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Uniqueness of solutions to degenerate elliptic problems with unbounded coefficients

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Abstract

We study well-posedness of the Dirichlet problem for linear degenerate elliptic equations under mild assumptions on the coefficients (in particular, they can be unbounded). We provide sufficient conditions both for uniqueness and nonuniqueness of solutions, which rely on the construction of suitable sub- and supersolutions to certain auxiliary problems.

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1. Introduction

In this paper we address linear *degenerate* elliptic equations of the form

$$\mathcal{L}u - cu = f \quad \text{in } \Omega. \tag{1.1}$$

Here $\Omega \subseteq \mathbb{R}^n$ is an open connected, possibly unbounded set with boundary $\partial\Omega$ and c, f are given functions, $c \geqslant 0$ in Ω; the operator $\mathcal{L}$ is formally defined as follows:

$$\mathcal{L}u \equiv \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i}.$$

We assume

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqslant 0 \quad \text{for any } x \in \Omega,\ (\xi_1, \dots, \xi_n) \in \mathbb{R}^n;$$

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in particular, for equations *degenerating at the boundary* we have

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j > 0 \quad \text{for any } x \in \Omega,\ (\xi_1, \ldots, \xi_n) \neq 0.$$

The coefficients a_{ij}, b_i, c and the function f may be *unbounded* (see assumptions (H_2)–(H_3) below).

We study existence and uniqueness of solutions to the Dirichlet boundary value problem for Eq. (1.1). Special attention will be paid to the case of bounded solutions.

(i) In the case of *bounded coefficients* much work has been devoted to this classical problem, using both analytical methods and stochastic calculus. For equations degenerating at the boundary, it was early recognized that the Dirichlet problem may be well posed prescribing boundary data only on a portion of the boundary, which depends on the behavior of the coefficients of the operator $\mathcal{L}$ ([19]; see also [26]). Introducing a classification of the boundary points based on such behavior, a general formulation of the Dirichlet problem for Eq. (1.1) was given in the pioneering paper [9]; existence, uniqueness and a priori estimates of solutions to the problem were also proved under suitable assumptions. A comprehensive account of such results can be found in [25] (see also [10,24]).

Clearly, uniqueness of solutions to the Dirichlet problem for Eq. (1.1) is related to the validity of the maximum principle for degenerate elliptic operators. Assume $a_{ij} \in C^2(\overline{\Omega})$, $b_i \in C^1(\overline{\Omega})$, $D^\alpha a_{ij} \in L^\infty(\Omega)$ for $|\alpha| \leqslant 2$, $D^\alpha b_i \in L^\infty(\Omega)$ for $|\alpha| \leqslant 1$; let $u \in C^2(\overline{\Omega})$ satisfy $\mathcal{L}u \geqslant cu$ in Ω. For any $x_0 \in \Omega$ such that $u(x_0) = \sup_\Omega u > 0$ consider the *propagation set* $\mathcal{P}(x_0) := \{x \in \Omega \mid u(x) = u(x_0)\}$. As proved in [32], $\mathcal{P}(x_0)$ contains the closure (in the relative topology) of the set $\mathcal{P}'(x_0)$ consisting of points, which can be joined to x_0 by a finite number of *subunitary* and/or *drift trajectories* (see [4,7,25,30] for the proof in particular cases; see also [1]). By a local version of the same result a sufficient condition for the uniqueness of solutions to the Dirichlet problem, as formulated in [9], can be derived (see [7]).

Remarkably, the above mechanism for propagation of maxima of subsolutions is closely related to the Markov process corresponding to the operator $\mathcal{L}$. In fact, the set $\mathcal{P}'(x_0)$ coincides with the *support* of this process, namely with the closure of the collection of all trajectories of a Markovian particle, starting at x_0, with generator $\mathcal{L}$ (see [31,32]). Hence, roughly speaking, the above uniqueness criterion for the Dirichlet problem can be rephrased by saying that the boundary data have to be specified only at *attainable* boundary points (see [11,12]).

The same idea underlies the so-called *refined maximum principle* in [3]. Consider the minimal positive solution U_0 of the *first exit time equation*

$$\mathcal{L}U = -1 \quad \text{in } \Omega \tag{1.2}$$

(*e.g.*, see [15]); consider those point of $\partial\Omega$ where U_0 can be prolonged to zero. It was proved in [3] that a sub- and a supersolution of Eq. (1.1) degenerating at the boundary are ordered in Ω, if they are ordered at these points; as a consequence, prescribing the boundary data at such points is sufficient for the uniqueness of the Dirichlet problem. Observe that prolonging U_0 to zero is possible at any point of $\partial\Omega$ where a *local barrier* for Eq. (1.2) exists, or, equivalently, at any *attracting* point of $\partial\Omega$ (see [20]; see also Definition 5.1 and Proposition 5.3 below).

Before discussing the results of the present paper, it is worth recalling the main assumptions made in the above literature:

- boundedness of the coefficients a_{ij}, b_i, c is always assumed;
- in [9,10,24,25] Ω is bounded; $\partial\Omega = \partial\overline{\Omega}$ is a finite union of smooth manifolds; $a_{ij} \in C^2(\overline{\Omega})$, $b_i \in C^1(\overline{\Omega})$, $c \in C(\overline{\Omega})$, $\min_{\overline{\Omega}} c > 0$;
- in [32] $a_{ij} \in C^2(\overline{\Omega})$, $b_i \in C^1(\overline{\Omega})$, $D^\alpha a_{ij} \in L^\infty(\Omega)$ for $|\alpha| \leqslant 2$, $D^\alpha b_i \in L^\infty(\Omega)$ for $|\alpha| \leqslant 1$. Moreover, subsolutions are meant in the classical sense;
- in [3] uniform ellipticity of the operator $\mathcal{L}$ is assumed.

(ii) In the present study the above assumptions are relaxed in several respects. In particular, as already remarked, we allow the coefficients of Eq. (1.1) to be *unbounded* (motivations for this hypothesis come from many problems; *e.g.*, think of the Ornstein–Uhlenbeck process). Elliptic equations with unbounded coefficients have been widely investigated in recent years—mostly in the case $\Omega = \mathbb{R}^n$—both by analytical and by probabilistic methods (see [5,23] and references therein). Also the corresponding parabolic equations have attracted much attention, particularly studying uniqueness of solutions to the Cauchy problem (*e.g.*, see [8,14,33] and references therein; see also [18,28] for different initial–boundary value problems).

We always think of the boundary $\partial\Omega$ as the disjoint union of the *regular boundary* $\mathcal{R}$ and the *singular boundary* $\mathcal{S}$ (see assumption (H_1)). In view of assumptions (H_2)–(H_3) below, it is natural to prescribe the Dirichlet boundary condition on $\mathcal{R}$. This leads to the problem

$$\begin{cases} \mathcal{L}u - cu = f & \text{in } \Omega, \\ u = g & \text{on } \mathcal{R}, \end{cases} \tag{1.3}$$

where the coefficients of $\mathcal{L}$ and the function c can either vanish or diverge, or need not have a limit, when $dist(x,\mathcal{S}) \to 0$ and/or $|x| \to \infty$, if Ω is unbounded. In addition, ellipticity is possibly lost in Ω and/or when $dist(x,\mathcal{S}) \to 0$, and/or when $|x| \to \infty$, if Ω is unbounded.

The assumptions concerning the regular boundary $\mathcal{R}$ and the singular boundary $\mathcal{S}$ are summarized as follows:

$$(H_1) \quad \begin{cases} \text{(i) } \partial\Omega = \mathcal{R} \cup \mathcal{S},\ \mathcal{R} \cap \mathcal{S} = \emptyset,\ \mathcal{S} \neq \emptyset; \\ \text{(ii) } \mathcal{R} \subseteq \partial\overline{\Omega} \text{ is open, } \Omega \text{ satisfies the outer sphere condition at } \mathcal{R}. \end{cases}$$

It is natural to choose $\mathcal{R}$ as *the largest subset* of $\partial\Omega$ where ellipticity of the operator $\mathcal{L}$ holds (see assumptions (H_2)(ii), (H_3)(iii) below), as we do in the following. Observe that no regularity assumption concerning $\mathcal{S}$ is made (see (H_1)(ii)).

Our nonuniqueness results only address the case of degeneracy at the boundary (see Section 2.1). To prove these results, we always assume the following about the coefficients a_{ij}, b_i and the functions c, f, g:

$$(H_2) \quad \begin{cases} \text{(i) } a_{ij} = a_{ji} \in C^{1,1}(\Omega \cup \mathcal{R}),\ b_i \in C^{0,1}(\Omega \cup \mathcal{R})\ (i,j = 1,\dots,n); \\ \text{(ii) } \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j > 0 \text{ for any } x \in \Omega \cup \mathcal{R} \text{ and } (\xi_1,\dots,\xi_n) \neq 0; \\ \text{(iii) } c \in C(\Omega \cup \mathcal{R}),\ c \geqslant 0; \\ \text{(iv) } f \in C(\Omega); \\ \text{(v) } g \in C(\mathcal{R}). \end{cases}$$

On the other hand, the uniqueness results in Section 2.2 hold for the general degenerate equation (1.1). In this case we replace assumption (H_2) by the following:

$$(H_3) \quad \begin{cases} \text{(i) } a_{ij} = a_{ji} \in C^{1,1}(\Omega \cup \mathcal{R}),\ \sigma_{i,j} \in C^1(\Omega), \\ \qquad b_i \in C^{0,1}(\Omega \cup \mathcal{R})\ (i,j = 1,\dots,n); \\ \text{(ii) } \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geqslant 0 \text{ for any } x \in \Omega \text{ and } (\xi_1,\dots,\xi_n) \in \mathbb{R}^n; \\ \text{(iii) } \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j > 0 \text{ for any } x \in \mathcal{R} \text{ and } (\xi_1,\dots,\xi_n) \neq 0; \\ \text{(iv) either } c > 0 \text{ in } \Omega \cup \mathcal{R}, \text{ or } c \geqslant 0,\ c + \sum_{i=1}^n \sigma_{ji}^2 > 0 \text{ in } \Omega \cup \mathcal{R} \\ \qquad \text{for some } j = 1,\dots,n \text{ and } c \in C(\Omega \cup \mathcal{R}); \\ \text{(v) } f \in C(\Omega); \\ \text{(vi) } g \in C(\mathcal{R}); \end{cases}$$

here $\sigma \equiv (\sigma_{ij})$ denotes the square root of the matrix $A \equiv (a_{ij})$ (namely, $A(x) = \sigma(x)\sigma(x)^T$; $x \in \Omega \cup \mathcal{R}$). Assumption (H_3) (in particular, (H_3)(iv)) enables us to use comparison results for *viscosity* sub- and supersolutions to second-order degenerate elliptic equations, via an equivalence result proved in [16] (see Propositions 2.3–2.4).

(iii) The results of the paper can be described as follows. First we prove sufficient conditions for nonuniqueness of solutions to problem (1.3), which require the existence of suitable supersolutions to the first exit time problem:

$$\begin{cases} \mathcal{L}U = -1 & \text{in } \Omega, \\ U = 0 & \text{on } \mathcal{R} \end{cases} \tag{1.4}$$

(in particular, see Theorem 2.5 below). Nonuniqueness depends on the need of prescribing the value of the solution of problem (1.3) at some point of the singular boundary $\mathcal{S}$, or at infinity if Ω is unbounded. Therefore, if uniqueness fails, it is natural to try and recover it by assigning boundary data on some subset $\mathcal{S}_1 \subseteq \mathcal{S}$ and/or a *condition at infinity*, if Ω is unbounded. Hence we study the problems:

$$\begin{cases} \mathcal{L}u - cu = f & \text{in } \Omega, \\ u = g & \text{on } \mathcal{R} \cup \mathcal{S}_1, \end{cases} \tag{1.5}$$

respectively

$$\begin{cases} \mathcal{L}u - cu = f & \text{in } \Omega, \\ u = g & \text{on } \mathcal{R} \cup \mathcal{S}_1, \\ \lim_{|x|\to\infty} u(x) = L & (L \in \mathbb{R}) \end{cases} \tag{1.6}$$

(where possibly $\mathcal{S}_1 = \emptyset$; see (2.9)). The following assumption will be made:

$$(H_4) \quad \begin{cases} \text{(i) } \mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2,\ \mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset; \\ \text{(ii) } \mathcal{S}_j = \bigcup_{k=1}^{k_j} \mathcal{S}_j^k, \text{ where every } \mathcal{S}_j^k \text{ is connected and, if } k_j \geqslant 2, \\ \quad \overline{\mathcal{S}_j^k} \cap \overline{\mathcal{S}_j^l} = \emptyset \text{ for any } k, l = 1, \ldots, k_j,\ k \neq l\ (k_j \in \mathbb{N};\ j = 1, 2). \end{cases}$$

We prove sufficient conditions for uniqueness of solutions to problems (1.5) and (1.6), extending the classical Phragmèn–Lindelöf principle to the present degenerate case (see Propositions 2.10, 2.11). Such conditions depend on the existence of subsolutions to the *homogeneous problem*:

$$\begin{cases} \mathcal{L}U = cU & \text{in } \Omega, \\ U = 0 & \text{on } \mathcal{R} \end{cases} \tag{1.7}$$

and on their behavior as $dist(x, \mathcal{S}_2) \to 0$ (*e.g.*, see Theorem 2.13).

