, +\infty), \\ \hat{\hat{h}}(0,\xi,\eta) = \hat{\hat{f}}_0(\xi,\eta), \end{cases}$$

which leads to the following explicit expression

$$\hat{\hat{h}}(t,\xi,\eta) = \left(\hat{\hat{f}}_0(\xi,\eta) + \int_0^t \hat{\hat{w}}(\tau,\xi,\eta) e^{\eta^2\tau - \eta\xi\tau^2 + \xi^2 \frac{\tau^3}{3}}\, d\tau \right) e^{-\eta^2 t + \eta\xi t^2 - \xi^2 \frac{t^3}{3}}. \tag{9}$$

Now, let us prove that the function f defined by (7), (9) is a solution of (2) in the sense of Definition 1. It is sufficient to prove that $h \in C^0([0,T], L^2(\Omega_1))$ and that for every $t^* \in [0,T]$ and $\psi \in C^2([0,T] \times \Omega_1, \mathbb{R}) \cap H^2((0,T) \times \Omega_1, \mathbb{R})$, one has

$$\int_{\Omega_1} \big[h(t^*,x,v)\psi(t^*,x,v) - f_0(x,v)\psi(0,x,v)\big]\, dx\, dv = \int_0^{t^*}\int_{\Omega_1} \big\{h\big[\partial_t + \partial_v^2 - 2t\partial_v\partial_x + t^2\partial_x^2\big]\psi + w\psi\big\}\, dx\, dv\, dt.$$

First, it is clear, from (9), that $h \in C^0([0,T], L^2(\Omega_1))$. Let $t^* \in [0,T]$ and $\psi \in C^2([0,T] \times \Omega_1, \mathbb{R}) \cap H^2((0,T) \times \Omega_1, \mathbb{R})$. Thanks to Plancherel theorem and (9), we have

$$\begin{aligned}
&\int_0^{t^*}\int_{\Omega_1} \big\{h\big[\partial_t + \partial_v^2 - 2t\partial_x\partial_v + t^2\partial_x^2\big]\psi + w\psi\big\}\, dx\, dv\, dt \\
&\quad = \Re \int_0^{t^*}\int_{\Omega_1} \big\{\hat{\hat{h}}\big[\partial_t - \eta^2 + 2t\eta\xi - t^2\xi^2\big]\bar{\hat{\hat{\psi}}} + \hat{\hat{w}}\bar{\hat{\hat{\psi}}}\big\}\, d\xi\, d\eta\, dt \\
&\quad = \Re \int_0^{t^*}\int_{\Omega_1} \frac{\partial}{\partial t}[\hat{\hat{h}}\bar{\hat{\hat{\psi}}}](t,\xi,\eta)\, d\xi\, d\eta\, dt \\
&\quad = \Re \int_{\Omega_1} \hat{\hat{h}}(t^*,\xi,\eta)\overline{\hat{\hat{\psi}}(t^*,\xi,\eta)} - \hat{\hat{h}}(0,\xi,\eta)\overline{\hat{\hat{\psi}}(0,\xi,\eta)}\, d\xi\, d\eta \\
&\quad = \int_{\Omega_1} \big[h(t^*,x,v)\psi(t^*,x,v) - h(0,x,v)\psi(0,x,v)\big]\, dx\, dv.
\end{aligned}$$

Let h be a weak solution of (8), in the sense above, associated to the initial condition $f_0 \equiv 0$ and the source term $w \equiv 0$. For every $t^* \in (0,T)$ and $\psi \in C^2([0,T]\times\Omega_1,\mathbb{R}) \cap H^2((0,T)\times\Omega_1,\mathbb{R})$, we have

$$\int_{\mathbb{R}^2} h(t^*,x,v)\psi(t^*,x,v)\,dx\,dv = \int_0^{t^*}\int_{\Omega_1} h(t,x,v)\big[\partial_t + \partial_v^2 - 2t\partial_v\partial_x + t^2\partial_x^2\big]\psi(t,x,v)\,dx\,dv\,dt.$$

Let $t^* \in [0,T]$ be fixed. We consider the sequence of functions $(g_n)_{n\in\mathbb{N}^*}$ defined by

$$\hat{\hat{g}}_n(\xi,\eta) := \hat{\hat{h}}(t^*,\xi,\eta)1_{[-n,n]}(\xi)1_{[-n,n]}(\eta), \quad \forall n \in \mathbb{N}^*.$$

Then g_n belongs to the Schwarz space $S(\mathbb{R}^2,\mathbb{R})$ for every $n \in \mathbb{N}^*$ because $\hat{\hat{g}}_n$ has compact support. Let ψ_n be the solution of

$$\begin{cases} \dfrac{\partial\psi_n}{\partial t} + \dfrac{\partial^2\psi_n}{\partial v^2} - 2t\dfrac{\partial^2\psi_n}{\partial v\partial x} + t^2\dfrac{\partial^2\psi_n}{\partial x^2} = 0, \quad (x,v)\in\Omega_1,\ t\in(0,T), \\ \psi_n(t^*,x,v) = g_n(x,v), \end{cases} \tag{10}$$

built with the previous construction. Since g_n is smooth, then ψ_n is also smooth and solves Eq. (10) pointwise. By definition, ψ_n belongs to $C^2([0,T]\times\Omega_1,\mathbb{R}) \cap H^2((0,T)\times\Omega_1,\mathbb{R})$. Thus, for every $n \in \mathbb{N}^*$,

$$\int_{\Omega_1} h(t^*,x,v)\psi_n(t^*,x,v)\,dx\,dv = \int_{\Omega_1} h(t^*,x,v)g_n(x,v)\,dx\,dv = 0.$$

By definition, $g_n \to h(t^*)$ in $L^2(\Omega_1,\mathbb{R})$ when $n \to +\infty$ thus, letting $n \to +\infty$ in the previous equality, we get

$$\int_{\Omega_1} \big|h(t^*,x,v)\big|^2\,dx\,dv = 0.$$

We have proved that, for every $t^* \in [0,T]$, $h(t^*) = 0$. This gives the uniqueness of the solution of (2).

Now let us prove the continuity with respect to the initial conditions. Let f_0, $\tilde{f}_0 \in L^2(\Omega_1)$ and f, $\tilde{f}$ be the solutions of (2) associated to these initial conditions, with the same source term u. With obvious notations, we have, for every $t^* \in [0,T]$,

$$\begin{aligned} \big\|(\tilde{f}-f)(t^*)\big\|^2_{L^2(\Omega_1)} &= \big\|(\tilde{h}-h)(t^*)\big\|^2_{L^2(\Omega_1)} \\ &= \frac{1}{2\pi}\big\|(\hat{\hat{\tilde{h}}}-\hat{\hat{h}})(t^*)\big\|^2_{L^2(\Omega_1)} \\ &= \frac{1}{2\pi}\int_{\Omega_1}\big|(\hat{\hat{\tilde{f}}}_0-\hat{\hat{f}}_0)(\xi,\eta)\big|^2 e^{-2\eta^2 t+2\eta\xi t^2-2\xi^2\frac{t^3}{3}}\,d\xi\,d\eta \\ &\leqslant \|f_0-\tilde{f}_0\|^2_{L^2(\Omega_1)}. \qquad \square \end{aligned}$$

2.2. Proof of Theorem 1

Let us consider a solution of (2) in the sense of Definition 1. The function f belongs to $C^0([0,T],L^2(\mathbb{R}^2,\mathbb{C}))$, so $x \mapsto f(t,x,v)$ belongs to $L^2(\mathbb{R},\mathbb{C})$, for almost every $(t,v) \in (0,T)\times\mathbb{R}$ and we can consider the Fourier transform of f in the variable x, denoted $\hat{f}(t,\xi,v)$, that solves

$$\begin{cases} \dfrac{\partial\hat{f}}{\partial t}(t,\xi,v) + i\xi v\hat{f}(t,\xi,v) - \dfrac{\partial^2\hat{f}}{\partial v^2}(t,\xi,v) = \hat{u}(t,\xi,v)1_{\mathbb{R}-[a_1,b_1]}(v), \quad (\xi,v)\in\mathbb{R}^2, \\ \hat{f}(0,\xi,v) = \hat{f}_0(\xi,v). \end{cases} \tag{11}$$

The strategy of the proof of Theorem 1 is standard and relies on two key ingredients. The first one is an explicit decay rate for the solutions of (11) without control, stated in the next lemma.

Lemma 1. *For every $f_0 \in L^2(\mathbb{R}^2, \mathbb{R})$, the solution of* (2) *with $u \equiv 0$ satisfies*

$$\left\| \hat{f}(t, \xi, \cdot) \right\|_{L^2(\mathbb{R})} \leqslant \left\| \hat{f}_0(\xi, \cdot) \right\|_{L^2(\mathbb{R})} e^{-\xi^2 t^3/12}, \quad \forall \xi \in \mathbb{R},\ \forall t \in \mathbb{R}_+. \tag{12}$$

Proof. We use an explicit expression of the solution of (11). Applying the Fourier transform in the variable v to (11), we get

$$\begin{cases} \dfrac{\partial \hat{\hat{f}}}{\partial t}(t, \xi, \eta) - \xi \dfrac{\partial \hat{\hat{f}}}{\partial \eta}(t, \xi, \eta) + \eta^2 \hat{\hat{f}} = 0, \quad (\xi, \eta) \in \mathbb{R}^2, \\ \hat{\hat{f}}(0, \xi, \eta) = \hat{\hat{f}}_0(\xi, \eta). \end{cases} \tag{13}$$

Let $\xi \in \mathbb{R}$ be fixed and $k(t, \tilde{\eta})$ be defined by $\hat{\hat{f}}(t, \xi, \eta) := k(t, \eta + \xi t)$. Then

$$\begin{cases} \dfrac{\partial k}{\partial t} + (\tilde{\eta} - \xi t)^2 k = 0, \quad \tilde{\eta} \in \mathbb{R}, \\ k(0, \tilde{\eta}) = \hat{\hat{f}}_0(\xi, \tilde{\eta}). \end{cases} \tag{14}$$

Thus,

$$k(t, \tilde{\eta}) = \hat{\hat{f}}_0(\xi, \tilde{\eta}) e^{-t\tilde{\eta}^2 + t^2 \tilde{\eta} \xi - \frac{t^3}{3} \xi^2},$$

from which we deduce

$$\hat{\hat{f}}(t, \xi, \eta) = \hat{\hat{f}}_0(\xi, \eta + \xi t) e^{-t\eta^2 - t^2 \eta \xi - \frac{t^3}{3} \xi^2}.$$

The inequality (12) is a consequence of

$$t\eta^2 + t^2 \eta \xi + \frac{t^3}{3} \xi^2 = t \left[\left(\eta + \frac{1}{2} \xi t \right)^2 + \frac{t^2}{12} \xi^2 \right] \geqslant \frac{t^3}{12} \xi^2. \qquad \square$$

The second key ingredient for the proof of Theorem 1 is the following result.

