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Existence and qualitative properties of solutions to a quasilinear elliptic equation involving the Hardy–Leray potential ☆

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Abstract

In this work we deal with the existence and qualitative properties of the solutions to a supercritical problem involving the $-\Delta_p(\cdot)$ operator and the Hardy–Leray potential. Assuming $0 \in \Omega$, we study the regularizing effect due to the addition of a first order nonlinear term, which provides the existence of solutions with a breaking of resonance. Once we have proved the existence of a solution, we study the qualitative properties of the solutions such as regularity, monotonicity and symmetry.

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1. Introduction

In this paper we shall study the existence and qualitative properties of weak positive solutions to the supercritical problem

$$\begin{cases} -\Delta_p u + |\nabla u|^p = \vartheta \dfrac{u^q}{|x|^p} + f & \text{in } \Omega, \\ u \geqslant 0 \quad \text{in } \Omega, \qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{P}$}$$

where Ω is a bounded domain in $\mathbb{R}^N$ such that $0 \in \Omega$, $\vartheta > 0$, $p - 1 < q < p$, $f \geqslant 0$, $f \in L^1(\Omega)$ and $1 < p < N$. The existence in the semilinear case $p = 2$ has been investigated in the recent work [15]. We start giving the following definition.

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Definition 1. We say that u is a weak solution to

$$-\Delta_p u + |\nabla u|^p = \vartheta \frac{u^q}{|x|^p} + f \quad \text{in } \Omega,$$

if $u \in W_0^{1,p}(\Omega)$ and

$$\int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla \phi) + \int_\Omega |\nabla u|^p \phi = \vartheta \int_\Omega \frac{u^q}{|x|^p}\phi + \int_\Omega f\phi \quad \forall \phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

The behavior of the supercritical problem with $0 \in \partial\Omega$ is quite different. See details in [10] and [15] for $p = 2$ and [16] for the p-Laplacian case $1 < p < N$.

In this work we consider $0 \in \Omega$ and, because of the regularizing effect due to the presence of the gradient term $|\nabla u|^p$ on the left-hand side of problem $(\mathcal{P})$, we are able to prove the existence of a weak solution u (see Definition 1) to problem $(\mathcal{P})$, remarkably for any $\vartheta > 0$ and for each $f \in L^1(\Omega)$, $f \geqslant 0$. As nowadays well understood, the solution obtained is called *solutions obtained as limits of approximations*, or simply SOLA, see [5]. By using the results in [6] in this case SOLA is equivalent to *entropy solution*, see [1], or *renormalized solution*.

We have the following result:

Theorem 1. *Consider problem $(\mathcal{P})$ with $1 < p < N$, $p - 1 < q < p$ and assume that $f \in L^1(\Omega)$ is a positive function. Then for all $\vartheta > 0$ there exists a weak solution $u \in W_0^{1,p}(\Omega)$ to $(\mathcal{P})$.*

This result emphasizes the fact that the term $|\nabla u|^p$ on the left-hand side of $(\mathcal{P})$ is enough to get a *resonance breaking result*. The scheme of the proof is the following:

(i) We prove the existence of a solution to the truncated problem

$$-\Delta_p u_k + |\nabla u_k|^p = \vartheta T_k\left(\frac{u_k^q}{|x|^p}\right) + T_k(f) \quad \text{in } \Omega,\ u_k \in W_0^{1,p}(\Omega).$$

where $T_k(s) = \max\{\min\{k, s\}, -k\}$, $k > 0$. This is done by solving the regularized problem (3) below and passing to the limit in $W_0^{1,p}(\Omega)$.

(ii) We show that the sequence of solutions to the truncated problem converges weakly in $W_0^{1,p}(\Omega)$ and then we deduce the a.e. convergence of the gradients. Finally we exploit it to deduce strong convergence in $W_0^{1,p}(\Omega)$.

(iii) We pass to the limit in the truncated problem and we obtain the existence of a solution to $(\mathcal{P})$.

Let us remark that, because of the presence of the gradient term (which causes the existence of solutions), to pass to the limit in the truncated problem it is necessary to deduce the convergence of u_k (solutions of the truncated problem) in $W_0^{1,p}(\Omega)$. A convergence in $W_0^{1,q}(\Omega)$ with $q < p$, in the spirit of [3], would not be sufficient to pass to the limit and get a weak formulation of the problem.

In the second part of this paper we deal with the study of the qualitative properties of weak solutions to $(\mathcal{P})$. First we point out some regularity properties of the solutions and then we prove the following result:

Theorem 2. *Let $u \in C^1(\overline{\Omega} \setminus \{0\})$ be a weak solution to $(\mathcal{P})$. Consider the domain Ω strictly convex w.r.t. the ν-direction ($\nu \in S^{N-1}$) and symmetric w.r.t. T_0^ν, where*

$$T_0^\nu = \{x \in \mathbb{R}^N \colon x \cdot \nu = 0\}.$$

Moreover, assume $f \in C^1(\overline{\Omega} \setminus \{0\})$ to be non-decreasing w.r.t. the ν-direction in the set

$$\Omega_0^\nu = \{x \in \Omega \colon x \cdot \nu < 0\}$$

and even w.r.t. T_0^ν. Then u is symmetric w.r.t. T_0^ν and non-decreasing w.r.t. the ν-direction in Ω_0^ν. Moreover, if Ω is a ball, then u is radially symmetric with $\frac{\partial u}{\partial r}(r) < 0$ for $r \neq 0$.

Remark 1. Notice that the extra regularity hypothesis on f is sufficient to have the corresponding regularity of a solution. See [20] for details on regularity.

We point out that Theorem 2 will be a consequence of a more general result, see Proposition 3 below, which states a monotonicity property of the solutions in general domains near strictly convex parts of the boundary. This can be useful for example in blow-up analysis.

Also, it will be clear from the proof, that the same technique could be applied to study the case of more general nonlinearities. In particular, we note that the nonlinearity in problem ($\mathcal{P}$) is in general locally Lipschitz continuous only in $(0, \infty)$.

The main ingredient in the proof of the symmetry result is the well known *Moving Plane Method* [21], that was used in a clever way in the celebrated paper [12] for the semilinear nondegenerate case. Actually our proof is more similar to the one of [2] and is based on the weak comparison principle in small domains. The *Moving Plane Method* was extended to the case of p-Laplace equations firstly in [7] for the case $1 < p < 2$ and later in [9] for the case $p \geqslant 2$. In the case $p \geqslant 2$ it is required the nonlinearity to be positive and as can be seen in some examples, this assumption is in general necessary.

The first crucial step is the proof of a weak comparison principle in small domains that we carry out in Proposition 2. This is based on some regularity results in the spirit of [9]. These results hold only away from the origin due to the presence of the Hardy potential in our problem. This will require more attention in the application of the moving plane procedure. Moreover, the presence of the gradient term $|\nabla u|^p$, leads to a proof of the weak comparison principle in small domains which makes use of the right choice of test functions.

Notation. Generic fixed numerical constants will be denoted by C (with subscript in some case) and will be allowed to vary within a single line or formula. Moreover f^+ and f^- will stand for the positive and negative part of a function, i.e. $f^+ = \max\{f, 0\}$ and $f^- = \min\{f, 0\}$. We also denote $|A|$ the Lebesgue measure of the set A.

2. Existence of an energy solution to the problem ($\mathcal{P}$)

It will be useful to refer to the following result.