Let us mention that the main step in the nonuniqueness proof concerning problem (1.3) is to prove existence of nontrivial solutions to the homogeneous problem (1.7) (see Section 2.1). Also observe that existence for problem (1.5) implies nonuniqueness for problem (1.3), if $\mathcal{S}_1 \neq \emptyset$; similarly for problems (1.6) and (2.9) below.

In Section 5 we apply our general results to some examples. The applicability of these results relies on the actual construction of suitable super- and subsolutions to problems (1.4), respectively (1.7) (or (2.22) below; see Section 2.2); in turn, this depends both on the behavior of the coefficients of the operator $\mathcal{L}$ at the boundary and on properties of the boundary itself (*e.g.*, the Hausdorff dimension of the subset $\mathcal{S}_2$). Concerning this point, we refer the reader to the paper [27].

2. Mathematical framework and results

Let us first make precise the definition of solution to the problems introduced above. Denote by $\mathcal{L}^*$ the formal adjoint of the operator $\mathcal{L}$, namely:

$$\mathcal{L}^* v \equiv \sum_{i,j=1}^{n} \frac{\partial^2(a_{ij}v)}{\partial x_i \partial x_j} - \sum_{i=1}^{n} \frac{\partial(b_i v)}{\partial x_i}.$$

Definition 2.1. By a *subsolution* to Eq. (1.1) we mean any function $u \in C(\Omega)$ such that

$$\int_\Omega u\{\mathcal{L}^*\psi - c\psi\}\,dx \geqslant \int_\Omega f\psi\,dx \tag{2.1}$$

for any $\psi \in C_0^\infty(\Omega)$, $\psi \geqslant 0$. *Supersolutions* of (1.1) are defined replacing "$\geqslant$" by "$\leqslant$" in (2.1). A function u is a *solution* of (1.1) if it is both a sub- and a supersolution.

Definition 2.2. Let $\mathcal{R} \subseteq \mathcal{E} \subseteq \partial\Omega$, $g \in C(\mathcal{E})$. By a *subsolution* to the problem

$$\begin{cases} \mathcal{L}u - cu = f & \text{in } \Omega, \\ u = g & \text{on } \mathcal{E} \end{cases} \tag{2.2}$$

we mean any function $u \in C(\Omega \cup \mathcal{E})$ such that:

(i) u is a subsolution of Eq. (1.1);
(ii) $u \leqslant g$ on $\mathcal{E}$.

Supersolutions and *solutions* of (2.2) are defined similarly.

Let us mention the following result (for the definition of viscosity subsolution of Eq. (1.1), see *e.g.* [6,16]).

Proposition 2.3. *Let either assumption* (H_2) *or* (H_3) *hold*; *let* $u \in C(\Omega)$. *Then the following statements are equivalent*:

(i) *u is a subsolution of Eq.* (1.1);
(ii) *u is a viscosity subsolution of Eq.* (1.1).

Proof. (i) $\Rightarrow$ (ii): Under the present regularity assumptions the square root σ of the matrix A is in $C^1(\Omega)$ (actually, assumptions (H_2)(i) and (H_2)(ii) imply $\sigma_{ij} \in C^{1,1}(\Omega)$; see [12, Ch. 6, Lemma 1.1]). Hence the claim follows from Theorem 2 in [16].

(ii) $\Rightarrow$ (i): Follows from Theorem 1 in [16], due to the present regularity assumptions. □

In view of the above proposition, we obtain the following comparison result (see [21,22] for a related maximum principle).

Proposition 2.4. *Let either assumption* (H_2) *or* (H_3) *hold*; *let* Ω_1 *be any open bounded subset of* Ω *such that* $\overline{\Omega}_1 \subseteq \Omega \cup \mathcal{R}$. *Let* $\underline{u} \in C(\overline{\Omega}_1)$ *be a subsolution,* $\bar{u} \in C(\overline{\Omega}_1)$ *a supersolution of the equation*

$$\mathcal{L}u - cu = f \quad \text{in } \Omega_1. \tag{2.3}$$

If $\underline{u} \leqslant \bar{u}$ *on* $\partial\Omega_1$, *then* $\underline{u} \leqslant \bar{u}$ *on* $\overline{\Omega}_1$.

Proof. By Proposition 2.3 $\underline{u}$ is a viscosity subsolution, $\bar{u}$ a viscosity supersolution of Eq. (2.3). Then the claim follows:

(a) by the comparison results in Section V.1 of [17], if (H_2) holds;
(b) by Theorem II.2 in [17], if (H_3) holds and $c > 0$ in $\Omega \cup \mathcal{R}$;
(c) by a slight refinement of Theorem 3.3 in [2], if (H_3) holds and $c \geqslant 0$, $c + \sum_{i=1}^n \sigma_{ji}^2 > 0$ in $\Omega \cup \mathcal{R}$ for some $j = 1, \ldots, n$. □

2.1. Existence and nonuniqueness results

Concerning problem (1.3), we shall prove the following

Theorem 2.5. *Let assumptions* (H_1)–(H_2) *be satisfied*; *suppose* $c \in L^\infty(\Omega)$. *Let there exist a supersolution* V *of problem* (1.4) *such that*

$$\inf_{\Omega\cup\mathcal{R}} V = 0 < \inf_{\mathcal{R}} V. \tag{2.4}$$

Then either no solutions, or infinitely many solutions of problem (1.3) *exist.*

The assumption $c \in L^\infty(\Omega)$ is necessary for the above theorem to hold (see Example (c) in Section 5.2). Let us also observe the following:

(a) if $c = 0$, in Theorem 2.5 we can assume V to be a supersolution of problem (1.7);
(b) if V is a supersolution of problem (1.4) bounded from below, then $\overline{V} := V - \inf_{\Omega\cup\mathcal{R}} V$ is a supersolution of the same problem with $\inf_{\Omega\cup\mathcal{R}} \overline{V} = 0$.

It is informative to outline the proof of Theorem 2.5. Suppose first Ω is bounded. The existence of a supersolution V of problem (1.4) satisfying (2.4) implies

$$\liminf_{dist(x,\mathcal{S})\to 0} V(x) = \inf_{\Omega\cup\mathcal{R}} V = 0 \tag{2.5}$$

(see Lemma 3.1). Then there exists a sequence $\{x_m\} \subseteq \Omega$ such that

$$\lim_{m\to\infty} dist(x_m, \mathcal{S}) = 0, \tag{2.6}$$

with the following property: for any $\beta \in \mathbb{R}$ there exists a bounded solution U_β of the homogeneous problem (1.7) such that

$$\lim_{m\to\infty} U_\beta(x_m) = \beta. \tag{2.7}$$

This gives the existence of infinitely many bounded solutions of the homogeneous problem (1.7) (see Proposition 3.4). Plainly, the existence of infinitely many nontrivial bounded solutions of (1.7) implies a corresponding nonuniqueness result for problem (1.3), if at least one solution of the latter exists (in this respect, see Proposition 2.7 below).

If Ω is unbounded, condition (2.4) implies either equality (2.5), or

$$\liminf_{|x|\to\infty} V(x) = \inf_{\Omega\cup\mathcal{R}} V = 0 \tag{2.8}$$

(see Lemma 3.2). In the latter case the limit (2.7) is attained along a *diverging* sequence $\{x_m\} \subseteq \Omega$, thus nonuniqueness depends on the absence of a "condition at infinity".

To rule out this possibility, it is natural to consider the problem:

$$\begin{cases} \mathcal{L}u - cu = f & \text{in } \Omega, \\ u = g & \text{on } \mathcal{R}, \\ \lim_{|x|\to\infty} u(x) = L & (L \in \mathbb{R}). \end{cases} \tag{2.9}$$

The following nonuniqueness result can be proved.

Theorem 2.6. *Let Ω be unbounded and assumptions* (H_1)–(H_2) *be satisfied*; *suppose $c \in L^\infty(\Omega)$. Let there exist a supersolution V of problem* (1.4) *such that*

$$\inf_{\Omega\cup\mathcal{R}} V = 0 < \min\Big\{\inf_{\mathcal{R}} V, \liminf_{|x|\to\infty} V(x)\Big\}. \tag{2.10}$$

Moreover, let there exist a positive supersolution F of the equation

$$\mathcal{L}u - cu = 0 \quad \text{in } \Omega \tag{2.11}$$

such that $\lim_{|x|\to\infty} F(x) = 0$. *Then either no solutions, or infinitely many solutions of problem* (2.9) *exist.*

The proof of Theorem 2.6 is analogous to that of Theorem 2.5. In this case the stricter inequality (2.10) implies the existence of a *bounded* sequence $\{x_m\} \subseteq \Omega$ satisfying (2.6), such that for any $\beta \in \mathbb{R}$ equality (2.7) holds. Now the bounded solution U_β of the homogeneous problem (1.7) satisfies the additional condition

$$\lim_{|x|\to\infty} U_\beta(x) = 0; \tag{2.12}$$

this follows from the properties of the function F, which plays the role of a *barrier at infinity*. This entails the existence of infinitely many bounded solutions to problem (2.9) with $f = g = L = 0$ (see Proposition 3.5), whence Theorem 2.6 follows.

Theorems 2.5 and 2.6 show that infinitely many solutions to problems (1.3), respectively (2.9) exist, if one does. Therefore the following existence results, combined with the above theorems, imply nonuniqueness for such problems.

Proposition 2.7. *Let assumptions* (H_1)–(H_2) *be satisfied*; *suppose $f \in L^\infty(\Omega)$, $g \in L^\infty(\mathcal{R})$. Let there exist a positive supersolution $F \in C(\Omega \cup \mathcal{R})$ of the equation*

$$\mathcal{L}u - cu = -1 \quad \text{in } \Omega. \tag{2.13}$$

Then there exists a solution of problem (1.3).

Remark 2.8. In connection with Proposition 2.7 observe that, if $c(x) \geqslant c_0 > 0$ for any $x \in \Omega \cup \mathcal{R}$, $F := 1/c_0$ is a bounded supersolution of Eq. (2.13).

Concerning problem (2.9), we have the following

Proposition 2.9. *Let Ω be unbounded and assumptions (H_1)–(H_2) be satisfied. Let $f \in L^\infty(\Omega)$, $g \in L^\infty(\mathcal{R})$, $c \in L^\infty(\Omega \setminus \overline{B_M})$ for some $M > 0$; if $\mathcal{R}$ is unbounded, suppose*

$$\lim_{|x|\to\infty} g(x) = L.$$

Let there exist a positive supersolution $F \in C(\Omega \cup \mathcal{R})$ of Eq. (2.13) *such that* $\lim_{|x|\to\infty} F(x) = 0$. *Then there exists a solution of problem* (2.9).

The proofs of Propositions 2.7, 2.9 make use of a local barrier at the points of $\mathcal{R}$ (which exists by assumptions (H_1)(ii), (H_2)(ii); *e.g.*, see [13]).

It is immediately seen that, if the supersolution F in the above statements is bounded, the solution u is bounded, too. Then by Propositions 3.4, 3.5 we obtain nonuniqueness in $L^\infty(\Omega)$ for problems (1.3), respectively (2.9).

