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Navier–Stokes equations with nonhomogeneous boundary conditions in a convex bi-dimensional domain

Vincent Girinon

Institut de Mathématiques, UMR CNRS 5219, Université Paul Sabatier, 31062 Toulouse Cedex 9, France

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Abstract

We prove the existence of a weak solution to Navier–Stokes equations describing the isentropic flow of a gas in a convex and bounded region, $\Omega \subset \mathbf{R}^2$, with nonhomogeneous Dirichlet boundary conditions on $\partial\Omega$. These results are also extended to flow domain surrounding an obstacle.

MSC: 35Q30; 35A05

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1. Introduction

1.1. Mathematical model

In the case of the isentropic flow of a gas in an open set $\Omega \subset \mathbf{R}^2$, Navier–Stokes equations can be written in the form:

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \mu \Delta \mathbf{u} - (\lambda + \mu)\nabla \operatorname{div} \mathbf{u} + \nabla p = \rho \mathbf{f}, \tag{1.2}$$

where the density $\rho = \rho(t, \mathbf{x})$, the velocity $\mathbf{u} = \mathbf{u}(t, \mathbf{x}) = (u^1(t, \mathbf{x}), u^2(t, \mathbf{x}))$ and the pressure $p = p(t, \mathbf{x})$ are functions of the time $t \in (0, T)$ and the spatial coordinate $\mathbf{x} \in \Omega$ ($T \in \mathbf{R}_+^*$). Moreover, the pressure only depends on the density according to the relation:

$$p = a\rho^\gamma \quad (a > 0), \tag{1.3}$$

where the adiabatic constant γ satisfies the condition: $\gamma > 1$.

The viscosity coefficients μ and λ are such that $\mu > 0$ and $\lambda + \mu \geqslant 0$.

Finally, $\mathbf{f} \in \mathbf{L}^\infty(\Omega \times (0, T))$ corresponds to external force density acting on the fluid.

E-mail address: girinon.vincent@neuf.fr.

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The initial conditions for the density and the velocity are

$$\begin{cases} \rho(0,\mathbf{x}) = \rho_0(\mathbf{x}) & \text{in } \Omega, \\ \mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) & \text{in } \Omega, \end{cases} \tag{1.4}$$

with

$$\begin{aligned} &\rho_0 \in L^\gamma(\Omega), \qquad \rho_0 \geqslant 0 \quad \text{a.e. on } \Omega, \\ &\mathbf{u}_0 \in \mathbf{L}^2(\Omega), \qquad \sqrt{\rho_0}\mathbf{u}_0 \in \mathbf{L}^2(\Omega). \end{aligned} \tag{1.5}$$

The system is completed by the following boundary conditions

$$\begin{cases} \rho(t,\mathbf{x}) = \rho_\infty(t,\mathbf{x}) & \text{on } \bigcup_{t\in(0,T)} \{t\} \times \Gamma_e^t, \\ \mathbf{u}(t,\mathbf{x}) = \mathbf{a}_\infty(t,\mathbf{x}) & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{1.6}$$

where

$$\Gamma_e^t := \{\mathbf{x} \in \partial\Omega\colon\ \mathbf{a}_\infty(t,\mathbf{x})\cdot\mathbf{n}(\mathbf{x}) < 0\}, \tag{1.7}$$

$\mathbf{n}(\mathbf{x})$ is the unit outward normal vector at $\mathbf{x} \in \partial\Omega$, with

$$\mathbf{a}_\infty \in \mathbf{C}^1([0,T]\times\mathbf{R}^2) \quad \text{and} \quad \rho_\infty \in L^\infty\Big(\bigcup_{t\in(0,T)} \{t\}\times\Gamma_e^t\Big). \tag{1.8}$$

1.2. Context

The first general proof giving the existence of a solution $(\rho, \mathbf{u})$ to the system (1.1), (1.2), (1.4), (1.6) was obtained by P.-L. Lions [6,7] (in three-dimensional space) for homogeneous boundary conditions: $\mathbf{a}_\infty(t,\mathbf{x}) = \mathbf{0}$ on $(0,T)\times\partial\Omega$ (and so $\Gamma_e^t = \emptyset$). This approach was improved by E. Feireisl (see [2–4]) who released the constraint concerning the adiabatic exponent γ in Lions' work.

The case of nonhomogeneous boundary conditions was studied by S. Novo [8] in the particular case of an open set $\Omega \subset \mathbf{R}^3$ defined by $\Omega = B(\mathbf{x}_0, R_0) \setminus S$, where $B(\mathbf{x}_0, R_0)$ is the open ball of center $\mathbf{x}_0$ and radius $R_0 > 0$, and where S is a compact set included in $B(\mathbf{x}_0, \frac{R_0}{2})$ satisfying the cone property. Novo considers the case of constant boundary conditions on $\partial B(\mathbf{x}_0, R_0)$: $\mathbf{a}_\infty \in \mathbf{R}^3 \setminus \{\mathbf{0}\}$, $\rho_\infty > 0$ (Γ_e^t is then independent of time and corresponds to a half-sphere included in $\partial B(\mathbf{x}_0, R_0)$).

1.3. Our purpose

The goal of this article is to prove the existence of a solution $(\rho, \mathbf{u})$ to the system (1.1), (1.2), (1.4), (1.6) under the following hypotheses:

(**H1**) **Description of Ω.**
The problem is studied in an open convex and bounded set Ω, included in $\mathbf{R}^2$, of type

$$\Omega = g^{-1}(]-\infty, 1[),$$

where $g : \mathbf{R}^2 \to \mathbf{R}$ is convex and C^1 on $\mathbf{R}^2$. We also suppose that $\mathbf{0} \in \Omega$.

(**H2**) **Description of the incoming border area.**
Γ_e^t is independent of t and is the intersection of $\partial\Omega$ with a cone of vertex $\mathbf{0}$. In the sequel, we simply write Γ_e rather than Γ_e^t.

Notations. More precisely, Γ_e is defined by

$$\Gamma_e = \partial\Omega \cap C_{\theta_1,\theta_2}, \tag{1.9}$$

where θ_1, θ_2 are real numbers such that $0 \leqslant \theta_1 < \theta_2 < 2\pi$, and where

$$C_{\theta_1,\theta_2} := \{(r\cos(\theta), r\sin(\theta)),\ (r,\theta) \in \mathbf{R}_+^* \times]\theta_1, \theta_2[\}. \tag{1.10}$$

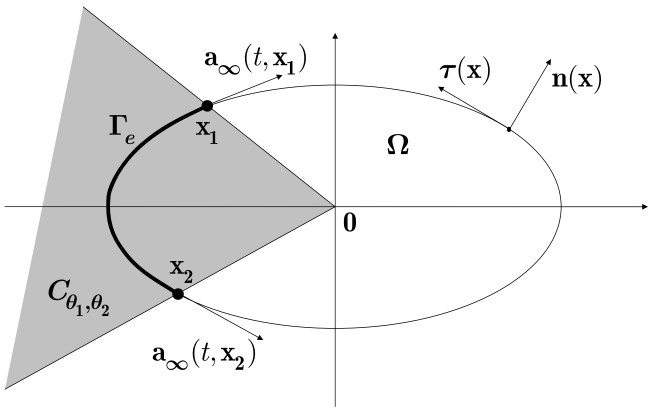

Fig. 1.

At any point $\mathbf{x}$ of $\partial\Omega$, let $\mathbf{n}(\mathbf{x})$ be the unit outward normal vector to Ω and $\tau(\mathbf{x})$ be the unit tangent vector (image of $\mathbf{n}(\mathbf{x})$ under rotation of angle $+\frac{\pi}{2}$). Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the points of $\partial\Omega$ belonging to the boundary of C_{θ_1,θ_2}. These points are of type:

$$\mathbf{x}_1 = \big(|\mathbf{x}_1|\cos(\theta_1), |\mathbf{x}_1|\sin(\theta_1)\big) \quad \text{and} \quad \mathbf{x}_2 = \big(|\mathbf{x}_2|\cos(\theta_2), |\mathbf{x}_2|\sin(\theta_2)\big). \tag{1.11}$$

Then, for $\mathbf{x} \in \{\mathbf{x}_1, \mathbf{x}_2\}$ and $t \in [0, T]$, the vector $\mathbf{a}_\infty(t, \mathbf{x})$ is colinear to $\tau(\mathbf{x})$. In addition, we suppose that:

(H3) Sufficient condition for no reflux.
(See Fig. 1.) For all $t \in [0, T]$,

$$\mathbf{a}_\infty(t, \mathbf{x}_1) = -\big|\mathbf{a}_\infty(t, \mathbf{x}_1)\big|\tau(\mathbf{x}_1) \quad \text{and} \quad \mathbf{a}_\infty(t, \mathbf{x}_2) = +\big|\mathbf{a}_\infty(t, \mathbf{x}_2)\big|\tau(\mathbf{x}_2).$$

Remark 1.1. The boundary condition on density is a *data* of our problem: the flow in Ω depends on this data but the opposite is false. From this point of view, we can understand assumption (**H3**) as a way to express the "independence" of Γ_e from the flow in Ω (intuitively, (**H3**) forbids fluid's particles which leave Ω to follow its boundary until Γ_e and next to go back in Ω). In our work, hypothesis (**H3**) appears in the proof of Lemma 2.5. This lemma gives the essential argument which enables us to construct an interesting extension of the initial density in order to recover $\rho = \rho_\infty$ on $(0, T) \times \Gamma_e$ (see Lemma 2.7 and Section 4.1).

In accordance with the terminology of [9, p. 413], we call *a bounded energy renormalized weak solution* of system (1.1), (1.2), (1.4), (1.6), any pair $(\rho, \mathbf{u})$ such that:

(i) *Regularity and initial conditions*.

$$\begin{cases} \rho \in L^\infty\big(0, T; L^\gamma(\Omega)\big) \quad \text{and} \quad \rho \geqslant 0 \quad \text{a.e. on } \Omega \times (0, T), \\ \mathbf{u} \in L^2\big(0, T; \mathbf{W}^{1,2}(\Omega)\big) \quad \text{with } \mathbf{u}(t, \mathbf{x}) = \mathbf{a}_\infty(t, \mathbf{x}) \text{ on } \partial\Omega \times (0, T), \\ \rho \in C\big([0, T]; L^\gamma_w(\Omega)\big) \quad \text{and} \quad \rho\mathbf{u} \in C\big([0, T]; \mathbf{L}_w^{\frac{2\gamma}{\gamma+1}}(\Omega)\big) \quad \text{with} \begin{cases} \rho(0) = \rho_0, \\ (\rho\mathbf{u})(0) = \rho_0\mathbf{u}_0. \end{cases} \end{cases} \tag{1.12}$$

(ii) *Equations' interpretation*. The continuity equation (1.1) and the momentum equation (1.2) hold in $\mathcal{D}'(\Omega \times (0, T))$.

(iii) *Boundary conditions for density*. For any function $\eta \in \mathcal{D}(\mathbf{R}^2 \times (0, T))$ such that $\eta = 0$ on Q_s, where

$$Q_s := \bigcup_{t \in (0,T)} \big\{\mathbf{x} \in \partial\Omega \colon \mathbf{a}_\infty(t, \mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) > 0\big\} \times \{t\}, \tag{1.13}$$

we have

$$\int_0^T \int_\Omega (\rho \partial_t \eta + \rho \mathbf{u} \cdot \nabla \eta)\, d\mathbf{x}\, dt = \int_0^T \int_{\Gamma_e} \rho_\infty \mathbf{a}_\infty \cdot \mathbf{n} \eta \, d\mathbf{S}\, dt. \tag{1.14}$$

Moreover, for any function $b \in C(\mathbf{R}_+) \cap C^1(\mathbf{R}_+^*)$ such that

$$\begin{cases} |b'(s)| \leqslant c s^{\lambda_0} & \text{if } s \in]0, 1], \\ |b'(s)| \leqslant c s^{\lambda_1} & \text{if } s \in [1, +\infty[, \end{cases} \tag{1.15}$$

with

$$c > 0, \qquad \lambda_0 \in]-1, +\infty[, \qquad \lambda_1 \in \left]-\infty, \frac{\gamma}{2} - 1\right], \tag{1.16}$$

we have

$$\int_0^T \int_\Omega \big(b(\rho)\partial_t \eta + b(\rho)\mathbf{u} \cdot \nabla \eta\big) - \int_0^T \int_\Omega \big(\rho b'(\rho) - b(\rho)\big) \operatorname{div} \mathbf{u} \eta = \int_0^T \int_{\Gamma_e} b(\rho_\infty)\mathbf{a}_\infty \cdot \mathbf{n}\eta. \tag{1.17}$$

(iv) *Energy inequality*. For almost every $t \in (0, T)$,

$$\begin{aligned} &E(t) + \mu \int_0^t \int_\Omega \big[\nabla(\mathbf{u} - \mathbf{u}_\infty)\big]^2 d\mathbf{x}\, ds + (\lambda + \mu) \int_0^t \int_\Omega \big[\operatorname{div}(\mathbf{u} - \mathbf{u}_\infty)\big]^2 d\mathbf{x}\, ds \\ &\quad \leqslant E_0 + \int_0^t \int_\Omega \rho \mathbf{f} \cdot (\mathbf{u} - \mathbf{u}_\infty)\, d\mathbf{x}\, ds - \int_0^t \int_\Omega \rho \partial_t \mathbf{u}_\infty \cdot (\mathbf{u} - \mathbf{u}_\infty)\, d\mathbf{x}\, ds - \mu \int_0^t \int_\Omega \nabla \mathbf{u}_\infty : \nabla(\mathbf{u} - \mathbf{u}_\infty)\, d\mathbf{x}\, ds \\ &\qquad - (\lambda + \mu) \int_0^t \int_\Omega \operatorname{div} \mathbf{u}_\infty \operatorname{div}(\mathbf{u} - \mathbf{u}_\infty)\, d\mathbf{x}\, ds - \int_0^t \int_\Omega (\rho \mathbf{u}) \cdot \big[\big((\mathbf{u} - \mathbf{u}_\infty) \cdot \nabla\big)\mathbf{u}_\infty\big]\, d\mathbf{x}\, ds \\ &\qquad - a \int_0^t \int_\Omega \rho^\gamma \operatorname{div} \mathbf{u}_\infty \, d\mathbf{x}\, ds - \int_0^t \int_{\Gamma_e} \frac{a}{\gamma - 1} \rho_\infty^\gamma \mathbf{u}_\infty \cdot \mathbf{n}\, d\mathbf{S}\, ds, \end{aligned} \tag{1.18}$$

with

$$\begin{cases} E(t) := \dfrac{1}{2} \displaystyle\int_\Omega \rho(t)(\mathbf{u} - \mathbf{u}_\infty)(t)^2 \, d\mathbf{x} + \int_\Omega \frac{a}{\gamma - 1} \rho(t)^\gamma \, d\mathbf{x}, \\ E_0 := \dfrac{1}{2} \displaystyle\int_\Omega \rho_0 \big(\mathbf{u}_0 - \mathbf{u}_\infty(0)\big)^2 \, d\mathbf{x} + \int_\Omega \frac{a}{\gamma - 1} \rho_0^\gamma \, d\mathbf{x}, \end{cases}$$

where $\mathbf{u}_\infty$ is a vector field defined on $[0, T] \times \mathbf{R}^2$ that is equal to $\mathbf{a}_\infty$ on $[0, T] \times \partial\Omega$ (it will be introduced in Section 2.3).

We are ready to state the main result of this paper:

Theorem 1.1. *For initial conditions ρ_0 and $\mathbf{u}_0$ which fulfil* (1.5)*, for boundary conditions $\mathbf{a}_\infty$ and ρ_∞ which fulfil* (1.8) *and under hypotheses* (**H1**)*,* (**H2**)*,* (**H3**)*, there exists a bounded energy renormalized weak solution to the problem* (1.1)*,* (1.2)*,* (1.4)*,* (1.6)*.*

1.4. Sketch of the proof

For proving Theorem 1.1, we follow a method similar to the one used by S. Novo in [8]. The strategy consists in extending the domain of study (we work on an open set D which contains Ω) and, consequently, in extending the data (initial conditions, density $\mathbf{f}$) in order to come back to a problem with Dirichlet homogeneous condition for the velocity (on the boundary of D). Then the problem is solved by the method of Lions and Feireisl. It consists in working with a "perturbed version" of Eqs. (1.1) and (1.2) on which Novo added a penalization's term. More precisely, the system is:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = \varepsilon \Delta \rho, \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \mu \Delta \mathbf{u} - (\lambda + \mu) \nabla \operatorname{div} \mathbf{u} + \nabla\big(a\rho^\gamma + \delta \rho^\beta\big) + \varepsilon \nabla \rho \cdot \nabla \mathbf{u} + m \chi_E (\mathbf{u} - \mathbf{u}_\infty) = \rho \mathbf{f}. \end{cases} \tag{1.19}$$

Here, $\varepsilon > 0$, $\delta > 0$, $m > 0$ and $\beta > \max\{6, \gamma\}$. χ_E is the characteristic function of the set $E := D \setminus \Omega$. The equations are studied on $D \times (0, T)$ and we consider the following boundary conditions:

$$\mathbf{u}(t, \mathbf{x}) = \mathbf{0} \quad \text{and} \quad \partial_{\mathbf{n}} \rho(t, \mathbf{x}) = 0 \quad \text{on } \partial D \times (0, T). \tag{1.20}$$

The system is completed by suitable initial conditions:

$$\rho(0, \mathbf{x}) = \widetilde{\rho}_{0,\varepsilon,\delta}(\mathbf{x}) \quad \text{and} \quad \mathbf{u}(0, \mathbf{x}) = \widetilde{\mathbf{u}}_{0,\varepsilon,\delta}(\mathbf{x}) \quad \text{in } D, \tag{1.21}$$

where $\widetilde{\rho}_0$ and $\widetilde{\mathbf{u}}_0$ are extensions of ρ_0 and $\mathbf{u}_0$ ($\widetilde{\mathbf{u}}_{0,\varepsilon,\delta}$, $\widetilde{\rho}_{0,\varepsilon,\delta}$ denote some "regularized versions" of $\widetilde{\mathbf{u}}_0$ and $\widetilde{\rho}_0$). We extend simply $\mathbf{u}_0$ by $\mathbf{u}_\infty(0)$ outside Ω. The extension of ρ_0 is more complex and depends on ρ_∞ in order to take the boundary conditions on density (1.6) into account (see Section 4.1). Let us present the main points of the proof of Theorem 1.1.

– *Step A: Construction of approximate solutions.*

By a Faedo–Galerkin method, we prove that, for any $n \in \mathbf{N}^*$, there exists an approximate solution $(\rho_n, \mathbf{u}_n)$ to the problem (1.19)–(1.21). The integer n is relative to the dimension of the vector space which belongs the vector field $\mathbf{u}_n(t, \cdot)$ for $t \in [0, T]$. By adding the diffusive term $\varepsilon \Delta \rho$ to the continuity equation, we obtain a parabolic equation which "regularizes" the density. Among other things, this regularization allows calculations which provide an energy estimate for the approximate solutions. Indeed, for $t \in [0, T]$, we have

$$\begin{aligned} &E[\rho_n, \mathbf{u}_n](t) + \varepsilon \int_0^t \int_D \delta \beta \rho_n^{\beta-2} |\nabla \rho_n|^2 + \mu \int_0^t \int_D |\nabla \mathbf{u}_n|^2 + (\lambda + \mu) \int_0^t \int_D |\operatorname{div} \mathbf{u}_n|^2 \\ &\quad \leqslant E[\rho_n, \mathbf{u}_n](0) + \int_0^t \int_D \rho_n \mathbf{f} \cdot \mathbf{u} - m \int_0^t \int_E (\mathbf{u}_n - \mathbf{u}_\infty) \cdot \mathbf{u}_n, \end{aligned} \tag{1.22}$$

where $E[\rho_n, \mathbf{u}_n](t) := \frac{1}{2} \int_D \rho_n \mathbf{u}_n^2(t) + \int_D (\frac{a}{\gamma-1} \rho_n^\gamma(t) + \frac{\delta}{\beta-1} \rho_n^\beta(t))$.