Lemma 1 *(Hardy–Sobolev inequality). Suppose $1 < p < N$ and $u \in W^{1,p}(\mathbb{R}^N)$. Then we have*

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} \leqslant C_{N,p} \int_{\mathbb{R}^N} |\nabla u|^p,$$

with $C_{N,p} = (\frac{p}{N-p})^p$ optimal and not achieved constant.

2.1. Existence of a solution to the truncated problem

First, we are going to study the existence of a solution to the truncated problem

$$-\Delta_p u_k + |\nabla u_k|^p = \vartheta T_k\left(\frac{u_k^q}{|x|^p}\right) + T_k(f) \quad \text{in } \Omega, \qquad u_k \in W_0^{1,p}(\Omega), \tag{1}$$

where $T_k(s) = \max\{\min\{k, s\}, -k\}$, $k > 0$.

Theorem 3. *There exists a positive solution to problem* (1).

Notice that $\phi \equiv 0$ is a subsolution to problem (1). Consider ψ the solution to

$$\begin{cases} -\Delta_p \psi = \vartheta \cdot k + T_k(f) & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

In fact ψ turns to be a supersolution to (1).

To prove Theorem 3 we will consider a sequence of approximated problems that we solve by iteration and by using some convenient comparison argument. We take as starting point $w_0 = 0$ and consider iteratively the problem,

$$\begin{cases} -\Delta_p w_n + \dfrac{|\nabla w_n|^p}{1+\frac{1}{n}|\nabla w_n|^p} = \vartheta T_k\left(\dfrac{w_{n-1}^q}{|x|^p}\right) + T_k(f) & \text{in } \Omega, \\ w_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

Notice that the subsolution $\phi \equiv 0$ and the supersolution ψ to problem (1) are subsolution and supersolution to the problem (3).

Next proposition follows using a comparison argument from [4].

Proposition 1. *There exists $w_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ solution to (3).*

Moreover, $0 \leqslant w_n \leqslant \psi$ $\forall n \in \mathbb{N}$.

Proof of Theorem 3. We proceed in two steps.

Step 1: Weak convergence of w_n in $W_0^{1,p}(\Omega)$. By simplicity let us set

$$H_n(\nabla w_n) = \frac{|\nabla w_n|^p}{1+\frac{1}{n}|\nabla w_n|^p}. \tag{4}$$

Taking w_n as a test function in the approximated problems (3), we obtain

$$\begin{aligned} \int_\Omega |\nabla w_n|^p\,dx + \int_\Omega H_n(\nabla w_n) w_n\,dx &= \vartheta \int_\Omega T_k\left(\frac{w_{n-1}^q}{|x|^p}\right) w_n\,dx + \int_\Omega T_k(f) w_n\,dx \\ &\leqslant \vartheta \int_\Omega k w_n\,dx + \int_\Omega f w_n\,dx. \end{aligned}$$

Since $w_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $f \in L^1(\Omega)$, there exists a positive constant $C(k, f, \psi, \vartheta, \Omega)$ such that

$$\int_\Omega |\nabla w_n|^p\,dx + \int_\Omega H_n(\nabla w_n) w_n\,dx \leqslant C(k, f, \psi, \vartheta, \Omega).$$

Moreover, since $\int_\Omega H_n(\nabla w_n) w_n\,dx \geqslant 0$, we have

$$\int_\Omega |\nabla w_n|^p\,dx \leqslant C(k, f, \psi, \vartheta, \Omega). \tag{5}$$

Therefore, up to a subsequence, $w_n \rightharpoonup u_k$ weakly in $W_0^{1,p}(\Omega)$ and $w_n \rightharpoonup u_k$ weakly-* in $L^\infty(\Omega)$, giving

$$u_k \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

Step 2: Strong convergence of w_n in $W_0^{1,p}(\Omega)$ and passing to the limit in (1). To get the strong convergence in $W_0^{1,p}(\Omega)$ first of all we notice that

$$\|w_n - u_k\|_{W_0^{1,p}(\Omega)} \leqslant \left\|(w_n - u_k)^+\right\|_{W_0^{1,p}(\Omega)} + \left\|(w_n - u_k)^-\right\|_{W_0^{1,p}(\Omega)}. \tag{6}$$

Thus, we proceed estimating each term on the right-hand side of (6).

Asymptotic behaviour of $\|(w_n - u_k)^+\|_{W_0^{1,p}(\Omega)}$. Choosing $(w_n - u_k)^+$ as a test function in (3) we obtain

$$\begin{aligned} &\int_\Omega |\nabla w_n|^{p-2}\left(\nabla w_n, \nabla (w_n - u_k)^+\right) dx + \int_\Omega H_n(\nabla w_n)(w_n - u_k)^+\,dx \\ &\quad = \vartheta \int_\Omega T_k\left(\frac{w_{n-1}^q}{|x|^p}\right)(w_n - u_k)^+\,dx + \int_\Omega T_k(f)(w_n - u_k)^+\,dx. \end{aligned} \tag{7}$$

Since $w_n \rightharpoonup u_k$ in $W_0^{1,p}(\Omega)$, one has $w_n \to u_k$ a.e. in Ω and thus $(w_n-u_k)^+ \to 0$ a.e. in Ω together with $(w_n-u_k)^+ \rightharpoonup 0$ in $W_0^{1,p}(\Omega)$ as well. Therefore, the right-hand side of (7) goes to zero when n goes to infinity.

Then, since $\int_\Omega H_n(\nabla w_n)(w_n-u_k)^+\,dx \geqslant 0$, (7) becomes

$$\int_\Omega |\nabla w_n|^{p-2}\big(\nabla w_n, \nabla(w_n-u_k)^+\big)\,dx = o(1) \quad \text{as } n\to+\infty. \tag{8}$$

Since

$$\int_\Omega |\nabla u_k|^{p-2}\big(\nabla u_k, \nabla(w_n-u_k)^+\big) = o(1) \quad \text{as } n\to+\infty,$$

it follows

$$\int_\Omega \big(|\nabla w_n|^{p-2}\nabla w_n - |\nabla u_k|^{p-2}\nabla u_k, \nabla(w_n-u_k)^+\big)\,dx = o(1). \tag{9}$$

Then, from (9) we have

$$o(1) = \begin{cases} C_1(p)\frac{|\nabla(w_n-u_k)^+|^2}{(|\nabla w_n|+|\nabla u_k|)^{2-p}} & \text{if } 1<p<2,\\ C_1(p)|\nabla(w_n-u_k)^+|^p & \text{if } p\geqslant 2,\end{cases} \tag{10}$$

with $C_1(p)$ a positive constant depending on p. In any case, since for $1<p<2$ using Hölder's inequality one has

$$\int_\Omega \big|\nabla(w_n-u_k)^+\big|^p \leqslant \left(\int_\Omega \frac{|\nabla(w_n-u_k)^+|^2}{(|\nabla w_n|+|\nabla u_k|)^{(2-p)}}\right)^{\frac{p}{2}} \left(\int_\Omega \big(|\nabla w_n|+|\nabla u_k|\big)^p\right)^{\frac{2-p}{2}}, \tag{11}$$

we obtain

$$\big\|(w_n-u_k)^+\big\|_{W_0^{1,p}(\Omega)} \to 0 \quad \text{as } n\to+\infty. \tag{12}$$