Existence results analogous to Propositions 2.7, 2.9 hold for problems (1.5) and (1.6), respectively (however, see Proposition 5.2 and Example (c) in Section 5.1).

2.2. Comparison and uniqueness results

In this subsection we address uniqueness of solutions to problem (1.5). In the particular case $\mathcal{S}_1 = \emptyset$, $\mathcal{S}_2 = \mathcal{S}$ we recover uniqueness criteria for problem (1.3).

Set $B_r(\bar{x}) := \{|x - \bar{x}| < r\}$ $(\bar{x} \in \mathbb{R}^n)$, $B_r(0) \equiv B_r$. We shall prove the following Phragmèn–Lindelöf principle (*e.g.*, see [29] for the classical case, where ellipticity of the operator, smoothness of the coefficients and a classical notion of supersolution are assumed).

Proposition 2.10. *Let assumptions (H_1) and (H_4) hold, and either (H_2) or (H_3) be satisfied*; *suppose $\mathcal{S}_2 \neq \emptyset$. Let there exist a subsolution $Z \leqslant H < 0$ of problem* (1.7). *Let u be a subsolution of problem* (1.5) *with $f = g = 0$, such that*

$$\liminf_{dist(x,\mathcal{S}_2)\to 0} \frac{u(x)}{Z(x)} \geqslant 0. \tag{2.14}$$

If Ω is unbounded, assume also

$$\liminf_{|x|\to\infty} \frac{u(x)}{Z(x)} \geqslant 0. \tag{2.15}$$

Then $u \leqslant 0$ in Ω.

If Ω is unbounded and condition (2.14) is satisfied, the same conclusion of Proposition 2.10 holds true if we “prescribe the sign at infinity”. In fact, the following result can be proved.

Proposition 2.11. *Let Ω be unbounded, assumptions (H_1) and (H_4) hold, and either (H_2) or (H_3) be satisfied*; *suppose $\mathcal{S}_2 \neq \emptyset$. Let there exist a subsolution $Z \leqslant H < 0$ of problem* (1.7) *Let u be a subsolution of problem* (1.5) *with $f = g = 0$ such that*

$$\liminf_{dist(x,\mathcal{S}_2)\to 0} \frac{u(x)}{Z(x)} \geqslant 0, \qquad \limsup_{|x|\to\infty} u(x) \leqslant 0. \tag{2.16}$$

Then $u \leqslant 0$ in Ω.

Remark 2.12. In the above propositions we can replace condition (2.14) by the weaker assumption

$$\limsup_{\varepsilon\to 0} \left\{ \inf_{\mathcal{A}_2^\varepsilon \setminus \mathcal{S}} \frac{u}{Z} \right\} \geqslant 0, \tag{2.17}$$

where

$$\mathcal{A}_2^\varepsilon := \{x \in \overline{\Omega} \mid dist(x, \mathcal{S}_2) = \varepsilon\}$$

($\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 > 0$ suitably small). Similarly, condition (2.15) can be replaced by the weaker assumption

$$\limsup_{\varepsilon\to 0} \left\{ \inf_{[\Omega\cup\mathcal{R}]\cap\partial B_{\frac{1}{\varepsilon}}} \frac{u}{Z} \right\} \geqslant 0, \tag{2.18}$$

and the second inequality in (2.16) by

$$\liminf_{\varepsilon\to 0} \left\{ \sup_{[\Omega\cup\mathcal{R}]\cap\partial B_{\frac{1}{\varepsilon}}} u \right\} \leqslant 0. \tag{2.19}$$

In fact, the proofs of Propositions 2.10–2.11 will be given using assumptions (2.17)–(2.19) instead of (2.14)–(2.16).

Observe that, if Z is a subsolution of problem (1.7) bounded from above with $H_0 := \sup_{\Omega\cup\mathcal{R}} Z \geqslant 0$ and $M > H_0$, then $\overline{Z} := Z - M \leqslant H_0 - M < 0$ is a subsolution of the same problem. The same remark holds for problems (1.4) and (2.22) below.

The following uniqueness result is an immediate consequence of Proposition 2.10.

Theorem 2.13. *Let assumptions (H_1) and (H_4) hold, and either (H_2) or (H_3) be satisfied. Suppose $\mathcal{S}_2 \neq \emptyset$, $g \in C(\mathcal{R} \cup \mathcal{S}_1)$. Let there exist a subsolution $Z \leqslant H < 0$ of problem* (1.7). *Then*:

(i) *if Ω is bounded, there exists at most one solution u of problem* (1.5) *such that*

$$\lim_{dist(x,\mathcal{S}_2)\to 0} \frac{u(x)}{Z(x)} = 0; \tag{2.20}$$

(ii) *if Ω is unbounded, there exists at most one solution u of problem* (1.5) *such that*

$$\lim_{dist(x,\mathcal{S}_2)\to 0} \frac{u(x)}{Z(x)} = \lim_{|x|\to +\infty} \frac{u(x)}{Z(x)} = 0. \tag{2.21}$$

Remark 2.14. (i) It is easily seen that in the proof of Proposition 2.10 (thus in Theorem 2.13) the homogeneous problem (1.7) can be replaced by the *eigenvalue problem*:

$$\begin{cases} \mathcal{L}U = \mu U & \text{in } \Omega, \\ U = 0 & \text{on } \mathcal{R} \end{cases} \tag{2.22}$$

with $\mu \in [0, \inf_{\Omega\cup\mathcal{R}} c]$.

(ii) If $c(x) \geqslant c_0 > 0$ for any $x \in \Omega \cup \mathcal{R}$, we can replace problem (2.22) by (1.4), obtaining uniqueness results analogous to Theorem 2.13. In fact, let Z be a subsolution of problem (1.4); it is not restrictive to assume

$$Z \leqslant -\frac{1}{c_0} \quad \text{in } \Omega \cup \mathcal{R}.$$

Then by Definition 2.2 we have

$$\int_\Omega Z\mathcal{L}^*\psi\, dx \geqslant -\int_\Omega \psi\, dx \geqslant c_0 \int_\Omega Z\psi\, dx \geqslant \int_\Omega cZ\psi\, dx$$

for any $\psi \in C_0^\infty(\Omega)$, $\psi \geqslant 0$; moreover, $Z \leqslant 0$ on $\mathcal{R}$. Hence Z is a subsolution of problem (1.7); thus by Theorem 2.13 and the above remark (i) the claim follows.

Concerning problem (1.6), from Proposition 2.11 we obtain the following uniqueness result. The elementary proof is omitted.

Theorem 2.15. *Let Ω be unbounded, assumptions (H_1) and (H_4) hold, and either (H_2) or (H_3) be satisfied*; *suppose $\mathcal{S}_2 \neq \emptyset$, $g \in C(\mathcal{R} \cup \mathcal{S}_1)$. Let there exist a subsolution $Z \leqslant H < 0$ of problem* (1.7). *Then there exists at most one solution u of problem* (1.6) *such that condition* (2.20) *is satisfied.*

Uniqueness results in $L^\infty(\Omega)$ for problems (1.5), (1.6) follow immediately from those above, if the subsolution Z diverges as $dist(x,\mathcal{S}_2)\to 0$. We state below such consequences of Theorems 2.13 and 2.15.

Proposition 2.16. *Let assumptions (H_1) and (H_4) hold, and either (H_2) or (H_3) be satisfied. Suppose $\mathcal{S}_2\neq\emptyset$, $g\in C(\mathcal{R}\cup\mathcal{S}_1)$. Let there exist a subsolution $Z\leqslant H<0$ of problem* (1.7) *such that*

$$\lim_{dist(x,\mathcal{S}_2)\to 0} Z(x)=-\infty \tag{2.23}$$

if Ω is bounded, or

$$\lim_{dist(x,\mathcal{S}_2)\to 0} Z(x)=\lim_{|x|\to\infty} Z(x)=-\infty \tag{2.24}$$

if Ω is unbounded. Then there exists at most one solution $u\in L^\infty(\Omega)$ of problem (1.5).

Proposition 2.17. *Let Ω be unbounded, assumptions (H_1) and (H_4) hold, and either (H_2) or (H_3) be satisfied; suppose $\mathcal{S}_2\neq\emptyset$, $g\in C(\mathcal{R}\cup\mathcal{S}_1)$. Let there exist a subsolution $Z\leqslant H<0$ of problem* (1.7)*, satisfying condition* (2.23)*. Then for any $L\in\mathbb{R}$ there exists at most one solution $u\in L^\infty(\Omega)$ of problem* (1.6).

3. Existence and nonuniqueness results: Proofs

To prove Theorem 2.5 we need a few preliminary results; the proofs are adapted from [28].

Lemma 3.1. *Let Ω be bounded and assumptions (H_1)–(H_2) be satisfied. Let V be a supersolution of problem* (1.7) *with $c=0$ satisfying condition* (2.4)*. Then equality* (2.5) *holds.*

Proof. By absurd, suppose

$$\liminf_{dist(x,\mathcal{S})\to 0} V(x)=:\gamma>0;$$

then $V(x)\geqslant\gamma/2$ for any $x\in\mathcal{S}^\varepsilon:=\{x\in\Omega\mid dist(x,\mathcal{S})<\varepsilon\}$ ($\varepsilon\in(0,\varepsilon_0)$ sufficiently small). It follows that

$$\inf_{\Omega\setminus\overline{\mathcal{S}^\varepsilon}} V=\inf_{\Omega\cup\mathcal{R}} V=0.$$

On the other hand, V is a supersolution of the problem

$$\begin{cases}\mathcal{L}U=0 & \text{in } \Omega\setminus\overline{\mathcal{S}^\varepsilon},\\ U=\alpha & \text{on } \partial[\Omega\setminus\overline{\mathcal{S}^\varepsilon}]\end{cases}$$

where $\alpha:=\min\{\frac{\gamma}{2},\inf_{\mathcal{R}\setminus\overline{\mathcal{S}^\varepsilon}} V\}$, while $V_1:=\alpha$ is a solution of the same problem. By Proposition 2.4 we obtain $V\geqslant\alpha>0$ in $\Omega\setminus\overline{\mathcal{S}^\varepsilon}$, a contradiction. Hence the conclusion follows. □

If Ω is unbounded, a slight modification of the previous proof gives the following[1]

Lemma 3.2. *Let Ω be unbounded and assumptions (H_1)–(H_2) be satisfied. Let V be a supersolution of problem* (1.7) *with $c=0$ satisfying condition* (2.4)*. Then either equality* (2.5)*, or equality* (2.8) *holds.*

Corollary 3.3. *Let Ω be unbounded and assumptions (H_1)–(H_2) be satisfied. Let V be a supersolution of problem* (1.7) *with $c=0$ satisfying condition* (2.10)*. Then equality* (2.5) *holds.*

Now we can prove the following result.

Proposition 3.4. *Let the assumptions of Theorem* 2.5 *be satisfied. Then there exist infinitely many bounded solutions of the homogeneous problem* (1.7).

[1] Lemmas 3.1–3.2 can be proved using the strong maximum principle in [21], if more regularity of the coefficients is assumed.

Proof. (i) If $c = 0$ and $\mathcal{R} = \emptyset$ any constant is a solution of problem (1.7), thus the conclusion follows in this case. Otherwise define

$$l := \begin{cases} \inf_{\mathcal{R}} V & \text{if } c = 0, \\ \min\{\inf_{\mathcal{R}} V, \frac{1}{\|c\|_\infty}\} & \text{if } c \neq 0; \end{cases}$$

then condition (2.4) implies $l \in (0, \infty)$.

Set

$$\mathcal{R}_j := \{x \in \mathcal{R} \mid dist(x, \mathcal{S}) > 1/j\} \quad (j \in \mathbb{N}).$$

Consider a sequence of bounded domains $\{H_j\}_{j\in\mathbb{N}}$ satisfying an exterior sphere condition at each point of the boundary ∂H_j, such that

$$\begin{cases} \overline{H}_j \subseteq \Omega \cup \overline{\mathcal{R}}_j, \quad H_j \subseteq H_{j+1}, \quad \bigcup_{j=1}^{\infty} H_j = \Omega \cup \mathcal{R}, \\ \partial H_j = \mathcal{R}_j \cup \mathcal{T}_j, \quad \mathcal{R}_j \cap \mathcal{T}_j = \emptyset; \end{cases} \tag{3.1}$$

observe that by assumption (H_2)(ii) the operator $\mathcal{L}$ is strictly elliptic in H_j $(j \in \mathbb{N})$.