– *Step B: Passage to the limit $n \to +\infty$.*

Choosing a suitable subsequence (still indexed by n), we prove that $\{(\rho_n, \mathbf{u}_n)\}$ converges to a pair $(\rho_m, \mathbf{u}_m)$ which is an "exact solution" of (1.19)–(1.21). Actually, the "perturbed continuity equation" is satisfied in the strong sense on $D \times (0, T)$, whereas the momentum equation holds in $\mathcal{D}'(D \times (0, T))$. At last, replacing ρ_n by ρ_m, $\mathbf{u}_n$ by $\mathbf{u}_m$ and $E[\rho_n, \mathbf{u}_n](0)$ by $\frac{1}{2} \int_D \widetilde{\rho}_{0,\varepsilon,\delta} \widetilde{\mathbf{u}}_{0,\varepsilon,\delta}^2 + \int_D (\frac{a}{\gamma-1} \widetilde{\rho}_{0,\varepsilon,\delta}^\gamma + \frac{\delta}{\beta-1} \widetilde{\rho}_{0,\varepsilon,\delta}^\beta)$, the energy inequality (1.22) holds a.e. on $(0, T)$.

Step C: Passage to the limit $m \to +\infty$.

This step consists in getting rid of the penalization term $m \chi_E (\mathbf{u} - \mathbf{u}_\infty)$ in order to "force" the velocity to take boundary conditions into account (recall that $\mathbf{u}_\infty$ coincides with $\mathbf{a}_\infty$ on $\partial \Omega$). More precisely, we want to prove that an extracted sequence of $\{(\rho_m, \mathbf{u}_m)\}$ converges to a pair $(\rho_\varepsilon, \mathbf{u}_\varepsilon)$ such that $\mathbf{u}_\varepsilon = \mathbf{u}_\infty$ on $E \times (0, T)$. The first difficulty comes from the energy inequality (1.22) satisfied by ρ_m and $\mathbf{u}_m$: it does not permit to obtain estimates independent of m because of the term $-m \int_0^t \int_E (\mathbf{u}_m - \mathbf{u}_\infty) \cdot \mathbf{u}_m$. It is solved by working with a new estimate deduced from (1.22) and from two new integral relations established by using some suitable test functions (depending on $\mathbf{u}_\infty$) in Eqs. (1.19).

– *Step D: Passage to the limit $\varepsilon \to 0$.*

Here, we prove that $\{(\rho_\varepsilon, \mathbf{u}_\varepsilon)\}$ converges to a couple of functions $(\rho_\delta, \mathbf{u}_\delta)$. In this step, the difficulty comes from the lack of suitable estimates on the sequence $\{\rho_\varepsilon\}$: whereas the strong convergence of the density was deduced from results of parabolic regularity in steps B and C, it comes here from properties of the *effective viscous flux* $a\rho_\varepsilon^\gamma + \delta\rho_\varepsilon^\beta - (\lambda + \mu)\operatorname{div}\mathbf{u}_\varepsilon$ discovered by Lions. Remark that the passage to the limit in the energy inequality is also a problem in view of the lack of information about $\{\rho_\varepsilon\}$. In our case, this matter will be solved thanks to the specific construction of the vector field $\mathbf{u}_\infty$ (see Section 3.5).

At the end of step C, we have $\mathbf{u}_\varepsilon = \mathbf{u}_\infty$ on $E \times (0, T)$, so we can deduce that $\mathbf{u}_\delta = \mathbf{u}_\infty$ on $E \times (0, T)$. This last result is used in order to "recover" the boundary condition for the density.

– *Step E: Passage to the limit $\delta \to 0$.*

The purpose of this final step is to show that the sequence $\{(\rho_\delta, \mathbf{u}_\delta)\}$ converges to a couple $(\rho, \mathbf{u})$ which is a bounded energy renormalized weak solution of system (1.1), (1.2), (1.4), (1.6). This part uses ideas developed by E. Feireisl, mainly the essential result about the oscillations amplitude of the density, that is

$$\sup_{k>0}\Big(\limsup_{\delta\to 0}\big\|T_k(\rho_\delta) - T_k(\rho)\big\|_{L^{\gamma+1}(\Omega\times(0,T))}\Big) \leqslant C, \tag{1.23}$$

where $C > 0$ is a constant (independent of δ and k) and T_k ($k \in \mathbb{N}^*$) a "cut-off" function defined by

$$T_k(s) := kT\left(\frac{s}{k}\right), \quad \text{where } \begin{cases} T \in C^\infty(\mathbf{R}), \quad T \text{ is increasing and concave on } \mathbf{R}, \\ T(s) = s \quad \text{for } s \leqslant 1, \qquad T(s) = 2 \quad \text{for } s \geqslant 3. \end{cases} \tag{1.24}$$

This result enables us to end the proof of Theorem 1.1.

1.5. *Outline of this paper*

In Section 2, we specify the "extended domain" D, the extensions of the data as well as the vector field $\mathbf{u}_\infty$. We do not detail the steps A, B and C because they can be found in [8] or in [9]. Sections 3 and 4 are respectively dedicated to steps D and E. Finally, Section 5 deals with other particular configurations as the presence of an internal obstacle or the case of a rectangular domain. In these cases, we indicate the modifications in the proof to obtain the existence of a weak solution.

2. Departure model

2.1. *Extension of domain*

According to hypothesis (**H1**), there exists $R_0 > 0$ such that

$$\overline{\Omega} \subset B(\mathbf{0}, R_0), \tag{2.1}$$

where $B(\mathbf{0}, R_0)$ denotes the open disk of center $\mathbf{0}$ and radius R_0. We define

$$D := B(\mathbf{0}, R_0 + 3). \tag{2.2}$$

2.2. *Extension of data*

We extend $\mathbf{f}$ by $\mathbf{0}$ outside Ω (and we still note $\mathbf{f}$ this extension) so that: $\mathbf{f} \in \mathbf{L}^\infty(D \times (0, T))$.

The regularity of the initial conditions plays an important role in the proof. For clarity, we will specify in the beginning of each step with what type of initial conditions we work.

- So, in the statement of Theorem 3.1, $\mathbf{u}_0$ and ρ_0 denote the initial conditions for the "extended model" (they are defined on D).
- In Section 3.3, where we start the description of the step D, initial conditions are still defined on D.
- Finally, in Section 4, $\mathbf{u}_0$ and ρ_0 will designate the initial conditions defined by (1.5). Then we specify the suitable extensions on D that permit to prove Theorem 1.1.

2.3. Vector field $\mathbf{u}_\infty$

The vector field $\mathbf{u}_\infty$, which appears in the energy inequality (1.18), plays an essential role in our proof: it is constructed in order to coincide with $\mathbf{a}_\infty$ on $(0,T)\times\partial\Omega$ and to obtain Lemmas 2.2 and 2.5. Indeed:

- Lemma 2.2 will be used in order to permit the passage to the limit in energy inequality at steps D and E (see Proposition 3.1),
- Lemma 2.5 conducts to Lemma 2.6 which enables to define an interesting extension of the initial density ρ_0. Thanks to this extension, we will be able to take the integral formulas (1.14) and (1.17) into account (see Section 4.1).

2.3.1. An auxiliary function

Lemma 2.1. *Under hypothesis* (**H1**):

(i) *Set* $\Omega^c=\mathbf{R}^2\setminus\Omega$. *Then, for every* $\mathbf{x}\in\Omega^c$, $\nabla g(\mathbf{x})\cdot\mathbf{x}\geqslant 1-g(\mathbf{0})>0$.
(ii) *There exists a mapping* $k:\mathbf{R}^2\setminus\{\mathbf{0}\}\to\mathbf{R}_+^*$ *of class* C^1 *on* $\mathbf{R}^2\setminus\{\mathbf{0}\}$ *such that, for any* $\mathbf{x}\in\mathbf{R}^2\setminus\{\mathbf{0}\}$, $g(k(\mathbf{x})\mathbf{x})=1$.
(iii) *The mapping* k *is* (-1)*-homogeneous.*

Proof. (i) Let $\mathbf{x}\in\Omega^c$ and $\varphi_\mathbf{x}:\mathbf{R}_+\to\mathbf{R}$ be the function defined by $\varphi_\mathbf{x}(t):=g(t\mathbf{x})$. The function $\varphi_\mathbf{x}$ is C^1 and convex on $\mathbf{R}_+$, thus: $\varphi'_\mathbf{x}(1)\geqslant\varphi_\mathbf{x}(1)-\varphi_\mathbf{x}(0)$, that is, $\nabla g(\mathbf{x})\cdot\mathbf{x}\geqslant g(\mathbf{x})-g(\mathbf{0})$. Since $\mathbf{x}\in\Omega^c$, $\mathbf{0}\in\Omega$, we have $g(\mathbf{x})\geqslant 1$ and $g(\mathbf{0})<1$. Therefore $\nabla g(\mathbf{x})\cdot\mathbf{x}\geqslant 1-g(\mathbf{0})>0$ and the proof of (i) is complete.

(ii) Let $\mathbf{x}\in\mathbf{R}^2\setminus\{\mathbf{0}\}$ and let us prove that the equation $(\mathcal{E}_\mathbf{x})$: $\varphi_\mathbf{x}(t)=1$ admits a unique solution in $\mathbf{R}_+$.

Existence. On the one hand, since Ω is bounded, there exists $t_0>0$ such that $t_0\mathbf{x}\notin\Omega$, hence $\varphi_\mathbf{x}(t_0)\geqslant 1$. On the other hand, due to (**H1**), we have $\varphi_\mathbf{x}(0)<1$. Thus, by continuity of $\varphi_\mathbf{x}$, there exists $t_\mathbf{x}>0$ such that $\varphi_\mathbf{x}(t_\mathbf{x})=1$.

Uniqueness. Assume the existence of two solutions of $(\mathcal{E}_\mathbf{x})$, $t_\mathbf{x}$ and $s_\mathbf{x}$ such that $0<s_\mathbf{x}<t_\mathbf{x}$. Thus, there exists $\lambda\in(0,1)$ such that $s_\mathbf{x}=\lambda t_\mathbf{x}$. But, due to the convexity of $\varphi_\mathbf{x}$,

$$\varphi_\mathbf{x}(s_\mathbf{x})\leqslant\lambda\varphi_\mathbf{x}(t_\mathbf{x})+(1-\lambda)\varphi_\mathbf{x}(0)=\lambda+(1-\lambda)g(\mathbf{0})<1,$$

which gives a contradiction.

Conclusion. For all $\mathbf{x}\in\mathbf{R}^2\setminus\{\mathbf{0}\}$, there exists a unique real $k(\mathbf{x})>0$ such that $g(k(\mathbf{x})\mathbf{x})=1$.

Regularity of mapping k. Consider the C^1-function $G:\mathbf{R}_+^*\times\mathbf{R}^2\to\mathbf{R}$ defined by $G(t,\mathbf{x})=g(t\mathbf{x})-1$. Let $(t_0,\mathbf{x}_0)\in\mathbf{R}_+^*\times(\mathbf{R}^2\setminus\{\mathbf{0}\})$ such that $G(t_0,\mathbf{x}_0)=0$ (thus $t_0=k(\mathbf{x}_0)$). We can write $\partial_t G(t_0,\mathbf{x}_0)=\frac{1}{t_0}\nabla g(t_0\mathbf{x}_0)\cdot t_0\mathbf{x}_0$ and, since $t_0\mathbf{x}_0\in\Omega^c$, due to (i), we have $\partial_t G(t_0,\mathbf{x}_0)>0$. According to the Implicit Mapping Theorem, k is C^1 on a neighborhood of $\mathbf{x}_0$.

(iii) Let $\mathbf{x}\in\mathbf{R}^2\setminus\{\mathbf{0}\}$ and $\lambda>0$. Since $\varphi_{\lambda\mathbf{x}}(t)=\varphi_\mathbf{x}(\lambda t)$ for all $t\in\mathbf{R}_+$, we deduce that $\lambda k(\lambda\mathbf{x})$ is a solution of $(\mathcal{E}_\mathbf{x})$. Due to the uniqueness of the solution, we have $\lambda k(\lambda\mathbf{x})=k(\mathbf{x})$. Consequently, k is (-1)-homogeneous. □

Remark 2.1.

- For $\mathbf{x}\in\mathbf{R}^2\setminus\{\mathbf{0}\}$, $k(\mathbf{x})$ is the ratio (strictly positive) of the homothety of center $\mathbf{0}$ that transforms $\mathbf{x}$ into some point of $\partial\Omega$ and $\frac{1}{k(\mathbf{x})}$ is the gauge of the convex Ω at $\mathbf{x}$.
- Setting $k(\mathbf{0})=+\infty$, we have $\Omega=\{\mathbf{x}\in\mathbf{R}^2\colon k(\mathbf{x})>1\}$ and $\partial\Omega=\{\mathbf{x}\in\mathbf{R}^2\colon k(\mathbf{x})=1\}$.

2.3.2. Definition of $\mathbf{u}_\infty$

Let $k_b\in\,]1,\frac{3}{2}[$ and choose $r_0>0$ such that

$$\overline{B(\mathbf{0},r_0)}\subset\Omega\setminus\Omega_b\quad\text{where }\Omega_b:=\left\{\mathbf{x}\in\mathbf{R}^2\colon 1<k(\mathbf{x})<k_b\right\}.\tag{2.3}$$

For $(t,\mathbf{x})\in[0,T]\times\mathbf{R}^2$, we set

$$\mathbf{u}_\infty(t,\mathbf{x})=\left(\mathbf{a}_\infty\big(t,k(\mathbf{x})\mathbf{x}\big)+C_\infty\left(\frac{1}{k(\mathbf{x})}-1\right)\mathbf{x}\right)\psi_\infty(\mathbf{x}),\tag{2.4}$$

where C_∞ is a strictly positive real constant and ψ_∞ is a C^∞-function on $\mathbf{R}^2$ such that

$$0 \leqslant \psi_\infty(\mathbf{x}) \leqslant 1 \quad \text{and} \quad \psi_\infty(\mathbf{x}) = \begin{cases} 1 & \text{if } r_0 \leqslant |\mathbf{x}| \leqslant R_0 + 2, \\ 0 & \text{if } |\mathbf{x}| \leqslant \frac{r_0}{2} \text{ or } |\mathbf{x}| \geqslant R_0 + 3. \end{cases} \tag{2.5}$$

Due to the regularity of $\mathbf{a}_\infty$ (see (1.8)) and k, we can claim that $\mathbf{u}_\infty$ is C^1 on $[0, T] \times \mathbf{R}^2$ and has a compact support in D. We complete these properties.

Lemma 2.2. *We can choose C_∞ such that*:

(i) *For $(t, \mathbf{x}) \in [0, T] \times \Omega^c$ satisfying $|\mathbf{x}| \leqslant R_0 + 2$,* $\operatorname{div} \mathbf{u}_\infty(t, \mathbf{x}) \geqslant 0$.
(ii) *For $(t, \mathbf{x}) \in [0, T] \times \Omega_b$,* $\operatorname{div} \mathbf{u}_\infty(t, \mathbf{x}) \geqslant 0$.
(iii) *For $(t, \mathbf{x}) \in [0, T] \times \Omega^c$ such that $R_0 + 1 \leqslant |\mathbf{x}| \leqslant R_0 + 2$,* $\mathbf{u}_\infty(t, \mathbf{x}) \cdot \mathbf{x} \geqslant 0$.

Proof. According to Lemma 2.1(iii), we know that $\nabla k(\mathbf{x}) \cdot \mathbf{x} = -k(\mathbf{x})$. Consequently, for $(t, \mathbf{x}) \in [0, T] \times \mathbf{R}^2$ satisfying $r_0 \leqslant |\mathbf{x}| \leqslant R_0 + 2$, we have

$$\operatorname{div} \mathbf{u}_\infty(t, \mathbf{x}) = \operatorname{div} \mathbf{A}_\infty(t, \mathbf{x}) + C_\infty \left[-\frac{\nabla k(\mathbf{x}) \cdot \mathbf{x}}{k(\mathbf{x})^2} + \left(\frac{2}{k(\mathbf{x})} - 2 \right) \right] = \operatorname{div} \mathbf{A}_\infty(t, \mathbf{x}) + C_\infty \left[\frac{3}{k(\mathbf{x})} - 2 \right],$$

where $\mathbf{A}_\infty(t, \mathbf{x}) := \mathbf{a}_\infty(t, k(\mathbf{x})\mathbf{x})$.

If $\mathbf{x} \in \Omega^c \cup \Omega_b$, then $k(\mathbf{x}) \in]0, k_b[$. Thus we have $\operatorname{div} \mathbf{u}_\infty(t, \mathbf{x}) \geqslant \operatorname{div} \mathbf{A}_\infty(t, \mathbf{x}) + C_\infty(\frac{3}{k_b} - 2)$. Set $\mathcal{C}_0 := \{\mathbf{x} \in \mathbf{R}^2\colon r_0 \leqslant |\mathbf{x}| \leqslant R_0 + 2\}$. Choosing C_∞ such that

$$C_\infty \geqslant \frac{k_b}{3 - 2k_b} \|\operatorname{div} \mathbf{A}_\infty\|_{L^\infty((0,T) \times \mathcal{C}_0)}, \tag{2.6}$$

then we get (i) and (ii).

On the other hand, for $(t, \mathbf{x}) \in [0, T] \times \mathbf{R}^2$ such that $r_0 \leqslant |\mathbf{x}| \leqslant R_0 + 2$,

$$\mathbf{u}_\infty(t, \mathbf{x}) \cdot \mathbf{x} = \mathbf{A}_\infty(t, \mathbf{x}) \cdot \mathbf{x} + C_\infty \left(\frac{1}{k(\mathbf{x})} - 1 \right) |\mathbf{x}|^2 \geqslant C_\infty \left(\frac{1}{k(\mathbf{x})} - 1 \right) |\mathbf{x}|^2 - \left| \mathbf{A}_\infty(t, \mathbf{x}) \right| |\mathbf{x}|.$$

But $k(\mathbf{x})\mathbf{x} \in \overline{\Omega} \subset B(\mathbf{0}, R_0)$ and, consequently, $\frac{1}{k(\mathbf{x})} \geqslant \frac{|\mathbf{x}|}{R_0}$. Hence

$$\mathbf{u}_\infty(t, \mathbf{x}) \cdot \mathbf{x} \geqslant |\mathbf{x}| \left(C_\infty \left(\frac{|\mathbf{x}|}{R_0} - 1 \right) |\mathbf{x}| - \left| \mathbf{A}_\infty(t, \mathbf{x}) \right| \right).$$

Moreover, if we have $R_0 + 1 \leqslant |\mathbf{x}| \leqslant R_0 + 2$, then: $\mathbf{u}_\infty(t, \mathbf{x}) \cdot \mathbf{x} \geqslant |\mathbf{x}|(C_\infty - |\mathbf{A}_\infty(t, \mathbf{x})|)$.

Therefore, if in addition to condition (2.6), we choose C_∞ so that

$$C_\infty \geqslant \left\| |\mathbf{A}_\infty| \right\|_{L^\infty((0,T) \times \mathcal{C}_0)}, \tag{2.7}$$

we obtain (iii). □

2.4. Invariance

Since $\mathbf{u}_\infty$ is C^1 with compact support in $[0, T] \times \mathbf{R}^2$, it is globally Lipschitz on $[0, T] \times \mathbf{R}^2$. Thus, for any $(t_0, \mathbf{x}_0) \in [0, T] \times \mathbf{R}^2$, the Cauchy problem:

$$\frac{d\mathbf{y}}{ds}(s) = \mathbf{u}_\infty\big(s, \mathbf{y}(s)\big) \quad \text{and} \quad \mathbf{y}(t_0) = \mathbf{x}_0,$$

admits a unique solution $\mathbf{Y}(\cdot; t_0, \mathbf{x}_0)$ on $[0, T]$. Furthermore, the mapping

$$\mathbf{Y} : [0, T] \times [0, T] \times \mathbf{R}^2 \to \mathbf{R}^2, \quad (s, t, \mathbf{x}) \mapsto \mathbf{Y}(s; t, \mathbf{x}), \tag{2.8}$$

is C^1 on $[0, T] \times [0, T] \times \mathbf{R}^2$.