Asymptotic behaviour of $\|(w_n-u_k)^-\|_{W_0^{1,p}(\Omega)}$. Let us consider $e^{-w_n}[(w_n-u_k)^-]$ as a test function in (3),

$$\int_\Omega e^{-w_n}|\nabla w_n|^{p-2}\big(\nabla w_n, \nabla(w_n-u_k)^-\big)\,dx + \int_\Omega e^{-w_n}\left(\frac{|\nabla w_n|^p}{1+\frac{1}{n}|\nabla w_n|^p} - |\nabla w_n|^p\right)(w_n-u_k)^-\,dx$$
$$= \vartheta\int_\Omega e^{-w_n}T_k\left(\frac{w_{n-1}^q}{|x|^p}\right)(w_n-u_k)^-\,dx + \int_\Omega e^{-w_n}T_k(f)(w_n-u_k)^-\,dx. \tag{13}$$

We point out that using this test function it follows

$$\int_\Omega e^{-w_n}\left(\frac{|\nabla w_n|^p}{1+\frac{1}{n}|\nabla w_n|^p} - |\nabla w_n|^p\right)(w_n-u_k)^- \geqslant 0. \tag{14}$$

As above, since $(w_n-u_k)^- \to 0$ a.e. in Ω, the right-hand side of (13) tends to zero as n goes to infinity. Being $w_n \leqslant \psi$ (see Proposition 1), one has $e^{-w_n} \geqslant \gamma > 0$ uniformly on n. Then Eq. (13) states as

$$\gamma\int_\Omega |\nabla w_n|^{p-2}\big(\nabla w_n, \nabla(w_n-u_k)^-\big)\,dx = o(1). \tag{15}$$

Arguing in the same way as we have done from Eq. (8) to (12), we obtain

$$\big\|(w_n-u_k)^-\big\|_{W_0^{1,p}(\Omega)} \to 0 \quad \text{as } n\to+\infty. \tag{16}$$

From Eq. (6), by using (12) and (16) we get

$$\big\|(w_n-u_k)\big\|_{W_0^{1,p}(\Omega)} \to 0 \quad \text{as } n\to+\infty$$

and consequently $\nabla w_n \to \nabla u_k$ a.e. in Ω. Then, by (4) follows $H_n(\nabla w_n) \to |\nabla u_k|^p$ a.e. in Ω and by Vitali's Lemma,

$$H_n(\nabla w_n) \to |\nabla u_k|^p \quad \text{in } L^1(\Omega).$$

Hence, $u_k \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ satisfies the problem in the following sense

$$\int_\Omega |\nabla u_k|^{p-2}(\nabla u_k, \nabla \phi) + \int_\Omega |\nabla u_k|^p \phi = \vartheta \int_\Omega T_k\left(\frac{u_k^p}{|x|^p}\right)\phi + \int_\Omega T_k(f)\phi \quad \forall \phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \tag{17}$$

concluding the proof. □

2.2. *Passing to the limit and convergence to a solution u of* ($\mathcal{P}$)

We want to show that $u_k \to u$ strongly in $W_0^{1,p}(\Omega)$ in order to prove the existence of a solution u to problem ($\mathcal{P}$).

Proof of Theorem 1. We perform the proof in different steps.

Step 1: Weak convergence of u_k in $W_0^{1,p}(\Omega)$. We start taking $T_n(u_k)$ as a test function in the truncated problem (1), obtaining

$$\int_\Omega \left|\nabla T_n(u_k)\right|^p dx + \int_\Omega |\nabla u_k|^p T_n(u_k)\, dx = \vartheta \int_\Omega T_k\left(\frac{u_k^q}{|x|^p}\right) T_n(u_k)\, dx + \int_\Omega T_k(f) T_n(u_k)\, dx.$$

Notice that, defining

$$\Psi_n(s) = \int_0^s T_n(t)^{\frac{1}{p}}\, dt, \tag{18}$$

one has

$$\int_\Omega \left|\nabla T_n(u_k)\right|^p dx + \int_\Omega \left|\nabla \Psi_n(u_k)\right|^p dx = \vartheta \int_\Omega T_k\left(\frac{u_k^q}{|x|^p}\right) T_n(u_k)\, dx + \int_\Omega T_k(f) T_n(u_k)\, dx$$
$$\leqslant \vartheta \int_\Omega \frac{u_k^q}{|x|^p} T_n(u_k) + n\|f\|_{L^1(\Omega)}. \tag{19}$$

By a straightforward calculation it is easy to check that for fixed $q \in [p-1, p)$, $\forall \varepsilon > 0$ and $\forall n > 0$, there exists C_ε such that

$$s^q T_n(s) \leqslant \varepsilon \Psi_n^p(s) + C_\varepsilon \quad s \geqslant 0. \tag{20}$$

Thanks to Lemma 1 and (20), Eq. (19) states as

$$\int_\Omega \left|\nabla T_n(u_k)\right|^p dx + \int_\Omega \left|\nabla \Psi_n(u_k)\right|^p dx \leqslant \varepsilon \frac{\vartheta}{C_{N,p}} \int_\Omega \left|\nabla \Psi_n(u_k)\right|^p + \vartheta C_\varepsilon \int_\Omega \frac{1}{|x|^p} + n\|f\|_{L^1(\Omega)}.$$

Then choosing $\varepsilon > 0$ such that $0 < \varepsilon \frac{\vartheta}{C_{N,p}} < 1$, for some positive C we get

$$\int_\Omega \left|\nabla T_n(u_k)\right|^p dx + C \int_\Omega \left|\nabla \Psi_n(u_k)\right|^p dx \leqslant \vartheta C_\varepsilon \int_\Omega \frac{1}{|x|^p} + n\|f\|_{L^1(\Omega)} \leqslant C(\vartheta, \varepsilon, f, p, n, \Omega). \tag{21}$$

Fixed $l \geqslant 1$, by definition (18) of Ψ_l and Eq. (21), one has

$$\int_\Omega |\nabla u_k|^p dx \leqslant \int_\Omega \left|\nabla T_l(u_k)\right|^p dx + \int_{\Omega \cap \{u_k \geqslant l\}} |\nabla u_k|^p dx \leqslant \int_\Omega \left|\nabla T_l(u_k)\right|^p + \frac{1}{l} \int_\Omega \left|\nabla \Psi_l(u_k)\right|^p \leqslant C, \tag{22}$$

uniformly on k. Therefore, up to a subsequence it follows $u_k \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$ and a.e. in Ω.