It is easily seen that $\underline{W} := \max\{l - V, 0\}$ is a subsolution of the problem

$$\begin{cases} \mathcal{L}u = cu & \text{in } H_j, \\ u = \underline{W} & \text{on } \partial H_j \end{cases} \tag{3.2}$$

for any $j \in \mathbb{N}$. In fact, for any $\psi \in C_0^\infty(H_j)$, $\psi \geqslant 0$ there holds:

$$\int_{H_j} (l - V)\{\mathcal{L}^*\psi - c\psi\}\,dx = -l\int_{H_j} c\psi\,dx - \int_{H_j} V\{\mathcal{L}^*\psi - c\psi\}\,dx \geqslant \int_{H_j} (1 - lc)\psi\,dx \geqslant 0.$$

Since $u = 0$ is also a (classical) subsolution, the claim follows. It is also immediately seen that $\overline{W} := \sup_{\Omega\cup\mathcal{R}} \underline{W} = l$ is a classical supersolution of problem (3.2).

(ii) By usual arguments (*e.g.*, see [13]) for any $j \in \mathbb{N}$ there exists a solution $W_j \in C(\overline{H}_j)$ $(\alpha \in (0, 1))$ of problem (3.2), such that

$$0 \leqslant \underline{W} \leqslant W_j \leqslant \overline{W} = l \quad \text{in } H_j; \tag{3.3}$$

observe that $\underline{W} = 0$, thus $W_j = 0$ on $\mathcal{R}_j$.

By compactness arguments there exists a subsequence $\{W_{j_k}\} \subseteq \{W_j\}$, which converges uniformly in any compact subset of Ω. Set

$$W := \lim_{k\to\infty} W_{j_k}. \tag{3.4}$$

We shall prove the following

Claim. *The function W is a bounded solution of the homogeneous problem* (1.7)*. Moreover, W is nontrivial, for there exists a sequence $\{x_m\} \subseteq \Omega$ such that*

$$\lim_{m\to\infty} W(x_m) = l > 0. \tag{3.5}$$

The above claim leads easily to the conclusion. In fact, define

$$U_\beta := \frac{\beta}{l} W. \tag{3.6}$$

Then U_β solves (1.7) and by (3.3)–(3.4)

$$|U_\beta| \leqslant |\beta| \quad \text{in } \Omega \cup \mathcal{R};$$

moreover, along the sequence $\{x_m\}$ there holds

$$\lim_{m\to\infty} U_\beta(x_m) = \beta.$$

Since β is arbitrary the result follows.

(iii) Let us now prove the claim. Clearly, by its very definition W is a solution of Eq. (2.11). We use a local barrier argument to prove that $W \in C(\Omega \cup \mathcal{R})$ and $W = 0$ on $\mathcal{R}$.

Let $x_0 \in \mathcal{R}$; take $j_0 \in \mathbb{N}$ so large that $x_0 \in \mathcal{R}_j$ for any $j \geqslant j_0$. Choose $\delta_0 > 0$ so small that

$$\overline{N_\delta(x_0)} \subseteq \overline{H}_j \subseteq \Omega \cup \mathcal{R}$$

for any $j \geqslant j_0$, where $N_\delta(x_0) := B_\delta(x_0) \cap \Omega$; observe that

$$\partial N_\delta(x_0) = \big[\partial B_\delta(x_0) \cap [\Omega \cup \mathcal{R}]\big] \cup \big[B_\delta(x_0) \cap \mathcal{R}\big] \quad \big(\delta \in (0, \delta_0)\big).$$

Since the operator $\mathcal{L}$ is strictly elliptic in $N_\delta(x_0)$ and an exterior sphere condition is satisfied at $x_0 \in \mathcal{R}$ by assumption (H_1)(ii), there exists a local barrier at x_0—namely, a function $h \in C^2(N_\delta(x_0)) \cap C(\overline{N_\delta(x_0)})$ such that

$$\mathcal{L}h - ch \leqslant -1 \quad \text{in } N_\delta(x_0), \tag{3.7}$$

$$h > 0 \quad \text{in } \overline{N_\delta(x_0)} \setminus \{x_0\}, \qquad h(x_0) = 0 \tag{3.8}$$

(*e.g.*, see [13]). Set

$$m := \min_{\partial B_\delta(x_0) \cap [\Omega \cup \mathcal{R}]} h > 0.$$

Plainly, from inequality (3.3) we obtain

$$0 \leqslant W_j \leqslant \frac{l}{m} h \quad \text{on } \partial N_\delta(x_0) \tag{3.9}$$

for any $j \geqslant j_0$ (recall that $W_j = 0$ on $\mathcal{R}_j$ for any $j \in \mathbb{N}$, thus $W_j = 0$ on $B_\delta(x_0) \cap \mathcal{R}$ if $j \geqslant j_0$). In view of inequality (3.9), it is easily seen that

$$F_j := -W_j + \frac{l}{m} h \quad (j \geqslant j_0)$$

is a supersolution of the problem

$$\begin{cases} \mathcal{L}u = cu & \text{in } N_\delta(x_0), \\ u = 0 & \text{on } \partial N_\delta(x_0); \end{cases} \tag{3.10}$$

then by Proposition 2.4 we obtain

$$0 \leqslant W_j \leqslant \frac{l}{m} h \quad \text{in } N_\delta(x_0)$$

for any $j \geqslant j_0$. Rewriting the above inequality with $j = j_k$ and letting $k \to \infty$, we obtain

$$0 \leqslant W(x) \leqslant \frac{l}{m} h(x) \quad \text{for any } x \in N_\delta(x_0),$$

whence $\lim_{x \to x_0} W(x) = 0$; then the claim follows.

It remains to prove equality (3.5). Let $\{x_m\} \subseteq \Omega$ be a sequence such that

$$\lim_{m \to \infty} V(x_m) = \inf_{\Omega \cup \mathcal{R}} V = 0; \tag{3.11}$$

such a sequence exists by Lemmas 3.1–3.2. By inequality (3.3) we have:

$$l - V \leqslant \underline{W} \leqslant W \leqslant \overline{W} = l \quad \text{in } \Omega \cup \mathcal{R}, \tag{3.12}$$

thus equality (3.11) implies (3.5). This completes the proof. $\square$

Proof of Theorem 2.5. Let $U_\beta \in L^\infty(\Omega)$ be the solution of problem (1.7) satisfying (2.7) constructed in the above proof ($\beta \in \mathbb{R}$). Since U_1 is nontrivial and $U_\beta = \beta U_1$ (see (3.6)), there exists $\bar{x} \in \Omega$ such that $U_1(\bar{x}) \neq 0$, thus $U_{\beta_1}(\bar{x}) \neq U_{\beta_2}(\bar{x})$ for any $\beta_1, \beta_2 \in \mathbb{R}$, $\beta_1 \neq \beta_2$.

Let there exist a solution $\bar{u}$ of problem (1.3). Then $u_\beta := \bar{u} + U_\beta$ is a solution of problem (1.3) for any $\beta \in \mathbb{R}$; moreover, $u_{\beta_1}(\bar{x}) \neq u_{\beta_2}(\bar{x})$ for any $\beta_1, \beta_2 \in \mathbb{R}$, $\beta_1 \neq \beta_2$. Hence the conclusion follows. $\square$

The proof of Theorem 2.6 is the same of Theorem 2.5, using the following proposition instead of Proposition 3.4.

Proposition 3.5. *Let the assumptions of Theorem* 2.6 *be satisfied. Then there exist infinitely many bounded solutions of problem* (2.9) *with* $f = g = L = 0$.

Proof. Define

$$l_\infty := \min\Big\{\inf_{\mathcal{R}} V, \liminf_{|x|\to\infty} V(x)\Big\} \quad \text{if } c = 0,$$
$$l_\infty := \min\Big\{\inf_{\mathcal{R}} V, \liminf_{|x|\to\infty} V(x), \frac{1}{\|c\|_\infty}\Big\} \quad \text{otherwise;}$$

then condition (2.10) implies $l_\infty > 0$.

Fix $l \in (0, l_\infty)$; consider the family of problems (3.2) with $H_j, \underline{W}$ defined as above. Arguing as in the proof of Proposition 3.4 (using Corollary 3.3 instead of Lemmas 3.1–3.2), we prove the following: there exist a sequence $\{x_m\} \subseteq \Omega$ satisfying (2.6) and a bounded solution $W \geqslant 0$ of problem (1.7), defined by (3.4), such that equality (3.5) holds. We prove below the additional property:

$$\lim_{|x|\to\infty} W(x) = 0; \tag{3.13}$$

then defining the family U_β ($\beta \in \mathbb{R}$) as in (3.6) the conclusion follows.

To prove equality (3.13), observe preliminarily that

$$\limsup_{|x|\to\infty} \underline{W}(x) = \lim_{|x|\to\infty} \underline{W}(x) = 0 \tag{3.14}$$

(this follows from the above definition of l, since $\underline{W} := \max\{l - V, 0\}$). Then for any $\sigma > 0$ there exists $M > 0$ such that

$$0 \leqslant \underline{W}(x) < \sigma \quad \text{in } [\Omega \cup \mathcal{R}] \setminus \overline{B_M}. \tag{3.15}$$

Consider the subsequence $\{j_k\} \subseteq \mathbb{N}$ such that (3.4) holds. Fix k so large that

$$N_k := H_{j_k} \cap \big[[\Omega \cup \mathcal{R}] \setminus \overline{B_M}\big] \neq \emptyset;$$

observe that

$$\partial N_k = \big[\partial H_{j_k} \cap \big[[\Omega \cup \mathcal{R}] \setminus \overline{B_M}\big]\big] \cup [\overline{H_{j_k}} \cap \partial B_M] \quad (k \in \mathbb{N}).$$

By (3.15) there holds

$$0 \leqslant W_{j_k} = \underline{W} < \sigma,$$

on $\partial H_{j_k} \cap [[\Omega \cup \mathcal{R}] \setminus \overline{B_M}]$ (see (3.2)). Besides, for any k sufficiently large in $\overline{H_{j_k}} \cap \partial B_M$ there holds (see inequality (3.3)):

$$0 \leqslant W_{j_k} \leqslant l \leqslant \frac{l}{m} F,$$

where

$$m := \min_{\partial B_M} F > 0.$$

From the above inequalities we get

$$0 \leqslant W_{j_k} < \sigma + \frac{l}{m} F \quad \text{on } \partial N_k \tag{3.16}$$

for any k sufficiently large.

It is easily seen that for such values of k the function

$$Z_k := W_{j_k} - \sigma - \frac{l}{m} F$$

is a subsolution of the problem

$$\begin{cases} \mathcal{L}u = cu & \text{in } N_k, \\ u = 0 & \text{on } \partial N_k. \end{cases}$$

In fact, for any $\psi \in C_0^\infty(N_k)$, $\psi \geqslant 0$ we have:

$$\int_{N_k} Z_k\{\mathcal{L}^*\psi - c\psi\}\,dx = \sigma \int_{N_k} c\psi\,dx - \frac{l}{m}\int_{N_k} F\{\mathcal{L}^*\psi - c\psi\}\,dx \geqslant 0;$$

moreover, $Z_k \leqslant 0$ on ∂N_k by (3.16), thus the claim follows.

In view of Proposition 2.4, this implies

$$0 \leqslant W_{j_k} < \sigma + \frac{l}{m}F \quad \text{in } N_k \tag{3.17}$$

for any $k \in \mathbb{N}$ sufficiently large. As $k \to \infty$ we obtain

$$0 \leqslant W(x) < \sigma + \frac{l}{m}F(x)$$

for any $x \in [\Omega \cup \mathcal{R}] \setminus \overline{B_M}$. This obtains

$$0 \leqslant \limsup_{|x|\to\infty} W(x) \leqslant \sigma;$$

since $\sigma > 0$ is arbitrary, equality (3.13) follows. This completes the proof. □

Let us now prove Proposition 2.7.