Theorem 6. *Let $T > 0$. There exists $C(T) > 0$ such that, for every $\xi \in \mathbb{R}$, for every $k_0 \in L^2(\mathbb{R}, \mathbb{C})$, there exists $\nu \in L^2((0, T) \times \mathbb{R}, \mathbb{C})$ such that the solution of*

$$\begin{cases} \dfrac{\partial k}{\partial t}(t, v) + i\xi v k(t, v) - \dfrac{\partial^2 k}{\partial v^2}(t, v) = \nu(t, v) 1_{\mathbb{R} - [a_1, b_1]}(v), \quad v \in \mathbb{R},\ t \in (0, T), \\ k(0, v) = k_0(v), \end{cases} \tag{15}$$

satisfies $k(T) = 0$ and

$$\|\nu\|_{L^2((0,T)\times\mathbb{R})} \leqslant e^{C(T) \max\{1, \sqrt{|\xi|}\}} \|k_0\|_{L^2(\mathbb{R})}.$$

This theorem is proved in the next subsection.

Proof of Theorem 1. Let $T > 0$ and $f_0 \in L^2(\mathbb{R}^2, \mathbb{R})$. On the time interval $[0, T/2]$, we apply the control $u \equiv 0$ in (2). Thanks to Lemma 1, we have

$$\left\| \hat{f}(T/2, \xi, \cdot) \right\|_{L^2(\mathbb{R}_v)} \leqslant e^{-\frac{\xi^2 T^3}{96}} \left\| \hat{f}_0(\xi, \cdot) \right\|_{L^2(\mathbb{R}_v)}, \quad \forall \xi \in \mathbb{R}. \tag{16}$$

Thanks to Theorem 6, for every $\xi \in \mathbb{R}_+$, there exists a control $\nu_\xi \in L^2((T/2, T) \times \mathbb{R}, \mathbb{C})$ such that the solution of (15) with $\nu(t) = \nu_\xi(T/2 + t)$ satisfies $k(T/2, \cdot) = 0$ and

$$\int_{T/2}^{T} \int_{\mathbb{R}} \left| \nu_\xi(t, v) \right|^2 dv\, dt \leqslant e^{2C(T) \max\{1, \sqrt{|\xi|}\}} \int_{\mathbb{R}} \left| \hat{f}(T/2, \xi, v) \right|^2 dv.$$

Then the function $v_{-\xi} := \overline{v_\xi}$ accomplishes the same purpose with ξ replaced by $-\xi$. Thanks to (16), we get

$$\int_{T/2}^{T}\int_{\mathbb{R}^2} \left|v_\xi(t,v)\right|^2 dv\,d\xi\,dt \leqslant \int_{\mathbb{R}^2} e^{2C(T)\max\{1,\sqrt{|\xi|}\}-\frac{\xi^2T^3}{48}} \left|\hat{f}_0(\xi,v)\right|^2 dv\,d\xi \leqslant M\int_{\mathbb{R}^2} \left|\hat{f}_0(\xi,v)\right|^2 dv\,d\xi,$$

where $M := \max\{e^{2C(T)\max\{1,\sqrt{|\xi|}\}-\frac{\xi^2T^3}{48}}; \xi\in\mathbb{R}\}$ is finite. Thus, there exists $u \in L^2((T/2,T)\times\mathbb{R}^2,\mathbb{R})$ such that $\hat{u}(t,\xi,v) = v_\xi(t,v)$ for almost every $(t,v)\in(T/2,T)\times\mathbb{R}$. On the time interval $[T/2,T]$, we apply this control u in (2). Then $\hat{f}(T,\xi,\cdot)=0$, for every $\xi\in\mathbb{R}$, so $f(T)=0$. □

Remark 1. By taking advantage of the dissipation of Eq. (15) we have proved that the cost for the null controllability of this equation can be bounded uniformly with respect to ξ.

Remark 2. In this section we have adopted the control strategy in two steps to show that the cost for the null controllability of system (11) is bounded uniformly with respect to ξ.

This, by duality, guarantees the uniform observability of the adjoint system (18) with respect to the parameter ξ (i.e. in Theorem 8 below, the obervability constant $e^{C(T)\max\{1,\sqrt{|\xi|}\}}$ can be replaced by a constant $C'(T)$).

This uniform observability property can be obtained directly working in the context of observability. We refer to Remark 4 below for a direct proof.

Remark 3. Let us recall that explicit bounds for the cost of the null controllability of heat equations with potentials are already known. For example in [12, Theorem 1.2]), one has the following result.

Theorem 7. *There exists $C>0$ such that, for every $T>0$, $a,b\in L^\infty((0,T)\times\mathbb{R},\mathbb{C})$, $y_0\in L^2(\mathbb{R},\mathbb{C})$, there exists a control $v\in L^2((0,T)\times\mathbb{R},\mathbb{C})$ such that the solution of*

$$\begin{cases} \dfrac{\partial y}{\partial t} - \dfrac{\partial^2 y}{\partial v^2} + b\dfrac{\partial y}{\partial v} + ay = v1_{\mathbb{R}-[a_1,b_1]}, \quad v\in\mathbb{R}, \\ y(0,v) = y_0(v), \end{cases} \tag{17}$$

satisfies $y(T)=0$ and

$$\|v\|_{L^2((0,T)\times\mathbb{R})} \leqslant e^{CH(T,\|a\|_\infty,\|b\|_\infty)}\|y_0\|_{L^2(\mathbb{R})},$$

where

$$H\big(T,\|a\|_\infty,\|b\|_\infty\big) := 1+\frac{1}{T}+T\|a\|_\infty+\|a\|_\infty^{2/3}+(1+T)\|b\|_\infty^2.$$

However, this result does not apply in our situation because our potential $a(v)=i\xi v$, does not belong to $L^\infty(\mathbb{R},\mathbb{C})$.

2.3. Proof of Theorem 6

It is well known that Theorem 6 is a consequence of the following observability estimate.

Theorem 8. *Let $T>0$. There exists $C>0$ such that, for every $\xi\in\mathbb{R}$, for every $g_0\in L^2(\mathbb{R},\mathbb{C})$, the solution of*

$$\begin{cases} \dfrac{\partial g}{\partial t} - i\xi v g - \dfrac{\partial^2 g}{\partial v^2} = 0, \quad v\in\mathbb{R},\ t\in(0,T), \\ g(0,v) = g_0(v), \end{cases} \tag{18}$$

satisfies

$$\int_{\mathbb{R}} \left|g(T,v)\right|^2 dv \leqslant e^{C(T)\max\{1,\sqrt{|\xi|}\}}\int_0^T\int_{\mathbb{R}-[a_1,b_1]} \left|g(t,v)\right|^2 dv\,dt. \tag{19}$$

Proof. First, notice that, thanks to the continuity of the solutions of (18) with respect to the initial condition (whose proof is the same as in Proposition 1), by density, it is sufficient to prove the inequality (19) when g_0 belongs to the Schwarz space $S(\mathbb{R}^2, \mathbb{R})$.

This assumption implies, in particular, that for every $k \in \mathbb{N}^*$, $\partial_v^k g$ belongs to $L^2(Q)$, where $Q := (0, T) \times \mathbb{R}$. Indeed, for every $t \in (0, T)$, one has

$$\begin{aligned}\int_{\mathbb{R}} \left|\partial_v^k g(t, v)\right|^2 dv &= \int_{\mathbb{R}} \left|\eta^k \hat{g}(t, \eta)\right|^2 d\eta \\ &= \int_{\mathbb{R}} \left|\eta^k \hat{g}_0(\eta - \xi t) e^{-t\eta^2 + t^2\eta\xi - \frac{t^3}{3}\xi^2}\right|^2 d\xi \\ &\leqslant \int_{\mathbb{R}} \left|(\eta + \xi t)^k \hat{g}_0(\eta)\right|^2 d\xi < +\infty.\end{aligned}$$

In the same way, one has $\partial_t g \in L^2(Q, \mathbb{R})$.

Let a, b be such that

$$-\infty < a < a_1 < b_1 < b < +\infty. \tag{20}$$

To obtain the relevant Carleman inequality, let us define a weight function, similar to the one introduced by Fursikov and Imanuvilov in [11],

$$\alpha(t, v) := \frac{M\beta(v)}{t(T - t)}, \quad (t, v) \in (0, T) \times \mathbb{R}, \tag{21}$$

where $\beta \in C^2(\mathbb{R}, \mathbb{R}_+)$ is such that

$$\beta \geqslant 1 \quad \text{on } \mathbb{R}, \tag{22}$$

$$|\beta'| > 0 \quad \text{on } [a, b], \tag{23}$$

$$\beta' = 0 \quad \text{on } (-\infty, a - 1) \cup (b + 1, +\infty), \tag{24}$$

$$\beta'' < 0 \quad \text{on } [a, b], \tag{25}$$

and $M > 0$ will be chosen later on. We also introduce the function $z(t, v)$ such that

$$z(t, v) = g(t, v) e^{-\alpha(t, v)}. \tag{26}$$

This function satisfies

$$P_1 z + P_2 z = P_3 z, \tag{27}$$

with

$$P_1 z := -\frac{\partial^2 z}{\partial v^2} + \left(\alpha_t - \alpha_v^2\right) z, \qquad P_2 z := \frac{\partial z}{\partial t} - 2\alpha_v \frac{\partial z}{\partial v} - i\xi v z, \qquad P_3 z := \alpha_{vv} z.$$

We develop the classical proof, consisting in taking the $L^2(Q, \mathbb{C})$-norm in the identity (27), then developing the double product. This leads to

$$\Re\left(\int_Q P_1 \overline{P_2}\right) \leqslant \frac{1}{2} \int_Q |P_3|^2, \tag{28}$$

where $Q := (0, T) \times \mathbb{R}$. And we compute precisely each term.