Step 2: Strong convergence in $L^1(\Omega)$ of the singular term. By Hölder inequality we have

$$\int_\Omega T_k\left(\frac{u_k^q}{|x|^p}\right) \leqslant \int_\Omega \frac{u_k^q}{|x|^p} \leqslant \left(\int_\Omega \frac{u_k^p}{|x|^p}\,dx\right)^{\frac{q}{p}}\left(\int_\Omega \frac{1}{|x|^p}\,dx\right)^{\frac{p-q}{p}} \leqslant C\left(\int_\Omega |\nabla u_k|^p\right)^{\frac{q}{p}} \leqslant C, \tag{23}$$

with C a positive constant that does not depend on k. It follows that $T_k(\frac{u_k^q}{|x|^p})$ is bounded in $L^1(\Omega)$ and converges almost everywhere to $\frac{u^q}{|x|^p}$. In particular Fatou's Lemma implies $\frac{u^q}{|x|^p} \in L^1(\Omega)$. Moreover, let $E \subset \Omega$ be a measurable set, by Fatou's Lemma we have

$$\int_E T_k\left(\frac{u_k^q}{|x|^p}\right) \leqslant \int_E \frac{u_k^q}{|x|^p} \leqslant \lim_{n\to+\infty}\int_E \frac{w_n^q}{|x|^p} \leqslant \int_E \frac{\psi^q}{|x|^p} \leqslant \delta\big(|E|\big),$$

uniformly in k where $\lim_{s\to 0}\delta(s) = 0$, w_n is as in the proof of Theorem 3 and ψ as in Proposition 1. Thus, from Vitali's Theorem it follows

$$T_k\left(\frac{u_k^q}{|x|^p}\right) \to \frac{u^q}{|x|^p} \quad \text{in } L^1(\Omega). \tag{24}$$

Step 3: Strong convergence of $|\nabla u_k|^p \to |\nabla u|^p$ in $L^1(\Omega)$. To show the strong convergence of the gradients we need some preliminary results. We have the following

Lemma 2. *Let u_k be defined by (1). Then*

$$\lim_{n\to\infty}\int_{\{u_k\geqslant n\}} |\nabla u_k|^p = 0 \tag{25}$$

uniformly in k.

Proof. Let us consider the functions

$$G_n(s) = s - T_n(s), \quad \text{and} \quad \psi_{n-1}(s) = T_1\big(G_{n-1}(s)\big).$$

Notice that, $\psi_{n-1}(u_k)|\nabla u_k|^p \geqslant |\nabla u_k|^p_{\chi\{u_k\geqslant n\}}$. Using $\psi_{n-1}(u_k)$ as a test function in (1) we get

$$\begin{aligned}\int_{\{u_k\geqslant n\}} |\nabla u_k|^p &\leqslant \int_\Omega \big|\nabla\psi_{n-1}(u_k)\big|^p + \int_\Omega |\nabla u_k|^p\psi_{n-1}(u_k)\\ &= \int_\Omega \vartheta T_k\left(\frac{u_k^q}{|x|^p}\right)\psi_{n-1}(u_k) + \int_\Omega T_k(f)\psi_{n-1}(u_k).\end{aligned} \tag{26}$$

Since $\{u_k\}$ is uniformly bounded in $W_0^{1,p}(\Omega)$, then up to a subsequence, $\{u_k\}$ strongly converges in $L^p(\Omega)$ for $1 \leqslant p < p^* = \frac{Np}{N-p}$ and a.e. in Ω. Thus we obtain that

$$\big|\big\{x\in\Omega:\ n-1 < u_k(x) < n\big\}\big| \to 0 \quad \text{if } n\to\infty,$$
$$\big|\big\{x\in\Omega:\ u_k(x) > n\big\}\big| \to 0 \quad \text{if } n\to\infty,$$

uniformly on k. Then, from (24) and (26) we have uniformly in k

$$\lim_{n\to\infty}\int_{\{u_k\geqslant n\}} |\nabla u_k|^p = 0. \qquad \square \tag{27}$$

Next lemma shows the strong convergence of the truncated terms.

Lemma 3. *Consider $u_k \rightharpoonup u$ as above. Then one has uniformly in m,*

$$T_m(u_k) \to T_m(u) \quad \text{in } W_0^{1,p}(\Omega) \text{ for } k \to +\infty.$$

Proof. Notice that

$$\big\|T_m(u_k) - T_m(u)\big\|_{W_0^{1,p}(\Omega)} \leqslant \big\|\big(T_m(u_k) - T_m(u)\big)^+\big\|_{W_0^{1,p}(\Omega)} + \big\|\big(T_m(u_k) - T_m(u)\big)^-\big\|_{W_0^{1,p}(\Omega)}. \tag{28}$$

We are going to estimate each term on the right-hand side of (28).

Asymptotic behaviour of $\|(T_m(u_k) - T_m(u))^+\|_{W_0^{1,p}(\Omega)}$. We take $(T_m(u_k) - T_m(u))^+$ as a test function in (1), obtaining

$$\int_\Omega \big(|\nabla u_k|^{p-2}\nabla u_k, \nabla\big(T_m(u_k) - T_m(u)\big)^+\big)\,dx + \int_\Omega |\nabla u_k|^p\big(T_m(u_k) - T_m(u)\big)^+\,dx$$
$$= \int_\Omega \left(\vartheta T_k\left(\frac{u_k^q}{|x|^p}\right) + T_k(f)\right)\big(T_m(u_k) - T_m(u)\big)^+\,dx. \tag{29}$$

Since $T_m(u_k) \rightharpoonup T_m(u)$ and $T_m(u_k) \to T_m(u)$ a.e. in Ω, we have $(T_m(u_k) - T_m(u))^+ \rightharpoonup 0$ in $W_0^{1,p}(\Omega)$ and $(T_m(u_k) - T_m(u))^+ \to 0$ a.e. in Ω. Thus, the right-hand side of (29), by dominated convergence, tends to zero as k goes to infinity. From (29) we have

$$\int_\Omega \big(|\nabla u_k|^{p-2}\nabla u_k, \nabla\big(T_m(u_k) - T_m(u)\big)^+\big)\,dx = o(1). \tag{30}$$

We estimate the left-hand side of (30) as

$$\int_\Omega \big(|\nabla u_k|^{p-2}\nabla u_k, \nabla\big(T_m(u_k) - T_m(u)\big)^+\big)\,dx$$
$$= \int_{\Omega\cap\{|u_k|\leqslant m\}} \big(\big|\nabla T_m(u_k)\big|^{p-2}\nabla T_m(u_k) - \big|\nabla T_m(u)\big|^{p-2}\nabla T_m(u), \nabla\big(T_m(u_k) - T_m(u)\big)^+\big)\,dx$$
$$+ \int_{\Omega\cap\{|u_k|\leqslant m\}} \big(\big|\nabla T_m(u)\big|^{p-2}\nabla T_m(u), \nabla\big(T_m(u_k) - T_m(u)\big)^+\big)\,dx$$
$$+ \int_{\Omega\cap\{|u_k|> m\}} \big(|\nabla u_k|^{p-2}\nabla u_k, \nabla\big(T_m(u_k) - T_m(u)\big)^+\big)\,dx. \tag{31}$$

Since $(T_m(u_k) - T_m(u))^+ \rightharpoonup 0$ weakly in $W_0^{1,p}(\Omega)$, denoting χ_m the characteristic function of the set $\{x \in \Omega\colon |u_k| > m\}$, the second term on the right-hand side of (31) becomes

$$\int_{\Omega\cap\{|u_k|\leqslant m\}} \big(\big|\nabla T_m(u)\big|^{p-2}\nabla T_m(u), \nabla\big(T_m(u_k) - T_m(u)\big)^+\big)\,dx$$
$$\leqslant \int_\Omega \big(\big|\nabla T_m(u)\big|^{p-2}\nabla T_m(u), \nabla\big(T_m(u_k) - T_m(u)\big)^+\big)\,dx$$
$$+ \left|\int_{\Omega\cap\{|u_k|> m\}} \big(\big|\nabla T_m(u)\big|^{p-2}\nabla T_m(u), \nabla\big(T_m(u_k) - T_m(u)\big)^+\big)\,dx\right|$$
$$\leqslant o(1) + C\|u\|_{W_0^{1,p}(\Omega)}^{p-1}\big\|\chi_m\nabla T_m(u)\big\|_{L^p(\Omega)} \to 0 \quad \text{as } k \to +\infty,$$

since, by dominate convergence again, $\chi_m \nabla T_m(u) \to 0$ strongly in $(L^p(\Omega))^N$. As above, the last term in (31) can be estimated as

$$\left| \int_\Omega \left(|\nabla u_k|^{p-2}\nabla u_k, \chi_m \nabla T_m(u)\right) dx \right| \leqslant C\|u_k\|^{p-1}_{W_0^{1,p}(\Omega)} \left\|\chi_m \nabla T_m(u)\right\|_{L^p(\Omega)} \to 0, \tag{32}$$

as $k \to +\infty$.