Proof of Proposition 2.7. (i) If $\overline{\mathcal{R}} \cap \overline{\mathcal{S}} \neq \emptyset$, let $\zeta_j \in C_0^\infty(\mathcal{R}_j)$, $0 \leqslant \zeta_j \leqslant 1$, $\zeta_j = 1$ in $\mathcal{R}_{j-1}$ ($j \in \mathbb{N}$; $\mathcal{R}_0 := \emptyset$). If $\overline{\mathcal{R}} \cap \overline{\mathcal{S}} = \emptyset$, we have that $\mathcal{R}_j = \mathcal{R}$, for any $j \geqslant j_0$, for some $j_0 \in \mathbb{N}$; in this case we set $\zeta_j \equiv 1$ on $\mathcal{R}_j = \mathcal{R}$, for any $j \geqslant j_0$.

For any $j \geqslant j_0$ consider the problem

$$\begin{cases} \mathcal{L}u - cu = f & \text{in } H_j, \\ u = \phi_j & \text{on } \partial H_j; \end{cases} \tag{3.18}$$

here $\{H_j\}$ is the sequence of domains used in the proof of Proposition 3.4 and the boundary data

$$\phi_j := \begin{cases} \zeta_j g + (1-\zeta_j)F & \text{on } \mathcal{R}_j, \\ F & \text{in } \mathcal{T}_j \end{cases} \tag{3.19}$$

are continuous on ∂H_j ($j \geqslant j_0$).

It is easily seen that the function

$$\tilde{F} := \max\bigl\{\|f\|_\infty, 1\bigr\}\bigl(F + \|g\|_\infty\bigr) \tag{3.20}$$

is a supersolution of problem (3.18)–(3.19) for any $j \geqslant j_0$. In fact, for any $\psi \in C_0^\infty(H_j)$, $\psi \geqslant 0$ we have:

$$\begin{aligned}\int_{H_j} \tilde{F}\{\mathcal{L}^*\psi - c\psi\}\,dx &= \max\bigl\{\|f\|_\infty, 1\bigr\}\Biggl\{\int_{H_j} F\{\mathcal{L}^*\psi - c\psi\}\,dx - \|g\|_\infty \int_{H_j} c\psi\,dx\Biggr\} \\ &\leqslant -\max\bigl\{\|f\|_\infty, 1\bigr\}\int_{H_j}\psi\,dx \leqslant \int_{H_j} f\psi\,dx;\end{aligned}$$

moreover, $\tilde{F} \geqslant F + \|g\|_\infty \geqslant \phi_j$ on ∂H_j. Hence the claim follows. It is similarly checked that $-\tilde{F}$ is a subsolution of the same problem.

(ii) In view of (i) above, there exists a solution $u_j \in C(\overline{H}_j)$ ($\alpha \in (0,1)$) of problem (3.18)–(3.19), such that

$$|u_j| \leqslant \tilde{F} \quad \text{in } H_j \tag{3.21}$$

for any $j \geqslant j_0$. By standard compactness arguments there exists a subsequence $\{u_{j_k}\} \subseteq \{u_j\}$, which converges uniformly in any compact subset of Ω. Clearly, $u := \lim_{k\to\infty} u_{j_k}$ is a solution of Eq. (1.1); moreover, $|u| \leqslant \tilde{F}$ in Ω.

(iii) It remains to prove that $u \in C(\Omega \cup \mathcal{R})$ and $u = g$ on $\mathcal{R}$. To this purpose, we use a local barrier argument as in the proof of Proposition 3.4.

Let $x_0 \in \mathcal{R}$ be arbitrarily fixed; take $j_0 \in \mathbb{N}$ so large that $x_0 \in \mathcal{R}_{j_0-1}$. Since each $\mathcal{R}_j$ is open and $\mathcal{R}_{j_0-1} \subseteq \mathcal{R}_j$ for $j \geqslant j_0$, there exists $\delta_0 > 0$ such that:

$$u_j = g \quad \text{in } B_{\delta_0}(x_0) \cap \mathcal{R} \tag{3.22}$$

for any $j \geqslant j_0$ (see (3.19)). Moreover, we can choose $\delta_0 > 0$ so small that

$$\overline{N_\delta(x_0)} \subseteq \overline{H}_j \subseteq \Omega \cup \mathcal{R} \tag{3.23}$$

for any $j \geqslant j_0$, where $N_\delta(x_0) := B_\delta(x_0) \cap \Omega$. Observe that

$$\partial N_\delta(x_0) = \big[\partial B_\delta(x_0) \cap [\Omega \cup \mathcal{R}]\big] \cup \big[B_\delta(x_0) \cap \mathcal{R}\big] \quad \big(\delta \in (0, \delta_0)\big).$$

Since $g \in C(\mathcal{R})$ (see (H_2)(v)), in view of (3.22) for any $\sigma > 0$ there exists $\delta \in (0, \delta_0)$ such that

$$\big|u_j(x) - g(x_0)\big| < \sigma \quad \text{for any } x \in B_{\delta_0}(x_0) \cap \mathcal{R},\ j \geqslant j_0. \tag{3.24}$$

Let $h \in C^2(N_\delta(x_0)) \cap C(\overline{N_\delta(x_0)})$ satisfy (3.7)–(3.8). For any $x \in \partial B_\delta(x_0) \cap [\Omega \cup \mathcal{R}]$ $(\delta \in (0, \delta_0))$ and $j \geqslant j_0$ there holds

$$\big|u_j(x) - g(x_0)\big| \leqslant \max_{N_\delta(x_0)} \tilde{F} + \big|g(x_0)\big| \leqslant mM \leqslant Mh(x), \tag{3.25}$$

where

$$m := \min_{\partial B_\delta(x_0) \cap [\Omega \cup \mathcal{R}]} h > 0,$$
$$M := \frac{2}{m} \max\Big\{ \max_{N_\delta(x_0)} \tilde{F}, \|g\|_\infty, m\|f\|_\infty, m\|g\|_\infty \max_{N_\delta(x_0)} c \Big\}$$

(see (3.21), (3.23)).

In view of inequalities (3.24)–(3.25), we conclude that for any $\sigma > 0$ there exists $\delta \in (0, \delta_0)$ such that

$$\big|u_j(x) - g(x_0)\big| < \sigma + Mh(x) \quad \text{for any } x \in \partial N_\delta(x_0),\ j \geqslant j_0.$$

Then it is easily seen that for such values of j

$$E_j := -u_j + g(x_0) - \sigma - Mh$$

is a subsolution,

$$F_j := -u_j + g(x_0) + \sigma + Mh$$

a supersolution of problem (3.10). By Proposition 2.4 this implies $E_j \leqslant 0 \leqslant F_j$ in $N_\delta(x_0)$, namely

$$\big|u_j(x) - g(x_0)\big| < \sigma + Mh(x) \quad \text{for any } x \in N_\delta(x_0),\ j \geqslant j_0. \tag{3.26}$$

Set $j = j_k$ in inequality (3.26), then let $k \to \infty$. This obtains the following: for any $\sigma > 0$ there exists $\delta \in (0, \delta_0)$ such that

$$\big|u(x) - g(x_0)\big| < \sigma + Mh(x) \quad \text{for any } x \in N_\delta(x_0),$$

whence

$$\limsup_{x \to x_0} \big|u(x) - g(x_0)\big| \leqslant \sigma$$

for any $\sigma > 0$. Then the conclusion follows. □

To prove Proposition 2.9, first a solution u of problem (1.3) is constructed as for Proposition 2.7. Then, arguing as in the proof of Proposition 3.5, it is proved that $\lim_{|x| \to \infty} u(x) = L$. We leave the details to the reader.

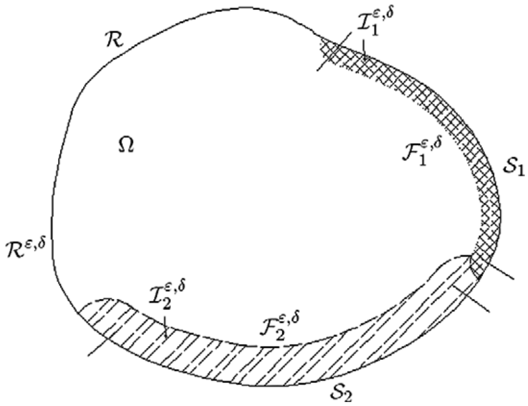

Fig. 1. Bounded Ω.

4. Comparison and uniqueness results: Proofs

Let us first introduce some notations. Set for any $\varepsilon \in (0, \varepsilon_0)$, $\delta \in (0, \frac{\varepsilon}{2})$ ($\varepsilon_0 > 0$ suitably small):

$$\mathcal{S}_{1,\varepsilon} := \{x \in \mathcal{S}_1 \mid dist(x, \mathcal{S}_2) \geqslant \varepsilon\},$$
$$\mathcal{S}^{\varepsilon} := \{x \in \Omega \mid dist(x, \mathcal{S}) < \varepsilon\},$$
$$\mathcal{S}_2^{\varepsilon} := \{x \in \Omega \mid dist(x, \mathcal{S}_2) < \varepsilon\},$$
$$\mathcal{A}_2^{\varepsilon} := \{x \in \overline{\Omega} \mid dist(x, \mathcal{S}_2) = \varepsilon\}.$$
$$\mathcal{R}^{\varepsilon,\delta} := \{x \in \mathcal{R} \mid dist(x, \mathcal{S}_1) > \delta,\ dist(x, \mathcal{S}_2) > \varepsilon\}.$$

If $\mathcal{S}_{1,\varepsilon} \neq \emptyset$, we also define:

$$\mathcal{I}_1^{\varepsilon,\delta} := \{x \in \Omega \mid dist(x, \mathcal{S}_{1,\varepsilon}) < \delta\},$$
$$\mathcal{F}_1^{\varepsilon,\delta} := \{x \in \overline{\Omega} \mid dist(x, \mathcal{S}_{1,\varepsilon}) = \delta,\ dist(x, \mathcal{S}_2) \geqslant \varepsilon\},$$
$$\mathcal{I}_2^{\varepsilon,\delta} := \{x \in \mathcal{S}_2^{\varepsilon} \mid dist(x, \mathcal{S}_{1,\varepsilon}) \geqslant \delta\},$$
$$\mathcal{F}_2^{\varepsilon,\delta} := \{x \in \mathcal{A}_2^{\varepsilon} \mid dist(x, \mathcal{S}_{1,\varepsilon}) > \delta\};$$

otherwise we set $\mathcal{I}_1^{\varepsilon,\delta} = \mathcal{F}_1^{\varepsilon,\delta} := \emptyset$, $\mathcal{I}_2^{\varepsilon,\delta} := \mathcal{S}_2^{\varepsilon}$, $\mathcal{F}_2^{\varepsilon,\delta} := \mathcal{A}_2^{\varepsilon}$. Finally, define:

$$\mathcal{I}^{\varepsilon,\delta} := \mathcal{I}_1^{\varepsilon,\delta} \cup \mathcal{I}_2^{\varepsilon,\delta}, \qquad \mathcal{F}^{\varepsilon,\delta} := \mathcal{F}_1^{\varepsilon,\delta} \cup \mathcal{F}_2^{\varepsilon,\delta}.$$

The above sets are depicted in Fig. 1 for the case of bounded Ω. Observe that $\mathcal{S}_1 \subseteq \overline{\mathcal{I}^{\varepsilon,\delta}}$.