2.4.1. Generalities

The results of this section are inspired from [1, pp. 211–220] but we repeat them under a form adapted to our purpose.

Definition 2.1. A subset $\mathcal{M}$ of $\mathbf{R}^2$ is called *negatively invariant with respect to* $\mathbf{Y}$ if, and only if, for every $\mathbf{x} \in \mathcal{M}$, for every $t \in [0,T]$, $\mathbf{Y}(s;t,\mathbf{x}) \in \mathcal{M}$ as soon as $s \in [0,t]$.

Here is a sufficient condition so that an open region is negatively invariant.

Lemma 2.3. *Let $m \in \mathbf{N}^*$ and let $f_1,\dots,f_m : \mathbf{R}^2 \to \mathbf{R}$ be C^1-functions on $\mathbf{R}^2$. Consider the open set $\mathcal{O} = \bigcap_{j=1}^{j=m} f_j^{-1}(]-\infty,0[)$ and suppose that, for every $\mathbf{x} \in \partial\mathcal{O}$,*

$$\begin{cases} \mathbf{u}_\infty(t,\mathbf{x}) = \mathbf{0} & \text{for all } t \in [0,T], \quad \text{or} \\ \text{there exists } j \in \{1,\dots,m\} \text{ such that } \mathbf{x} \in f_j^{-1}(\{0\}) \text{ and } \nabla f_j(\mathbf{x}) \cdot \mathbf{u}_\infty(t,\mathbf{x}) > 0 \text{ for all } t \in [0,T]. \end{cases}$$

Then $\mathcal{O}$ is negatively invariant.

The following result will permit us to prove that the mapping $\mathbf{G}_e$ introduced in Lemma 2.7 is a diffeomorphism.

Lemma 2.4. *Let $m \in \mathbf{N}^*$ and let $f_1,\dots,f_m : \mathbf{R}^2 \to \mathbf{R}$ be C^1-functions on $\mathbf{R}^2$. Assume the closed set $\mathcal{F} = \bigcap_{j=1}^{m} f_j^{-1}(]-\infty,0])$ is negatively invariant. If we set*

$$\Gamma := \left\{\mathbf{x} \in \partial\mathcal{F}\colon\ \exists j \in \{1,\dots,m\} \;\middle|\; f_j(\mathbf{x}) = 0 \text{ and } \left(\forall t \in [0,T],\ \nabla f_j(\mathbf{x}) \cdot \mathbf{u}_\infty(t,\mathbf{x}) > 0\right)\right\},$$

then, for all $\mathbf{x} \in \Gamma$ and for $s,t \in [0,T]$ such that $s > t$, $\mathbf{Y}(s;t,\mathbf{x}) \notin \mathcal{F}$.

2.4.2. A negatively invariant region

Lemma 2.5. *We define the open set D_e by*

$$D_e := \left\{\mathbf{x} \in C_{\theta_1,\theta_2} \cap D\colon\ g(\mathbf{x}) > 1\right\}. \tag{2.9}$$

Then $\overline{D_e}$ is negatively invariant with respect to $\mathbf{Y}$.

Proof. *Step 1. We prove that $\mathcal{M} = \overline{C_{\theta_1,\theta_2}} \cap \Omega^c$ is negatively invariant with respect to* $\mathbf{Y}$.

First of all, we define an open set $\mathcal{O}$ by

$$\mathcal{O} = f_1^{-1}\left(]-\infty,0[\right) \cap f_2^{-1}\left(]-\infty,0[\right) \cap f_3^{-1}\left(]-\infty,0[\right), \tag{2.10}$$

where the functions f_1, f_2 and f_3 are defined on $\mathbf{R}^2$ by

$$f_1(\mathbf{x}) = -\mathbf{x}\cdot\mathbf{x}_1^\perp, \qquad f_2(\mathbf{x}) = \mathbf{x}\cdot\mathbf{x}_2^\perp, \qquad f_3(\mathbf{x}) = 1 - g(\mathbf{x}). \tag{2.11}$$

($\mathbf{v}^\perp$ is the image of $\mathbf{v}$ under the rotation of angle $+\frac{\pi}{2}$ and $\mathbf{x}_1$, $\mathbf{x}_2$ are still given by (1.11).)

For $\varepsilon > 0$, let $\mathbf{u}_\infty^\varepsilon$ be the vector field defined on $[0,T] \times \mathbf{R}^2$ by

$$\mathbf{u}_\infty^\varepsilon(t,\mathbf{x}) = \mathbf{u}_\infty(t,\mathbf{x}) - \varepsilon\mathbf{x} + \varepsilon^2\left(\frac{\mathbf{x}_2^\perp}{|\mathbf{x}_2|} - \frac{\mathbf{x}_1^\perp}{|\mathbf{x}_1|}\right). \tag{2.12}$$

This vector field is C^1 on $[0,T] \times \mathbf{R}^2$ and globally Lipschitz on $[0,T] \times \mathbf{R}^2$. Thus, the associated flow $\mathbf{Y}^\varepsilon : [0,T] \times [0,T] \times \mathbf{R}^2 \to \mathbf{R}^2$, $(s,t,\mathbf{x}) \mapsto \mathbf{Y}^\varepsilon(s;t,\mathbf{x})$, is also C^1.

For $\mathbf{x} \in \partial\mathcal{O} \cap f_1^{-1}(\{0\})$ and $t \in [0,T]$, we have

$$\begin{aligned} \nabla f_1(\mathbf{x}) \cdot \mathbf{u}_\infty^\varepsilon(t,\mathbf{x}) &= -\mathbf{x}_1^\perp \cdot \mathbf{a}_\infty(t,\mathbf{x}_1)\psi_\infty(\mathbf{x}) - \frac{\varepsilon^2}{|\mathbf{x}_2|}\mathbf{x}_1\cdot\mathbf{x}_2 + \varepsilon^2|\mathbf{x}_1|, \\ &= -\mathbf{x}_1^\perp \cdot \mathbf{a}_\infty(t,\mathbf{x}_1)\psi_\infty(\mathbf{x}) + \varepsilon^2|\mathbf{x}_1|\left(1 - \cos(\theta_2 - \theta_1)\right). \end{aligned} \tag{2.13}$$

But, in view of (**H3**), $\mathbf{a}_\infty(t,\mathbf{x}_1) = -|\mathbf{a}_\infty(t,\mathbf{x}_1)|\tau(\mathbf{x}_1) = -|\mathbf{a}_\infty(t,\mathbf{x}_1)|\mathbf{n}(\mathbf{x}_1)^\perp$, thus

$$-\mathbf{x}_1^\perp\cdot\mathbf{a}_\infty(t,\mathbf{x}_1) = \left|\mathbf{a}_\infty(t,\mathbf{x}_1)\right|\mathbf{x}_1^\perp\cdot\mathbf{n}(\mathbf{x}_1)^\perp = \left|\mathbf{a}_\infty(t,\mathbf{x}_1)\right|\mathbf{x}_1\cdot\mathbf{n}(\mathbf{x}_1) = \frac{|\mathbf{a}_\infty(t,\mathbf{x}_1)|}{|\nabla g(\mathbf{x}_1)|}\mathbf{x}_1\cdot\nabla g(\mathbf{x}_1).$$

According to Lemma 2.1(i), $\mathbf{x}_1\cdot\nabla g(\mathbf{x}_1)>0$, therefore $-\mathbf{x}_1^\perp\cdot\mathbf{a}_\infty(t,\mathbf{x}_1)\geqslant 0$. Furthermore, since $\psi_\infty\geqslant 0$ and $\theta_2-\theta_1\in\,]0,2\pi[$, we deduce from (2.13):

$$\nabla f_1(\mathbf{x})\cdot\mathbf{u}_\infty^\varepsilon(t,\mathbf{x})>0. \tag{2.14}$$

An analogous calculation shows that, for $\mathbf{x}\in\partial\mathcal{O}\cap f_2^{-1}(\{0\})$ and $t\in[0,T]$,

$$\nabla f_2(\mathbf{x})\cdot\mathbf{u}_\infty^\varepsilon(t,\mathbf{x})>0. \tag{2.15}$$

Now, for every $\mathbf{x}\in\partial\mathcal{O}\cap f_3^{-1}(\{0\})$ and $t\in[0,T]$,

$$\begin{aligned}\nabla f_3(\mathbf{x})\cdot\mathbf{u}_\infty^\varepsilon(t,\mathbf{x}) &= -\nabla g(\mathbf{x})\cdot\mathbf{a}_\infty(t,\mathbf{x})+\varepsilon\nabla g(\mathbf{x})\cdot\mathbf{x}-\varepsilon^2\nabla g(\mathbf{x})\cdot\left(\frac{\mathbf{x}_2^\perp}{|\mathbf{x}_2|}-\frac{\mathbf{x}_1^\perp}{|\mathbf{x}_1|}\right)\\ &\geqslant -\nabla g(\mathbf{x})\cdot\mathbf{a}_\infty(t,\mathbf{x})+\varepsilon\nabla g(\mathbf{x})\cdot\mathbf{x}-2\varepsilon^2\left|\nabla g(\mathbf{x})\right|.\end{aligned}$$

But $\nabla g(\mathbf{x})\cdot\mathbf{a}_\infty(t,\mathbf{x})\leqslant 0$ (indeed $\mathbf{x}\in\overline{\Gamma_e}$) and $\nabla g(\mathbf{x})\cdot\mathbf{x}\geqslant 1-g(\mathbf{0})$ (according to Lemma 2.1(i)), thus: $\nabla f_3(\mathbf{x})\cdot\mathbf{u}_\infty^\varepsilon(t,\mathbf{x})\geqslant\varepsilon(1-g(\mathbf{0}))-2\varepsilon^2\||\nabla g|\|_{L^\infty(\partial\Omega)}$. Consequently, for $\varepsilon>0$ small enough,

$$\nabla f_3(\mathbf{x})\cdot\mathbf{u}_\infty^\varepsilon(t,\mathbf{x})>0. \tag{2.16}$$

Due to (2.14)–(2.16) and Lemma 2.3, $\mathcal{O}$ is negatively invariant with respect to $\mathbf{Y}^\varepsilon$ (for $\varepsilon>0$ small enough). Thus, for all $\mathbf{x}\in\mathcal{O}$ and for every $s,t\in[0,T]$ such that $s\leqslant t$, $\mathbf{Y}^\varepsilon(s;t,\mathbf{x})\in\mathcal{O}$. Since $\mathbf{u}_\infty^\varepsilon$ uniformly converges to $\mathbf{u}_\infty$ on $[0,T]\times\mathbf{R}^2$, according to theorems dealing with differential equations with parameters, we can claim that $\mathbf{Y}^\varepsilon(s;t,\mathbf{x})\xrightarrow{\varepsilon\to 0}\mathbf{Y}(s;t,\mathbf{x})$. We deduce that, for all $\mathbf{x}\in\mathcal{O}$,

$$\mathbf{Y}(s;t,\mathbf{x})\in\overline{\mathcal{O}}\quad(0\leqslant s\leqslant t\leqslant T). \tag{2.17}$$

Since $\mathcal{M}=\overline{\mathcal{O}}$ and due to the continuity of $\mathbf{Y}$, the above result is still valid for $\mathbf{x}\in\mathcal{M}$. This proves $\mathcal{M}$ is negatively invariant with respect to $\mathbf{Y}$.

Step 2. We show that $\overline{D_e}$ is negatively invariant with respect to $\mathbf{Y}$.

Since $\mathbf{u}_\infty$ is equal to zero on $\partial D\times[0,T]$, it is obvious that $\overline{D}$ is negatively invariant with respect to $\mathbf{Y}$. Consequently, $\overline{D_e}=\mathcal{M}\cap\overline{D}$ is negatively invariant (as intersection of two negatively invariant sets). □

2.4.3. A diffeomorphism

Let us start with some classical properties of the flow $\mathbf{Y}$ associated with $\mathbf{u}_\infty$.

Lemma 2.6.

(i) *The mapping* $\mathbf{Y}:[0,T]\times[0,T]\times\mathbf{R}^2\to\mathbf{R}^2$, $(s,t,\mathbf{x})\mapsto\mathbf{Y}(s;t,\mathbf{x})$ *is* C^1 *and the following partial derivatives exist and are continuous on* $[0,T]\times[0,T]\times\mathbf{R}^2$:

$$\frac{\partial^2\mathbf{Y}}{\partial s\partial t}=\frac{\partial^2\mathbf{Y}}{\partial t\partial s}\quad\textit{and,}\quad\textit{for } i=1,2,\quad\frac{\partial^2\mathbf{Y}}{\partial s\partial x_i}=\frac{\partial^2\mathbf{Y}}{\partial x_i\partial s}.$$

(ii) *The mapping* $\mathbf{X}:[0,T]\times\mathbf{R}^2\to\mathbf{R}^2$, $(t,\mathbf{x})\mapsto\mathbf{X}(t,\mathbf{x}):=\mathbf{Y}(t;0,\mathbf{x})$ *is* C^1 *on* $[0,T]\times\mathbf{R}^2$ *and, for all* $t\in[0,T]$, $\mathbf{X}(t,\cdot)$ *is a* C^1*-diffeomorphism on* $\mathbf{R}^2$ *onto itself whose inverse mapping is* $\mathbf{X}(t,\cdot)^{-1}:\mathbf{R}^2\to\mathbf{R}^2$, $\mathbf{x}\mapsto\mathbf{Y}(0;t,\mathbf{x})$. *Moreover, for all* $\mathbf{x}\in\mathbf{R}^2$,

$$\det\bigl(\nabla_{\mathbf{x}}\mathbf{X}(t,\mathbf{x})\bigr)=\exp\left(\int_0^t\operatorname{div}\mathbf{u}_\infty\bigl(s,\mathbf{X}(s,\mathbf{x})\bigr)\,ds\right).$$

Now, we are able to prove the following result:

Lemma 2.7. *The mapping* $\mathbf{G}_e : (0,T)\times\Gamma_e \to \mathbf{R}^2$, $(t,\mathbf{x})\mapsto \mathbf{Y}(0;t,\mathbf{x})$ *is a* C^1*-diffeomorphism from* $(0,T)\times\Gamma_e$ *onto an open set* $\mathcal{G}_e$ *included in* D_e.

Remark 2.2. $\mathcal{G}_e$ is the set of all the initial positions of fluid's particles which, under the action of the flow generated by $\mathbf{u}_\infty$, cross Γ_e at one time.

Proof of Lemma 2.7. Remark that Γ_e admits a C^1 parametrization:

$$\Phi_e :]\theta_1,\theta_2[\to \mathbf{R}^2, \quad \theta\mapsto \Phi_e(\theta) = k\big(\mathbf{m}(\theta)\big)\mathbf{m}(\theta), \tag{2.18}$$

where $\mathbf{m}(\theta) := (\cos(\theta),\sin(\theta))$.

Φ_e is injective and regular on $]\theta_1,\theta_2[$ (for all $\theta\in]\theta_1,\theta_2[$, $\Phi_e'(\theta)\neq \mathbf{0}$). Indeed, a quick calculation yields for every $\theta\in]\theta_1,\theta_2[$, $\Phi_e'(\theta) = -(\nabla k(\mathbf{m}(\theta)))^\perp$. Now, consider the mapping

$$\widehat{\mathbf{G}}_e : (0,T)\times(\theta_1,\theta_2)\to\mathbf{R}^2, \quad (t,\theta)\mapsto \mathbf{G}_e\big(t,\Phi_e(\theta)\big).$$

Step 1. We prove that $\widehat{\mathbf{G}}_e$ *is a* C^1 *local diffeomorphism from* $(0,T)\times(\theta_1,\theta_2)$ *onto an open set* $\mathcal{G}_e\subset D_e$.

The C^1 regularity of $\widehat{\mathbf{G}}_e$ results immediately from Lemma 2.6(i) and from the regularity of Φ_e. Furthermore, for all $(t,\theta)\in(0,T)\times(\theta_1,\theta_2)$,

$$\operatorname{Jac}\widehat{\mathbf{G}}_e(t,\theta) = \det\left(\frac{\partial\mathbf{Y}}{\partial t}\big(0;t,\Phi_e(\theta)\big), \nabla_\mathbf{x}\mathbf{Y}\big(0;t,\Phi_e(\theta)\big)\Phi_e'(\theta)\right).$$

But, observing that for every $(t,\xi)\in[0,T]\times\mathbf{R}^2$, we have $\mathbf{Y}(0;t,\mathbf{X}(t,\xi))=\xi$ and differentiating this relation with respect to t, one obtains

$$\frac{\partial\mathbf{Y}}{\partial t}\big(0;t,\mathbf{X}(t,\xi)\big) + \nabla_\mathbf{x}\mathbf{Y}\big(0;t,\mathbf{X}(t,\xi)\big)\partial_t\mathbf{X}(t,\xi) = \mathbf{0}.$$

Since $\partial_t\mathbf{X}(t,\xi) = \mathbf{u}_\infty(t,\mathbf{X}(t,\xi))$, we have $\frac{\partial\mathbf{Y}}{\partial t}(0;t,\mathbf{X}(t,\xi)) = -\nabla_\mathbf{x}\mathbf{Y}(0;t,\mathbf{X}(t,\xi))\mathbf{u}_\infty(t,\mathbf{X}(t,\xi))$. When ξ describes $\mathbf{R}^2$, $\mathbf{x} := \mathbf{X}(t,\xi)$ describes also $\mathbf{R}^2$. Thus, for all $(t,\mathbf{x})\in[0,T]\times\mathbf{R}^2$,

$$\frac{\partial\mathbf{Y}}{\partial t}(0;t,\mathbf{x}) = -\nabla_\mathbf{x}\mathbf{Y}(0;t,\mathbf{x})\mathbf{u}_\infty(t,\mathbf{x}).$$

Consequently, thanks to Lemma 2.6(ii), for $(t,\theta)\in(0,T)\times(\theta_1,\theta_2)$, we get

$$\begin{aligned}\operatorname{Jac}\widehat{\mathbf{G}}_e(t,\theta) &= \det\big[\nabla_\mathbf{x}\mathbf{Y}\big(0;t,\Phi_e(\theta)\big)\big]\times\det\big[\Phi_e'(\theta),\mathbf{u}_\infty\big(t,\Phi_e(\theta)\big)\big]\\ &= \frac{\det[\Phi_e'(\theta),\mathbf{u}_\infty(t,\Phi_e(\theta))]}{\exp(\int_0^t \operatorname{div}\mathbf{u}_\infty(s,\mathbf{X}(s,\Phi_e(\theta)))\,ds)}.\end{aligned}$$

But $\det[\Phi_e'(\theta),\mathbf{u}_\infty(t,\Phi_e(\theta))] = \Phi_e'(\theta)^\perp\cdot\mathbf{u}_\infty(t,\Phi_e(\theta))$ so, for all $(t,\theta)\in(0,T)\times(\theta_1,\theta_2)$,

$$\det\big[\Phi_e'(\theta),\mathbf{u}_\infty\big(t,\Phi_e(\theta)\big)\big] = -\big|\Phi_e'(\theta)\big|\mathbf{n}\big(\Phi_e(\theta)\big)\cdot\mathbf{u}_\infty\big(t,\Phi_e(\theta)\big) > 0,$$

and therefore $\operatorname{Jac}\widehat{\mathbf{G}}_e(t,\theta)\neq 0$.

Due to the Inverse Mapping Theorem, $\widehat{\mathbf{G}}_e$ is a C^1 local diffeomorphism from $(0,T)\times(\theta_1,\theta_2)$ onto $\mathcal{G}_e := \widehat{\mathbf{G}}_e((0,T)\times(\theta_1,\theta_2))$. Since Γ_e is included in $\overline{D_e}$ which is negatively invariant relatively to $\mathbf{Y}$, we can claim than $\mathcal{G}_e$ is an open set included in $\overline{D_e}$, hence $\mathcal{G}_e\subset D_e$.

Step 2. We prove that $\mathbf{G}_e$ *is a (global) diffeomorphism from* $(0,T)\times\Gamma_e$ *onto* $\mathcal{G}_e$.

At this point, it is sufficient to show that $\mathbf{G}_e$ is injective.