First, let us justify that all the terms in (27) belong to $L^2(Q, \mathbb{C})$. At this step, the assumption $g_0 \in S(\mathbb{R}^2, \mathbb{R})$ is used. Let us justify it, for example, with the term $\alpha_v \partial_v z$ (the arguments for the other terms are similar). We have

$$\begin{aligned}\left|\alpha_v\frac{\partial z}{\partial v}\right| &\leqslant \left|\left(\alpha_v\frac{\partial g}{\partial v}-\alpha_v^2 g\right)e^{-\alpha}\right| \\ &\leqslant \left|\frac{M\beta'}{t(T-t)}e^{-\frac{M\beta}{t(T-t)}}\frac{\partial g}{\partial v}\right| + \left|\left(\frac{M\beta'}{t(T-t)}\right)^2 e^{-\frac{M\beta}{t(T-t)}} g\right| \\ &\leqslant c_1\frac{|\beta'|}{\beta}\left|\frac{\partial g}{\partial v}\right| + c_2\left(\frac{\beta'}{\beta}\right)^2|g|^2, \end{aligned} \tag{29}$$

where $c_1 := \sup\{xe^{-x}, x\in\mathbb{R}_+\}$, $c_2 := \sup\{x^2e^{-x}, x\in\mathbb{R}_+\}$. Thanks to the assumptions (22) and (24), the function β'/β is bounded on $\mathbb{R}$. Moreover g and $\partial_v g$ belong to $L^2(Q,\mathbb{R})$ (because $g_0\in S(\mathbb{R}^2,\mathbb{R})$) thus $\alpha_v\partial_v z$ belongs to $L^2(Q,\mathbb{R})$.

Let us compute the left-hand side of the inequality (28). In the following computations, we use integrations by parts in the space variable, in which the boundary terms at $v=\pm\infty$ vanish because z, $\partial_v z$, $\partial_t z$ vanish at $v=\pm\infty$, for every $t\in(0,T)$ and α, α_t, α_v are bounded on $\mathbb{R}$, for every $t\in(0,T)$.

Terms concerning $-\partial^2 z/\partial v^2$: First, one has

$$\Re\left(\int_Q -\frac{\partial^2 z}{\partial v^2}\frac{\partial\bar z}{\partial t}\right) = \int_0^T \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\left|\frac{\partial z}{\partial v}\right|^2 dv\,dt = 0,$$

where the first equality comes from an integration by parts in the space variable and the second one is due to $z(0)\equiv z(T)\equiv 0$, which is a consequence of (26), (21) and (22). Then, one has

$$\Re\left(\int_Q \frac{\partial^2 z}{\partial v^2}2\alpha_v\frac{\partial\bar z}{\partial v}\right) = -\int_Q\left|\frac{\partial z}{\partial v}\right|^2\alpha_{vv},$$

thanks to an integration by parts in the space variable. Finally, one has

$$\Re\left(-\int_Q \frac{\partial^2 z}{\partial v^2}i\xi v\bar z\right) = -\xi\Im\left(\int_Q\frac{\partial z}{\partial v}\bar z\right),$$

thanks to an integration by parts in the space variable.

Terms concerning $(\alpha_t-\alpha_v^2)z$: First, one has

$$\Re\left(\int_Q(\alpha_t-\alpha_v^2)z\frac{\partial\bar z}{\partial t}\right) = -\int_Q\frac{1}{2}(\alpha_t-\alpha_v^2)_t|z|^2,$$

thanks to an integration by parts in the time variable. The boundary terms at $t=0$ and $t=T$ vanish because, thanks to (26), (21), (22),

$$\left|(\alpha_t-\alpha_v^2)|z|^2\right| \leqslant \frac{1}{[t(T-t)]^2}e^{\frac{-M}{t(T-t)}}\left|M(T-2t)\beta+(M\beta')^2\right||g|^2 \tag{30}$$

tends to zero when $t\to 0$ and $t\to T$, for every $v\in\mathbb{R}$. Then, one has

$$\Re\left(-\int_Q(\alpha_t-\alpha_v^2)z2\alpha_v\frac{\partial\bar z}{\partial v}\right) = \int_Q\left[(\alpha_t-\alpha_v^2)\alpha_v\right]_v|z|^2,$$

thanks to an integration by parts in the space variable. Finally, one has

$$\Re\left(\int_Q(\alpha_t-\alpha_v^2)zi\xi v\bar z\right) = 0.$$

Putting all these computations together and using (28), we get

$$\int_Q -\left|\frac{\partial z}{\partial v}\right|^2\alpha_{vv} - \xi\Im\left(\frac{\partial z}{\partial v}\bar z\right) + |z|^2\left\{-\frac{1}{2}(\alpha_t-\alpha_v^2)_t + \left[(\alpha_t-\alpha_v^2)\alpha_v\right]_v - \frac{1}{2}\alpha_{vv}^2\right\}\leqslant 0. \tag{31}$$

Now, we separate in (31) the terms concerning $(0,T)\times\mathbb{R}-[a,b]$ and the terms concerning $(0,T)\times(a,b)$. First, one has

$$-\alpha_{vv}(t,v)=-\frac{M\beta''(v)}{t(T-t)}\geqslant\frac{C_1M}{t(T-t)},\quad\forall v\in[a,b]\text{ and }\forall t\in(0,T),$$

where $C_1:=\min\{-\beta''(v);\ v\in[a,b]\}$ is positive thanks to (25). One also has

$$\left|\alpha_{vv}(t,v)\right|=\left|\frac{M\beta''(v)}{t(T-t)}\right|\leqslant\frac{C_2M}{t(T-t)},\quad\forall v\in\mathbb{R}-[a,b]\text{ and }\forall t\in(0,T),$$

where $C_2:=\sup\{|\beta''(v)|;\ v\in\mathbb{R}-[a,b]\}$ is finite thanks to (24). Thus,

$$\int\limits_Q-\left|\frac{\partial z}{\partial v}\right|^2\alpha_{vv}\geqslant\int\limits_0^T\int\limits_{(a,b)}\frac{C_1M}{t(T-t)}\left|\frac{\partial z}{\partial v}\right|^2-\int\limits_0^T\int\limits_{\mathbb{R}-[a,b]}\frac{C_2M}{t(T-t)}\left|\frac{\partial z}{\partial v}\right|^2.\tag{32}$$

Then, one has

$$\begin{aligned}&-\frac12\big(\alpha_t-\alpha_v^2\big)_t+\big[\big(\alpha_t-\alpha_v^2\big)\alpha_v\big]_v-\frac12\alpha_{vv}^2\\&\quad=\frac{1}{(t(T-t))^3}\left(-3M^3\beta''\beta'^2-M^2\left[(T-2t)\beta''\beta'+\frac12t(T-t)\beta''\right]+M\big(T^2-5Tt+5t^2\big)\beta\right).\end{aligned}$$

Thus, using (23) and (25), there exists $M_1=M_1(T,\beta)>0$, $C_3=C_3(T,\beta)>0$, $C_4=C_4(T,\beta)>0$, such that, for every $M\geqslant M_1$,

$$-\frac12\big(\alpha_t-\alpha_v^2\big)_t+\big[\big(\alpha_t-\alpha_v^2\big)\alpha_v\big]_v-\frac12\alpha_{vv}^2\geqslant\frac{M^3C_3}{[t(T-t)]^3},\quad\forall v\in(a,b),\ \forall t\in(0,T)\tag{33}$$

and

$$\left|-\frac12\big(\alpha_t-\alpha_v^2\big)_t+\big[\big(\alpha_t-\alpha_v^2\big)\alpha_v\big]_v-\frac12\alpha_{vv}^2\right|\leqslant\frac{M^3C_4}{[t(T-t)]^3},\quad\forall v\in\mathbb{R}-(a,b),\ \forall t\in(0,T).\tag{34}$$

Thus,

$$\begin{aligned}&\int\limits_Q|z|^2\left\{-\frac12\big(\alpha_t-\alpha_v^2\big)_t+\big[\big(\alpha_t-\alpha_v^2\big)\alpha_v\big]_v-\frac12\alpha_{vv}^2\right\}\\&\quad\geqslant\int\limits_0^T\int\limits_{(a,b)}\frac{M^3C_3}{[t(T-t)]^3}|z|^2-\int\limits_0^T\int\limits_{\mathbb{R}-(a,b)}\frac{M^3C_4}{[t(T-t)]^3}|z|^2.\end{aligned}\tag{35}$$

Using (31), (32) and (35), we get, for every $M\geqslant M_1$,

$$\begin{aligned}&\int\limits_0^T\int\limits_{(a,b)}\frac{C_1M}{t(T-t)}\left|\frac{\partial z}{\partial v}\right|^2-\xi\Im\left(\frac{\partial z}{\partial v}\bar z\right)+\frac{C_3M^3}{(t(T-t))^3}|z|^2\\&\quad\leqslant\int\limits_0^T\int\limits_{\mathbb{R}-[a,b]}\frac{C_2M}{t(T-t)}\left|\frac{\partial z}{\partial v}\right|^2+\xi\Im\left(\frac{\partial z}{\partial v}\bar z\right)+\frac{C_4M^3}{(t(T-t))^3}|z|^2.\end{aligned}\tag{36}$$