Considering that, by the dominated convergence theorem, we have

$$\left| \int_{\Omega\cap\{|u_k|>m\}} \left(\left|\nabla T_m(u_k)\right|^{p-2}\nabla T_m(u_k) - \left|\nabla T_m(u)\right|^{p-2}\nabla T_m(u), \nabla\left(T_m(u_k) - T_m(u)\right)^+\right) dx \right|$$
$$\leqslant \int_\Omega \chi_m \left|\nabla T_m(u)\right|^p \to 0 \quad \text{as } k \to +\infty,$$

Eq. (30) becomes

$$\int_\Omega \left(|\nabla u_k|^{p-2}\nabla u_k, \nabla\left(T_m(u_k) - T_m(u)\right)^+\right) dx$$
$$= \int_\Omega \left(\left|\nabla T_m(u_k)\right|^{p-2}\nabla T_m(u_k) - \left|\nabla T_m(u)\right|^{p-2}\nabla T_m(u), \nabla\left(T_m(u_k) - T_m(u)\right)^+\right) dx + o(1).$$

Finally we obtain

$$\int_\Omega \left(|\nabla u_k|^{p-2}\nabla u_k, \nabla\left(T_m(u_k) - T_m(u)\right)^+\right) dx$$
$$\geqslant \begin{cases} C_1(p)\frac{|\nabla(T_m(u_k)-T_m(u))^+|^2}{(|\nabla T_m(u_k)|+|\nabla T_m(u)|)^{2-p}} + o(1) & \text{if } 1 < p < 2, \\ C_1(p)|\nabla(T_m(u_k) - T_m(u))^+|^p + o(1) & \text{if } p \geqslant 2, \end{cases} \tag{33}$$

with $C_1(p)$ a positive constant depending on p, which implies (together with (11))

$$\left\|\left(T_m(u_k) - T_m(u)\right)^+\right\|_{W_0^{1,p}(\Omega)} \to 0 \quad \text{as } k \to +\infty. \tag{34}$$

Asymptotic behaviour of $\|(T_m(u_k) - T_m(u))^-\|_{W_0^{1,p}(\Omega)}$. We use $e^{-T_m(u_k)}(T_m(u_k) - T_m(u))^-$ as a test function in (1) (see Section 2.1) obtaining

$$\int_\Omega e^{-T_m(u_k)}\left(|\nabla u_k|^{p-2}\nabla u_k, \nabla\left(T_m(u_k) - T_m(u)\right)^-\right) dx$$
$$- \int_\Omega e^{-T_m(u_k)}\left(T_m(u_k) - T_m(u)\right)^-\left(|\nabla u_k|^{p-2}\nabla u_k, \nabla T_m(u_k)\right) dx$$
$$+ \int_\Omega |\nabla u_k|^p e^{-T_m(u_k)}\left(T_m(u_k) - T_m(u)\right)^- dx$$
$$= \int_\Omega \left(\vartheta T_k\left(\frac{u_k^q}{|x|^p}\right) + T_k(f)\right)e^{-T_m(u_k)}\left(T_m(u_k) - T_m(u)\right)^- dx. \tag{35}$$

In this case as well, since $(T_m(u_k) - T_m(u))^- \rightharpoonup 0$ weakly in $W_0^{1,p}(\Omega)$ and $(T_m(u_k) - T_m(u))^- \to 0$ a.e. in Ω, the right-hand side of (35) tends to zero as k goes to infinity.

The first term on the left-hand side of (35), being $(\nabla T_m(u_k))\chi_m = 0$, states as

$$\int_\Omega e^{-T_m(u_k)}\big(|\nabla u_k|^{p-2}\nabla u_k, \nabla\big(T_m(u_k)-T_m(u)\big)^-\big)\,dx + \int_{\Omega\cap\{|u_k|>m\}} |\nabla u_k|^p e^{-T_m(u_k)}\big(T_m(u_k)-T_m(u)\big)^-\,dx = o(1). \tag{36}$$

We point out that

$$\big(T_m(u_k)-T_m(u)\big)^-\chi_m = 0,$$

hence (36) becomes

$$\int_\Omega \big(|\nabla u_k|^{p-2}\nabla u_k, \nabla\big(T_m(u_k)-T_m(u)\big)^-\big) = C_m o(1) \quad \text{as } k\to+\infty, \tag{37}$$

with C_m a positive constant depending on m.

The choice and use of $e^{-T_m(u_k)}(T_m(u_k)-T_m(u))^-$ as a test function allows to simplify conveniently Eq. (35) in order to obtain the desired result. In fact, we proceed writing the left-hand side of (37) as

$$\begin{aligned}
&\int_\Omega \big(|\nabla u_k|^{p-2}\nabla u_k, \nabla\big(T_m(u_k)-T_m(u)\big)^-\big)\,dx \\
&\quad= \int_{\Omega\cap\{|u_k|\leqslant m\}} \big(\big|\nabla T_m(u_k)\big|^{p-2}\nabla T_m(u_k) - \big|\nabla T_m(u)\big|^{p-2}\nabla T_m(u), \nabla\big(T_m(u_k)-T_m(u)\big)^-\big)\,dx \\
&\qquad+ \int_{\Omega\cap\{|u_k|\leqslant m\}} \big|\nabla T_m(u)\big|^{p-2}\big(\nabla T_m(u), \nabla\big(T_m(u_k)-T_m(u)\big)^-\big)\,dx \\
&\qquad+ \int_{\Omega\cap\{|u_k|>m\}} \big(|\nabla u_k|^{p-2}\nabla u_k, \nabla\big(T_m(u_k)-T_m(u)\big)^-\big)\,dx.
\end{aligned} \tag{38}$$