Let $(t,\mathbf{x})$ and $(t',\mathbf{x}')\in(0,T)\times\Gamma_e$ such that $\mathbf{G}_e(t,\mathbf{x}) = \mathbf{G}_e(t',\mathbf{x}') = \mathbf{y}_0$. Therefore, the functions $\mathbf{Y}(\cdot;t,\mathbf{x})$ and $\mathbf{Y}(\cdot;t',\mathbf{x}')$ coincide with the solution $\mathbf{y}$ of the following Cauchy problem:

$$\frac{d\mathbf{y}}{ds}(s) = \mathbf{u}_\infty\big(s,\mathbf{y}(s)\big) \quad \text{on } [0,T], \qquad \mathbf{y}(0)=\mathbf{y}_0.$$

Suppose that $t \neq t'$ and, for example, let us deal with the case $t < t'$. Since $\mathbf{y}(t) = \mathbf{x} \in \Gamma_e$, according to Lemma 2.4 applied to $\overline{D_e}$, for all $s \in]t, T]$, $\mathbf{y}(s) \notin \overline{D_e}$. Thus, in particular, $\mathbf{y}(t') \notin \Gamma_e$, that is $\mathbf{x}' \notin \Gamma_e$ which is absurd. Consequently, we have $t = t'$, and thus $\mathbf{x} = \mathbf{y}(t) = \mathbf{y}(t') = \mathbf{x}'$. □

3. Step D: Passage to the limit on ε

3.1. Conclusion of step C

The details about steps A, B and C, can be found in [8] or [9, pp. 415–418]. The following theorem sums up the results of step C.

Theorem 3.1. *For any $\varepsilon > 0$, $\delta > 0$ and for every couple $(\rho_0, \mathbf{u}_0)$ such that*

$$\begin{cases} \mathbf{u}_0 \in \mathbf{L}^2(D), \\ \rho_0 \in W^{1,\infty}(D) \cap W^{2,2}(D) \quad \text{with } \inf_D \rho_0 > 0 \quad \text{and} \quad \partial_{\mathbf{n}} \rho_0(\mathbf{x}) = 0 \quad \text{on } \partial D, \end{cases} \tag{3.1}$$

there exists a pair of functions $(\rho, \mathbf{u})$ such that

$$\begin{cases} \rho \in L^\infty\big(0, T; L^\beta(D)\big) \quad \text{and} \quad \rho \geqslant 0 \quad \text{a.e. on } D \times (0,T), \\ \mathbf{u} \in L^2\big(0, T; \mathbf{W}_0^{1,2}(D)\big) \quad \text{with } \mathbf{u} = \mathbf{u}_\infty \text{ on } E \times (0,T), \\ \rho \in C\big([0,T]; L^\beta(D)\big) \quad \text{and} \quad \rho\mathbf{u} \in C\big([0,T]; \mathbf{L}_w^{\frac{2\beta}{\beta+1}}(\Omega)\big) \quad \text{with } \begin{cases} \rho(0) = \rho_0, \\ (\rho\mathbf{u})(0) = \rho_0 \mathbf{u}_0. \end{cases} \end{cases}$$

These functions satisfy in the strong sense

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho\mathbf{u}) = \varepsilon \Delta\rho & \text{in } D \times (0,T), \\ \partial_{\mathbf{n}} \rho(t, \mathbf{x}) = 0 & \text{on } \partial D \times (0,T), \\ \rho(0, \mathbf{x}) = \rho_0(\mathbf{x}) & \text{in } D, \end{cases}$$

and in the sense of distributions in $\Omega \times (0,T)$:

$$\partial_t(\rho\mathbf{u}) + \operatorname{div}(\rho\mathbf{u} \otimes \mathbf{u}) - \mu\Delta\mathbf{u} - (\lambda+\mu)\nabla \operatorname{div}\mathbf{u} + \nabla\big(a\rho^\gamma + \delta\rho^\beta\big) + \varepsilon(\nabla\rho \cdot \nabla)\mathbf{u} = \rho\mathbf{f}.$$

Moreover, setting

$$\begin{cases} P_\delta(\rho) := \dfrac{a}{\gamma-1}\rho^\gamma + \dfrac{\delta}{\beta-1}\rho^\beta, \\ E(t) := \displaystyle\int_D \left(\frac{1}{2}\rho(t)\big(\mathbf{u}(t) - \mathbf{u}_\infty(t)\big)^2 + P_\delta\big(\rho(t)\big)\right) d\mathbf{x}, \\ E_0 := \displaystyle\int_D \left(\frac{1}{2}\rho_0\big(\mathbf{u}_0 - \mathbf{u}_\infty(0)\big)^2 + P_\delta(\rho_0)\right) d\mathbf{x}, \end{cases}$$

we have for almost every $t \in (0,T)$,

$$\begin{aligned} &E(t) + \int_0^t \int_D \big(\mu\big[\nabla(\mathbf{u} - \mathbf{u}_\infty)\big]^2 + (\lambda+\mu)\big[\operatorname{div}(\mathbf{u} - \mathbf{u}_\infty)\big]^2\big)\, d\mathbf{x}\, ds \\ &\quad + \varepsilon \int_0^t \int_D \delta\beta\rho^{\beta-2}|\nabla\rho|^2\, d\mathbf{x}\, ds + \int_0^t \int_D \big(a\rho^\gamma + \delta\rho^\beta\big) \operatorname{div}\mathbf{u}_\infty\, d\mathbf{x} \\ &\leqslant E_0 + \int_0^t \int_D \rho\mathbf{f} \cdot (\mathbf{u} - \mathbf{u}_\infty)\, d\mathbf{x}\, ds - \int_0^t \int_D \rho\partial_t\mathbf{u}_\infty \cdot (\mathbf{u} - \mathbf{u}_\infty)\, d\mathbf{x}\, ds \end{aligned}$$

$$- \mu \int_0^t \int_D \nabla \mathbf{u}_\infty : \nabla(\mathbf{u} - \mathbf{u}_\infty)\, d\mathbf{x}\, ds - (\lambda + \mu) \int_0^t \int_D \operatorname{div} \mathbf{u}_\infty \operatorname{div}(\mathbf{u} - \mathbf{u}_\infty)\, d\mathbf{x}\, ds$$

$$+ \varepsilon \int_0^t \int_D \big[(\nabla\rho \cdot \nabla)(\mathbf{u} - \mathbf{u}_\infty)\big] \cdot \mathbf{u}_\infty\, d\mathbf{x}\, ds - \int_0^t \int_D \rho\mathbf{u} \cdot \big[\big((\mathbf{u} - \mathbf{u}_\infty) \cdot \nabla\big)\mathbf{u}_\infty\big]\, d\mathbf{x}\, ds. \tag{3.2}$$

3.2. Aim of step D

The purpose of this section is to establish the following result:

Theorem 3.2. *For any $\delta > 0$ and for every couple $(\rho_0, \mathbf{u}_0)$ such that*

$$\begin{cases} \mathbf{u}_0 \in \mathbf{L}^2(D) \quad \text{with } \mathbf{u}_0 = \mathbf{u}_\infty(0) \text{ on } E, \\ \rho_0 \in L^\beta(D) \quad \text{with } \rho_0 \geqslant 0 \text{ on } D \quad \text{and} \quad \rho_0|_{\overline{D_e}} \in C(\overline{D_e}), \\ \sqrt{\rho_0}\mathbf{u}_0 \in \mathbf{L}^2(D), \end{cases} \tag{3.3}$$

there exists a pair $(\rho, \mathbf{u})$ such that

$$\begin{cases} \rho \in L^\infty\big(0, T; L^\beta(D)\big) \quad \text{and} \quad \rho \geqslant 0 \quad \text{a.e. on } D \times (0, T), \\ \mathbf{u} \in L^2\big(0, T; \mathbf{W}_0^{1,2}(D)\big) \quad \text{and} \quad \mathbf{u} = \mathbf{u}_\infty \quad \text{a.e. on } E \times (0, T), \\ \rho \in C\big([0, T]; L_w^\beta(D)\big) \quad \text{and} \quad \rho\mathbf{u} \in C\big([0, T]; \mathbf{L}_w^{\frac{2\beta}{\beta+1}}(\Omega)\big) \quad \text{with} \begin{cases} \rho(0) = \rho_0, \\ (\rho\mathbf{u})(0) = \rho_0\mathbf{u}_0. \end{cases} \end{cases} \tag{3.4}$$

Moreover, the pair $(\rho, \mathbf{u})$ satisfies the continuity equation in the sense of distributions in $D \times (0, T)$ and the momentum equation is satisfied in the sense of distributions in $\Omega \times (0, T)$:

$$\begin{aligned} &\partial_t \rho + \operatorname{div}(\rho\mathbf{u}) = 0, \\ &\partial_t(\rho\mathbf{u}) + \operatorname{div}(\rho\mathbf{u} \otimes \mathbf{u}) - \mu\Delta\mathbf{u} - (\lambda + \mu)\nabla \operatorname{div} \mathbf{u} + \nabla\big(a\rho^\gamma + \delta\rho^\beta\big) = \rho\mathbf{f}. \end{aligned} \tag{3.5}$$

These functions also have the following properties

$$\begin{aligned} &\rho \in C\big([0, T]; L^p(D)\big) \quad (1 \leqslant p < \beta), \qquad \rho \in L^{\beta+1}\big(0, T; L_{loc}^{\beta+1}(\Omega)\big), \\ &\rho\mathbf{u} \in L^2\big(0, T; \mathbf{L}^m(D)\big) \quad (1 \leqslant m < \beta), \qquad \rho|\mathbf{u}|^2 \in L^2\big(0, T; L^r(D)\big) \quad \left(1 \leqslant r < \frac{2\beta}{\beta + 1}\right). \end{aligned} \tag{3.6}$$

Moreover, for any function $\eta \in \mathcal{D}(\mathbf{R}^2 \times (0, T))$ such that $\eta|_{Q_s} = 0$ (where Q_s is defined by (1.13)*), and for any function $b \in C(\mathbf{R}_+) \cap C^1(\mathbf{R}_+^*)$ satisfying* (1.15) *with*

$$c > 0, \qquad \lambda_0 \in]-1, +\infty[, \qquad \lambda_1 \in \left]-\infty, \frac{\beta}{2} - 1\right], \tag{3.7}$$

we have

$$\begin{aligned} &\int_0^T \int_\Omega \big[b(\rho)\partial_t \eta + b(\rho)\mathbf{u} \cdot \nabla\eta\big]\, d\mathbf{x}\, dt - \int_0^T \int_\Omega \big[\rho b'(\rho) - b(\rho)\big] \operatorname{div} \mathbf{u}\eta\, d\mathbf{x}\, dt \\ &\quad = \int_0^T \int_{\Gamma_e} b\left(\frac{\rho_0(\mathbf{Y}(0; t, \mathbf{x}))}{J(t, \mathbf{x})}\right) \eta(t, \mathbf{x})\mathbf{u}_\infty(t, \mathbf{x}) \cdot \mathbf{n}(\mathbf{x})\, d\mathbf{S}(\mathbf{x})\, dt, \end{aligned} \tag{3.8}$$

where $J(t, \mathbf{x})$ is defined by

$$J(t, \mathbf{x}) = \exp\left(\int_0^t \operatorname{div} \mathbf{u}_\infty\big(s, \mathbf{Y}(s; t, \mathbf{x})\big)\, ds\right). \tag{3.9}$$

Finally, we obtain two energy inequalities.

– *On the one hand, setting*

$$\begin{cases} E(t) := \int\limits_D \left(\frac{1}{2}\rho(t)\big(\mathbf{u}(t) - \mathbf{u}_\infty(t)\big)^2 + P_\delta\big(\rho(t)\big)\right) d\mathbf{x}, \\ E_0 := \int\limits_D \left(\frac{1}{2}\rho_0\big(\mathbf{u}_0 - \mathbf{u}_\infty(0)\big)^2 + P_\delta(\rho_0)\right) d\mathbf{x}, \end{cases}$$

we have for almost every $t \in (0, T)$,

$$\begin{aligned} &E(t) + \int_0^t \int_D \big(\mu\big[\nabla(\mathbf{u} - \mathbf{u}_\infty)\big]^2 + (\lambda + \mu)\big[\operatorname{div}(\mathbf{u} - \mathbf{u}_\infty)\big]^2\big)\, d\mathbf{x}\, ds \\ &\quad \leqslant E_0 + \int_0^t \int_D \rho\mathbf{f} \cdot (\mathbf{u} - \mathbf{u}_\infty)\, d\mathbf{x}\, ds - \int_0^t \int_D \rho\partial_t \mathbf{u}_\infty \cdot (\mathbf{u} - \mathbf{u}_\infty)\, d\mathbf{x}\, ds \\ &\qquad - \mu \int_0^t \int_D \nabla\mathbf{u}_\infty : \nabla(\mathbf{u} - \mathbf{u}_\infty)\, d\mathbf{x}\, ds - (\lambda + \mu) \int_0^t \int_D \operatorname{div}\mathbf{u}_\infty \operatorname{div}(\mathbf{u} - \mathbf{u}_\infty)\, d\mathbf{x}\, ds \\ &\qquad - \int_0^t \int_D \rho\mathbf{u} \cdot \big[\big((\mathbf{u} - \mathbf{u}_\infty) \cdot \nabla\big)\mathbf{u}_\infty\big]\, d\mathbf{x}\, ds - \int_0^t \int_D \big(a\rho^\gamma + \delta\rho^\beta\big) \operatorname{div}\mathbf{u}_\infty\, d\mathbf{x}\, ds. \end{aligned} \tag{3.10}$$

– *On the other hand, setting*

$$\begin{cases} E^\Omega(t) := \int\limits_\Omega \left(\frac{1}{2}\rho(t)\big(\mathbf{u}(t) - \mathbf{u}_\infty(t)\big)^2 + P_\delta\big(\rho(t)\big)\right) d\mathbf{x}, \\ E_0^\Omega := \int\limits_\Omega \left(\frac{1}{2}\rho_0\big(\mathbf{u}_0 - \mathbf{u}_\infty(0)\big)^2 + P_\delta(\rho_0)\right) d\mathbf{x}, \end{cases}$$

we have for almost every $t \in (0, T)$,

$$\begin{aligned} &E^\Omega(t) + \int_0^t \int_\Omega \big(\mu\big[\nabla(\mathbf{u} - \mathbf{u}_\infty)\big]^2 + (\lambda + \mu)\big[\operatorname{div}(\mathbf{u} - \mathbf{u}_\infty)\big]^2\big)\, d\mathbf{x}\, ds \\ &\quad \leqslant E_0^\Omega + \int_0^t \int_\Omega \rho\mathbf{f} \cdot (\mathbf{u} - \mathbf{u}_\infty)\, d\mathbf{x}\, ds - \int_0^t \int_\Omega \rho\partial_t \mathbf{u}_\infty \cdot (\mathbf{u} - \mathbf{u}_\infty)\, d\mathbf{x}\, ds - \mu \int_0^t \int_\Omega \nabla\mathbf{u}_\infty : \nabla(\mathbf{u} - \mathbf{u}_\infty)\, d\mathbf{x}\, ds \\ &\qquad - (\lambda + \mu) \int_0^t \int_\Omega \operatorname{div}\mathbf{u}_\infty \operatorname{div}(\mathbf{u} - \mathbf{u}_\infty)\, d\mathbf{x}\, ds - \int_0^t \int_\Omega \rho\mathbf{u} \cdot \big[\big((\mathbf{u} - \mathbf{u}_\infty) \cdot \nabla\big)\mathbf{u}_\infty\big]\, d\mathbf{x}\, ds \\ &\qquad - \int_0^t \int_\Omega \big(a\rho^\gamma + \delta\rho^\beta\big) \operatorname{div}\mathbf{u}_\infty\, d\mathbf{x}\, ds - \int_0^t \int_{\Gamma_e} P_\delta\left(\frac{\rho_0(\mathbf{Y}(0; s, \mathbf{x}))}{J(s, \mathbf{x})}\right)\mathbf{u}_\infty \cdot \mathbf{n}\, d\mathbf{S}\, ds. \end{aligned} \tag{3.11}$$

3.3. Choice of initial conditions

Here, we consider initial conditions $(\rho_0, \mathbf{u}_0)$ satisfying (3.3). For every $\varepsilon > 0$, $(\rho_\varepsilon, \mathbf{u}_\varepsilon)$ denotes a couple of functions deduced from Theorem 3.1 for some initial conditions $(\rho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon})$ defined as follows:

• **Definition of** $\rho_{0,\varepsilon}$**.** We set

$$\rho_{0,\varepsilon} := S_\varepsilon(\rho_0 \chi_{D_\varepsilon}) + \varepsilon, \tag{3.12}$$

where $D_\varepsilon := \{\mathbf{x} \in D \mid d(\mathbf{x}, \partial D) > 2\varepsilon\}$ and where S_ε is the standard mollifier operator over space variable (see [9, pp. 37–38]). Then, one checks that:

$$\begin{cases} \rho_{0,\varepsilon} \in C^\infty(\mathbf{R}^2), \qquad \nabla\rho_{0,\varepsilon} = \mathbf{0} \quad \text{on } \partial D, \\ \varepsilon \leqslant \rho_{0,\varepsilon}(\mathbf{x}) \leqslant \sup_D \rho_{0,\varepsilon} < \infty \quad \text{for all } \mathbf{x} \in D, \\ \rho_{0,\varepsilon} \xrightarrow{\varepsilon\to 0} \rho_0 \quad \text{in } L^\beta(D). \end{cases} \tag{3.13}$$

• **Definition of** $\mathbf{u}_{0,\varepsilon}$**.** Since $[\mathcal{D}(D)]^2$ is dense in $\mathbf{L}^2(D)$, there exists $\mathbf{v}_\varepsilon \in [\mathcal{D}(D)]^2$ such that $\|\sqrt{\rho_0}\mathbf{u}_0 - \mathbf{v}_\varepsilon\|_{\mathbf{L}^2(D)} \leqslant \varepsilon$. Then we set

$$\mathbf{u}_{0,\varepsilon} := \frac{\mathbf{v}_\varepsilon}{\sqrt{\rho_{0,\varepsilon}}}. \tag{3.14}$$

• **Consequences.** We have

$$\sqrt{\rho_{0,\varepsilon}}\mathbf{u}_{0,\varepsilon} \xrightarrow{\varepsilon\to 0} \sqrt{\rho_0}\mathbf{u}_0 \quad \text{in } \mathbf{L}^2(D), \tag{3.15}$$

$$\rho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon} \xrightarrow{\varepsilon\to 0} \rho_0\mathbf{u}_0 \quad \text{in } \mathbf{L}^{\frac{2\beta}{\beta+1}}(D), \tag{3.16}$$

and

$$\int_D \left(\frac{1}{2}\rho_{0,\varepsilon}\big(\mathbf{u}_{0,\varepsilon} - \mathbf{u}_\infty(0)\big)^2 + P_\delta(\rho_{0,\varepsilon})\right) d\mathbf{x} \xrightarrow{\varepsilon\to 0} \int_D \left(\frac{1}{2}\rho_0\big(\mathbf{u}_0 - \mathbf{u}_\infty(0)\big)^2 + P_\delta(\rho_0)\right) d\mathbf{x}. \tag{3.17}$$

3.4. The first results of the passage to the limit

Here, the method used coincides with the one developed by Novo (see [8] or [9, pp. 418–420]) and the only differences with these references come from Sobolev's injection theorems (we work in dimension 2 whereas the author works in $\mathbf{R}^3$). We only mention the essential results.