Lemma 4.1. *For any $\varepsilon \in (0, \varepsilon_0)$, $\delta \in (0, \frac{\varepsilon}{2})$:*

(i) *there hold*

$$\overline{\Omega \setminus \overline{\mathcal{I}^{\varepsilon,\delta}}} \subseteq \Omega \cup \mathcal{R}, \tag{4.1}$$
$$\partial\big[\Omega \setminus \overline{\mathcal{I}^{\varepsilon,\delta}}\big] = \mathcal{R}^{\varepsilon,\delta} \cup \mathcal{F}^{\varepsilon,\delta}; \tag{4.2}$$

(ii) *for any open subset $\Omega_1 \subseteq \Omega \setminus \overline{\mathcal{I}^{\varepsilon,\delta}}$ there holds*

$$\partial\Omega_1 \setminus [\mathcal{R} \cup \mathcal{S}_1] = \partial\Omega_1 \setminus \mathcal{R} = \partial\Omega_1 \setminus \mathcal{R}^{\varepsilon,\delta}. \tag{4.3}$$

Proof. We only check (4.1), since equalities (4.2)–(4.3) are clear. In fact, there holds:

$$\overline{\Omega \setminus \overline{\mathcal{I}^{\varepsilon,\delta}}} \subseteq \big[\overline{\Omega} \setminus \overline{\mathcal{I}^{\varepsilon,\delta}}\big] \cup \mathcal{F}^{\varepsilon,\delta} \subseteq \overline{\Omega} \setminus \mathcal{S} = \Omega \cup \mathcal{R}, \tag{4.4}$$

since $\mathcal{S} \subseteq \overline{\mathcal{I}^{\varepsilon,\delta}}$, $\mathcal{F}^{\varepsilon,\delta} \subseteq \overline{\Omega} \setminus \mathcal{S}$. $\square$

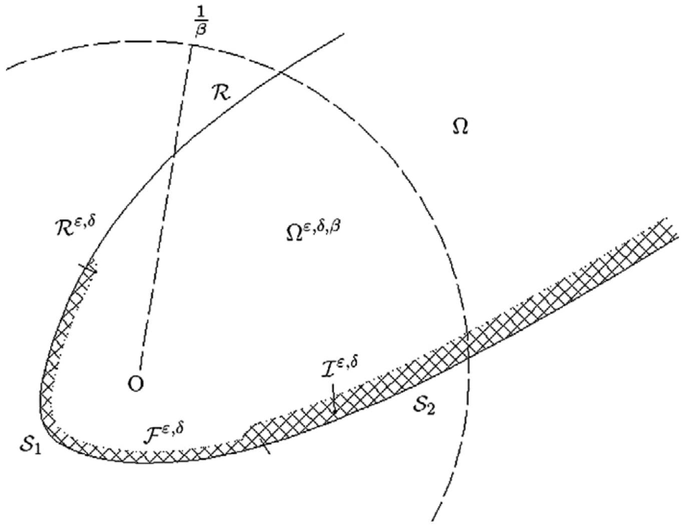

Fig. 2. Unbounded Ω.

When Ω is unbounded, we also use the following family of subsets of Ω (see Fig. 2):

$$\Omega^{\varepsilon,\delta,\beta} := \big(\Omega \setminus \overline{\mathcal{I}^{\varepsilon,\delta}}\big) \cap B_{\frac{1}{\beta}}$$

($\varepsilon \in (0, \varepsilon_0)$, $\delta \in (0, \frac{\varepsilon}{2})$, $\beta > 0$); observe that by (4.2)

$$\begin{aligned}\partial\Omega^{\varepsilon,\delta,\beta} &= \big[\partial\big[\Omega \setminus \overline{\mathcal{I}^{\varepsilon,\delta}}\big] \cap \overline{B_{\frac{1}{\beta}}}\big] \cup \big[\overline{\Omega \setminus \overline{\mathcal{I}^{\varepsilon,\delta}}} \cap \partial B_{\frac{1}{\beta}}\big] \\ &= \big[\big[\mathcal{R}^{\varepsilon,\delta} \cup \mathcal{F}^{\varepsilon,\delta}\big] \cap \overline{B_{\frac{1}{\beta}}}\big] \cup \big[\overline{\Omega \setminus \overline{\mathcal{I}^{\varepsilon,\delta}}} \cap \partial B_{\frac{1}{\beta}}\big].\end{aligned}$$

Now we can prove Proposition 2.10.

Proof of Proposition 2.10. Let us distinguish two cases: (a) Ω bounded, and (b) Ω unbounded.

(a) *Ω bounded*: (i) In view of inequality (2.17), there exists a sequence $\{\varepsilon_k\} \subseteq (0, \varepsilon_0)$, $\varepsilon_k \to 0$ as $k \to \infty$, such that

$$\lim_{k\to+\infty}\Big\{\inf_{\mathcal{A}_2^{\varepsilon_k}\setminus\mathcal{S}} \frac{u}{Z}\Big\} \geqslant 0. \tag{4.5}$$

Then for any $\alpha > 0$ there exists $\bar{k} = \bar{k}(\alpha) \in \mathbb{N}$ such that for any $k > \bar{k}$ there holds

$$\frac{u}{Z} > -\alpha \quad \text{in } \mathcal{A}_2^{\varepsilon_k} \setminus \mathcal{S}. \tag{4.6}$$

(ii) Define for any $\alpha > 0$

$$V_\alpha(x) := -\alpha Z(x) = \alpha\big|Z(x)\big| \quad (x \in \Omega \cup \mathcal{R}). \tag{4.7}$$

Observe that

$$\alpha|H| \leqslant V_\alpha \quad \text{in } \Omega \cup \mathcal{R}. \tag{4.8}$$

In view of (4.1)–(4.2) and (4.8), the following claim is easily seen to hold.

Claim 1. *For any $\alpha > 0$, $\varepsilon \in (0, \varepsilon_0)$, $\delta \in (0, \frac{\varepsilon}{2})$ the function V_α defined in* (4.7) *is a supersolution of the problem*

$$\begin{cases} \mathcal{L}u - cu = 0 & \text{in } \Omega \setminus \overline{\mathcal{I}^{\varepsilon,\delta}}, \\ u = 0 & \text{on } \mathcal{R}^{\varepsilon,\delta}, \\ u = V_\alpha & \text{on } \mathcal{F}^{\varepsilon,\delta}. \end{cases} \tag{4.9}$$

(iii) We shall prove the following

Claim 2. *For any $\alpha > 0$ there exists $\bar{k} = \bar{k}(\alpha) \in \mathbb{N}$ with the following property: for any $k > \bar{k}$ there exists $\delta_k \in (0, \frac{\varepsilon_k}{2})$ such that the function u is a subsolution of problem* (4.9) *with $\varepsilon = \varepsilon_k$, $\delta = \delta_k$, where $\{\varepsilon_k\}$ is the infinitesimal sequence of inequality* (4.6).

From Claims 1 and 2 the conclusion follows immediately. In fact, by Proposition 2.4 we obtain for any $\alpha > 0$, $k > \bar{k}$

$$u \leqslant V_\alpha \quad \text{in } \Omega \setminus \overline{\mathcal{I}^{\varepsilon_k,\delta_k}}.$$

Letting $\alpha \to 0$ in the latter inequality we obtain $u \leqslant 0$ in any compact subset of Ω (observe that $\bar{k} \to \infty$, thus $\varepsilon_k \to 0$ as $\alpha \to 0$); hence the result follows.

To prove Claim 2 we use the following facts:

- for any $\alpha > 0$, $\varepsilon \in (0, \varepsilon_0)$ there exists $\bar{\delta} \in (0, \frac{\varepsilon}{2})$ such that for any $\delta \in (0, \bar{\delta})$ there holds

$$u < \alpha|H| \quad \text{in } \mathcal{F}_1^{\varepsilon,\delta}; \tag{4.10}$$

- for any $\alpha > 0$ there exists $\bar{k} = \bar{k}(\alpha) \in \mathbb{N}$ such that for any $k > \bar{k}$ and for any $\delta \in (0, \frac{\varepsilon_k}{2})$ the function V_α satisfies

$$u < V_\alpha \quad \text{in } \mathcal{F}_2^{\varepsilon_k,\delta}. \tag{4.11}$$

Let us put off the proof of (4.10)–(4.11) and complete the proof of Claim 2. Plainly, from (4.8) and (4.10)–(4.11) we obtain

$$u < V_\alpha \quad \text{in } \mathcal{F}^{\varepsilon_k,\delta_k} \tag{4.12}$$

for any $\alpha > 0$, $k > \bar{k}$ and some $\delta_k \in (0, \frac{\varepsilon_k}{2})$. On the other hand, the function u is by assumption a subsolution of the problem

$$\begin{cases} \mathcal{L}u = cu & \text{in } \Omega, \\ u = 0 & \text{on } \mathcal{R} \cup \mathcal{S}_1, \end{cases} \tag{4.13}$$

thus in particular $u \leqslant 0$ on $\mathcal{R}^{\varepsilon_k,\delta_k} \subseteq \mathcal{R}$. Hence Claim 2 follows.

It remains to prove inequalities (4.10)–(4.11). Concerning (4.10), observe that $u \leqslant 0$ on $\mathcal{S}_1$ and $u \in C(\mathcal{S}_1)$, thus in particular $u \leqslant 0$ on $\mathcal{S}_{1,\varepsilon}$ and $u \in C(\mathcal{S}_{1,\varepsilon})$ (recall that u is a subsolution of (4.13), hence $u \in C(\Omega \cup \mathcal{R} \cup \mathcal{S}_1)$ by Definition 2.2). As a consequence, for any $\bar{x} \in \mathcal{S}_{1,\varepsilon}$ and any $\sigma > 0$ there exists $\delta = \delta(\bar{x}, \sigma) > 0$ such that

$$u(x) < \sigma \quad \text{for any } x \in [\Omega \cup \mathcal{R}] \cap B_\delta(\bar{x}).$$

It is immediately seen that $\mathcal{S}_{1,\varepsilon}$ is closed, thus compact. Hence from the covering $\{B_\delta(\bar{x})\}_{\bar{x} \in \mathcal{S}_{1,\varepsilon}}$ we can extract a finite covering $\{B_{\delta_n}(\bar{x}_n)\}_{n=1,\ldots,\bar{n}}$ ($\bar{n} \in \mathbb{N}$), namely

$$\mathcal{S}_{1,\varepsilon} \subseteq \bigcup_{n=1}^{\bar{n}} B_{\delta_n}(\bar{x}_n) =: \mathcal{B}_{\varepsilon,\sigma}.$$

Set

$$\bar{\delta} := \min\left\{\delta_1, \ldots, \delta_{\bar{n}}, \frac{\varepsilon}{3}\right\};$$

then

$$\left\{x \in \Omega \cup \mathcal{R} \;\middle|\; dist(x, \mathcal{S}_{1,\varepsilon}) \leqslant \bar{\delta}\right\} \subseteq [\Omega \cup \mathcal{R}] \cap \mathcal{B}_{\varepsilon,\sigma},$$

thus in particular

$$\mathcal{F}_1^{\varepsilon,\delta} \subseteq [\Omega \cup \mathcal{R}] \cap \mathcal{B}_{\varepsilon,\sigma} \quad \text{for any } \delta \in (0, \bar{\delta}).$$

This shows that for any $\sigma > 0$, $\varepsilon \in (0, \varepsilon_0)$ and $\delta \in (0, \bar\delta)$ there holds

$$u < \sigma \quad \text{in } \mathcal{F}_1^{\varepsilon,\delta};$$

choosing $\sigma = \alpha|H|$ we obtain (4.10).

Inequality (4.11) follows immediately from (4.6), since $\mathcal{F}_2^{\varepsilon_k,\delta} \subseteq \mathcal{A}_2^{\varepsilon_k} \setminus \mathcal{S}$ for any $\delta \in (0, \frac{\varepsilon_k}{2})$. This completes the proof when Ω is bounded.

(b) Ω *unbounded*: (i) In view of inequalities (2.17) and (2.18), there exist two sequences $\{\varepsilon_k\} \subseteq (0, \varepsilon_0)$, $\varepsilon_k \to 0$ as $k \to \infty$ and $\{\beta_k\} \subseteq (0, \infty)$, $\beta_k \to 0$ as $k \to \infty$, such that

$$\lim_{k\to+\infty}\left\{\inf_{\mathcal{A}_2^{\varepsilon_k}\setminus\mathcal{S}} \frac{u}{Z}\right\} \geqslant 0, \qquad \lim_{k\to+\infty}\left\{\inf_{[\Omega\cup\mathcal{R}]\cap\partial B_{\frac{1}{\beta_k}}} \frac{u}{Z}\right\} \geqslant 0. \tag{4.14}$$

Then for any $\alpha > 0$ there exists $\bar k = \bar k(\alpha) \in \mathbb{N}$ such that for any $k > \bar k$

$$\frac{u}{Z} > -\alpha \quad \text{in } \mathcal{A}_2^{\varepsilon_k} \setminus \mathcal{S}, \qquad \frac{u}{Z} \geqslant -\alpha \quad \text{on } [\Omega \cup \mathcal{R}] \cap \partial B_{\frac{1}{\beta_k}}. \tag{4.15}$$

(ii) As in the above case of bounded Ω, it is easily seen that the function $V_\alpha := -\alpha Z$ is a supersolution of the problem

$$\begin{cases} \mathcal{L}u - cu = 0 & \text{in } \Omega^{\varepsilon,\delta,\beta}, \\ u = 0 & \text{on } \mathcal{R}^{\varepsilon,\delta} \cap \overline{B_{\frac{1}{\beta}}}, \\ u = V_\alpha & \text{on } [\mathcal{F}^{\varepsilon,\delta} \cap \overline{B_{\frac{1}{\beta}}}] \cup [\overline{\Omega \setminus \overline{\mathcal{I}^{\varepsilon,\delta}}} \cap \partial B_{\frac{1}{\beta}}] \end{cases} \tag{4.16}$$

for any $\alpha > 0$, $\varepsilon \in (0, \varepsilon_0)$, $\delta \in (0, \frac{\varepsilon}{2})$, $\beta > 0$.