Let $M_2=M_2(T,\beta,\xi)$ be defined by

$$M_2:=\frac{T^2\sqrt{|\xi|}}{4(2C_1C_3)^{1/4}}.\tag{37}$$

From now on, we take

$$M=M(T,\beta,\xi):=\max\big(1,M_1(T,\beta),M_2(T,\beta,\xi)\big).\tag{38}$$

Then, we have

$$\left|\xi\Im\left(\frac{\partial z}{\partial v}\bar{z}\right)\right| \leqslant \frac{1}{2}\frac{C_3M^3}{(t(T-t))^3}|z|^2 + \frac{1}{2}\frac{(t(T-t))^3}{C_3M^3}\xi^2\left|\frac{\partial z}{\partial v}\right|^2 \leqslant \frac{1}{2}\frac{C_3M^3}{(t(T-t))^3}|z|^2 + \frac{C_1M}{t(T-t)}\left|\frac{\partial z}{\partial v}\right|^2, \tag{39}$$

because

$$\frac{1}{2}\frac{(t(T-t))^3}{C_3M^3}\xi^2 = \frac{C_1M}{t(T-t)}\frac{(t(T-t))^4\xi^2}{2C_1C_3M^4} \leqslant \frac{C_1M}{t(T-t)}\frac{(T^2/4)^4\xi^2}{2C_1C_3M^4} = \frac{C_1M}{t(T-t)}\frac{M_2^4}{M^4}$$

and $M \geqslant M_2$. Using (36) and (39), we get

$$\int_0^T\int_{(a,b)} \frac{C_3M^3}{2(t(T-t))^3}|z|^2 \leqslant \int_0^T\int_{\mathbb{R}-[a,b]} \frac{C_5M^3}{(t(T-t))^3}|z|^2 + \frac{C_6M}{t(T-t)}\left|\frac{\partial z}{\partial v}\right|^2 \tag{40}$$

where $C_5 = C_5(T,\beta) := C_4 + C_3/2$ and $C_6 = C_6(T,\beta) := C_2 + C_1$. Coming back to our original variable thanks to (26) the inequality (40) can be written

$$\int_0^T\int_{(a,b)} \frac{C_3M^3|g|^2e^{-2\alpha}}{2(t(T-t))^3} \leqslant \int_0^T\int_{\mathbb{R}-[a,b]} \left(\frac{C_5M^3|g|^2}{(t(T-t))^3} + \frac{C_6M}{t(T-t)}\left|\frac{\partial g}{\partial v} - \alpha_v g\right|^2\right)e^{-2\alpha} \tag{41}$$

thus

$$\int_0^T\int_{(a,b)} \frac{C_3M^3|g|^2e^{-2\alpha}}{2(t(T-t))^3} \leqslant \int_0^T\int_{\mathbb{R}-[a,b]} \left(\frac{C_7M^3|g|^2}{(t(T-t))^3} + \frac{C_8M}{t(T-t)}\left|\frac{\partial g}{\partial v}\right|^2\right)e^{-2\alpha} \tag{42}$$

where $C_8 := C_8(T,\beta) = 2C_6$ and $C_7 = C_7(T,\beta) := C_5 + 2C_6\sup\{\beta'(v)^2;\ v \in \mathbb{R}-(a,b)\}$ is finite thanks to (24). Thanks to (22), and the assumption $M \geqslant 1$ (see (38)) we have, for every $v \in \mathbb{R}-(a,b)$,

$$\frac{C_7M^3}{(t(T-t))^3}e^{-2\alpha} \leqslant \frac{C_7M^3}{(t(T-t))^3}e^{-\frac{2M}{t(T-t)}} \leqslant C_9,$$
$$\frac{C_8M}{t(T-t)}e^{-2\alpha} \leqslant \frac{C_8t(T-t)}{M}\left(\frac{M}{t(T-t)}\right)^2 e^{-\frac{2M}{t(T-t)}} \leqslant C_{10}t(T-t)$$

where $C_9 = C_9(T,\beta) := C_7\sup\{x^3e^{-2x};\ x \in \mathbb{R}_+\}$ and $C_{10} = C_{10}(T,\beta) := C_8\sup\{x^2e^{-2x};\ x \in \mathbb{R}_+\}$. Therefore, using the two previous inequalities and (42), we get

$$\int_0^T\int_{(a,b)} \frac{C_3M^3|g|^2e^{-2\alpha}}{2(t(T-t))^3} \leqslant \int_0^T\int_{\mathbb{R}-[a,b]} \left(C_9|g|^2 + C_{10}t(T-t)\left|\frac{\partial g}{\partial v}\right|^2\right). \tag{43}$$

Now, let us prove that the last term can be bounded by a first order term in g on $\mathbb{R}-[a_1,b_1]$. We consider $\rho \in C^\infty(\mathbb{R},\mathbb{R}_+)$ such that

$$\rho \equiv 1 \quad \text{on } (-\infty,a)\cup(b,+\infty), \tag{44}$$
$$\rho \equiv 0 \quad \text{on } (a_1,b_1). \tag{45}$$

Multiplying the first equation of (18) by $\bar{g}\rho t(T-t)$, integrating over $(0,T)\times\mathbb{R}$ and considering the real part of the resulting equality, we get

$$\int_0^T\int_{\mathbb{R}}\left(\frac{1}{2}\frac{d}{dt}\big[|g|^2\big]\rho t(T-t) - \Re\left(\frac{\partial^2 g}{\partial v^2}\bar{g}\right)\rho t(T-t)\right)dv\,dt = 0. \tag{46}$$

Performing integrations by parts with respect to the space and time variables, we get

$$\int_0^T\int_{\mathbb{R}} \frac{1}{2}\frac{d}{dt}\big[|g|^2\big]\rho t(T-t)\,dv\,dt = -\int_0^T\int_{\mathbb{R}} \frac{1}{2}|g|^2\rho(T-2t)\,dv\,dt, \tag{47}$$

$$\begin{aligned}-\Re\int_0^T\int_{\mathbb{R}} \frac{\partial^2 g}{\partial v^2}\bar{g}\rho t(T-t)\,dv\,dt &= \Re\int_0^T\int_{\mathbb{R}} \left|\frac{\partial g}{\partial v}\right|^2\rho t(T-t) + \frac{\partial g}{\partial v}\bar{g}\rho' t(T-t)\,dv\,dt\\ &= \int_0^T\int_{\mathbb{R}} \left|\frac{\partial g}{\partial v}\right|^2\rho t(T-t) - \frac{1}{2}|g|^2\rho'' t(T-t)\,dv\,dt.\end{aligned} \tag{48}$$

Indeed, the boundary terms at $t=0$ and $t=T$ in (47) vanish thanks to the factor $t(T-t)$ and the boundary terms at $v=\pm\infty$ in (48) vanish because $g(t)\in S(\mathbb{R},\mathbb{R})$ for every $t\in(0,T)$. Thanks to (46), (47) and (48), we get

$$\int_0^T\int_{\mathbb{R}} \left(-\frac{1}{2}|g|^2\rho(T-2t) + \left|\frac{\partial g}{\partial v}\right|^2\rho t(T-t) - \frac{1}{2}|g|^2\rho'' t(T-t)\right)dv\,dt = 0. \tag{49}$$

Thus, using (44), (49) and (45), we get

$$\begin{aligned}\int_0^T\int_{\mathbb{R}-[a,b]} \left|\frac{\partial g}{\partial v}\right|^2 t(T-t)\,dv\,dt &\leqslant \int_0^T\int_{\mathbb{R}} \left|\frac{\partial g}{\partial v}\right|^2\rho t(T-t)\,dv\,dt\\ &= \int_0^T\int_{\mathbb{R}} \frac{1}{2}|g|^2\big[\rho(T-2t)+\rho'' t(T-t)\big]\,dv\,dt\\ &\leqslant C_{11}\int_0^T\int_{\mathbb{R}-(a_1,b_1)} |g|^2\,dv\,dt,\end{aligned} \tag{50}$$

where $C_{11}=C_{11}(T,\rho):=T\|\rho\|_{L^\infty(\mathbb{R})}+\frac{T^2}{2}\|\rho''\|_{L^\infty(\mathbb{R})}$. The previous inequality and (43) lead to

$$\int_0^T\int_{(a,b)} \frac{C_3M^3|g|^2e^{-2\alpha}}{(t(T-t))^3}\,dv\,dt \leqslant \int_0^T\int_{\mathbb{R}-(a_1,b_1)} C_{12}|g|^2\,dv\,dt, \tag{51}$$

where $C_{12}=C_{12}(T,\beta,\rho):=2[C_9+C_{10}C_{11}]$. We have

$$t(T-t)\geqslant\frac{2T^2}{9},\quad \forall t\in\left[\frac{T}{3},\frac{2T}{3}\right],$$

$$t(T-t)\leqslant\frac{T^2}{4},\quad \forall t\in\left[\frac{T}{3},\frac{2T}{3}\right],$$

thus

$$\frac{e^{-2\alpha(t,v)}}{(t(T-t))^3}\geqslant\frac{e^{-9c_3M/T^2}}{(T^2/4)^3},\quad \forall v\in(a,b),\ \forall t\in(T/3,2T/3),$$

where $c_3:=\sup\{\beta(v);\ v\in[a,b]\}$. Thus (51) leads to

$$\frac{C_3M^3}{(T^2/4)^3}e^{-\frac{9c_3M}{T^2}}\int_{T/3}^{2T/3}\int_{(a,b)} |g|^2\,dv\,dt \leqslant \int_0^T\int_{\mathbb{R}-(a_1,b_1)} C_{12}|g|^2\,dv\,dt. \tag{52}$$