There exists a suitable subsequence of $\{(\rho_\varepsilon, \mathbf{u}_\varepsilon)\}$ (not relabeled) and a couple $(\rho, \mathbf{u})$ satisfying (3.4)–(3.6) such that:

$$\begin{cases} \mathbf{u}_\varepsilon \rightharpoonup \mathbf{u} \quad \text{in } L^2(0, T; \mathbf{W}_0^{1,2}(D)), \\ \rho_\varepsilon \overset{*}{\rightharpoonup} \rho \quad \text{in } L^\infty(0, T; L^\beta(D)), \end{cases} \tag{3.18}$$

$$\begin{cases} \rho_\varepsilon \to \rho \quad \text{in } L^p(D \times (0, T)) \quad (1 \leqslant p < \beta), \\ \rho_\varepsilon \to \rho \quad \text{in } L^p(0, T; L^p_{loc}(\Omega)) \quad (1 \leqslant p < \beta + 1), \end{cases} \tag{3.19}$$

$$\begin{cases} \rho_\varepsilon\mathbf{u}_\varepsilon \rightharpoonup \rho\mathbf{u} \quad & \text{in } L^2(0, T; \mathbf{L}^m(D)) \quad (1 \leqslant m < \beta), \\ \rho_\varepsilon\mathbf{u}_\varepsilon \overset{*}{\rightharpoonup} \rho\mathbf{u} \quad & \text{in } L^\infty(0, T; \mathbf{L}^{\frac{2\beta}{\beta+1}}(D)), \\ \rho_\varepsilon u_\varepsilon^i u_\varepsilon^j \rightharpoonup \rho u^i u^j \quad & \text{in } L^2(0, T; L^r(D)) \quad \left(1 \leqslant r < \dfrac{2\beta}{\beta+1},\ i, j \in \{1, 2\}\right). \end{cases} \tag{3.20}$$

One can also prove that

$$\begin{aligned} &\varepsilon\nabla\rho_\varepsilon \to \mathbf{0} \quad \text{in } L^2(0, T; \mathbf{L}^2(D)), \\ &\varepsilon(\nabla\rho_\varepsilon \cdot \nabla)\mathbf{u}_\varepsilon \to \mathbf{0} \quad \text{in } L^1(0, T; \mathbf{L}^1(D)). \end{aligned} \tag{3.21}$$

At last, $(\rho, \mathbf{u})$ is a renormalized solution of the continuity equation (see [9, p. 304]). It means that, for any function $b \in C(\mathbf{R}_+) \cap C^1(\mathbf{R}_+^*)$ satisfying (1.15), (3.7), we have, extending ρ and $\mathbf{u}$ by 0 outside D,

$$\partial_t b(\rho) + \operatorname{div}\big(b(\rho)\mathbf{u}\big) + \big(\rho b'(\rho) - b(\rho)\big)\operatorname{div}\mathbf{u} = 0 \quad \text{in } \mathcal{D}'(\mathbf{R}^2 \times (0, T)). \tag{3.22}$$

3.5. Passage to the limit in the energy inequality

Proposition 3.1. $(\rho, \mathbf{u})$ *satisfies energy inequality* (3.10).

Proof. Let ψ be in $\mathcal{D}(0,T)$ such that, for all $t \in [0,T]$, $\psi(t) \geqslant 0$. We multiply (3.2) (which is satisfied by ρ_ε and $\mathbf{u}_\varepsilon$) by ψ, take the integral over $(0,T)$ and pass to the inferior limit in this inequality in order to prove that the inequality (3.10) holds in $\mathcal{D}'(0,T)$. Since each member of this relation is, at least, in $L^1(0,T)$, this inequality holds almost everywhere in $(0,T)$, which gives the desired conclusion. Actually, thanks to the results of Section 3.4, the only difficulty is to prove that

$$\liminf_{\varepsilon\to 0} I_\varepsilon \geqslant \int_0^T \psi(t)\int_D \frac{\delta}{\beta-1}\rho(t)^\beta\,d\mathbf{x}\,dt + \int_0^T \psi(t)\int_0^t\int_D \delta\rho^\beta \operatorname{div}\mathbf{u}_\infty\,d\mathbf{x}\,ds\,dt,$$

where

$$I_\varepsilon := \int_0^T \psi(t)\left(\int_D \frac{\delta}{\beta-1}\rho_\varepsilon^\beta(t)\,d\mathbf{x} + \int_0^t\int_D \delta\rho_\varepsilon^\beta \operatorname{div}\mathbf{u}_\infty\,d\mathbf{x}\,ds + \varepsilon\int_0^t\int_D \delta\beta\rho_\varepsilon^{\beta-2}|\nabla\rho_\varepsilon|^2\,d\mathbf{x}\,ds\right)dt.$$

To achieve this goal, we write $I_\varepsilon = I_\varepsilon^E + I_\varepsilon^\Omega$, with $E = D\setminus\Omega$ and

$$\begin{cases} I_\varepsilon^E := \displaystyle\int_0^T \psi(t)\left(\int_E \frac{\delta}{\beta-1}\rho_\varepsilon^\beta(t) + \varepsilon\int_0^t\int_E \delta\beta\rho_\varepsilon^{\beta-2}|\nabla\rho_\varepsilon|^2 + \int_0^t\int_E \delta\rho_\varepsilon^\beta \operatorname{div}\mathbf{u}_\infty\right)dt,\\ I_\varepsilon^\Omega := \displaystyle\int_0^T \psi(t)\left(\int_\Omega \frac{\delta}{\beta-1}\rho_\varepsilon^\beta(t) + \varepsilon\int_0^t\int_\Omega \delta\beta\rho_\varepsilon^{\beta-2}|\nabla\rho_\varepsilon|^2 + \int_0^t\int_\Omega \delta\rho_\varepsilon^\beta \operatorname{div}\mathbf{u}_\infty\right)dt. \end{cases}$$

Step 1. We prove that $\liminf_{\varepsilon\to 0} I_\varepsilon^E \geqslant \int_0^T \psi(t)(\int_E \frac{\delta}{\beta-1}\rho^\beta(t) + \int_0^t\int_E \delta\rho^\beta \operatorname{div}\mathbf{u}_\infty)\,dt$.

In the sequel, $\eta \in \mathcal{D}(\mathbf{R}^2)$ denotes a function such that $\eta(\mathbf{x}) = w(|\mathbf{x}|)$, where $w \in C^\infty(\mathbf{R})$ satisfies $0 \leqslant w(s) \leqslant 1$ for all $s \in \mathbf{R}$ and:

$$\begin{cases} w(s) = 0 \quad \text{for } s \in\,]-\infty, R_0+1], & w'(s) \geqslant 0 \quad \text{for } s \in [R_0+1, R_0+2],\\ w(s) = 1 \quad \text{for } s \in [R_0+2, R_0+3], & w(s) = 0 \quad \text{for } s \geqslant R_0+4. \end{cases} \tag{3.23}$$

1.1. An integral formula for ρ.

Consider the function b_k defined by $b_k(s) = \frac{\delta}{\beta-1}T_k(s)^\beta$, where k is a nonnegative integer and T_k is the cut-off function given by (1.24). As b_k fulfils conditions (1.15), (3.7), we can use the relation (3.22) with $b = b_k$. Since $b_k(0) = 0$ and ρ and $\mathbf{u}$ are zero outside D, we obtain (in $\mathcal{D}'(0,T)$):

$$\frac{d}{dt}\left(\int_E b_k(\rho)\eta\,d\mathbf{x}\right) = -\int_E \big(\rho b_k'(\rho) - b_k(\rho)\big)\operatorname{div}\mathbf{u}\eta\,d\mathbf{x} + \int_E b_k(\rho)\mathbf{u}\cdot\nabla\eta\,d\mathbf{x}. \tag{3.24}$$

Thanks to the regularity of ρ, we can integrate (3.24) between 0 and $t \in [0,T]$. In view of $\mathbf{u} = \mathbf{u}_\infty$ on $E \times (0,T)$, we obtain

$$\int_E b_k\big(\rho(t)\big)\eta + \int_0^t\int_E \big[\rho(s)b_k'\big(\rho(s)\big) - b_k\big(\rho(s)\big)\big]\operatorname{div}\mathbf{u}_\infty\eta = \int_E b_k(\rho_0)\eta + \int_0^t\int_E b_k\big(\rho(s)\big)\mathbf{u}_\infty\cdot\nabla\eta.$$

Since $\rho \in L^\infty(0,T;L^\beta(D))$, the Dominated Convergence Theorem enables the passage to the limit $k\to\infty$ and yields, for any $t \in [0,T]$,

$$\int_E \frac{\delta}{\beta-1}\rho(t)^\beta\eta + \int_0^t\int_E \delta\rho^\beta \operatorname{div}\mathbf{u}_\infty\eta = \int_E \frac{\delta}{\beta-1}\rho_0^\beta\eta + \int_0^t\int_E \frac{\delta}{\beta-1}\rho^\beta\mathbf{u}_\infty\cdot\nabla\eta. \tag{3.25}$$

1.2. An integral formula for ρ_ε.

First, we multiply the equation $\partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon \mathbf{u}_\infty) = \varepsilon \Delta \rho_\varepsilon$ (satisfied in the strong sense) by the function $b_k'(\rho_\varepsilon)\eta$ and then, we take the integral over $E \times (0,t)$. Since $\mathbf{u}_\varepsilon = \mathbf{u}_\infty$ on $E \times (0,t)$, after some integrations by parts, one arrives at

$$\int_E b_k(\rho_\varepsilon(t))\eta\,d\mathbf{x} + \int_0^t \int_E \left[\rho_\varepsilon b_k'(\rho_\varepsilon) - b_k(\rho_\varepsilon)\right] \operatorname{div} \mathbf{u}_\infty \eta\,d\mathbf{x}\,ds + \varepsilon \int_0^t \int_E b_k''(\rho_\varepsilon)|\nabla \rho_\varepsilon|^2 \eta\,d\mathbf{x}\,ds$$
$$= \int_E b_k(\rho_{0,\varepsilon})\eta\,d\mathbf{x} + \int_0^t \int_E b_k(\rho_\varepsilon)\mathbf{u}_\infty \cdot \nabla \eta\,d\mathbf{x}\,ds + \varepsilon \int_0^t \int_E b_k(\rho_\varepsilon)\Delta \eta\,d\mathbf{x}\,ds.$$

Thanks to the Dominated Convergence Theorem, we can pass to the limit as $k \to \infty$. That yields

$$\int_E \frac{\delta}{\beta-1}\rho_\varepsilon^\beta(t)\eta\,d\mathbf{x} + \int_0^t \int_E \delta\rho_\varepsilon^\beta \operatorname{div} \mathbf{u}_\infty \eta\,d\mathbf{x}\,ds + \varepsilon \int_0^t \int_E \delta\beta\rho_\varepsilon^{\beta-2}|\nabla \rho_\varepsilon|^2 \eta\,d\mathbf{x}\,ds$$
$$= \int_E \frac{\delta}{\beta-1}\rho_{0,\varepsilon}^\beta \eta\,d\mathbf{x} + \int_0^t \int_E \frac{\delta}{\beta-1}\rho_\varepsilon^\beta \mathbf{u}_\infty \cdot \nabla \eta\,d\mathbf{x}\,ds + \varepsilon \int_0^t \int_E \frac{\delta}{\beta-1}\rho_\varepsilon^\beta \Delta \eta\,d\mathbf{x}\,ds. \tag{3.26}$$

1.3. Passage to the limit on ε.

We write I_ε^E in the form $I_\varepsilon^E = I_\varepsilon^{(\eta)} + I_\varepsilon^{(1-\eta)}$ where

$$\begin{cases} I_\varepsilon^{(\eta)} := \displaystyle\int_0^T \psi(t)\left(\int_E \frac{\delta}{\beta-1}\rho_\varepsilon(t)^\beta \eta + \int_0^t \int_E \delta\rho_\varepsilon^\beta \operatorname{div} \mathbf{u}_\infty \eta + \int_0^t \int_E \varepsilon\delta\beta\rho_\varepsilon^{\beta-2}|\nabla \rho_\varepsilon|^2 \eta\right) dt, \\ I_\varepsilon^{(1-\eta)} := \displaystyle\int_0^T \psi(t)\left(\int_E \frac{\delta}{\beta-1}\rho_\varepsilon(t)^\beta (1-\eta) + \int_0^t \int_E \delta\rho_\varepsilon^\beta \operatorname{div} \mathbf{u}_\infty (1-\eta) + \int_0^t \int_E \varepsilon\delta\beta\rho_\varepsilon^{\beta-2}|\nabla \rho_\varepsilon|^2 (1-\eta)\right) dt. \end{cases}$$

1.3.1. First, we study $\liminf_{\varepsilon\to 0} I_\varepsilon^{(\eta)}$.

– According to (3.13), $\rho_{0,\varepsilon}^\beta$ converges to ρ_0^β in $L^1(D)$, thus

$$\int_0^T \psi(t)\,dt \int_E \frac{\delta}{\beta-1}\rho_{0,\varepsilon}^\beta \eta \xrightarrow{\varepsilon\to 0} \int_0^T \psi(t)\,dt \int_E \frac{\delta}{\beta-1}\rho_0^\beta \eta. \tag{3.27}$$

– For $\mathbf{x} \in E$ such that $R_0 + 1 \leqslant |\mathbf{x}| \leqslant R_0 + 2$, we have $\nabla \eta(\mathbf{x}) = w'(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}$ and $\mathbf{u}_\infty(t,\mathbf{x}) \cdot \mathbf{x} \geqslant 0$ (see Lemma 2.2(iii) and (3.23)), thus $\mathbf{u}_\infty(t,\mathbf{x}) \cdot \nabla \eta(\mathbf{x}) \geqslant 0$. For $\mathbf{x} \in E$ such that $|\mathbf{x}| < R_0 + 1$ or $|\mathbf{x}| > R_0 + 2$, we have $\nabla \eta(\mathbf{x}) = \mathbf{0}$ and thus we also get $\mathbf{u}_\infty(t,\mathbf{x}) \cdot \nabla \eta(\mathbf{x}) \geqslant 0$.

To sum up, $\mathbf{u}_\infty(t,\mathbf{x}) \cdot \nabla \eta(\mathbf{x}) \geqslant 0$ on $E \times (0,T)$. Since ρ_ε converges to ρ a.e. on $D \times (0,T)$, we deduce from Fatou's Lemma

$$\liminf_{\varepsilon\to 0} \int_0^T \psi(t)\left(\int_0^t \int_E \frac{\delta}{\beta-1}\rho_\varepsilon^\beta \mathbf{u}_\infty \cdot \nabla \eta\right) \geqslant \int_0^T \psi(t)\left(\int_0^t \int_E \frac{\delta}{\beta-1}\rho^\beta \mathbf{u}_\infty \cdot \nabla \eta\right). \tag{3.28}$$

– Finally, since $\{\rho_\varepsilon\}$ is bounded in $L^\infty(0,T;L^\beta(D))$,

$$\int_0^T \psi(t)\left(\varepsilon \int_0^t \int_E \frac{\delta}{\beta-1}\rho_\varepsilon^\beta \Delta \eta\,d\mathbf{x}\,ds\right) dt \xrightarrow{\varepsilon\to 0} 0. \tag{3.29}$$

Consequently, we deduce from (3.26)–(3.29):

$$\liminf_{\varepsilon\to 0} I_\varepsilon^{(\eta)} \geqslant \int_0^T \psi(t)\,dt \int_E \frac{\delta}{\beta-1}\rho_0^\beta \eta + \int_0^T \psi(t)\left(\int_0^t \int_E \frac{\delta}{\beta-1}\rho^\beta \mathbf{u}_\infty \cdot \nabla\eta\right) dt.$$

Next, thanks to (3.25), the above inequality gives

$$\liminf_{\varepsilon\to 0} I_\varepsilon^{(\eta)} \geqslant \int_0^T \psi(t) \int_E \frac{\delta}{\beta-1}\rho(t)^\beta \eta + \int_0^T \psi(t)\left(\int_0^t \int_E \delta\rho^\beta \operatorname{div}\mathbf{u}_\infty \eta\right). \tag{3.30}$$

1.3.2. Now, we study $\liminf_{\varepsilon\to 0} I_\varepsilon^{(1-\eta)}$.

– On the one hand, thanks to Fatou's Lemma, one obtains

$$\liminf_{\varepsilon\to 0} \int_0^T \psi(t) \int_E \frac{\delta}{\beta-1}\rho_\varepsilon(t)^\beta (1-\eta)\,d\mathbf{x}\,dt \geqslant \int_0^T \psi(t) \int_E \frac{\delta}{\beta-1}\rho(t)^\beta (1-\eta)\,d\mathbf{x}\,dt. \tag{3.31}$$

– On the other hand, obviously, we have

$$\liminf_{\varepsilon\to 0} \int_0^T \psi(t) \int_0^t \int_E \varepsilon\delta\beta\rho_\varepsilon^{\beta-2}|\nabla\rho_\varepsilon|^2(1-\eta)\,d\mathbf{x}\,ds\,dt \geqslant 0. \tag{3.32}$$

– Observe that $1-\eta(\mathbf{x}) \geqslant 0$ for $\mathbf{x} \in \Omega^c \cap B(\mathbf{0}, R_0+2)$, and $1-\eta(\mathbf{x}) = 0$ for $\mathbf{x} \in E \setminus (\Omega^c \cap B(\mathbf{0}, R_0+2))$. Since $\mathbf{u}_\infty$ has a positive divergence on $\Omega^c \cap B(\mathbf{0}, R_0+2)$ (see Lemma 2.2(i)), thanks to Fatou's Lemma, we have

$$\liminf_{\varepsilon\to 0} \int_0^T \psi(t) \int_0^t \int_E \delta\rho_\varepsilon^\beta \operatorname{div}\mathbf{u}_\infty (1-\eta) \geqslant \int_0^T \psi(t) \int_0^t \int_E \delta\rho^\beta \operatorname{div}\mathbf{u}_\infty (1-\eta). \tag{3.33}$$

Due to (3.31)–(3.33), we obtain

$$\liminf_{\varepsilon\to 0} I_\varepsilon^{(1-\eta)} \geqslant \int_0^T \psi(t) \int_E \frac{\delta}{\beta-1}\rho(t)^\beta (1-\eta) + \int_0^T \psi(t) \int_0^t \int_E \delta\rho^\beta \operatorname{div}\mathbf{u}_\infty (1-\eta). \tag{3.34}$$

Thus we can conclude by combining (3.30) and (3.34).

Step 2. We prove that $\liminf_{\varepsilon\to 0} I_\varepsilon^\Omega \geqslant \int_0^T \psi(t) \int_\Omega \frac{\delta}{\beta-1}\rho^\beta(t) + \int_0^T \psi(t) \int_0^t \int_\Omega \delta\rho^\beta \operatorname{div}\mathbf{u}_\infty$.

Of course, we have

$$\liminf_{\varepsilon\to 0} \int_0^T \psi(t) \int_0^t \int_\Omega \varepsilon\delta\beta\rho_\varepsilon^{\beta-2}|\nabla\rho_\varepsilon|^2\,d\mathbf{x}\,ds \geqslant 0. \tag{3.35}$$

Thanks to Fatou and (3.19), one obtains

$$\liminf_{\varepsilon\to 0} \int_0^T \psi(t) \int_\Omega \frac{\delta}{\beta-1}\rho_\varepsilon(t)^\beta\,d\mathbf{x}\,ds\,dt \geqslant \int_0^T \psi(t) \int_\Omega \frac{\delta}{\beta-1}\rho(t)^\beta\,d\mathbf{x}\,dt, \tag{3.36}$$

and, since $\operatorname{div}\mathbf{u}_\infty \geqslant 0$ on $[0,T] \times \Omega_b$ (see Lemma 2.2(ii)), we also have

$$\liminf_{\varepsilon\to 0} \int_0^T \psi(t) \int_0^t \int_{\Omega_b} \delta\rho_\varepsilon^\beta \operatorname{div}\mathbf{u}_\infty\,d\mathbf{x}\,ds\,dt \geqslant \int_0^T \psi(t) \int_0^t \int_{\Omega_b} \delta\rho^\beta \operatorname{div}\mathbf{u}_\infty\,d\mathbf{x}\,ds\,dt. \tag{3.37}$$

Finally, according to (3.19), ρ_ε strongly converges to ρ in, say, $L^\beta(0,T;L^\beta(\Omega\setminus\Omega_b))$, thus

$$\liminf_{\varepsilon\to 0}\int_0^T \psi(t)\int_0^t\int_{\Omega\setminus\Omega_b}\delta\rho_\varepsilon^\beta \operatorname{div}\mathbf{u}_\infty\,d\mathbf{x}\,ds\,dt=\int_0^T\psi(t)\int_0^t\int_{\Omega\setminus\Omega_b}\delta\rho^\beta\operatorname{div}\mathbf{u}_\infty\,d\mathbf{x}\,ds\,dt. \tag{3.38}$$

The combination of (3.35)–(3.38) finishes step 2.