Arguing as in (a) above the conclusion follows from

Claim 3. *For any $\alpha > 0$ there exists $\bar k = \bar k(\alpha) \in \mathbb{N}$ with the following property: for any $k > \bar k$ there exists $\delta_k \in (0, \frac{\varepsilon_k}{2})$ such that the function u is a subsolution of problem* (4.16) *with $\varepsilon = \varepsilon_k$, $\delta = \delta_k$, $\beta = \beta_k$, where $\{\varepsilon_k\}$ and $\{\beta_k\}$ are the infinitesimal sequences of inequalities* (4.15).

To prove Claim 3, it suffices to prove that

$$u < V_\alpha \quad \text{on } \big[\mathcal{F}^{\varepsilon_k,\delta_k} \cap \overline{B_{\frac{1}{\beta_k}}}\big] \cup \big[\overline{\Omega \setminus \overline{\mathcal{I}^{\varepsilon_k,\delta_k}}} \cap \partial B_{\frac{1}{\beta_k}}\big] \tag{4.17}$$

with $\alpha, k, \varepsilon_k, \delta_k, \beta_k$ as above. Notice that (4.8) and (4.11) are still valid. Moreover, in view of the compactness of $\mathcal{S}_{1,\varepsilon} \cap \overline{B_{\frac{1}{\beta}}}$ ($\varepsilon \in (0, \varepsilon_0)$, $\beta > 0$), arguing as in the proof of (4.10), we get that

- for any $\alpha > 0$, $\varepsilon \in (0, \varepsilon_0)$, $\beta > 0$ there exists $\bar\delta \in (0, \frac{\varepsilon}{2})$ such that for any $\delta \in (0, \bar\delta)$ there holds

$$u < \alpha|H| \quad \text{in } \mathcal{F}_1^{\varepsilon,\delta} \cap \overline{B_{\frac{1}{\beta}}}. \tag{4.18}$$

Then by (4.8), (4.11), (4.18), the inequality

$$u < V_\alpha \quad \text{in } \mathcal{F}^{\varepsilon_k,\delta_k} \cap \overline{B_{\frac{1}{\beta_k}}} \tag{4.19}$$

follows. Concerning the inequality

$$u < V_\alpha \quad \text{in } \overline{\Omega \setminus \overline{\mathcal{I}^{\varepsilon_k,\delta_k}}} \cap \partial B_{\frac{1}{\beta_k}}, \tag{4.20}$$

it follows immediately from (4.15) since $\overline{\Omega \setminus \overline{\mathcal{I}^{\varepsilon_k,\delta}}} \subseteq \Omega \cup \mathcal{R}$ for any $\delta \in (0, \frac{\varepsilon_k}{2})$ (see (4.1)). Then inequality (4.17) and the conclusion for unbounded Ω follow. This completes the proof. □

Proof of Proposition 2.11. The proof is the same of Proposition 2.10 in the case of unbounded Ω, the only difference being that to prove (4.20) we use (4.8) and the following inequality:

$$u < \alpha|H| \quad \text{in } \overline{\Omega \setminus \overline{\mathcal{I}^{\varepsilon_k,\delta_k}}} \cap \partial B_{\frac{1}{\beta_k}}. \tag{4.21}$$

As for the latter, by the second inequality in (2.16) there exists a sequence $\{\beta_k\} \subseteq (0,\infty)$, $\beta_k \to 0$ as $k \to \infty$, such that

$$\lim_{k\to+\infty}\Big\{\sup_{[\Omega\cup\mathcal{R}]\cap\partial B_{\frac{1}{\beta_k}}} u\Big\} \leqslant 0.$$

Then for any $\alpha > 0$ there exists $\bar{k} = \bar{k}(\alpha) \in \mathbb{N}$ such that for any $k > \bar{k}$ there holds

$$u < \alpha|H| \quad \text{in } [\Omega \cup \mathcal{R}] \cap \partial B_{\frac{1}{\beta_k}},$$

which implies (4.21). Then the conclusion follows. □

Proof of Theorem 2.13. Let u_1, u_2 solve problem (1.5); then both $u_1 - u_2$ and $u_2 - u_1$ are solutions of the same problem with $f = g = 0$. In view of Proposition 2.10 and Remark 2.12, conditions (2.14), (2.15) with $u = u_1 - u_2$, $u = u_2 - u_1$ yield $u_1 \leqslant u_2$, respectively $u_2 \leqslant u_1$. Then the conclusion follows. □

5. Examples and remarks

In this section we discuss some applications of the above general results.

5.1. Nonuniqueness and existence

According to the assumptions made in Section 2.1, only degeneracy at the boundary is allowed in the examples of this subsection.

(a) Consider the problem

$$\begin{cases} x^2u_{xx} + y^2u_{yy} - u_y - u = f & \text{in } (0,\infty)\times(0,1) = \Omega, \\ u = g & \text{on } (0,\infty)\times\{1\} = \mathcal{R} \end{cases} \tag{5.22}$$

with $f \in C(\Omega) \cap L^\infty(\Omega)$, $g \in C(\mathcal{R}) \cap L^\infty(\mathcal{R})$. The function $V(x,y) = y$ satisfies

$$\mathcal{L}V = -1 \quad \text{in } \Omega, \qquad \inf_{\Omega\cup\mathcal{R}} V = 0 < \inf_{\mathcal{R}} V = 1;$$

moreover,

$$\mathcal{L}V - V = -1 - y \leqslant -1 \quad \text{in } \Omega.$$

By Theorem 2.5 and Proposition 2.7 (applied with $F = V$) problem (5.22) has infinitely many solutions in $L^\infty(\Omega)$.

(b) Consider the problem

$$\begin{cases} \frac{1}{2}x^2u_{xx} + (x-1)^2y^2u_{yy} + 2x^2u_x - (2x^2+1)u_y - u = f & \text{in } (1,\infty)\times(0,1) = \Omega, \\ u = g & \text{on } (1,\infty)\times\{1\} = \mathcal{R}, \\ \lim_{x\to\infty} u(x,y) = L & (y \in (0,1)) \end{cases} \tag{5.23}$$

with $f \in C(\Omega) \cap L^\infty(\Omega)$, $g \in C(\mathcal{R}) \cap L^\infty(\mathcal{R})$, $L \in \mathbb{R}$ and

$$\lim_{x\to\infty} g(x) = L.$$

The function $V(x,y) = x + y - 1$ satisfies

$$\mathcal{L}V = -1 \quad \text{in } \Omega, \qquad \inf_{\Omega\cup\mathcal{R}} V = 0 < \min\Big\{\inf_{\mathcal{R}} V, \lim_{x\to\infty} V(x,y)\Big\} = 1 \quad \big(y \in (0,1)\big).$$

Moreover, the function $F \in L^\infty(\Omega)$, $F(x,y) = \frac{1}{x}$ satisfies

$$\mathcal{L}F - F \leqslant -1 \quad \text{in } \Omega, \qquad \lim_{x\to\infty} F(x) = 0.$$

In view of Theorem 2.6 and Proposition 2.9, problem (5.23) has infinitely many solutions in $L^\infty(\Omega)$.

Let us show by an example that general Dirichlet boundary data cannot be prescribed on a portion of the boundary, which is *attracting* in the sense of the following definition (see [20]).

Let $\Sigma \subseteq \partial\Omega$; for any $\varepsilon \in (0, \varepsilon_0)$ ($\varepsilon_0 > 0$ suitably small) set

$$\Sigma^\varepsilon := \{x \in \Omega \mid dist(x, \Sigma) < \varepsilon\}.$$

Definition 5.1. A subset $\Sigma \subseteq \partial\Omega$ is *attracting* if there exist $\varepsilon \in (0, \varepsilon_0)$ and a supersolution $V \in C(\overline{\Sigma^\varepsilon})$ of the equation:

$$\mathcal{L}V - cV = -1 \quad \text{in } \Sigma^\varepsilon, \tag{5.24}$$

such that

$$V > 0 \quad \text{in } \overline{\Sigma^\varepsilon} \setminus \Sigma, \qquad V = 0 \quad \text{on } \Sigma.$$

Sufficient conditions for the attractivity of Σ can be given adapting results in [12,25]. The proof of the following result is very similar to that of Proposition 2.7, thus we omit it.

Proposition 5.2. *Let $\mathcal{S}_1 \subseteq \partial\Omega$. Let assumptions* (H_1), (H_2) *and* (H_4) *be satisfied; suppose $f \in L^\infty(\Omega)$, $g \in C(\overline{\mathcal{R} \cup \mathcal{S}_1}) \cap L^\infty(\mathcal{R} \cup \mathcal{S}_1)$, $c \in L^\infty(\mathcal{S}_1^\varepsilon)$ for some $\varepsilon \in (0, \varepsilon_0)$. Let there exist a positive supersolution $F \in C(\Omega \cup \mathcal{R}) \cap L^\infty(\mathcal{S}_1^\varepsilon)$ of Eq.* (2.13). *If $\mathcal{S}_1$ is attracting and bounded, there exists a solution $u \in C(\Omega \cup \mathcal{R} \cup \mathcal{S}_1)$ of problem* (1.5), *provided that*

$$g = constant \quad on\ \mathcal{S}_1. \tag{5.25}$$

Condition (5.25) *and the boundedness of $\mathcal{S}_1$ are unnecessary, if a local barrier exists at any point $x_0 \in \mathcal{S}_1$.*

In view of the above proposition, the function V can be regarded as a barrier *for the whole of* $\mathcal{S}_1$, if the latter is bounded (clearly, V is a local barrier at some point $x_0 \in \Sigma$ if and only if x_0 is isolated in the relative topology of $\partial\Omega$). In such case *constant* Dirichlet data can be prescribed on $\mathcal{S}_1$. However, this need not be the case for general Dirichlet data, as the following example shows.

(c) Consider the problem

$$\begin{cases} \frac{1}{y \sin x}(u_{xx} + y^2 u_{yy}) = f & \text{in } (\frac{\pi}{4}, \frac{3\pi}{4}) \times (0,1) = \Omega, \\ u = g & \text{on } \partial\Omega \setminus ([\frac{\pi}{4}, \frac{3\pi}{4}] \times \{0\}) = \mathcal{R} \end{cases} \tag{5.26}$$

with $f \in C(\Omega) \cap L^\infty(\Omega)$, $g \in C(\mathcal{R}) \cap L^\infty(\mathcal{R})$. Here we take $\mathcal{S}_1 = \mathcal{S} = [\frac{\pi}{4}, \frac{3\pi}{4}] \times \{0\}$, $\mathcal{S}_2 = \emptyset$.

It is easily checked that the function $Z(x, y) := x^2 + \log y - \pi^2$ satisfies

$$Z < 0 \quad \text{in } \Omega, \qquad \mathcal{L}Z = \frac{1}{y \sin x} > 0 \quad \text{in } \Omega, \qquad \lim_{y \to 0} Z(x, y) = -\infty.$$

Then by Proposition 2.16 there exists at most one solution $u \in L^\infty(\Omega)$ of problem (5.26).