Adding the same quantity in both sides and using the inclusion $[\mathbb{R}-(a,b)]\subset[\mathbb{R}-(a_1,b_1)]$ (which is a consequence of (20)), we get

$$\frac{C_3M^3}{(T^2/4)^3}e^{-\frac{9c_3M}{T^2}}\int_{T/3}^{2T/3}\int_{\mathbb{R}}|g|^2\,dv\,dt\leqslant\left(C_{12}+\frac{C_3M^3}{(T^2/4)^3}e^{-\frac{9c_3M}{T^2}}\right)\int_0^T\int_{\mathbb{R}-(a_1,b_1)}|g|^2\,dv\,dt, \tag{53}$$

which can also be written

$$\int_{T/3}^{2T/3}\int_{\mathbb{R}}|g|^2\,dv\,dt\leqslant\left(C_{12}\frac{(T^2/4)^3}{C_3M^3}e^{\frac{9c_3M}{T^2}}+1\right)\int_0^T\int_{\mathbb{R}-(a_1,b_1)}|g|^2\,dv\,dt. \tag{54}$$

Using the decreasing property of $t\mapsto\|g(t)\|_{L^2}$ and the inequality $M\geqslant 1$ (see (38)), we get

$$\int_{\mathbb{R}}\left|g(T,v)\right|^2dv\leqslant\frac{3}{T}e^{\frac{9c_3M}{T^2}}\left(C_{12}\frac{(T^2/4)^3}{C_3}+1\right)\int_0^T\int_{\mathbb{R}-(a_1,b_1)}|g|^2\,dv\,dt$$

which gives the conclusion thanks to (38). □

Remark 4. Let us propose an alternative strategy for the direct proof of the uniform observability of (18) with respect to the parameter ξ, i.e. the existence of a constant $C(T)>0$ such that, for every $\xi\in\mathbb{R}$, the solution of (18) satisfies

$$\int_{\mathbb{R}}\left|g(T,v)\right|^2dv\leqslant C(T)\int_0^T\int_{\mathbb{R}-(a_1,b_1)}\left|g(t,v)\right|^2dv\,dt. \tag{55}$$

This proof is the same as the one of Theorem 8 above, until the formula (54). At this step, instead of using the decreasing behavior of $t\mapsto\|g(t)\|_{L^2}$, one may use the inequality

$$\left\|g(t)\right\|_{L^2(\mathbb{R})}\geqslant\left\|g(T)\right\|_{L^2(\mathbb{R})}e^{\frac{\xi^2(T-t)^3}{12}}. \tag{56}$$

This inequality can be proved in the same way as Lemma 1.

This strategy has already been used in [3].

3. Control in the square

In this section, we prove Theorem 1 on the square.

As mentioned in the introduction, the boundary conditions in (3) have been chosen to ensure that the function $h(t,x,v):=f(t,x+vt,v)$ is 2π periodic with respect to x and v. Then, one has explicit solutions of (3), for which one can prove an explicit decay rate in the same spirit as in Lemma 1, which allows to use the strategy of the previous section for the proof of Theorem 1.

We adopt the following convention: for any function $\varphi:(0,2\pi)\times(0,2\pi)\to\mathbb{C}$, $\varphi=\varphi(x,v)$, such that $\varphi(0,v)=\varphi(2\pi,v)$ for every $v\in(0,2\pi)$, φ denotes indifferently the function $\varphi:(0,2\pi)\times(0,2\pi)\to\mathbb{C}$ or the function $\varphi:\mathbb{R}\times(0,2\pi)\to\mathbb{C}$ which is 2π-periodic with respect to x. In order to simplify the notations, in this section, we write (Ω,ω) instead of (Ω_2,ω_2).

3.1. Well posedness of the Cauchy problem

First, let us define a concept of solution for (3).

Definition 2. Let $T>0$, $f_0\in L^2(\Omega,\mathbb{R})$ and $u\in L^2((0,T)\times\Omega,\mathbb{R})$. A solution of the Cauchy problem (3) is a function $f\in C^0([0,T],L^2(\Omega,\mathbb{R}))$ such that $f(0)=f_0$ in $L^2(\Omega,\mathbb{R})$ and for every $t^*\in[0,T]$ and $\varphi\in C^2([0,T]\times\Omega,\mathbb{R})$ with

$$\begin{cases}\varphi(t,0,v)=\varphi(t,2\pi,v),\quad\forall(t,v)\in[0,T]\times(0,2\pi),\\ \varphi(t,x,0)=\varphi(t,x+2\pi t,2\pi),\quad\forall(t,x)\in[0,T]\times(0,2\pi),\\ \partial_v\varphi(t,x,0)=\partial_v\varphi(t,x+2\pi t,2\pi),\quad\forall(t,x)\in[0,T]\times(0,2\pi),\end{cases}$$

one has

$$\int_\Omega \big[f(t^*,x,v)\varphi(t^*,x,v) - f_0(x,v)\varphi(0,x,v)\big]\,dx\,dv$$
$$= \int_0^{t^*}\int_\Omega \big\{f(t,x,v)\big(\partial_t + v\partial_x + \partial_v^2\big)\varphi(t,x,v) + u(t,x,v)1_\omega(x,v)\varphi(t,x,v)\big\}\,dx\,dv\,dt.$$

With this definition, one has the following result.

Proposition 2. *Let $T>0$, $f_0 \in L^2(\Omega,\mathbb{R})$ and $u \in L^2((0,T)\times\Omega,\mathbb{R})$. There exists a unique solution of the Cauchy problem* (3). *Moreover, the solutions are continuous with respect to the initial conditions for the $C^0([0,T],L^2(\Omega))$-topology.*

Proof. The proof is similar to the one of Proposition 1. We perform the heuristic part, because the explicit expression will be useful in the end of the article. For $\varphi \in L^2(\Omega,\mathbb{C})$, $\varphi=\varphi(x,v)$, we denote by $\hat{\hat{\varphi}}(p,n)$ its Fourier coefficients

$$\hat{\hat{\varphi}}(p,n) = \frac{1}{(2\pi)^2}\int_{(0,2\pi)}\int_{(0,2\pi)} \varphi(x,v)e^{-i(px+nv)}\,dx\,dv, \quad \forall n,\, p\in\mathbb{Z}.$$

Let $w\in L^2((0,T)\times\Omega,\mathbb{R})$ be defined by $u(t,x,v)1_\omega(x,v) = w(t,x-vt,v)$ and $h\in C^0([0,T],L^2(\Omega,\mathbb{R}))$ be defined by its Fourier coefficients,

$$\hat{\hat{h}}(t,p,n) := \hat{\hat{f}}_0(p,n)e^{-n^2t+npt^2-p^2\frac{t^3}{3}} + \left(\int_0^t \hat{\hat{w}}(\tau,p,n)e^{n^2\tau-np\tau^2+p^2\frac{\tau^3}{3}}\,d\tau\right)e^{-n^2t+npt^2-p^2\frac{t^3}{3}}. \tag{57}$$

Then h is a solution of

$$\begin{cases} \dfrac{\partial h}{\partial t} - \dfrac{\partial^2 h}{\partial v^2} + 2t\dfrac{\partial^2 h}{\partial x\partial v} - t^2\dfrac{\partial^2 h}{\partial x^2} = w, \quad (x,v)\in\Omega,\ t\in(0,+\infty),\\ h(t,0,v) = h(t,2\pi,v),\\ h(t,x,0) = h(t,x,2\pi),\\ \partial_v h(t,x,0) = \partial_v h(t,x,2\pi),\\ h(0,x,v) = f_0(x,v). \end{cases} \tag{58}$$

Let f be defined by

$$f(t,x,v) := h(t,x-vt,v). \tag{59}$$

Then, f is a solution of (3). □

3.2. Proof of Theorem 1

The strategy for the proof of Theorem 1 is the same as in the previous section. We consider a solution f of (3). The Fourier components

$$\hat{f}(t,p,v) := \frac{1}{2\pi}\int_0^{2\pi} f(t,x,v)e^{-ipx}\,dx, \quad t\in(0,+\infty),\ p\in\mathbb{Z},\ v\in(0,2\pi),$$

solve

$$\begin{cases} \dfrac{\partial \hat{f}}{\partial t}(t,p,v) + ipv\hat{f}(t,p,v) - \dfrac{\partial^2 \hat{f}}{\partial v^2}(t,p,v) = \hat{u}(t,p,v)1_{(a_2,b_2)}(v), \quad v\in(0,2\pi),\\ \hat{f}(t,p,0) = \hat{f}(t,p,2\pi)e^{i2\pi pt},\\ \partial_v\hat{f}(t,p,0) = \partial_v\hat{f}(t,p,2\pi)e^{i2\pi pt},\\ \hat{f}(0,p,v) = \hat{f}_0(p,v). \end{cases} \tag{60}$$

The key ingredients for the proof of Theorem 1 are the following lemma and the following theorem.

Lemma 2. *For every $f_0 \in L^2(\Omega, \mathbb{C})$, the solution of* (3) *with $u \equiv 0$ satisfies*

$$\left\| \hat{f}(t, p, \cdot) \right\|_{L^2((0,2\pi),\mathbb{C})} \leqslant \left\| \hat{f}_0(p, \cdot) \right\|_{L^2((0,2\pi),\mathbb{C})} e^{-\frac{p^2 t^3}{12}}, \quad \forall p \in \mathbb{Z},\ \forall t \in \mathbb{R}_+.$$

Remark 5. Notice that we have the same decay rate as in Lemma 1.