Thanks to the conclusions of steps 1 and 2, the proof is complete. □

3.6. Boundary conditions for density

Until now, we did not need the continuity of ρ_0 on $\overline{D_e}$. In the sequel, this hypothesis will be used in order to prove (3.8).

3.6.1. Transport of particles

The following result is a direct consequence of Lemma 2.5 ($\overline{D_e}$ is negatively invariant).

Lemma 3.1. *For every $t\in[0,T]$, $\rho(t,\mathbf{x})=\frac{\rho_0(\mathbf{Y}(0;t,\mathbf{x}))}{J(t,\mathbf{x})}$ a.e. in D_e.*

Proof. Let t be in $[0,T]$, $\mathbf{x}$ be in D_e and choose $r>0$ such that $B=B(\mathbf{x},r)\subset D_e$. Then, for $s\in[0,t]$, we set

$$B(s):=\{\mathbf{Y}(s;t,\mathbf{z}),\ \mathbf{z}\in B\}=\mathbf{X}(s,\cdot)\circ\mathbf{X}(t,\cdot)^{-1}(B).$$

For any $s\in[0,T]$, the mappings $\mathbf{X}(s,\cdot)$ are C^1-diffeomorphisms from $\mathbf{R}^2$ onto itself. Thus $B(s)$ in an open set and, according to Lemma 2.5, it is included in $\overline{D_e}$, thus in D_e.

Due to (3.22), the pair $(\rho,\mathbf{u})$, extended by 0 outside D, satisfies $\partial_t\rho+\operatorname{div}(\rho\mathbf{u})=0$ in $\mathcal{D}'(\mathbf{R}^2\times(0,T))$. Let S_α ($\alpha>0$) be a mollifier operator over space variable. Then, using the Friedrichs' Lemma [9, Lemma 6.7, p. 305], we can claim that the function $S_\alpha(\rho)$ defined by $S_\alpha(\rho)(t,\mathbf{x})=\int_{\mathbf{R}^2}\omega_\alpha(\mathbf{x}-\mathbf{y})\rho(t,\mathbf{y})\,d\mathbf{y}$, satisfies

$$\partial_t S_\alpha(\rho)+\operatorname{div}\big(S_\alpha(\rho)\mathbf{u}\big)=r_\alpha\quad\text{a.e. in }\mathbf{R}^2\times(0,T),$$

with $r_\alpha=\operatorname{div}(S_\alpha(\rho)\mathbf{u})-\operatorname{div}S_\alpha(\rho\mathbf{u})$ and $r_\alpha\xrightarrow{\alpha\to 0}0$ in $L^{\frac{2\beta}{\beta+2}}(0,T;L^{\frac{2\beta}{\beta+2}}_{loc}(\mathbf{R}^2))$. In view of the regularity of $S_\alpha(\rho)$, we can write:

$$\int_{B(t)}S_\alpha(\rho)(t,\mathbf{y})\,d\mathbf{y}=\int_{B(0)}S_\alpha(\rho)(0,\mathbf{y})\,d\mathbf{y}+\int_0^t\int_{B(s)}\big[\partial_t S_\alpha(\rho)(s,\mathbf{y})+\operatorname{div}\big(S_\alpha(\rho)\mathbf{u}_\infty\big)(s,\mathbf{y})\big]\,d\mathbf{y}\,ds.$$

Since $B(s)$ is included in D_e (thus in E) and $\mathbf{u}=\mathbf{u}_\infty$ on $E\times(0,T)$, one arrives at

$$\begin{aligned}\int_{B(t)}S_\alpha(\rho)(t,\mathbf{y})\,d\mathbf{y}&=\int_{B(0)}S_\alpha(\rho)(0,\mathbf{y})\,d\mathbf{y}+\int_0^t\int_{B(s)}\big[\partial_t S_\alpha(\rho)(s,\mathbf{y})+\operatorname{div}\big(S_\alpha(\rho)\mathbf{u}\big)(s,\mathbf{y})\big]\,d\mathbf{y}\,ds\\&=\int_{B(0)}S_\alpha(\rho)(0,\mathbf{y})\,d\mathbf{y}+\int_0^t\int_{B(s)}r_\alpha(s,\mathbf{y})\,d\mathbf{y}\,ds.\end{aligned}$$

Letting $\alpha\to 0^+$, one obtains: $\int_{B(t)}\rho(t,\mathbf{y})\,d\mathbf{y}=\int_{B(0)}\rho_0(\mathbf{y})\,d\mathbf{y}$. In the integral where appears ρ_0, we make the change of variable $\mathbf{y}:=\mathbf{Y}(0;t,\mathbf{z})=\mathbf{X}(t,\cdot)^{-1}(\mathbf{z})$, with $\mathbf{z}\in B=B(t)$. According to Lemma 2.6(ii), we have

$$\operatorname{Jac}\big[\mathbf{X}(t,\cdot)^{-1}\big](\mathbf{z})=\frac{1}{\exp(\int_0^t\operatorname{div}\mathbf{u}_\infty(s,\mathbf{X}(s,\mathbf{Y}(0;t,\mathbf{z})))\,ds)}.$$

Observing that $\mathbf{X}(s,\mathbf{Y}(0;t,\mathbf{z}))=\mathbf{Y}(s;t,\mathbf{z})$ and using J (defined by (3.9)), one finally obtains

$$\int_B\rho(t,\mathbf{y})\,d\mathbf{y}=\int_B\frac{\rho_0(\mathbf{Y}(0;t,\mathbf{z}))}{J(t,\mathbf{z})}\,d\mathbf{z}.$$

The function $\frac{\rho_0(\mathbf{Y}(0;t,\cdot))}{J(t,\cdot)}$ is continuous on D_e because ρ_0 is continuous on D_e and the mapping $\mathbf{z} \in D_e \mapsto \mathbf{Y}(0;t,\mathbf{z})$ is continuous with values in D_e (see Lemma 2.5). Therefore

$$\lim_{r\to 0} \frac{1}{\pi r^2} \int\limits_{B(\mathbf{x},r)} \frac{\rho_0(\mathbf{Y}(0;t,\mathbf{z}))}{J(t,\mathbf{z})}\, d\mathbf{z} = \frac{\rho_0(\mathbf{Y}(0;t,\mathbf{x}))}{J(t,\mathbf{x})} \quad \text{for all } \mathbf{x} \in D_e.$$

However, thanks to the Lebesgue's points theorem, for almost every $\mathbf{x} \in D_e$, we have

$$\lim_{r\to 0} \frac{1}{\pi r^2} \int\limits_{B(\mathbf{x},r)} \rho(t,\mathbf{y})\, d\mathbf{y} = \rho(t,\mathbf{x}),$$

which finishes the proof. □

3.6.2. Integral form for the continuity equation

We present a collection of open sets $(\Omega_\sigma)_{\sigma\in[0,1]}$ that will appears in the proofs of Lemmas 3.4 and 3.5.

- **Description of an open sets family.** For any $\sigma \in [0,1]$, we set

$$\Omega_\sigma := \Omega \cup (C_{\theta_1,\theta_2} \cap V_\sigma) \quad \text{with } V_\sigma := \left\{ \mathbf{x} \in \mathbf{R}^2\colon\ k(\mathbf{x}) > \frac{1}{1+\frac{3\sigma}{R_0}} \right\}. \tag{3.39}$$

The open sets Ω_σ, $\sigma \in [0,1]$, are bounded with Lipschitz boundary and included in D. Moreover, we have the following property:

$$\chi_{\Omega_\sigma} \text{ converges } \big(\text{almost everywhere on } \mathbf{R}^2\big) \text{ to } \chi_\Omega \text{ as } \sigma \to 0^+. \tag{3.40}$$

- **Description of the boundary of Ω_σ.** Consider the mapping

$$\mathbf{H}\colon [\theta_1,\theta_2] \times [0,1] \to \mathbf{R}^2, \quad (\theta,\sigma) \mapsto \mathbf{H}(\theta,\sigma) := \left(1 + \frac{3\sigma}{R_0}\right) \Phi_e(\theta),$$

where Φ_e is defined by (2.18). Then $\mathbf{H}$ is C^1 on $[\theta_1,\theta_2] \times [0,1]$.

For every $\sigma \in [0,1]$, $\mathbf{H}(\cdot,\sigma)$ is a regular (and injective) parametrization. We set

$$\Gamma_e^\sigma := \mathbf{H}(\cdot,\sigma)\big(]\theta_1,\theta_2[\big), \qquad \mathbf{x}_1^\sigma := \mathbf{H}(\theta_1,\sigma) \quad \text{and} \quad \mathbf{x}_2^\sigma := \mathbf{H}(\theta_2,\sigma). \tag{3.41}$$

So the boundary of Ω_σ can be written in the form

$$\partial\Omega_\sigma = \Gamma_e^\sigma \cup \Gamma_1^\sigma \cup \Gamma_2^\sigma \cup (\partial\Omega \setminus \Gamma_e), \tag{3.42}$$

with $\Gamma_1^\sigma = [\mathbf{x}_1, \mathbf{x}_1^\sigma]$ (segment of extremities $\mathbf{x}_1$ and $\mathbf{x}_1^\sigma$), $\Gamma_2^\sigma = [\mathbf{x}_2, \mathbf{x}_2^\sigma]$ (segment of extremities $\mathbf{x}_2$ and $\mathbf{x}_2^\sigma$) and $\partial\Omega \setminus \Gamma_e := \{\mathbf{x} \in \partial\Omega\colon\ \mathbf{u}_\infty(t,\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \geqslant 0 \text{ for all } t \in [0,T]\}$.

Lemma 3.2. *For any function $\eta \in \mathcal{D}(\mathbf{R}^2 \times (0,T))$ such that $\eta|_{Q_s} = 0$ (where Q_s is defined by* (1.13)*), and for any function $b \in C(\mathbf{R}_+) \cap C^1(\mathbf{R}_+^*)$ satisfying* (1.15), (3.7), (3.8) *holds.*

Remark 3.1. Equality (3.8), related with the integral formulas (1.14) and (1.17) which take the boundary condition on density into account, enables to guess that will be done in step E: identify $\rho_\infty(t,\cdot)$ with $\frac{\rho_0(\mathbf{Y}(0;t,\cdot))}{J(t,\cdot)}$ on Γ_e.

Proof of Lemma 3.2. Since $(\rho,\mathbf{u})$ is a renormalized solution of the continuity equation, we have in $\mathcal{D}'(\mathbf{R}^2 \times (0,T))$: $\partial_t b_k(\rho) + \operatorname{div}(b_k(\rho)\mathbf{u}) + B_k(\rho) \operatorname{div} \mathbf{u} = 0$ where $b_k(s) := b(T_k(s))$, $B_k(s) := s b_k'(s) - b_k(s)$, and where T_k still denotes the function defined by (1.24). As in the proof of Lemma 3.1, regularizing by convolution in space, we obtain

$$\partial_t\big(S_\alpha\big[b_k(\rho)\big]\big) + \operatorname{div}\big(S_\alpha\big[b_k(\rho)\big]\mathbf{u}\big) + S_\alpha\big[B_k(\rho) \operatorname{div} \mathbf{u}\big] = r_\alpha \quad \text{in } \mathcal{D}'\big(\mathbf{R}^2 \times (0,T)\big),$$

with $r_\alpha = \operatorname{div}(S_\alpha[b_k(\rho)]\mathbf{u}) - \operatorname{div} S_\alpha[b_k(\rho)\mathbf{u}]$ and $r_\alpha \xrightarrow{\alpha\to 0} 0$ in $L^1(0,T;L^1_{loc}(\mathbf{R}^2))$. In view of the regularity of each term, this equation holds almost everywhere in $(0,T)\times\mathbf{R}^2$. Thus we can multiply by η and then take the integral over $(0,T)\times\Omega_\sigma$ (with $\sigma>0$). We obtain

$$\int_0^T\int_{\Omega_\sigma}\big(\partial_t S_\alpha\big[b_k(\rho)\big]+\operatorname{div}\big(S_\alpha\big[b_k(\rho)\big]\mathbf{u}\big)+S_\alpha\big[B_k(\rho)\operatorname{div}\mathbf{u}\big]\big)\eta\,d\mathbf{x}\,dt=\int_0^T\int_{\Omega_\sigma} r_\alpha\eta\,d\mathbf{x}\,dt.$$

We write the boundary Γ_e^σ (defined by (3.41)) in the form:

$$\Gamma_e^\sigma=\Gamma_{e,1}^\sigma\cup\widetilde{\Gamma}_e^\sigma\cup\Gamma_{e,2}^\sigma, \tag{3.43}$$

where

$$\begin{aligned}&\Gamma_{e,1}^\sigma:=\big\{\mathbf{H}(\theta,\sigma),\ \theta\in\,]\theta_1,\theta_1+\sigma[\big\},\qquad \widetilde{\Gamma}_e^\sigma:=\big\{\mathbf{H}(\theta,\sigma),\ \theta\in[\theta_1+\sigma,\theta_2-\sigma]\big\},\\&\Gamma_{e,2}^\sigma:=\big\{\mathbf{H}(\theta,\sigma),\ \sigma\in\,]\theta_2-\sigma,\theta_2[\big\}.\end{aligned} \tag{3.44}$$

(This decomposition is valid as soon as $\sigma<\frac{\theta_2-\theta_1}{2}$. It is sufficient for our goal since σ will tend to 0.)

After some integrations by parts, denoting $\mathbf{n}_\sigma$ the outer unit normal vector to Ω_σ and taking $\eta|_{Q_s}=0$ into account, we obtain:

$$\begin{aligned}&\int_0^T\int_{\widetilde{\Gamma}_e^\sigma} S_\alpha\big[b_k(\rho)\big]\mathbf{u}\cdot\mathbf{n}_\sigma\eta\,d\mathbf{S}\,dt+\int_0^T\int_{\Omega_\sigma} S_\alpha\big[B_k(\rho)\operatorname{div}\mathbf{u}\big]\eta\,d\mathbf{x}\,dt\\&\quad-\int_0^T\int_{\Omega_\sigma}\big(S_\alpha\big[b_k(\rho)\big]\partial_t\eta+S_\alpha\big[b_k(\rho)\big]\mathbf{u}\cdot\nabla\eta\big)\,d\mathbf{x}\,dt-\int_0^T\int_{\Omega_\sigma} r_\alpha\eta\,d\mathbf{x}\,dt\\&\quad=-\int_0^T\int_{\Gamma_1^\sigma\cup\Gamma_{e,1}^\sigma\cup\Gamma_{e,2}^\sigma\cup\Gamma_2^\sigma} S_\alpha\big[b_k(\rho)\big]\mathbf{u}\cdot\mathbf{n}_\sigma\eta\,d\mathbf{S}\,dt.\end{aligned} \tag{3.45}$$

Thanks to the convergence's properties of S_α and r_α, we have

$$\begin{cases}\displaystyle\int_0^T\int_{\Omega_\sigma}\big(S_\alpha\big[b_k(\rho)\big]\partial_t\eta+S_\alpha\big[b_k(\rho)\big]\mathbf{u}\cdot\nabla\eta\big)\,d\mathbf{x}\,dt\xrightarrow{\alpha\to 0}\int_0^T\int_{\Omega_\sigma}\big(b_k(\rho)\partial_t\eta+b(\rho)\mathbf{u}\cdot\nabla\eta\big)\,d\mathbf{x}\,dt,\\ \displaystyle\int_0^T\int_{\Omega_\sigma} S_\alpha\big[B_k(\rho)\operatorname{div}\mathbf{u}\big]\eta\,d\mathbf{x}\,dt\xrightarrow{\alpha\to 0}\int_0^T\int_{\Omega_\sigma} B_k(\rho)\operatorname{div}\mathbf{u}\eta\,d\mathbf{x}\,dt,\\ \displaystyle\int_0^T\int_{\Omega_\sigma} r_\alpha\eta\,d\mathbf{x}\,dt\xrightarrow{\alpha\to 0}0.\end{cases} \tag{3.46}$$

For $\alpha\in\,]0,d(\widetilde{\Gamma}_e^\sigma,D_e^c)[$, $\mathbf{x}\in\widetilde{\Gamma}_e^\sigma$ and $\mathbf{z}\in\mathbf{R}^2$ such that $|\mathbf{z}|\leqslant 1$, we have $\mathbf{x}-\alpha\mathbf{z}\in D_e$ and so (Lemma 3.1), $\rho(t,\mathbf{x}-\alpha\mathbf{z})=\frac{\rho_0(\mathbf{Y}(0;t,\mathbf{x}-\alpha\mathbf{z}))}{J(t,\mathbf{x}-\alpha\mathbf{z})}$. Hence, thanks to the continuity of ρ_0 (on $\overline{D_e}$) and b_k (on $\mathbf{R}^+$), we can claim that, for all $(t,\mathbf{x})\in(0,T)\times\widetilde{\Gamma}_e^\sigma$,

$$S_\alpha\big[b_k(\rho)\big](t,\mathbf{x})=\int_{|\mathbf{z}|\leqslant 1}\omega_0(\mathbf{z})b_k\big(\rho(t,\mathbf{x}-\alpha\mathbf{z})\big)\,d\mathbf{z}\xrightarrow{\alpha\to 0} b_k\left(\frac{\rho_0(\mathbf{Y}(0;t,\mathbf{x}))}{J(t,\mathbf{x})}\right).$$

Then, another use of the Dominated Convergence Theorem yields

$$\int_0^T \int_{\widetilde{\Gamma}_e^\sigma} S_\alpha\big[b_k(\rho)\big]\mathbf{u}\cdot\mathbf{n}_\sigma\eta\,d\mathbf{S}\,dt \xrightarrow{\alpha\to 0} \int_0^T \int_{\widetilde{\Gamma}_e^\sigma} b_k\bigg(\frac{\rho_0(\mathbf{Y}(0;t,\mathbf{x}))}{J(t,\mathbf{x})}\bigg)\eta(t,\mathbf{x})\mathbf{u}_\infty(t,\mathbf{x})\cdot\mathbf{n}_\sigma\,d\mathbf{S}\,dt. \tag{3.47}$$

Observing that the length of each curve Γ_1^σ, $\Gamma_{e,1}^\sigma$, $\Gamma_{e,2}^\sigma$ and Γ_2^σ is estimated from above by $c\sigma$ (for some constant $c > 0$), one obtains for all $\alpha > 0$:

$$\left|-\int_0^T \int_{\Gamma_1^\sigma\cup\Gamma_{e,1}^\sigma\cup\Gamma_{e,2}^\sigma\cup\Gamma_2^\sigma} S_\alpha\big[b_k(\rho)\big]\mathbf{u}\cdot\mathbf{n}_\sigma\eta\,d\mathbf{S}\,dt\right| \leqslant C_k\sigma, \tag{3.48}$$

where $C_k := 4cT(\sup_{\mathbf{R}_+}|b_k|)\times\||\mathbf{u}_\infty|\eta\|_{L^\infty((0,T)\times D_e)}$ (recall that $\mathbf{u}=\mathbf{u}_\infty$ on $(0,T)\times E$). Consequently, combining (3.46)–(3.48) and passing to the limit as $\alpha\to 0$, we get

$$\left|\int_0^T \int_{\widetilde{\Gamma}_e^\sigma} b_k\bigg(\frac{\rho_0(\mathbf{Y}(0;t,\mathbf{x}))}{J(t,\mathbf{x})}\bigg)\eta(t,\mathbf{x})\mathbf{u}_\infty(t,\mathbf{x})\cdot\mathbf{n}_\sigma(\mathbf{x})\,d\mathbf{S}(\mathbf{x})\,dt - \int_0^T \int_{\Omega_\sigma} \big[b_k(\rho)\partial_t\eta + b_k(\rho)\mathbf{u}\cdot\nabla\eta - B_k(\rho)\operatorname{div}\mathbf{u}\eta\big]\,d\mathbf{x}\,dt\right| \leqslant C_k\sigma. \tag{3.49}$$