On the other hand, the function $V \in C(\overline{\Omega})$, $V(x, y) := y \sin x$ satisfies

$$V > 0 \quad \text{in } \Omega \cup \mathcal{R}, \qquad V = 0 \quad \text{on } \mathcal{S}_1, \qquad \mathcal{L}V = -1 \quad \text{in } \Omega,$$

thus $\mathcal{S}_1$ is attracting (see Definition 5.1). By Proposition 5.2 there exists a solution $u_0 \in L^\infty(\Omega)$ of the problem

$$\begin{cases} \frac{1}{y \sin x}(u_{xx} + y^2 u_{yy}) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In view of the above uniqueness result, this implies that there exists no solution $u_g \in L^\infty(\Omega)$ of the problem

$$\begin{cases} \frac{1}{y \sin x}(u_{xx} + y^2 u_{yy}) = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases}$$

with $g \in C(\partial\Omega)$, $g = 0$ on $\mathcal{R}$, $g(\bar{x}) \neq 0$ at some point $\bar{x} \in \mathcal{S}_1$.

Let us add some remarks concerning Proposition 5.2. If some subset $\Sigma \subseteq \partial\overline{\Omega}$ is attracting and the coefficients $a_{i,j}, b_i$ are bounded in Σ^ε for some $\varepsilon > 0$, for any $x_0 \in \Sigma$ a local barrier does exist, thus general Dirichlet data g can be assigned on Σ. This is the content of the following proposition (see [20]).

Proposition 5.3. *Let assumptions* (H_1), (H_2) *and* (H_4) *be satisfied. Let* $\Sigma \subseteq \partial\overline{\Omega}$ *be attracting*; *suppose* $a_{i,j}, b_i \in L^\infty(\Sigma^\varepsilon)$ *for some* $\varepsilon \in (0, \varepsilon_0)$ $(i, j = 1, \dots, n)$. *Then for any* $x_0 \in \Sigma$ *there exists a local barrier.*

Finally, let us mention a nonuniqueness result for problem (1.3), which immediately follows from Proposition 5.2.

Corollary 5.4. *Let the assumptions of Proposition* 5.2 *be satisfied*; *suppose* $\overline{\mathcal{R}} \cap \overline{\mathcal{S}_1} = \emptyset$. *Then there exist infinitely many solutions of problem* (1.3).

5.2. Uniqueness

(a) Consider the problem

$$\begin{cases} u_{xx} + y^2 u_{yy} + y u_y = f & \text{in } (0,1)\times(0,1) = \Omega, \\ u = g & \text{on } \partial\Omega \setminus ([0,1]\times\{0\}) = \mathcal{R} \end{cases} \tag{5.27}$$

with $f \in C(\Omega) \cap L^\infty(\Omega)$, $g \in C(\mathcal{R}) \cap L^\infty(\mathcal{R})$. Here we take $\mathcal{S}_1 = \emptyset$, $\mathcal{S}_2 = \mathcal{S} = [0,1]\times\{0\}$.

It is easily checked that the function $Z(x,y) := x^2 + \log y - 2$ satisfies

$$Z \leqslant -1 \quad \text{in } \Omega, \qquad \mathcal{L}Z = 0 \quad \text{in } \Omega, \qquad \lim_{y\to 0} Z(x,y) = -\infty.$$

Then by Proposition 2.16 there exists at most one solution $u \in L^\infty(\Omega)$ of problem (5.27). Moreover, observe that the function $F(x,y) = -x^2 + 1$ satisfies

$$F > 0 \quad \text{in } \Omega, \qquad \mathcal{L}F < -1 \quad \text{in } \Omega.$$

Then by Proposition 2.7 and the above uniqueness result problem (5.27) is well posed in $L^\infty(\Omega)$.

(b) Consider the problem

$$\begin{cases} (x - \frac{1}{2})^4 [u_{xx} + y^2(1-y) u_{yy}] - u = f & \text{in } (0,1)\times(0,1) = \Omega, \\ u = g & \text{on } \mathcal{R} \cup \mathcal{S}_1 \end{cases} \tag{5.28}$$

with $f \in C(\Omega)$, $g \in C(\mathcal{R} \cup \mathcal{S}_1)$; here $\mathcal{R} = [\{0\}\times(0,1)] \cup [\{1\}\times(0,1)]$, $\mathcal{S}_1 = [0,1]\times\{1\}$, $\mathcal{S}_2 = [0,1]\times\{0\}$. Observe that the operator $\mathcal{L}$ degenerates on $\{\frac{1}{2}\}\times(0,1) \subseteq \Omega$.

The function $Z(x,y) := x^2 + \log y - 2$ satisfies

$$Z \leqslant -1 \quad \text{in } \Omega, \qquad \mathcal{L}Z \geqslant Z \quad \text{in } \Omega, \qquad \lim_{y\to 0} Z(x,y) = -\infty.$$

Then by Proposition 2.16 there exists at most one solution $u \in L^\infty(\Omega)$ of problem (5.28).

(c) Consider the problem

$$\begin{cases} u_{xx} + y^2 u_{yy} - u_y - \frac{y+1}{y|\log y|} u = f & \text{in } (0,1)\times(0,1) = \Omega, \\ u = g & \text{on } \partial\Omega \setminus ([0,1]\times\{0\}) = \mathcal{R} \end{cases} \tag{5.29}$$

with $f \in C(\Omega) \cap L^\infty(\Omega)$, $g \in C(\mathcal{R}) \cap L^\infty(\mathcal{R})$. Here we take $\mathcal{S}_1 = \emptyset$, $\mathcal{S}_2 = \mathcal{S} = [0,1]\times\{0\}$. Observe that the coefficient c does not belong to $L^\infty(\Omega)$.

The function $Z(x,y) = \log y - 1$ satisfies

$$Z \leqslant -1 \quad \text{in } \Omega \cup \mathcal{R}, \qquad \mathcal{L}Z - cZ = c \geqslant 0 \quad \text{in } \Omega, \qquad \lim_{y\to 0} Z(x,y) = -\infty;$$

hence by Proposition 2.16 there exists at most one bounded solution u of problem (5.29). Further observe that the function $V(x,y) := (x - \frac{1}{2})^2 + 3y$ is a positive supersolution of Eq. (2.13). Then, in view of Proposition 2.7 and the above uniqueness result, problem (5.29) is well posed in $L^\infty(\Omega)$.

It is worth observing that

$$\inf_{\Omega\cup\mathcal{R}} V = 0 < \inf_{\mathcal{R}} V = \frac{1}{4}, \qquad \mathcal{L}V = -1 \quad \text{in } \Omega;$$

however, Theorem 2.5 does not apply since the coefficient c is unbounded.

(d) Consider the problem

$$\begin{cases} u_{xx} + y^2(1-y)^2 u_{yy} - \frac{(y+1)^2(y^2+2)}{x} u_x = f & \text{in } (0,\infty) \times (0,1) = \Omega, \\ u = g & \text{on } \{0\} \times (0,1) = \mathcal{S}_1 \end{cases} \tag{5.30}$$

with $f \in C(\Omega)$, $g \in C(\mathcal{S}_1)$; here we take $\mathcal{R} = \emptyset$, $\mathcal{S} = \partial\Omega$, $\mathcal{S}_1 = \{0\} \times (0,1)$, $\mathcal{S}_2 = [0,\infty) \times \{0,1\}$.

The function

$$Z(x,y) = -\left(x^2 + y^2\right) + \log\left[y(1-y)\right] - 1$$

satisfies

$$Z \leqslant -1 \quad \text{in } \Omega, \qquad \mathcal{L}Z \geqslant 0 \quad \text{in } \Omega,$$
$$\lim_{y\to 0} Z(x,y) = \lim_{y\to 1} Z(x,y) = \lim_{x\to\infty} Z(x,y) = -\infty.$$

In view of Proposition 2.16, there exists at most one solution in $L^\infty(\Omega)$ of problem (5.30).

(e) Consider the problem[2]

$$\begin{cases} x^2 u_{xx} + u_{yy} + 3x u_x = f & \text{in } (0,\infty) \times \mathbb{R} = \Omega, \\ \lim_{|x|+|y|\to\infty} u(x,y) = L \end{cases} \tag{5.31}$$

with $f \in C(\Omega)$, $L \in \mathbb{R}$. In this case $\mathcal{R} = \mathcal{S}_1 = \emptyset$, $\mathcal{S} = \mathcal{S}_2 = \{0\} \times \mathbb{R}$. Consider the function

$$Z(x,y) = -\frac{1}{dist((x,y),\mathcal{S})} - 1 = -\frac{1}{x} - 1 \quad \big((x,y) \in \Omega\big).$$

It is easily seen that

$$Z \leqslant -1 \quad \text{in } \Omega, \qquad \mathcal{L}Z > 0 \quad \text{in } \Omega, \qquad \lim_{x\to 0} Z(x,y) = -\infty.$$

In view of Proposition 2.17, for any $L \in \mathbb{R}$ problem (5.31) admits at most one bounded solution. Observe that the "condition at infinity" is necessary to ensure uniqueness: in fact, any constant is a bounded solution of the differential equation in (5.31) with $f = 0$.

The sub- and supersolutions constructed in the above examples are smooth in Ω; however, less regularity is needed for the general results to hold (see Definition 2.2). This is expedient in several respects; for instance, the subsolution used to prove uniqueness is often a function of the distance from the boundary, thus its smoothness depends on that of the latter. A simple example is given below.

(f) Consider the problem

$$\begin{cases} a_{11} u_{xx} + a_{22} u_{yy} + 2x u_x + 2y u_y - |\log|1-x^2||u = f & \text{in } \Omega, \\ u = g & \text{on } \mathcal{R} \cup \mathcal{S}_1, \end{cases} \tag{5.32}$$

where

$$\Omega = \big((-1,1) \times [0,1)\big) \cup \big((-1,0) \times (-1,0)\big),$$
$$\mathcal{R} = \big([-1,0] \times \{-1\}\big) \cup \big([-1,1] \times \{1\}\big),$$
$$\mathcal{S}_1 = \big(\{-1\} \times (-1,1)\big) \cup \big(\{1\} \times (0,1)\big),$$
$$\mathcal{S}_2 = \big(\{0\} \times (-1,0)\big) \cup \big([0,1] \times \{0\}\big);$$

$$a_{11}(x,y) = \begin{cases} x^2 + y^2 & \text{if } x \in (-1,1),\ y \in [0,1), \\ x^2 & \text{if } x \in (-1,0),\ y \in (-1,0), \end{cases}$$
$$a_{22}(x,y) = \begin{cases} x^2 + y^2 & \text{if } x \in (-1,0),\ y \in (-1,1), \\ y^2 & \text{if } x \in [0,1),\ y \in (0,1) \end{cases}$$

[2] This example was suggested by X. Cabré.

and $f \in C(\Omega)$, $g \in C(\mathcal{R} \cup \mathcal{S}_1)$. Since

$$dist\big((x,y),\mathcal{S}_2\big) = \begin{cases} y & \text{if } x \in [0,1),\ y \in (0,1), \\ \sqrt{x^2+y^2} & \text{if } x \in (-1,0),\ y \in (0,1), \\ -x & \text{if } x \in (-1,0),\ y \in (-1,0] \end{cases}$$

for any $(x,y) \in \Omega$), it is easily seen that the function

$$Z(x,y) = 2\log\big[dist\big((x,y),\mathcal{S}_2\big)\big] - \log 3 \quad \big((x,y) \in \Omega\big)$$

belongs to $C^1(\Omega) \cap C(\Omega \cup \mathcal{R})$, but not to $C^2(\Omega)$. However, $Z \in C^2(\Omega \setminus [\{0\} \times (0,1) \cup (-1,0) \times \{0\}])$ and there holds

$$Z \leqslant \log\frac{2}{3} \quad \text{in } \Omega, \qquad \mathcal{L}Z \geqslant 0 \quad \text{a.e. in } \Omega;$$

hence we have

$$\int_\Omega Z\{\mathcal{L}^*\psi - c\psi\}\,dx = \int_\Omega \{\mathcal{L}Z - cZ\}\psi\,dx \geqslant 0$$

for any $\psi \in C_0^\infty(\Omega)$, $\psi \geqslant 0$. Clearly, condition (2.23) is satisfied; hence by Proposition 2.16 problem (5.32) admits at most one bounded solution.

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