Proof of Lemma 2. Let h be defined by (59). Thanks to (57) and Bessel Parseval equality, we have

$$\frac{1}{2\pi} \int_0^{2\pi} \left| \hat{f}(t, p, v) \right|^2 dv = \frac{1}{2\pi} \int_0^{2\pi} \left| \hat{h}(t, p, v) e^{-ipvt} \right|^2 dv = \sum_{n \in \mathbb{Z}} \left| \hat{\hat{h}}(t, p, n) \right|^2$$

$$= \sum_{n \in \mathbb{Z}} \left| \hat{\hat{f}}_0(p, n) \right|^2 e^{-2t(n - \frac{pt}{2})^2} e^{-\frac{p^2 t^3}{6}} \leqslant e^{-\frac{p^2 t^3}{6}} \frac{1}{2\pi} \int_0^{2\pi} \left| \hat{f}_0(p, v) \right|^2 dv. \qquad \square$$

Theorem 9. *Let $T > 0$. There exists $C(T) > 0$ such that, for every $p \in \mathbb{Z}$ and $k_0 \in L^2((0, 2\pi), \mathbb{C})$, there exists $v \in L^2((0, T) \times (0, 2\pi), \mathbb{C})$ such that the solution of*

$$\begin{cases} \dfrac{\partial k}{\partial t}(t, v) + ipvk(t, v) - \dfrac{\partial^2 k}{\partial v^2}(t, v) = v(t, v) 1_{(a_2, b_2)}(v), \quad v \in (0, 2\pi), \\ k(t, 0) = k(t, 2\pi) e^{i2\pi pt}, \\ \partial_v k(t, 0) = \partial_v k(t, 2\pi) e^{i2\pi pt}, \\ k(0, v) = k_0(v), \end{cases} \tag{61}$$

satisfies $k(T) = 0$ and

$$\|v\|_{L^2((0,T)\times(0,2\pi))} \leqslant e^{C(T) \max\{1, \sqrt{|p|}\}} \|k_0\|_{L^2(0,2\pi)}.$$

This theorem is proved in the next subsection.

Remark 6. The analogue of Theorem 7 when the system is posed on the bounded domain $v \in (0, 2\pi)$, with Dirichlet boundary conditions at $v = 0$ and $v = 2\pi$ is proved in [9, Theorem 1.3] with $b \equiv 0$ and in [4, Theorem 2.3] with $b \neq 0$. Moreover, it has been proved in [5] that, in that case, the power $2/3$ of the norm of the potential a appearing in the exponential factor (of Theorem 7) is optimal.

However, the analogue of Theorem 7 for system (61), in which the boundary conditions are not of Dirichlet type, is unknown.

Notice that, if it was known with the boundary conditions of (61), it would be sufficient to conclude.

Instead of checking that the proof of [9] can indeed be generalized in our context, we have preferred to adapt it in order to emphasize that, in particular cases, the same technics may lead to a better bound for the cost (here $e^{C(T)\sqrt{|p|}} \ll e^{CT|p|}$ for large p).

Note however that, our main results show that, by taking advantage of the dissipativity of the systems, the cost for the null controllability of the 1D heat equation (60) can be made independent of the frequency parameter p.

Remark 7. Another strategy to prove Theorem 1 in the case of the square domain consists in considering the Fourier series of the function h solution of (58),

$$\begin{cases} \dfrac{\partial \hat{h}}{\partial t}(t, p, v) - \dfrac{\partial^2 \hat{h}}{\partial v^2}(t, p, v) + 2ipt \dfrac{\partial \hat{h}}{\partial v}(t, p, v) + p^2 t^2 \hat{h}(t, p, v) = 0, \quad v \in (0, 2\pi), \\ \hat{h}(t, p, 0) = \hat{h}(t, p, 2\pi), \\ \partial_v \hat{h}(t, p, 0) = \partial_v \hat{h}(t, p, 2\pi), \\ \hat{h}(0, p, v) = \hat{f}_0(p, v). \end{cases} \tag{62}$$

Indeed, one has (see the proof of Lemma 2)

$$\left\|\hat{h}(t,p,\cdot)\right\|_{L^2(0,2\pi)} \leqslant e^{-p^2\frac{t^3}{12}}\left\|\hat{f}_0(p,\cdot)\right\|_{L^2(0,2\pi)}.$$

Assuming that the analogue of Theorem 7 holds in the bounded domain $v \in (0,2\pi)$ with periodic boundary conditions, one would have the following bound for the cost of the null controllability of Eq. (62)

$$e^{C[1+\frac{1}{T}+Tp^2+|p|^{4/3}+(1+T)p^2]}.$$

However, the function

$$p \mapsto e^{C[1+\frac{1}{T}+Tp^2+|p|^{4/3}+(1+T)p^2]-p^2\frac{T^3}{96}}$$

is not necessarily bounded on $\mathbb{Z}$ (it depends on the values of C and T). For this strategy to work, one would need a better bound for the cost of the null controllability of (62) than the one given in Theorem 7.

3.3. Proof of Theorem 9

It is well known that Theorem 9 is a consequence of the following observability estimate.

Theorem 10. *Let $T > 0$. There exists $C(T) > 0$ such that, for every $p \in \mathbb{Z}$, and $g_0 \in L^2(\Omega,\mathbb{R})$, the solution of*

$$\begin{cases} \dfrac{\partial g}{\partial t} - ipvg - \dfrac{\partial^2 g}{\partial v^2} = 0, \quad v \in (0,2\pi),\ t \in (0,T), \\ g(t,0) = g(t,2\pi)e^{ip2\pi(T-t)}, \\ \partial_v g(t,0) = \partial_v g(t,2\pi)e^{ip2\pi(T-t)}, \\ g(0,v) = g_0(v), \end{cases} \tag{63}$$

satisfies

$$\int_{(0,2\pi)} \left|g(T,v)\right|^2 dv \leqslant e^{C(T)\max\{1,\sqrt{|p|}\}} \int_0^T \int_{(a_2,b_2)} \left|g(t,v)\right|^2 dv\,dt. \tag{64}$$

Remark 8. Note however that, as a consequence of our uniform (in p) controllability result, the observability constant in (64) can be made uniform on the frequency parameter p.

The proof of Theorem 10 relies on a new Carleman estimate for the solutions of (63).

Proof of Theorem 10. Let $p \in \mathbb{Z}$ be fixed in all the proof and a, b be such that

$$0 \leqslant a_2 < a < b < b_2 \leqslant 2\pi. \tag{65}$$

To obtain the relevant Carleman inequality, let us define a weight function, similar to the one introduced by Fursikov and Imanuvilov in [11],

$$\alpha(t,v) := \frac{M\beta(v)}{t(T-t)}, \quad (t,v) \in (0,T)\times\mathbb{R}, \tag{66}$$

where $\beta \in C^2(\mathbb{R},\mathbb{R}_+)$ is 2π periodic and

$$\beta \geqslant 1 \quad \text{on } (0,2\pi), \tag{67}$$

$$|\beta'| > 0 \quad \text{on } [0,a]\cup[b,2\pi], \tag{68}$$

$$\beta'' < 0 \quad \text{on } [0,a]\cup[b,2\pi], \tag{69}$$

and $M > 0$ will be chosen later on. We also introduce the function

$$z(t,v) := g(t,v)e^{-\alpha(t,v)} \tag{70}$$

that satisfies

$$P_1 z + P_2 z = P_3 z, \tag{71}$$

with

$$P_1 z := -\frac{\partial^2 z}{\partial v^2} + \big(\alpha_t - \alpha_v^2\big) z, \qquad P_2 z := \frac{\partial z}{\partial t} - 2\alpha_v \frac{\partial z}{\partial v} - ipvz, \qquad P_3 z := \alpha_{vv} z. \tag{72}$$

We develop the classical proof, consisting in taking the $L^2(Q, \mathbb{C})$-norm in the identity (71), then developing the double product, which leads to

$$\Re\bigg(\int_Q P_1 \overline{P_2} \bigg) \leqslant \frac{1}{2} \int_Q |P_3|^2, \tag{73}$$

where $Q := (0, T) \times (0, 2\pi)$ and we compute precisely each term. Notice that, since β is 2π-periodic, one has

$$z(t, 0) = z(t, 2\pi) e^{ip2\pi(T-t)}, \quad \partial_v z(t, 0) = \partial_v z(t, 2\pi) e^{ip2\pi(T-t)}. \tag{74}$$

Terms concerning $-\partial^2 z/\partial v^2$: First, thanks to an integration by parts, one has

$$\Re\bigg(\int_Q -\frac{\partial^2 z}{\partial v^2} \frac{\partial \bar z}{\partial t} \bigg) = \Re\bigg(\int_0^T \bigg(-\frac{\partial z}{\partial v}(t, 2\pi) \frac{\partial \bar z}{\partial t}(t, 2\pi) + \frac{\partial z}{\partial v}(t, 0) \frac{\partial \bar z}{\partial t}(t, 0) \bigg) dt \bigg) + \Re\bigg(\int_Q \frac{\partial z}{\partial v} \frac{\partial^2 \bar z}{\partial t \partial v} \bigg).$$

We have

$$\Re\bigg(\int_Q \frac{\partial z}{\partial v} \frac{\partial^2 \bar z}{\partial t \partial v} \bigg) = \int_0^T \frac{1}{2} \frac{d}{dt} \int_{(0,2\pi)} \bigg| \frac{\partial z}{\partial v} \bigg|^2 dv\, dt = 0,$$

because $z(0) \equiv z(T) \equiv 0$, which is a consequence of (70), (66) and (67). Thanks to (74), we have

$$\frac{\partial z}{\partial t}(t, 0) = \bigg(\frac{\partial z}{\partial t}(t, 2\pi) - i2\pi p z(t, 2\pi) \bigg) e^{ip2\pi(T-t)},$$

so

$$\frac{\partial z}{\partial v}(t, 0) \frac{\partial \bar z}{\partial t}(t, 0) - \frac{\partial z}{\partial v}(t, 2\pi) \frac{\partial \bar z}{\partial t}(t, 2\pi) = i2\pi p \frac{\partial z}{\partial v}(t, 2\pi) \overline{z(t, 2\pi)}.$$

Therefore

$$\Re\bigg(\int_Q -\frac{\partial^2 z}{\partial v^2} \frac{\partial \bar z}{\partial t} \bigg) = -2\pi p \Im\bigg(\int_0^T \frac{\partial z}{\partial v}(t, 2\pi) \overline{z(t, 2\pi)}\, dt \bigg). \tag{75}$$