The second term on the right-hand side of (38) can be estimated as follows

$$\begin{aligned}
&\int_{\Omega\cap\{|u_k|\leqslant m\}} \big|\nabla T_m(u)\big|^{p-2}\big(\nabla T_m(u), \nabla\big(T_m(u_k)-T_m(u)\big)^-\big)\,dx \\
&\quad= \int_\Omega \big|\nabla T_m(u)\big|^{p-2}\big(\nabla T_m(u), \nabla\big(T_m(u_k)-T_m(u)\big)^-\big)\,dx \\
&\qquad- \int_{\Omega\cap\{|u_k|>m\}} \big|\nabla T_m(u)\big|^{p-2}\big(\nabla T_m(u), \nabla\big(T_m(u_k)-T_m(u)\big)^-\big) \\
&\quad\leqslant o(1) + C\|u\|^{p-1}_{W_0^{1,p}(\Omega)}\big\|\chi_m\nabla T_m(u)\big\|_{L^p(\Omega)} \to 0 \quad \text{as } k\to+\infty,
\end{aligned} \tag{39}$$

since by weak convergence the first term on the right-hand side of (39) goes to zero, while the second one goes to zero using (22) and the fact that, for dominated convergence, $\chi_m\nabla T_m(u)\to 0$ strongly in $L^p(\Omega)$. Moreover, we observe that the last term in (38) is zero since $(T_m(u_k)-T_m(u))^-\chi_m = 0$. Finally as above, Eq. (38) becomes

$$\begin{aligned}
o(1) &= \int_\Omega \big(|\nabla u_k|^{p-2}\nabla u_k, \nabla\big(T_m(u_k)-T_m(u)\big)^-\big)\,dx \\
&\geqslant \begin{cases} C_1(p)\frac{|\nabla(T_m(u_k)-T_m(u))^-|^2}{(|\nabla T_m(u_k)|+|\nabla T_m(u)|)^{2-p}} + o(1) & \text{if } 1<p<2, \\ C_1(p)|\nabla(T_m(u_k)-T_m(u))^-|^p + o(1) & \text{if } p\geqslant 2, \end{cases}
\end{aligned} \tag{40}$$

with $C_1(p)$ a positive constant depending on p. By (37) and (40) (using (11) again) we get

$$\big\|\big(T_m(u_k)-T_m(u)\big)^-\big\|_{W_0^{1,p}(\Omega)} \to 0 \quad \text{as } k\to+\infty. \tag{41}$$

From (28), (34) and (41) we have the desired result, i.e.

$$\big\|\big(T_m(u_k) - T_m(u)\big)\big\|_{W_0^{1,p}(\Omega)} \to 0 \quad \text{as } k \to +\infty. \qquad \square$$

Now we prove that $|\nabla u_k|^p \to |\nabla u|^p$ strongly in $L^1(\Omega)$. By Lemma 3 the sequence of the gradients converges a.e. In order to use again Vitali's Theorem we need to prove the equi-integrability of $|\nabla u_k|^p$. Let $E \subset \Omega$ be a measurable set, then

$$\int_E |\nabla u_k|^p\,dx \leqslant \int_E \big|\nabla T_m(u_k)\big|^p\,dx + \int_{\{u_k \geqslant m\}\cap E} |\nabla u_k|^p\,dx.$$

By Lemma 3, $T_m(u_k) \to T_m(u)$ in $W_0^{1,p}(\Omega)$ $\forall m > 0$ and therefore $\int_E |\nabla T_m(u_k)|^p\,dx$ is uniformly small for $|E|$ small enough. Moreover, by Lemma 2 we obtain

$$\int_{\{u_k \geqslant m\}\cap E} |\nabla u_k|^p\,dx \leqslant \int_{\{u_k \geqslant m\}} |\nabla u_k|^p\,dx \to 0 \quad \text{as } m \to \infty,$$

uniformly in k. Then Vitali's Theorem implies that

$$|\nabla u_k|^p \to |\nabla u|^p \quad \text{in } L^1(\Omega). \tag{42}$$

Step 4: Passing to the limit in (1). Finally, since $\|u_k - u\|_{W_0^{1,p}(\Omega)} \to 0$ as $k \to +\infty$, we conclude that u is a distributional solution to the problem

$$\begin{cases} -\Delta_p u + |\nabla u|^p = \vartheta \dfrac{u^q}{|x|^p} + f & \text{in } \Omega, \\ u \geqslant 0 \quad \text{in } \partial\Omega, & u = 0 \quad \text{on } \Omega. \end{cases}$$

In particular, we point out that the equation is verified even in a stronger way, that is

$$\int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla\phi) + \int_\Omega |\nabla u|^p\phi = \vartheta \int_\Omega \frac{u^q}{|x|^p}\phi + \int_\Omega f\phi \quad \forall\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),$$

concluding the proof. $\square$

3. Symmetry of solutions

To study the qualitative properties of the solutions u to the problem ($\mathcal{P}$) we need some preliminary results about their regularity.

3.1. *Local regularity of solutions of* ($\mathcal{P}$)

Given any solution $u \in W^{1,p}(\Omega)$, the $C^{1,\alpha}_{\mathrm{loc}}(\Omega \setminus \{0\})$ regularity of u follows by the results in [11,18].

We will use the notation $C^{1,\alpha}(\Omega \setminus \{0\})$ to refer the continuity in the local sense and outside the origin. The reader will guess that the arguments in [11,18] generally do not work up to the origin, because of the lack of regularity of the nonlinearity.

Moreover, if one assumes (as in our case) that the domain is smooth, the $C^{1,\alpha}(\overline{\Omega} \setminus \{0\})$ regularity up to the boundary follows by [14].

The fact that solutions to p-Laplace equations are not in general $C^2(\Omega)$, leads to the study of the summability properties of the second derivatives of the solutions. This is important in some issues such as the study of the qualitative properties of these solutions. The results in [9] (and in [17] where a more general equation with a gradient term as in ($\mathcal{P}$) appears) hold outside the singularity and therefore we have the following theorem:

Theorem 4. *Assume $1 < p < N$ and consider $u \in C^{1,\alpha}(\overline{\Omega} \setminus \{0\})$ a solution of ($\mathcal{P}$), with $f \in C^1(\overline{\Omega} \setminus \{0\})$. Denoting $u_i = \frac{\partial u}{\partial x_i}$, we have*

$$\int_{\tilde{\Omega}} \frac{|\nabla u|^{p-2-\beta}|\nabla u_i|^2}{|x-y|^\gamma}\,dx \leqslant \mathcal{C} \quad \forall i = 1, \ldots, N, \tag{43}$$

for any $\tilde{\Omega} \Subset \Omega \setminus \{0\}$ and uniformly for any $y \in \tilde{\Omega}$, with

$$\mathcal{C} := \mathcal{C}\big(p, \gamma, \beta, f, q, \vartheta, \|u\|_{L^\infty(\tilde{\Omega})}, \|\nabla u\|_{L^\infty(\tilde{\Omega})}, \operatorname{dist}\big(\tilde{\Omega}, \{0\}\big)\big),$$

for $0 \leqslant \beta < 1$ and $\gamma < (N-2)$ if $N \geqslant 3$ ($\gamma = 0$ if $N = 2$).

If we also assume that f is nonnegative in Ω then it follows that actually $\vartheta \frac{u^q}{|x|^p} + f$ is strictly positive in the interior of Ω and for any $\tilde{\Omega} \Subset \Omega \setminus \{0\}$, uniformly for any $y \in \tilde{\Omega}$, we have that

$$\int_{\tilde{\Omega}} \frac{1}{|\nabla u|^t} \frac{1}{|x-y|^\gamma}\,dx \leqslant \mathcal{C}^*, \tag{44}$$

with $\max\{(p-2), 0\} \leqslant t < p-1$ and $\gamma < (N-2)$ if $N \geqslant 3$ ($\gamma = 0$ if $N = 2$). Moreover $\mathcal{C}^*$ depends on $\mathcal{C}$.

See [9,17] for a detailed proof.

Remark 2. Let $Z_u = \{x \in \Omega\colon\ \nabla u(x) = 0\}$. It is clear that Z_u is a closed set in Ω and moreover, by (44) it follows implicitly that the Lebesgue measure

$$|Z_u| = 0,$$

provided that f is nonnegative.