Now, we want to pass to the limit $\sigma\to 0^+$ in the above relation. First, thanks to (3.40),

$$\int_0^T \int_{\Omega_\sigma} \big[b_k(\rho)\partial_t\eta + b(\rho)\mathbf{u}\cdot\nabla\eta - B_k(\rho)\operatorname{div}\mathbf{u}\eta\big]\,d\mathbf{x}\,dt \xrightarrow{\sigma\to 0} \int_0^T \int_{\Omega} \big[b_k(\rho)\partial_t\eta + b(\rho)\mathbf{u}\cdot\nabla\eta - B_k(\rho)\operatorname{div}\mathbf{u}\eta\big]\,d\mathbf{x}\,dt. \tag{3.50}$$

Moreover, each $\mathbf{x}\in\widetilde{\Gamma}_e^\sigma$ can be written in the form $\mathbf{x}=\mathbf{H}(\theta,\sigma)$ for some $\theta\in[\theta_1+\sigma,\theta_2-\sigma]$, so that $d\mathbf{S}(\mathbf{x}) = |\partial_\theta\mathbf{H}(\theta,\sigma)|\,d\theta$ and $\mathbf{n}_\sigma(\mathbf{x}) = -(\frac{\partial_\theta\mathbf{H}(\theta,\sigma)}{|\partial_\theta\mathbf{H}(\theta,\sigma)|})^\perp$ ($\mathbf{v}^\perp$ still denotes the image of $\mathbf{v}$ under the rotation of angle $+\pi/2$). Therefore we have

$$\int_0^T \int_{\widetilde{\Gamma}_e^\sigma} b_k\bigg(\frac{\rho_0(\mathbf{Y}(0;t,\mathbf{x}))}{J(t,\mathbf{x})}\bigg)\eta(t,\mathbf{x})\mathbf{u}_\infty(t,\mathbf{x})\cdot\mathbf{n}_\sigma(\mathbf{x})\,d\mathbf{S}(\mathbf{x})\,dt$$
$$= -\int_0^T \int_{\theta_1+\sigma}^{\theta_2-\sigma} b_k\bigg(\frac{\rho_0(\mathbf{Y}(0;t,\mathbf{H}(\theta,\sigma)))}{J(t,\mathbf{H}(\theta,\sigma))}\bigg)\eta\big(t,\mathbf{H}(\theta,\sigma)\big)\mathbf{u}_\infty\big(t,\mathbf{H}(\theta,\sigma)\big)\cdot\bigg(\frac{\partial\mathbf{H}}{\partial\theta}(\theta,\sigma)\bigg)^\perp d\theta\,dt.$$

All the functions that appear in the above integral are continuous on $[\theta_1,\theta_2]\times[0,T]$. Thus, thanks to the Dominated Convergence Theorem, one arrives at

$$\int_0^T \int_{\widetilde{\Gamma}_e^\sigma} b_k\bigg(\frac{\rho_0(\mathbf{Y}(0;t,\mathbf{x}))}{J(t,\mathbf{x})}\bigg)\eta(t,\mathbf{x})\mathbf{u}_\infty(t,\mathbf{x})\cdot\mathbf{n}_\sigma(\mathbf{x})\,d\mathbf{S}(\mathbf{x})\,dt$$
$$\xrightarrow{\sigma\to 0} -\int_0^T \int_{\theta_1}^{\theta_2} b_k\bigg(\frac{\rho_0(\mathbf{Y}(0;t,\mathbf{H}(\theta,0)))}{J(t,\mathbf{H}(\theta,0))}\bigg)\eta\big(t,\mathbf{H}(\theta,0)\big)\mathbf{u}_\infty\big(t,\mathbf{H}(\theta,0)\big)\cdot\bigg(\frac{\partial\mathbf{H}}{\partial\theta}(\theta,0)\bigg)^\perp d\theta\,dt. \tag{3.51}$$

The above integral corresponds to $\int_0^T\int_{\Gamma_e} b_k(\frac{\rho_0(\mathbf{Y}(0;t,\mathbf{x}))}{J(t,\mathbf{x})})\eta(t,\mathbf{x})\mathbf{u}_\infty(t,\mathbf{x})\cdot\mathbf{n}(\mathbf{x})\,d\mathbf{S}(\mathbf{x})\,dt$. In conclusion, by combining (3.50) and (3.51), the passage to the limit $\sigma\to 0$ yields

$$\int_0^T \int_{\Gamma_e} b_k\left(\frac{\rho_0(\mathbf{Y}(0;t,\mathbf{x}))}{J(t,\mathbf{x})}\right)\eta \mathbf{u}_\infty \cdot \mathbf{n}\, d\mathbf{S}\, dt = \int_0^T \int_\Omega \left[b_k(\rho)\partial_t \eta + b_k(\rho)\mathbf{u}\cdot\nabla\eta - B_k(\rho)\operatorname{div}\mathbf{u}\eta\right].$$

At last, the Dominated Convergence Theorem enables us to pass to the limit as $k \to \infty$ and the proof is complete. □

In the same way, we can establish an integral relation which connects the boundary conditions to the energy estimate:

Lemma 3.3. *For all $t \in [0,T]$,*

$$\int_E P_\delta\left(\rho(t)\right) + \int_0^t \int_E \left(a\rho^\gamma + \delta\rho^\beta\right)\operatorname{div}\mathbf{u}_\infty \geqslant \int_E P_\delta(\rho_0) + \int_0^t \int_{\Gamma_e} P_\delta\left(\frac{\rho_0(\mathbf{Y}(0;s,\mathbf{x}))}{J(s,\mathbf{x})}\right)\mathbf{u}_\infty \cdot \mathbf{n}\, d\mathbf{S}\, ds.$$

Proof. Consider the function $b_k \in C^1(\mathbf{R}_+)$ defined by $b_k(s) = P_\delta(T_k(s))$, with $P_\delta(s) := \frac{as^\gamma}{\gamma-1} + \frac{\delta s^\beta}{\beta-1}$. As $(\rho,\mathbf{u})$ is a renormalized solution of the continuity equation, by regularizing in space, one obtains

$$\partial_t S_\alpha\left[b_k(\rho)\right] + \operatorname{div}\left(S_\alpha\left[b_k(\rho)\right]\mathbf{u}\right) + S_\alpha\left[\left(\rho b_k'(\rho) - b_k(\rho)\right)\operatorname{div}\mathbf{u}\right] = r_\alpha \quad \text{a.e. in } \mathbf{R}^2 \times (0,T),$$

where $S_\alpha[b_k(\rho)] \in L^\infty(0,T;C^\infty(\mathbf{R}^2))$, $S_\alpha[(\rho b_k'(\rho) - b_k(\rho))\operatorname{div}\mathbf{u}] \in L^2(0,T;C^\infty(\mathbf{R}^2))$ and, according to Friedrichs' Lemma [9, Lemma 6.7, p. 304], $r_\alpha \xrightarrow{\alpha\to 0} 0$ in, say, $L^1(0,T;L^1_{loc}(\mathbf{R}^2))$.

We deduce that $\partial_t S_\alpha[b_k(\rho)] \in L^1(0,T;L^1_{loc}(\mathbf{R}^2))$ and thus $S_\alpha[b_k(\rho)] \in C([0,T],L^1_{loc}(\mathbf{R}^2))$.

Consequently, we can integrate the above equality over $E_\sigma \times (0,t)$ where

$$E_\sigma := D \setminus \overline{\Omega_\sigma} \quad (\text{for } \sigma > 0). \tag{3.52}$$

One obtains

$$\int_{E_\sigma} S_\alpha\left[b_k(\rho)\right](t) + \int_0^t \int_{E_\sigma} \operatorname{div}\left(S_\alpha\left[b_k(\rho)\right]\mathbf{u}\right) + \int_0^t \int_{E_\sigma} S_\alpha\left[\left(\rho b_k'(\rho) - b_k(\rho)\right)\operatorname{div}\mathbf{u}\right]$$
$$= \int_{E_\sigma} S_\alpha\left[b_k(\rho)\right](0) + \int_0^t \int_{E_\sigma} r_\alpha. \tag{3.53}$$

Since $\mathbf{u} = \mathbf{0}$ on $\partial D \times (0,T)$, we have

$$\int_0^t \int_{E_\sigma} \operatorname{div}\left(S_\alpha\left[b_k(\rho)\right]\mathbf{u}\right) = \int_0^t \int_{\partial E_\sigma} S_\alpha\left[b_k(\rho)\right]\mathbf{u}\cdot\mathbf{n}\, d\mathbf{S}\, ds = -\int_0^t \int_{\partial\Omega_\sigma} S_\alpha\left[b_k(\rho)\right]\mathbf{u}\cdot\mathbf{n}_\sigma\, d\mathbf{S}\, ds,$$

where $\mathbf{n}_\sigma$ is the outer normal vector to Ω_σ. We know that $\mathbf{u}\cdot\mathbf{n}_\sigma \geqslant 0$ on $\partial\Omega \setminus \Gamma_e$, hence, using the decomposition of $\partial\Omega_\sigma$ in the form: $\partial\Omega_\sigma = (\partial\Omega \setminus \Gamma_e) \cup \widetilde{\Gamma_e^\sigma} \cup (\Gamma_1^\sigma \cup \Gamma_{e,1}^\sigma \cup \Gamma_{e,2}^\sigma \cup \Gamma_2^\sigma)$, we get

$$\int_0^t \int_{\partial\Omega_\sigma} S_\alpha\left[b_k(\rho)\right]\mathbf{u}\cdot\mathbf{n}_\sigma \geqslant \int_0^t \int_{\widetilde{\Gamma_e^\sigma}} S_\alpha\left[b_k(\rho)\right]\mathbf{u}\cdot\mathbf{n}_\sigma + \int_0^t \int_{\Gamma_1^\sigma \cup \Gamma_{e,1}^\sigma \cup \Gamma_{e,2}^\sigma \cup \Gamma_2^\sigma} S_\alpha\left[b_k(\rho)\right]\mathbf{u}\cdot\mathbf{n}_\sigma.$$

As in the proof of Lemma 3.2, we can prove there exists $C_k > 0$ such that, for all $\alpha > 0$,

$$\left|\int_0^t \int_{\Gamma_1^\sigma \cup \Gamma_{e,1}^\sigma \cup \Gamma_{e,2}^\sigma \cup \Gamma_2^\sigma} S_\alpha\left[b_k(\rho)\right]\mathbf{u}\cdot\mathbf{n}_\sigma\, d\mathbf{S}\, ds\right| \leqslant C_k\sigma.$$

So, we can deduce from (3.53):

$$\int_{E_\sigma} S_\alpha\big[b_k(\rho)\big](t) + \int_0^t \int_{E_\sigma} S_\alpha\big[\big(\rho b_k'(\rho) - b_k(\rho)\big)\operatorname{div}\mathbf{u}\big]$$
$$\geqslant \int_{E_\sigma} S_\alpha\big[b_k(\rho_0)\big] + \int_0^t \int_{\widetilde{\Gamma}_e^\sigma} S_\alpha\big[b_k(\rho)\big]\mathbf{u}\cdot\mathbf{n}_\sigma\, d\mathbf{S}\, ds + \int_0^t \int_{E_\sigma} r_\alpha - C_k\sigma.$$

Then, the successive passages to the limit: $\alpha \to 0^+$, $\sigma \to 0^+$ and $k \to \infty$ give the desired inequality. □

Remark 3.2. Since $\mathbf{u}_0 = \mathbf{u}_\infty(0)$ on E and $\mathbf{u} = \mathbf{u}_\infty$ on $E \times (0,T)$, we deduce the second energy inequality (3.11) from Lemma 3.3 and from the energy inequality (3.10) (which has been obtained in Proposition 3.1). Then the proof of Theorem 3.2 is complete.

4. Step E: Passage to the limit on δ

In this section, we prove Theorem 1.1 thanks to Theorem 3.2.

4.1. Choice of initial conditions

Let $(\rho_0, \mathbf{u}_0)$ be a pair satisfying (1.5) and ρ_∞ be in $L^\infty((0,T)\times\Gamma_e)$.

For $\delta > 0$, we note $(\rho_\delta, \mathbf{u}_\delta)$ a couple of functions deduced from Theorem 3.2 with some initial conditions $(\rho_{0,\delta}, \mathbf{u}_{0,\delta})$ defined as follows:

- **Definition of $\rho_{0,\delta}$.** We define $\rho_{0,\delta} : D \to \mathbf{R}$ by

$$\rho_{0,\delta}(\mathbf{x}) = \begin{cases} \rho_0(\mathbf{x})\chi_{A_\delta}(\mathbf{x}) & \text{if } \mathbf{x}\in\Omega, \\ \rho_{e,\delta}(\mathbf{x}) & \text{if } \mathbf{x}\in E, \end{cases} \tag{4.1}$$

where $A_\delta := \{\mathbf{x}\in\Omega \mid \rho_0(\mathbf{x}) \leqslant \delta^{\frac{-1}{\beta+1}}\}$ and $\rho_{e,\delta} := S_\delta(\rho_e)$, S_δ denoting a mollifier operator over space variable. The function $\rho_e : \mathbf{R}^2 \to \mathbf{R}$ is given by

$$\rho_e(\mathbf{x}) = \begin{cases} \rho_\infty(\mathbf{G}_e^{-1}(\mathbf{x}))J(\mathbf{G}_e^{-1}(\mathbf{x})) & \text{if } \mathbf{x}\in\mathcal{G}_e, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathbf{G}_e$ is the diffeomorphism of Lemma 2.7. Then ρ_e is a nonnegative function, has a compact support ($\mathcal{G}_e$ is an open set included in D_e) and, in addition, $|\rho_e(\mathbf{x})| \leqslant M_\infty$ a.e. in $\mathbf{R}^2$, with $M_\infty := \|\rho_\infty\|_{L^\infty((0,T)\times\Gamma_e)}\|J\|_{L^\infty(\mathbf{R}^2\times[0,T])}$. This implies

$$\rho_{e,\delta} \in C^\infty\big(\mathbf{R}^2\big)\cap L^\infty\big(\mathbf{R}^2\big) \quad \text{with } \|\rho_{e,\delta}\|_{L^\infty(\mathbf{R}^2)} \leqslant M_\infty, \qquad \rho_{e,\delta} \xrightarrow{\delta\to 0} \rho_e \quad \text{in } L^p\big(\mathbf{R}^2\big)\ (1\leqslant p<\infty).$$

Choosing a suitable subsequence, we can suppose that $\rho_{e,\delta}$ converge a.e. to ρ_e on $\mathbf{R}^2$. Since $\mathcal{G}_e$ have non-zero measure, for almost every $\mathbf{y}\in\mathcal{G}_e$, $\rho_{e,\delta}(\mathbf{y}) \xrightarrow{\delta\to 0} \rho_\infty(\mathbf{G}_e^{-1}(\mathbf{y}))J(\mathbf{G}_e^{-1}(\mathbf{y}))$. Since $\mathbf{G}_e$ is a C^1-diffeomorphism from $(0,T)\times\Gamma_e$ onto $\mathcal{G}_e$, we can claim that, for almost every $(t,\mathbf{x})\in(0,T)\times\Gamma_e$, $\rho_{e,\delta}(\mathbf{G}_e(t,\mathbf{x})) \xrightarrow{\delta\to 0} \rho_\infty(t,\mathbf{x})J(t,\mathbf{x})$. Finally, we have

$$\begin{cases} \rho_{0,\delta}\in L^\infty(D), \qquad \rho_{0,\delta}|_{\overline{D_e}} \in C(\overline{D_e}), \\ \rho_{0,\delta} \xrightarrow{\delta\to 0} \rho_0 \quad \text{in } L^\gamma(\Omega), \end{cases} \tag{4.2}$$

and

$$\begin{cases} \rho_{0,\delta}\big(\mathbf{Y}(0;t,\mathbf{x})\big) \xrightarrow{\delta\to 0} \rho_\infty(t,\mathbf{x})J(t,\mathbf{x}) & \text{a.e. on } (0,T)\times\Gamma_e, \\ \big|\rho_{0,\delta}\big(\mathbf{Y}(0;t,\mathbf{x})\big)\big| \leqslant M_\infty & \text{a.e. on } (0,T)\times\Gamma_e. \end{cases} \tag{4.3}$$

- **Definition of $\mathbf{u}_{0,\delta}$.** We set

$$\mathbf{u}_{0,\delta}(\mathbf{x}) = \begin{cases} \mathbf{u}_0(\mathbf{x}) & \text{if } \mathbf{x}\in\Omega, \\ \mathbf{u}_\infty(0,\mathbf{x}) & \text{if } \mathbf{x}\in E. \end{cases} \tag{4.4}$$

• **Consequences.** We deduce from (4.1), (4.2) and (4.4) that

$$\sqrt{\rho_{0,\delta}}\mathbf{u}_{0,\delta} \xrightarrow{\delta\to 0} \sqrt{\rho_0}\mathbf{u}_0 \quad \text{in } \mathbf{L}^2(\Omega), \tag{4.5}$$

$$\rho_{0,\delta}\mathbf{u}_{0,\delta} \xrightarrow{\delta\to 0} \rho_0\mathbf{u}_0 \quad \text{in } \mathbf{L}^{\frac{2\gamma}{\gamma+1}}(\Omega). \tag{4.6}$$

For $\mathbf{x} \in \Omega$, $0 \leqslant \delta\rho_{0,\delta}^{\beta}(\mathbf{x}) \leqslant \delta^{\frac{1}{\beta+1}}$, thus $\delta\rho_{0,\delta}^{\beta} \xrightarrow{\delta\to 0} 0$ in $L^1(\Omega)$. The previous results enable us to justify that

$$\int_{\Omega}\left(\frac{1}{2}\rho_{0,\delta}\big(\mathbf{u}_{0,\delta}-\mathbf{u}_\infty(0)\big)^2 + P_\delta(\rho_{0,\delta})\right) \xrightarrow{\delta\to 0} \int_{\Omega}\left(\frac{1}{2}\rho_0\big(\mathbf{u}_0-\mathbf{u}_\infty(0)\big)^2 + \frac{a}{\gamma-1}\rho_0^{\gamma}\right). \tag{4.7}$$

At last, since $\rho_{0,\delta} \xrightarrow{\delta\to 0} \rho_e$ in $L^p(E)$ $(1 \leqslant p < \infty)$, the sequence

$$\left\{E_{0,\delta} := \frac{1}{2}\int_D \rho_{0,\delta}\big(\mathbf{u}_{0,\delta}-\mathbf{u}_\infty(0)\big)^2\,d\mathbf{x} + \int_D\left(\frac{a}{\gamma-1}\rho_{0,\delta}^{\gamma} + \frac{\delta}{\beta-1}\rho_{0,\delta}^{\beta}\right)d\mathbf{x}\right\} \text{ is bounded.} \tag{4.8}$$

4.2. The results of the passage to the limit

The departure's point of this step is a series of upper bounds (for the sequence $\{(\rho_\delta, \mathbf{u}_\delta)\}$) deduced from the energy inequality (3.10) (and (4.8)). Then, we follow the method described in [8] or [9, pp. 422–424]. As in Section 3.4, we only give the main results.