Then, thanks to an integration by parts, one has

$$\Re\bigg(\int_Q \frac{\partial^2 z}{\partial v^2} 2\alpha_v \frac{\partial \bar z}{\partial v} \bigg) = \Re\bigg(\int_0^T \bigg(\alpha_v(t, 2\pi) \bigg| \frac{\partial z}{\partial v}(t, 2\pi) \bigg|^2 - \alpha_v(t, 0) \bigg| \frac{\partial z}{\partial v}(t, 0) \bigg|^2 \bigg) dt \bigg) - \Re\bigg(\int_Q \bigg| \frac{\partial z}{\partial v} \bigg|^2 \alpha_{vv}\, dv\, dt \bigg),$$

where the boundary term vanishes thanks to (74) and the 2π-periodicity of the function β'. Thus

$$\Re\bigg(\int_Q \frac{\partial^2 z}{\partial v^2} 2\alpha_v \frac{\partial \bar z}{\partial v} \bigg) = -\int_Q \bigg| \frac{\partial z}{\partial v} \bigg|^2 \alpha_{vv}. \tag{76}$$

Finally, thanks to an integration by parts, one has

$$\Re\bigg(-\int_Q \frac{\partial^2 z}{\partial v^2} ipv \bar z \bigg) = 2\pi p \Im\bigg(\int_0^T \frac{\partial z}{\partial v}(t, 2\pi) \overline{z(t, 2\pi)}\, dt \bigg) - p \Im\bigg(\int_Q \frac{\partial z}{\partial v} \bar z\, dv\, dt \bigg). \tag{77}$$

Terms concerning $(\alpha_t - \alpha_v^2)z$: First, one has

$$\Re\left(\int_Q \left(\alpha_t - \alpha_v^2\right)z\frac{\partial\bar{z}}{\partial t}\right) = -\int_Q \frac{1}{2}\left(\alpha_t - \alpha_v^2\right)_t|z|^2, \tag{78}$$

thanks to an integration by parts in the time variable. The boundary terms at $t = 0$ and $t = T$ vanish because, thanks to (70), (66), (67),

$$\left|\left(\alpha_t - \alpha_v^2\right)|z|^2\right| \leqslant \frac{1}{[t(T-t)]^2}e^{\frac{-M}{t(T-t)}}\left|M(T-2t)\beta + (M\beta')^2\right||g|^2$$

tends to zero when $t \to 0$ and $t \to T$, for every $v \in [0, 2\pi]$. Then, one has

$$\Re\left(-\int_Q \left(\alpha_t - \alpha_v^2\right)z2\alpha_v\frac{\partial\bar{z}}{\partial v}\right) = \int_Q \left[\left(\alpha_t - \alpha_v^2\right)\alpha_v\right]_v|z|^2, \tag{79}$$

thanks to an integration by parts in the space variable. The boundary terms vanish thanks to (74) and the 2π-periodicity of the functions β and β'. Finally, one has

$$\Re\left(\int_Q \left(\alpha_t - \alpha_v^2\right)zipv\bar{z}\right) = 0. \tag{80}$$

Putting together (73), (75)–(80) and noticing that the boundary terms in (75) and (77) compensate each other, we get

$$\int_Q -\left|\frac{\partial z}{\partial v}\right|^2\alpha_{vv} - p\Im\left(\frac{\partial z}{\partial v}\bar{z}\right) + |z|^2\left\{-\frac{1}{2}\left(\alpha_t - \alpha_v^2\right)_t + \left[\left(\alpha_t - \alpha_v^2\right)\alpha_v\right]_v - \frac{1}{2}\alpha_{vv}^2\right\} \leqslant 0. \tag{81}$$

Now, we separate in (81) the terms in $(0, T) \times [(0, a) \cup (b, 2\pi)]$ and those in $(0, T) \times (a, b)$. First, one has

$$\begin{aligned}
&-\alpha_{vv}(t, v) = -\frac{M\beta''(v)}{t(T-t)} \geqslant \frac{C_1 M}{t(T-t)}, \quad \forall v \in [0, a] \cup [b, 2\pi],\ \forall t \in (0, T),\\
&\left|\alpha_{vv}(t, v)\right| = \left|\frac{M\beta''(v)}{t(T-t)}\right| \leqslant \frac{C_2 M}{t(T-t)}, \quad \forall v \in [a, b],\ \forall t \in (0, T),
\end{aligned} \tag{82}$$

where $C_1 := \min\{-\beta''(v);\ v \in [0, a] \cup [b, 2\pi]\}$ is positive thanks to (69) and $C_2 := \sup\{|\beta''(v)|;\ v \in [a, b]\}$. Then, one has

$$\begin{aligned}
&-\frac{1}{2}\left(\alpha_t - \alpha_v^2\right)_t + \left[\left(\alpha_t - \alpha_v^2\right)\alpha_v\right]_v - \frac{1}{2}\alpha_{vv}^2\\
&\quad = \frac{1}{(t(T-t))^3}\left\{M\left(T^2 - 5Tt + 5t^2\right) - M^2\left[(T-2t)\beta''\beta' + \frac{t(T-t)\beta''}{2}\right] - 3M^3\beta''\beta'^2\beta\right\}.
\end{aligned}$$

Thus, using (68) and (69), there exists $M_1 = M_1(T, \beta) > 0$, $C_3 = C_3(T, \beta) > 0$, $C_4 = C_4(T, \beta) > 0$, such that, for every $M \geqslant M_1$ and $t \in (0, T)$,

$$\begin{aligned}
&-\frac{1}{2}\left(\alpha_t - \alpha_v^2\right)_t + \left[\left(\alpha_t - \alpha_v^2\right)\alpha_v\right]_v - \frac{1}{2}\alpha_{vv}^2 \geqslant \frac{M^3 C_3}{[t(T-t)]^3}, \quad \forall v \in [0, a] \cup [b, 2\pi],\\
&\left|-\frac{1}{2}\left(\alpha_t - \alpha_v^2\right)_t + \left[\left(\alpha_t - \alpha_v^2\right)\alpha_v\right]_v - \frac{1}{2}\alpha_{vv}^2\right| \leqslant \frac{M^3 C_4}{[t(T-t)]^3}, \quad \forall v \in [a, b].
\end{aligned} \tag{83}$$

Using (81), (82) and (83), we get, for every $M \geqslant M_1$,

$$\begin{aligned}
&\int_0^T \int_{(0,a)\cup(b,2\pi)} \frac{C_1 M}{t(T-t)}\left|\frac{\partial z}{\partial v}\right|^2 - p\Im\left(\frac{\partial z}{\partial v}\bar{z}\right) + \frac{C_3 M^3}{(t(T-t))^3}|z|^2\\
&\quad \leqslant \int_0^T \int_{(a,b)} \frac{C_2 M}{t(T-t)}\left|\frac{\partial z}{\partial v}\right|^2 + p\Im\left(\frac{\partial z}{\partial v}\bar{z}\right) + \frac{C_4 M^3}{(t(T-t))^3}|z|^2.
\end{aligned} \tag{84}$$

Let $M_2 = M_2(T, \beta, p)$ be defined by

$$M_2 := \frac{T^2\sqrt{|p|}}{4(2C_1C_3)^{1/4}}. \tag{85}$$

From now on, we take

$$M = M(T, \beta, p) := \max\bigl(1, M_1(T, \beta), M_2(T, \beta, p)\bigr). \tag{86}$$

We have

$$\left|p\Im\left(\frac{\partial z}{\partial v}\bar{z}\right)\right| \leqslant \frac{1}{2}\frac{C_3M^3}{(t(T-t))^3}|z|^2 + \frac{1}{2}\frac{(t(T-t))^3}{C_3M^3}p^2\left|\frac{\partial z}{\partial v}\right|^2 \leqslant \frac{1}{2}\frac{C_3M^3}{(t(T-t))^3}|z|^2 + \frac{C_1M}{t(T-t)}\left|\frac{\partial z}{\partial v}\right|^2, \tag{87}$$

because

$$\frac{1}{2}\frac{(t(T-t))^3}{C_3M^3}p^2 = \frac{C_1M}{t(T-t)}\frac{(t(T-t))^4p^2}{2C_1C_3M^4} \leqslant \frac{C_1M}{t(T-t)}\frac{(T^2/4)^4p^2}{2C_1C_3M^4} = \frac{C_1M}{t(T-t)}\frac{M_2^4}{M^4}$$

and $M \geqslant M_2$. Using (84) and (87), we get

$$\int_0^T \int_{(0,a)\cup(b,2\pi)} \frac{C_3M^3}{2(t(T-t))^3}|z|^2 \leqslant \int_0^T \int_{(a,b)} \frac{C_5M^3}{(t(T-t))^3}|z|^2 + \frac{C_6M}{t(T-t)}\left|\frac{\partial z}{\partial v}\right|^2 \tag{88}$$

where $C_5 = C_5(T, \beta) := C_4 + C_3/2$ and $C_6 = C_6(T, \beta) := C_2 + C_1$. Coming back to our original variables thanks to (70) the inequality (88) provides

$$\int_0^T \int_{(0,a)\cup(b,2\pi)} \frac{C_3M^3|g|^2e^{-2\alpha}}{2(t(T-t))^3} \leqslant \int_0^T \int_{(a,b)} \left(\frac{C_7M^3|g|^2}{(t(T-t))^3} + \frac{C_8M}{t(T-t)}\left|\frac{\partial g}{\partial v}\right|^2\right)e^{-2\alpha} \tag{89}$$

where $C_8 := C_8(T, \beta) = 2C_6$ and $C_7 = C_7(T, \beta) := C_5 + 2C_6\sup\{\beta'(v)^2;\ v \in [a, b]\}$. Thanks to (67), and the assumption $M \geqslant 1$ (see (86)) we have, for every $v \in [a, b]$,

$$\frac{C_7M^3}{(t(T-t))^3}e^{-2\alpha} \leqslant \frac{C_7M^3}{(t(T-t))^3}e^{-\frac{2M}{t(T-t)}} \leqslant C_9,$$

$$\frac{C_8M}{t(T-t)}e^{-2\alpha} \leqslant \frac{C_8t(T-t)}{M}\left(\frac{M}{t(T-t)}\right)^2 e^{-\frac{2M}{t(T-t)}} \leqslant C_{10}t(T-t)$$

where $C_9 = C_9(T, \beta) := C_7\sup\{x^3e^{-2x};\ x \in \mathbb{R}_+\}$ and $C_{10} = C_{10}(T, \beta) := C_8\sup\{x^2e^{-2x};\ x \in \mathbb{R}_+\}$. Therefore, using the two previous inequalities and (89), we get

$$\int_0^T \int_{(0,a)\cup(b,2\pi)} \frac{C_3M^3|g|^2e^{-2\alpha}}{2(t(T-t))^3} \leqslant \int_0^T \int_{(a,b)} \left(C_9|g|^2 + C_{10}t(T-t)\left|\frac{\partial g}{\partial v}\right|^2\right). \tag{90}$$