Assume $\tilde{\Omega} \Subset \Omega \setminus \{0\}$ and recall the following:

Definition 2. Let $\rho \in L^1(\tilde{\Omega})$ and $1 \leqslant q < \infty$. The space $H^{1,q}_\rho(\tilde{\Omega})$ is defined as the completion of $C^1(\tilde{\Omega})$ (or $C^\infty(\tilde{\Omega})$) with the norm

$$\|v\|_{H^{1,q}_\rho} = \|v\|_{L^q(\tilde{\Omega})} + \|\nabla v\|_{L^q(\tilde{\Omega},\rho)}, \tag{45}$$

where

$$\|\nabla v\|^q_{L^p(\tilde{\Omega},\rho)} := \int_{\tilde{\Omega}} \rho(x)\big|\nabla v(x)\big|^q\,dx.$$

We also recall that $H^{1,q}_\rho(\tilde{\Omega})$ may be equivalently defined as the space of functions with distributional derivatives represented by a function for which the norm defined in (45) is bounded. These two definitions are equivalent if the domain has piecewise regular boundary.

The space $H^{1,q}_{0,\rho}(\tilde{\Omega})$ is consequently defined as the completion of $C^1_c(\tilde{\Omega})$ (or $C^\infty_c(\tilde{\Omega})$), w.r.t. the norm (45).

A short, but quite complete, reference for weighted Sobolev spaces in [13, Chapter 1], and the references therein.

We have the following result (see [9]).

Theorem 5 *(Weighted Poincaré inequality). Let $p \geqslant 2$ and $u \in C^{1,\alpha}(\overline{\Omega} \setminus \{0\})$ be a solution of ($\mathcal{P}$). Setting $\rho = |\nabla u|^{p-2}$ and $\tilde{\Omega} \Subset \Omega \setminus \{0\}$ as above, we have that $H^{1,2}_0(\tilde{\Omega}, \rho)$ is continuously embedded in $L^q(\tilde{\Omega})$ for $1 \leqslant q < \hat{2}^*$ where*

$$\frac{1}{\hat{2}^*} = \frac{1}{2} - \frac{1}{N} + \frac{p-2}{p-1}\frac{1}{N}.$$

Consequently, since $\hat{2}^* > 2$*, for* $w \in H_0^{1,2}(\tilde{\Omega}, \rho)$ *we have*

$$\|w\|_{L^2(\tilde{\Omega})} \leqslant \mathcal{C}_S \|\nabla w\|_{L^2(\tilde{\Omega},\rho)} = \mathcal{C}_S \left(\int_{\tilde{\Omega}} \rho |\nabla w|^2 \right)^{\frac{1}{2}}, \tag{46}$$

with $\mathcal{C}_S = \mathcal{C}_S(\tilde{\Omega}) \to 0$ *if* $|\tilde{\Omega}| \to 0$.

Notice that Theorem 5 holds for $p \geqslant 2$. If $1 < p < 2$ and $|\nabla u|$ is bounded, the weighted Poincaré inequality (46) follows at once by the classic Poincaré inequality.

3.2. Some preliminaries and useful tools

To state the next results we need some notations. Let ν be a direction in $\mathbb{R}^N$ with $|\nu| = 1$. As customary, for a real number λ we set

$$T_\lambda^\nu = \{x \in \mathbb{R}^N\colon\ x \cdot \nu = \lambda\} \tag{47}$$

and observe that $0 \in T_0^\nu$. Moreover, let us denote

$$\Omega_\lambda^\nu = \{x \in \Omega\colon\ x \cdot \nu < \lambda\}, \tag{48}$$

$$x_\lambda^\nu = R_\lambda^\nu(x) = x + 2(\lambda - x \cdot \nu)\nu \tag{49}$$

(which is the reflection trough the hyperplane T_λ^ν),

$$u_\lambda^\nu(x) = u(x_\lambda^\nu), \tag{50}$$

$$a(\nu) = \inf_{x \in \Omega} x \cdot \nu. \tag{51}$$

When $\lambda > a(\nu)$, since Ω_λ^ν is nonempty, we set

$$(\Omega_\lambda^\nu)' := R_\lambda^\nu(\Omega_\lambda^\nu) \tag{52}$$

and finally for $\lambda > a(\nu)$ we denote

$$\lambda_1(\nu) = \sup\{\lambda\colon\ (\Omega_\lambda^\nu)' \subset \Omega\}. \tag{53}$$

Here below we are going to prove a couple of useful results. We have

Lemma 4. *Assume* $\vartheta > 0$ *and* $f \geqslant 0$*. Consider* $u \in W_0^{1,p}(\Omega)$ *a nonnegative weak solution to problem* ($\mathcal{P}$)*. Then*

$$\lim_{|x| \to 0} u(x) = +\infty.$$

Proof. We consider the test function $\varphi = e^{-u}\psi$, with $\psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ so that $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Then putting φ as test function in ($\mathcal{P}$) one has

$$\int_\Omega \left|e^{-\frac{u}{p-1}} \nabla u\right|^{p-2} \left(e^{-\frac{u}{p-1}} \nabla u, \nabla \psi\right) dx \geqslant \int_\Omega \frac{u^q}{|x|^p} \left(e^{-\frac{u}{p-1}}\right)^{p-1} \psi\, dx, \tag{54}$$

being $f(\cdot)$ nonnegative. Defining $v = 1 - e^{-\frac{u}{p-1}}$, from (54) we get

$$C_p \int_\Omega |\nabla v|^{p-2} (\nabla v, \nabla \psi)\, dx \geqslant \int_\Omega \frac{u^q}{|x|^p} (1 - v)^{p-1} \psi\, dx. \tag{55}$$

Let us consider u_R the radial solution to the problem

$$\begin{cases} -\Delta_p u + |\nabla u|^p = \dfrac{C}{|x|^p} & \text{in } B_R, \\ u \geqslant 0 \quad \text{in } B_R, \qquad u = 0 & \text{on } \partial B_R, \end{cases} \tag{56}$$

constructed as limit of the solutions, say $u_{R,k}$, to the truncated problems, in the same way as we did in Section 2 but setting here $\vartheta = 0$, with C, R some positive constants to be chosen later. Moreover since, for k fixed the solution $u_{R,k}$ is unique, it follows that $u_{R,k}$ must be radial for all k. Finally the strong convergence in $W_0^{1,p}(\Omega)$ (and thus pointwise $u_R(x) = \lim_{k\to\infty} u_{R,k}(x)$) implies that $u_R(x) = u_R(|x|)$.