• **Convergence.** There exists a suitable subsequence of $\{(\rho_\delta, \mathbf{u}_\delta)\}$ (not relabeled) and a pair $(\rho, \mathbf{u})$ satisfying (1.12) such that

$$\begin{cases} \mathbf{u}_\delta \rightharpoonup \mathbf{u} & \text{in } L^2\big(0,T;\mathbf{W}_0^{1,2}(D)\big) \text{ with } \mathbf{u} = \mathbf{u}_\infty \text{ on } E\times(0,T), \\ \rho_\delta \overset{*}{\rightharpoonup} \rho & \text{in } L^\infty\big(0,T;L^\gamma(D)\big) \text{ with } \rho \geqslant 0 \text{ on } D\times(0,T), \end{cases} \tag{4.9}$$

$$\begin{cases} \rho_\delta\mathbf{u}_\delta \rightharpoonup \rho\mathbf{u} & \text{in } L^2\big(0,T;\mathbf{L}^m(D)\big)\ (1\leqslant m<\gamma), \\ \rho_\delta\mathbf{u}_\delta \overset{*}{\rightharpoonup} \rho\mathbf{u} & \text{in } L^\infty\big(0,T;\mathbf{L}^{\frac{2\gamma}{\gamma+1}}(D)\big), \\ \rho_\delta u_\delta^i u_\delta^j \rightharpoonup \rho u^i u^j & \text{in } L^2\big(0,T;L^r(D)\big)\ \left(1\leqslant r<\frac{2\gamma}{\gamma+1},\ i,j\in\{1,2\}\right), \end{cases} \tag{4.10}$$

and

$$\begin{cases} \rho_\delta \to \rho & \text{in } L^p\big(\Omega\times(0,T)\big)\ (1\leqslant p<\gamma), \\ \rho_\delta \to \rho & \text{in } L^p\big(0,T;L^p_{loc}(\Omega)\big)\ \left(1\leqslant p<\gamma+\frac{\gamma-1}{2}\right), \\ \delta\rho_\delta^\beta \to 0 & \text{in } L^{1+\frac{\gamma-1}{2\beta}}\big(0,T;L_{loc}^{1+\frac{\gamma-1}{2\beta}}(\Omega)\big). \end{cases} \tag{4.11}$$

• **Equations.** The momentum equation (1.2) holds in $\mathcal{D}'(\Omega\times(0,T))$. Moreover, $(\rho, \mathbf{u})$ satisfies the continuity equation (1.1) in the sense of distributions in $\mathbf{R}^2\times(0,T)$ (provided ρ and $\mathbf{u}$ are extended by 0 outside D) and, with the same conventions of prolongation, $(\rho, \mathbf{u})$ also satisfies (3.22) for any function $b \in C(\mathbf{R}_+)\cap C^1(\mathbf{R}_+^*)$ verifying (1.15), (1.16).

• **Boundary conditions.** The integral formulations of the boundary condition for density (1.14), (1.17) are directly deduced from (3.8) (see Theorem 3.2), from the convergence's results and from (4.3).

• **Energy inequality.** As we have proved the strong convergence of ρ_δ to ρ in Ω and not in D, we are only able to pass to the limit in the energy inequality (3.11). The passage to the limit is similar to the one used in step D: we multiply the inequality by a nonnegative function $\psi \in \mathcal{D}(0,T)$, and, after integration, we pass to the inferior limit. The treatment of each term is classical except for $\liminf_{\delta\to 0}\int_0^T \psi(t)(\int_0^t\int_\Omega (a\rho_\delta^\gamma + \delta\rho_\delta^\beta)\operatorname{div}\mathbf{u}_\infty\,d\mathbf{x}\,ds)\,dt$. Indeed, in this case, we adapt the proof of Proposition 3.1 (step 2). First of all, using (4.11), Lemma 2.2(ii) and Fatou's Lemma, one obtains

$$\liminf_{\delta\to 0}\int_0^T \psi(t)\int_0^t\int_{\Omega_b}\big(a\rho_\delta^\gamma+\delta\rho_\delta^\beta\big)\operatorname{div}\mathbf{u}_\infty \geqslant \int_0^T \psi(t)\int_0^t\int_{\Omega_b} a\rho^\gamma\operatorname{div}\mathbf{u}_\infty.$$

Next, due to (4.11), we have

$$\lim_{\delta\to 0}\int_0^T \psi(t)\int_0^t\int_{\Omega\setminus\Omega_b}\big(a\rho_\delta^\gamma+\delta\rho_\delta^\beta\big)\operatorname{div}\mathbf{u}_\infty = \int_0^T \psi(t)\int_0^t\int_{\Omega\setminus\Omega_b} a\rho^\gamma\operatorname{div}\mathbf{u}_\infty.$$

Therefore

$$\liminf_{\delta\to 0}\int_0^T \psi(t)\int_0^t\int_{\Omega}\big(a\rho_\delta^\gamma+\delta\rho_\delta^\beta\big)\operatorname{div}\mathbf{u}_\infty \geqslant \int_0^T \psi(t)\int_0^t\int_{\Omega} a\rho^\gamma\operatorname{div}\mathbf{u}_\infty. \tag{4.12}$$

Then, one arrives at the expected inequality (1.18) (which first holds in $\mathcal{D}'(0,T)$ then, in view of the regularity of each term, a.e. on $(0,T)$).

The proof of Theorem 1.1 is complete.

5. Possible extensions

This section, based on the ideas previously developed, deals with some particular situations which result from modifications of assumptions (**H1**), (**H2**) and (**H3**). Then we detail the adaptations to bring into the proof of Theorem 1.1 in order to establish the existence of a bounded energy renormalized weak solution.

5.1. Presence of an internal obstacle

- **Hypotheses.** We modify hypothesis (**H1**) in the following way:

(**H1**) The problem is studied in an open and bounded set Ω, included in $\mathbf{R}^2$, of type

$$\Omega = g^{-1}\big(]-\infty,1[\big)\setminus S$$

where $g:\mathbf{R}^2\to\mathbf{R}$ is a convex and C^1-function on $\mathbf{R}^2$ and where S is a Lipschitz compact set included in $\Omega' := g^{-1}(]-\infty,1[)$. We also suppose that $\mathbf{0}\in\Omega$.

Assumption (**H2**) consists in defining the incoming border area by $\Gamma_e := \partial\Omega'\cap C_{\theta_1,\theta_2}$ and assumption (**H3**) remains identical.

The set S is interpreted as a physical obstacle to the flow. Thus, the boundary conditions (1.6) take the form:

$$\begin{cases}\rho(t,\mathbf{x})=\rho_\infty(t,\mathbf{x}) & \text{on } (0,T)\times\Gamma_e,\\ \mathbf{u}(t,\mathbf{x})=\mathbf{a}_\infty(t,\mathbf{x}) & \text{on } (0,T)\times\partial\Omega',\\ \mathbf{u}(t,\mathbf{x})=\mathbf{0} & \text{on } (0,T)\times\partial S.\end{cases}$$

- **Definition of $\mathbf{u}_\infty$.** (See Fig. 2.) First, we choose real numbers k_S and $k_b\in\,]1,\frac{3}{2}[$ so that

$$S\subset\Omega'_S := \big\{\mathbf{x}\in\mathbf{R}^2\colon\ k(\mathbf{x})>k_S\big\}\quad\text{and}\quad k_S>k_b.$$

It ensures the existence of a function ϕ_S of class C^∞ on $\mathbf{R}^2$ such that

$$0\leqslant\phi_S(\mathbf{x})\leqslant 1\quad\text{and}\quad \phi_S(\mathbf{x})=\begin{cases}1 & \text{for } \mathbf{x}\in\mathbf{R}^2 \text{ such that } k(\mathbf{x})\leqslant k_b,\\ 0 & \text{for } \mathbf{x}\in\mathbf{R}^2 \text{ such that } k(\mathbf{x})\geqslant k_S.\end{cases}$$

Then, we define $\mathbf{u}_\infty$ by

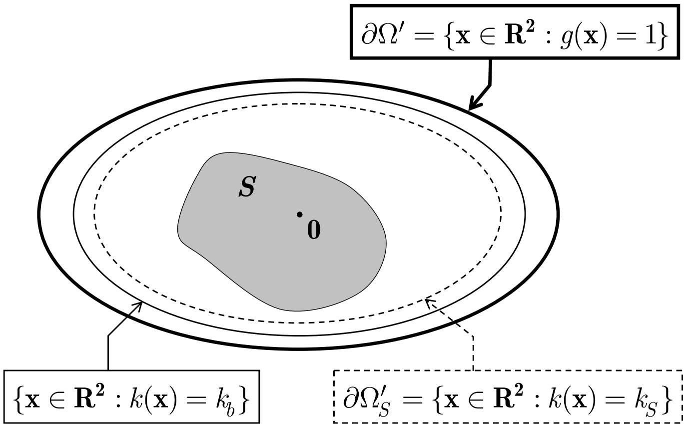

Fig. 2.

$$\mathbf{u}_\infty(t,\mathbf{x}) = \left(\mathbf{a}_\infty\big(t, k(\mathbf{x})\mathbf{x}\big) + C_\infty\left(\frac{1}{k(\mathbf{x})} - 1\right)\mathbf{x}\right)\psi_\infty(\mathbf{x})\phi_S(\mathbf{x}),$$

where C_∞ and ψ_∞ are defined as in Section 2.3.2. Thus, we observe that $\mathbf{u}_\infty$ is zero on $(0,T) \times \Omega'_S$ (so on a neighborhood of S).

• **Adaptation of the proof.**

– About the proof of Proposition 3.1 (step 1), we write I_ε^E in the form $I_\varepsilon^E = I_\varepsilon^{E'} + I_\varepsilon^S$, with $E' = D \setminus \Omega'$ and

$$\begin{cases} I_\varepsilon^{E'} := \displaystyle\int_0^T \psi(t)\left(\int_{E'} \frac{\delta}{\beta-1}\rho_\varepsilon^\beta(t) + \varepsilon\int_0^t\int_{E'} \delta\beta\rho_\varepsilon^{\beta-2}|\nabla\rho_\varepsilon|^2 + \int_0^t\int_{E'} \delta\rho_\varepsilon^\beta \operatorname{div}\mathbf{u}_\infty\right)dt, \\ I_\varepsilon^S := \displaystyle\int_0^T \psi(t)\left(\int_S \frac{\delta}{\beta-1}\rho_\varepsilon^\beta(t) + \varepsilon\int_0^t\int_S \delta\beta\rho_\varepsilon^{\beta-2}|\nabla\rho_\varepsilon|^2 + \int_0^t\int_S \delta\rho_\varepsilon^\beta \operatorname{div}\mathbf{u}_\infty\right)dt. \end{cases}$$

The proof of $\liminf_{\varepsilon\to0} I_\varepsilon^{E'} \geqslant \int_0^T \psi(t)(\int_{E'} \frac{\delta}{\beta-1}\rho^\beta(t) + \int_0^t\int_{E'} \delta\rho^\beta \operatorname{div}\mathbf{u}_\infty)\,dt$ is identical to the one already detailed whereas $\liminf_{\varepsilon\to0} I_\varepsilon^S \geqslant \int_0^T \psi(t)(\int_S \frac{\delta}{\beta-1}\rho^\beta(t) + \int_0^t\int_S \delta\rho^\beta \operatorname{div}\mathbf{u}_\infty)\,dt$ is a consequence of Fatou's Lemma and the fact that $\operatorname{div}\mathbf{u}_\infty = 0$ on $S \times (0,T)$.

– In the proof of Lemmas 3.2 and 3.3, the open sets Ω_σ and E_σ are given by

$$\Omega_\sigma := \Omega \cup \big(C_{\theta_1,\theta_2} \cap V_\sigma \cap S^c\big) \quad \text{and} \quad E_\sigma := E \setminus \overline{\Omega_\sigma}.$$

Making the same calculations, the integrations by parts yield curvilinear integrals over $\partial\Omega_\sigma$ which, in the present case, contains ∂S. Since $\mathbf{u}_\infty = \mathbf{0}$ on $\partial S \times (0,T)$, these integrals only carry over $\Gamma_e^\sigma \cup \Gamma_1^\sigma \cup \Gamma_2^\sigma \cup (\partial\Omega' \setminus \Gamma_e)$, which takes us back to the situation already treated.

– In step E, for the passage to the limit in the energy inequality, in order to obtain (4.12), it is sufficient to write that

$$\int_0^T \psi(t)\int_0^t\int_\Omega \big(a\rho_\delta^\gamma + \delta\rho_\delta^\beta\big)\operatorname{div}\mathbf{u}_\infty = \int_0^T \psi(t)\int_0^t \int_{\Omega_b\cup(\Omega'_S\setminus S)} \big(a\rho_\delta^\gamma + \delta\rho_\delta^\beta\big)\operatorname{div}\mathbf{u}_\infty$$
$$+ \int_0^T \psi(t)\int_0^t \int_{\Omega\setminus(\Omega_b\cup\Omega'_S)} \big(a\rho_\delta^\gamma + \delta\rho_\delta^\beta\big)\operatorname{div}\mathbf{u}_\infty.$$

In the first integral, the passage to the inferior limit results from Fatou's Lemma ($\operatorname{div}\mathbf{u}_\infty \geqslant 0$ on $(0,T) \times \Omega_b$ and $\mathbf{u}_\infty = \mathbf{0}$ on $(0,T) \times \Omega'_S$) whereas, in the second one, we can apply the Dominated Convergence Theorem thanks to the strong convergence on the compact $\Omega \setminus (\Omega_b \cup \Omega'_S)$ (see (4.11)).

5.2. *Case of an open convex set with a C^1 piecewise boundary*

In the case of an open convex set Ω which has a C^1 piecewise boundary, we can take up again the construction of the vector field $\mathbf{u}_\infty$ in Section 2.3.2. Indeed, the function k of Lemma 2.1 is then piecewise-defined on each angular sector (with origin $\mathbf{0}$) whose sides intercept the corners of $\partial\Omega$: the function obtained is then continuous (and even locally Lipschitz) and C^1 on each angular sector. We give an example inspired by [5].

• **Hypotheses.** Consider a rectangular domain

(H1) $\qquad \Omega =]-1, 1[\times]-h, h[\quad (h > 0).$

Thus, we have $\overline{\Omega} \subset B(\mathbf{0}, R_0)$ with, say, $R_0 := 1 + h$. We can also claim that

$$\Omega := \bigcap_{i=1}^{i=4} g_i^{-1}\big(]-\infty, 1[\big),$$

where g_1, g_2, g_3 and g_4 are linear functions (thus convex) defined, for $\mathbf{x} = (x_1, x_2) \in \mathbf{R}^2$, by

$$g_1(\mathbf{x}) := x_1, \qquad g_2(\mathbf{x}) := \frac{x_2}{h}, \qquad g_3(\mathbf{x}) := -x_1 \quad \text{and} \quad g_4(\mathbf{x}) := -\frac{x_2}{h}.$$

We assume the incoming border area is given by

(H2) $\qquad \Gamma_e := \{-1\} \times]-h, +h[.$

Setting $\mathbf{e}_1 := (1, 0)$ and $\mathbf{e}_2 := (0, 1)$, it means that the vector field $\mathbf{a}_\infty$ is such that

$$\mathbf{a}_\infty(t, \mathbf{x}) \cdot \mathbf{e}_1 > 0 \quad \text{for } (t, \mathbf{x}) \in [0, T] \times \Gamma_e.$$

Since $\mathbf{a}_\infty \cdot \mathbf{n} \geqslant 0$ a.e. on $\partial\Omega \setminus \Gamma_e$, it also implies that:

$$\text{For all } t \in [0, T], \quad \begin{cases} \mathbf{a}_\infty(t, \mathbf{x}) \cdot \mathbf{e}_1 \geqslant 0 & \text{for } \mathbf{x} \in \partial\Omega \text{ such that } x_1 = +1 \text{ and } x_2 \in]-h, h[, \\ \mathbf{a}_\infty(t, \mathbf{x}) \cdot \mathbf{e}_2 \geqslant 0 & \text{for } \mathbf{x} \in \partial\Omega \text{ such that } x_1 \in]-1, +1[\text{ and } x_2 = h, \\ \mathbf{a}_\infty(t, \mathbf{x}) \cdot \mathbf{e}_2 \leqslant 0 & \text{for } \mathbf{x} \in \partial\Omega \text{ such that } x_1 \in]-1, +1[\text{ and } x_2 = -h. \end{cases}$$

According to the notations of Section 1.3, setting $\mathbf{x}_1 := (-1, h)$ and $\mathbf{x}_2 := (-1, -h)$, we deduce from the continuity of $\mathbf{a}_\infty$ that

$$\text{For all } t \in [0, T], \quad \begin{cases} \mathbf{a}_\infty(t, \mathbf{x}_1) \cdot \mathbf{e}_1 \geqslant 0, \\ \mathbf{a}_\infty(t, \mathbf{x}_1) \cdot \mathbf{e}_2 \geqslant 0, \end{cases} \quad \text{and} \quad \begin{cases} \mathbf{a}_\infty(t, \mathbf{x}_2) \cdot \mathbf{e}_1 \geqslant 0, \\ \mathbf{a}_\infty(t, \mathbf{x}_2) \cdot \mathbf{e}_2 \leqslant 0. \end{cases}$$

In these conditions, we do not need any supplementary hypothesis to establish Theorem 1.1.

• **Definition of $\mathbf{u}_\infty$.** We divide $\mathbf{R}^2$ in four sectors

$$S_1 := \{\mathbf{x} \in \mathbf{R}^2 \colon |x_2| \leqslant h x_1\}, \qquad S_2 := \{\mathbf{x} \in \mathbf{R}^2 \colon x_2 \geqslant h|x_1|\},$$
$$S_3 := \{\mathbf{x} \in \mathbf{R}^2 \colon |x_2| \leqslant -h x_1\}, \qquad S_4 := \{\mathbf{x} \in \mathbf{R}^2 \colon x_2 \leqslant -h|x_1|\}.$$

Then, for $\mathbf{x} = (x_1, x_2) \in \mathbf{R}^2 \setminus \{\mathbf{0}\}$,

$$\begin{cases} k(\mathbf{x}) = \dfrac{1}{x_1} & \text{if } \mathbf{x} \in S_1, \qquad k(\mathbf{x}) = \dfrac{h}{x_2} \quad \text{if } \mathbf{x} \in S_2, \\ k(\mathbf{x}) = -\dfrac{1}{x_1} & \text{if } \mathbf{x} \in S_3, \qquad k(\mathbf{x}) = -\dfrac{h}{x_2} \quad \text{if } \mathbf{x} \in S_4. \end{cases}$$

We define $\mathbf{u}_\infty$ by (2.4) and thus, $\mathbf{u}_\infty$ is globally Lipschitz on $[0, T] \times \mathbf{R}^2$ and C^1 on $[0, T] \times S_i$ $(1 \leqslant i \leqslant 4)$. This regularity is sufficient to apply the method described in steps A, B and C. In particular, we can use $\mathbf{u}_\infty$ as a test function in the momentum equation (1.19) and $|\mathbf{u}_\infty|^2$ as test function in the continuity equation (1.19) in order to recover the new energy inequality which allows us to start the step C.

• **Adaptation of the proof.**

– To recover the results of Lemma 2.2, it is sufficient to make the calculations over each sector S_i $(1 \leqslant i \leqslant 4)$ in order to choose suitably the constant C_∞.

– In the statement of Lemma 2.5, the region D_e becomes

$$D_e := \{\mathbf{x} = (x_1, x_2) \in D\colon\ |x_2| < -hx_1 \text{ and } x_1 < -1\},$$

and, in the proof, the function f_3 is given by $f_3(\mathbf{x}) = 1 - g_3(\mathbf{x})$.

– In order to prove Lemma 2.7, it is easy to build a vector field $\mathbf{u}_\infty^{(3)}$ which is C^1 (with compact support) on $[0, T] \times \mathbf{R}^2$ and coincides with $\mathbf{u}_\infty$ on $[0, T] \times S_3$. Denoting $\mathbf{Y}^{(3)}$ the flow associated to $\mathbf{u}_\infty^{(3)}$, since $\mathbf{u}_\infty^{(3)} = \mathbf{u}_\infty$ on $[0, T] \times S_3$, it is clear that

(i) $\overline{D_e}$ is negatively invariant with respect to $\mathbf{Y}^{(3)}$,
(ii) $\mathbf{Y} = \mathbf{Y}^{(3)}$ on the closed set $\{(s, t, \mathbf{x}) \in [0, T] \times [0, T] \times \overline{D_e}\colon\ s \leqslant t\}$.

However, thanks to the regularity of $\mathbf{u}_\infty^{(3)}$, $\mathbf{Y}^{(3)}$ is C^1 and thus, we can obtain the results of Lemmas 2.6 and 2.7 "for $\mathbf{Y}^{(3)}$". In particular, the mapping $[0, T] \times \Gamma_e \to \mathbf{R}^2$, $(t, \mathbf{x}) \mapsto \mathbf{Y}^{(3)}(0; t, \mathbf{x})$ is a C^1-diffeomorphism from $(0, T) \times \Gamma_e$ onto an open set $\mathcal{G}_e$ included in D_e. Hence, thanks to (ii), Lemma 2.7 is still valid.

– Likewise, the fact that $\mathbf{u}_\infty$ is C^1 on $[0, T] \times S_3$ is sufficient to obtain Lemma 3.1.

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