Now, let us prove that the right-hand side of the previous inequality can be bounded by a first order term in g on $(0, T) \times (a_2, b_2)$. We consider $\rho \in C^\infty(\mathbb{R}, \mathbb{R}_+)$ 2π-periodic, such that

$$\rho \equiv 1 \quad \text{on } (a, b), \tag{91}$$

$$\rho \equiv 0 \quad \text{on } (0, a_2) \cup (b_2, 2\pi). \tag{92}$$

Multiplying the first equation of (63) by $\bar{g}\rho t(T-t)$, integrating over $(0, T) \times (0, 2\pi)$ and considering the real part of the resulting equality, we get

$$\int_0^T \int_{(0,2\pi)} \left(\frac{1}{2}\frac{d}{dt}\bigl[|g|^2\bigr]\rho t(T-t) - \Re\left(\frac{\partial^2 g}{\partial v^2}\bar{g}\right)\rho t(T-t)\right)dv\,dt = 0. \tag{93}$$

Performing integrations by parts with respect to the space and time variables, we get

$$\int_0^T \int_{(0,2\pi)} \frac{1}{2}\frac{d}{dt}\big[|g|^2\big]\rho t(T-t)\,dv\,dt = -\int_0^T \int_{(0,2\pi)} \frac{1}{2}|g|^2\rho(T-2t)\,dv\,dt, \tag{94}$$

$$\begin{aligned}-\Re\int_0^T \int_{(0,2\pi)} \frac{\partial^2 g}{\partial v^2}\bar{g}\rho t(T-t)\,dv\,dt &= \Re\int_0^T \int_{(0,2\pi)} \left|\frac{\partial g}{\partial v}\right|^2 \rho t(T-t) + \frac{\partial g}{\partial v}\bar{g}\rho' t(T-t)\,dv\,dt \\ &= \int_0^T \int_{(0,2\pi)} \left|\frac{\partial g}{\partial v}\right|^2 \rho t(T-t) - \frac{1}{2}|g|^2\rho'' t(T-t)\,dv\,dt. \end{aligned} \tag{95}$$

Indeed, the boundary terms at $t=0$ and $t=T$ in (94) vanish thanks to the factor $t(T-t)$ and the boundary terms at $v=0$, $v=2\pi$ in (95) vanish thanks to the boundary conditions satisfied by g and the 2π-periodicity of the function ρ. Thanks to (93), (94) and (95), we get

$$\int_0^T \int_{(0,2\pi)} \left(-\frac{1}{2}|g|^2\rho(T-2t) + \left|\frac{\partial g}{\partial v}\right|^2 \rho t(T-t) - \frac{1}{2}|g|^2\rho'' t(T-t)\right) dv\,dt = 0. \tag{96}$$

Thus, using (91), (92) and (96), we get

$$\begin{aligned}\int_0^T \int_{(a,b)} \left|\frac{\partial g}{\partial v}\right|^2 t(T-t)\,dv\,dt &\leqslant \int_0^T \int_{(0,2\pi)} \left|\frac{\partial g}{\partial v}\right|^2 \rho t(T-t)\,dv\,dt \\ &= \int_0^T \int_{(0,2\pi)} \frac{1}{2}|g|^2\big[\rho(T-2t) + \rho'' t(T-t)\big]\,dv\,dt \leqslant C_{11}\int_0^T \int_{(a_2,b_2)} |g|^2\,dv\,dt,\end{aligned}$$

where $C_{11} = C_{11}(T,\rho) := T\|\rho\|_{L^\infty} + \frac{T^2}{2}\|\rho''\|_{L^\infty}$. The previous inequality and (90) lead to

$$\int_0^T \int_{(0,a)\cup(b,2\pi)} \frac{C_3 M^3 |g|^2 e^{-2\alpha}}{(t(T-t))^3}\,dv\,dt \leqslant \int_0^T \int_{(a_2,b_2)} C_{12}|g|^2\,dv\,dt, \tag{97}$$

where $C_{12} = C_{12}(T,\beta,\rho) := 2[C_9 + C_{10}C_{11}]$. We have

$$t(T-t) \geqslant \frac{2T^2}{9}, \quad \forall t \in \left[\frac{T}{3}, \frac{2T}{3}\right],$$

$$t(T-t) \leqslant \frac{T^2}{4}, \quad \forall t \in \left[\frac{T}{3}, \frac{2T}{3}\right],$$

thus

$$\frac{e^{-2\alpha(t,v)}}{(t(T-t))^3} \geqslant \frac{e^{-9c_3 M/T^2}}{(T^2/4)^3}, \quad \forall v \in [0,a]\cup[b,2\pi],\ \forall t \in (T/3, 2T/3),$$

where $c_3 := \sup\{\beta(v);\ v \in [0,a]\cup[b,2\pi]\}$. Thus (97) leads to

$$\frac{C_3 M^3}{(T^2/4)^3} e^{-\frac{9c_3 M}{T^2}} \int_{T/3}^{2T/3} \int_{(0,a)\cup(b,2\pi)} |g|^2\,dv\,dt \leqslant \int_0^T \int_{(a_2,b_2)} C_{12}|g|^2\,dv\,dt. \tag{98}$$

Adding the same quantity in both sides and using the inclusion $(a,b)\subset(a_2,b_2)$ (see (65)), we get

$$\frac{C_3M^3}{(T^2/4)^3}e^{-\frac{9c_3M}{T^2}}\int_{T/3}^{2T/3}\int_{(0,2\pi)}|g|^2\,dv\,dt\leqslant\left(C_{12}+\frac{C_3M^3}{(T^2/4)^3}e^{-\frac{9c_3M}{T^2}}\right)\int_0^T\int_{(a_2,b_2)}|g|^2\,dv\,dt, \tag{99}$$

which can also be written

$$\int_{T/3}^{2T/3}\int_{(0,2\pi)}|g|^2\,dv\,dt\leqslant\left(C_{12}\frac{(T^2/4)^3}{C_3M^3}e^{\frac{9c_3M}{T^2}}+1\right)\int_0^T\int_{(a_2,b_2)}|g|^2\,dv\,dt. \tag{100}$$

Using the decreasing property of $t\mapsto\|g(t)\|_{L^2}$, and the inequality $M\geqslant 1$ (see (86)) we get

$$\int_{(0,2\pi)}|g(T,v)|^2\,dv\leqslant\frac{3}{T}e^{\frac{9c_3M}{T^2}}\left(C_{12}\frac{(T^2/4)^3}{C_3}+1\right)\int_0^T\int_{(a_2,b_2)}|g|^2\,dv\,dt, \tag{101}$$

which gives the conclusion, thanks to (86). □

4. Conclusion and open problems

In this article, we have proved the null controllability of the 2D Kolmogorov equation with a control domain ω that is

- either the complementary of a strip in the whole space ($\Omega=\mathbb{R}^2$),
- or a strip in a square domain ($\Omega=(0,2\pi)^2$).

4.1. On the whole space

Our result on the control of the Kolmogorov equation in the whole space with control in the exterior of a finite band implies in particular the controllability with control in the exterior of any bounded domain. In this sense the result coincides with the well known one on the heat equation that we recalled in Theorem 5.

However, in the case of the heat equation on the whole space, it is well known that there are other geometric situations in which the null controllability holds (see, for instance, [12] and [19]). It would be desirable to explore this issue further for the Kolmogorov equation.

4.2. On bounded domains

In the case of the square domain, the null controllability of the Kolmogorov equation in any time $T>0$, with an arbitrarily small control domain $\omega\subset(0,2\pi)^2$ (i.e. the analogue of Theorem 4 for the Kolmogorov equation) stays an open problem. Is the hypoellipticity or the hypocoercivity property of the Kolmogorov equation sufficient to prove the same controllability result as for the heat equation?

For more general domains Ω, the analysis of the control domains ω for which the null controllability holds is also an open problem. In that cases, one has additional difficulties. Which boundary conditions ensure the hypocoercivity of the Kolmogorov equation? How to use this hypocoercivity in the proof of the null controllability? In other words, what are the analogues of Lemmas 1 and 2 when the Fourier technic cannot be used?

In the case of bounded domains Ω, for the Kolmogorov equation, as far as we know, the only existing result is the one we have given above for the square domain. However, we can use the result above on the control of the Kolmogorov equation in the whole space to derive null controllability results for the same equation in an arbitrary domain Ω. This can be done by the classical extension–restriction argument, that we recall for brevity. Given an initial datum to be controlled in Ω, we extend it by zero to the whole space and then build a control for the Cauchy problem in the whole space with support in the exterior of Ω. The restriction of the solution of the controlled Cauchy problem to the boundary of Ω, $\partial\Omega$, yields a boundary control for the Kolmogorov equation in Ω. Note however that this

argument, that applies in any bounded domain Ω, yields controls that are distributed everywhere on the boundary of Ω. Whether the same holds with controls localized in some subset of the boundary or more general internal controls than the ones we have built for the equation in the square are interesting open problems.

4.3. More general hypoelliptic operators

Finally, it would also be of interest to analyze to which extent the results of this paper extend to more general linear hypoelliptic equations as those analyzed in [13] and the more general linear and nonlinear kinetic models in [22].

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