Then, by setting $\varphi = e^{-u_R}\psi$, $v_R = 1 - e^{-\frac{u_R}{p-1}}$ (as in Eqs. (54) and (55)), we have

$$C_p \int_{B_R} |\nabla v_R|^{p-2}(\nabla v_R, \nabla\psi)\,dx = \int_{B_R} \frac{C}{|x|^p}(1-v_R)^{p-1}\psi\,dx. \tag{57}$$

We note that v (resp. v_R) belongs to $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ (to $W_0^{1,p}(B_R)\cap L^\infty(B_R)$). Using (55) and (57) with $\psi = (v_R - v)^+$, R small such that $B_R \Subset \Omega$ and in particular, noting that since $v_R < v$ on ∂B_R one has that $\psi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, we have

$$\begin{aligned} C_p \int_{B_R} |\nabla v|^{p-2}\big(\nabla v, \nabla(v_R - v)^+\big)\,dx &\geqslant \int_{B_R} \frac{u^q}{|x|^p}(1-v)^{p-1}(v_R-v)^+\,dx \\ &\geqslant \int_{B_R} \frac{C_R}{|x|^p}(1-v)^{p-1}(v_R-v)^+\,dx, \end{aligned} \tag{58}$$

with $C_R = \inf_{B_R} u(x) > 0$ by the strong maximum principle and

$$C_p \int_{B_R} |\nabla v_R|^{p-2}\big(\nabla v_R, \nabla(v_R - v)^+\big)\,dx = \int_{B_R} \frac{C_R}{|x|^p}(1-v_R)^{p-1}(v_R-v)^+\,dx, \tag{59}$$

where in (56) we choose $C = C_R$. Thus subtracting (58) and (59) we obtain

$$\begin{aligned} &C_p \int_{B_R} \big(|\nabla v_R|^{p-2}\nabla v_R - |\nabla v|^{p-2}\nabla v, \nabla(v_R - v)^+\big)\,dx \\ &\quad = \int_{B_R} \frac{C_R}{|x|^p}\big((1-v_R)^{p-1} - (1-v)^{p-1}\big)(v_R - v)^+\,dx. \end{aligned} \tag{60}$$

On the set $B_R \cap \{v_R \geqslant v\}$ the right-hand side of (60) is nonpositive and therefore, by

$$\int_{B_R} \big(|\nabla v_R|^{p-2}\nabla v_R - |\nabla v|^{p-2}\nabla v, \nabla(v_R - v)^+\big)\,dx \leqslant 0,$$

we have that $v \geqslant v_R$ on B_R, that is (using the definition of v and v_R and the monotonicity of $s = 1 - e^{-\frac{s}{p-1}}$),

$$u \geqslant u_R. \tag{61}$$

Let us now study the qualitative behaviour of u_R and therefore consider the test function $\varphi = e^{-u_R}\psi$, with $\psi = \psi(|x|)$ belonging to $W_0^{1,p}(B_R)\cap L^\infty(B_R)$. Then by (56) we have

$$\int_0^R e^{-u_R}|u_R'|^{p-2}(u_R', \psi')\rho^{N-1}\,d\rho = \int_0^R C_R e^{-u_R}\psi\rho^{N-1-p}\,d\rho,$$

with $\rho = |x|$. By classical regularity results and Hopf's Lemma we have $u_R \in C^2(\overline{B}_R \setminus \{0\})$ and thus

$$\big(e^{-u_R}|u_R'|^{p-2}(-u_R')\rho^{N-1}\big)' = C_R e^{-u_R}\rho^{N-1-p} \quad \forall \rho \neq 0.$$

Since $u_R(\rho)$ is positive and monotone decreasing w.r.t. ρ, we have the two following cases:

(i) either $\lim_{\rho\to 0} u_R(\rho) = C > 0$;
(ii) or $\lim_{\rho\to 0} u_R(\rho) = +\infty$.

If we assume the case (i) we have $(e^{-u_R}|u_R'|^{p-2}(-u_R')\rho^{N-1})'/(\rho^{N-p})' \to C$ as $\rho \to 0$, for some positive constant C. It is standard to see that $-u_R' \geqslant C/\rho + o(1)$ for $\rho \to 0$, getting a contradiction with the case (i). Then the case (ii) holds and together with (61) it concludes the proof. □ □

From now on we shall assume the following hypotheses:

(hp.1) $f(x) \in C^1(\overline{\Omega} \setminus \{0\})$ and $f(x) \geqslant 0$;
(hp.2) Monotonicity of $f(\cdot)$ in the ν-direction: $f(x) \leqslant f(x_\lambda^\nu)$, $\forall \lambda \in (a(\nu), \lambda_1(\nu))$.

Define $\phi_\rho(x) \in C_c^\infty(\Omega)$, $\phi \geqslant 0$ such that

$$\begin{cases} \phi \equiv 1 & \text{in } \Omega \setminus B_{2\rho}, \\ \phi \equiv 0 & \text{in } B_\rho, \\ |\nabla \phi| \leqslant \dfrac{C}{\rho} & \text{in } B_{2\rho} \setminus B_\rho, \end{cases} \tag{62}$$

where B_ρ denotes the open ball with center 0 and radius $\rho > 0$.

Lemma 5. *Let $u \in C^1(\overline{\Omega} \setminus \{0\})$ a solution to ($\mathcal{P}$) and let us define the critical set*

$$Z_u = \{x \in \Omega\colon\ \nabla u(x) = 0\}.$$

Then, the set $\Omega \setminus Z_u$ does not contain any connected component $\mathcal{C}$ such that $\overline{\mathcal{C}} \subset \Omega$. Moreover, if we assume that Ω is a smooth bounded domain with connected boundary, it follows that $\Omega \setminus Z_u$ is connected.

Proof. We proceed by contradiction. Let us assume that such component exists, namely

$$\mathcal{C} \subset \Omega \quad \text{such that} \quad \partial\mathcal{C} \subset Z_u.$$

By Remark 2, we have

$$|Z_u| = 0.$$

Thus

$$-\Delta_p u + |\nabla u|^p = \vartheta \frac{u^q}{|x|^p} + f(x) \quad \text{a.e. in } \Omega. \tag{63}$$

For all $\varepsilon > 0$, let us define $J_\varepsilon : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ by setting

$$J_\varepsilon(t) = \begin{cases} t & \text{if } t \geqslant 2\varepsilon, \\ 2t - 2\varepsilon & \text{if } \varepsilon \leqslant t \leqslant 2\varepsilon, \\ 0 & \text{if } 0 \leqslant t \leqslant \varepsilon. \end{cases} \tag{64}$$

We shall use

$$\Psi = e^{-u}\phi_\rho(x)\frac{J_\varepsilon(|\nabla u|)}{|\nabla u|}\chi_{\mathcal{C}} \tag{65}$$

as a test function in (63), where $\phi_\rho(x)$ as in (62). We point out that the supp $\Psi \subset \mathcal{C}$, which implies $\Psi \in W_0^{1,p}(\mathcal{C})$. Integrating by parts we get

$$\begin{aligned} &\int_{\mathcal{C}} e^{-u}\left(|\nabla u|^{p-2}\nabla u, \nabla\left(\frac{J_\varepsilon(|\nabla u|)}{|\nabla u|}\right)\right)\phi_\rho\, dx + \int_{\mathcal{C}} e^{-u}\left(|\nabla u|^{p-2}\nabla u, \nabla\phi_\rho\right)\frac{J_\varepsilon(|\nabla u|)}{|\nabla u|}\, dx \\ &\quad - \int_{\mathcal{C}} e^{-u}|\nabla u|^p\phi_\rho\frac{J_\varepsilon(|\nabla u|)}{|\nabla u|}\, dx + \int_{\mathcal{C}} e^{-u}|\nabla u|^p\phi_\rho\frac{J_\varepsilon(|\nabla u|)}{|\nabla u|}\, dx \\ &= \vartheta\int_{\mathcal{C}} \frac{u^q}{|x|^p}e^{-u}\phi_\rho\frac{J_\varepsilon(|\nabla u|)}{|\nabla u|}\, dx + \int_{\mathcal{C}} f e^{-u}\phi_\rho\frac{J_\varepsilon(|\nabla u|)}{|\nabla u|}\, dx, \end{aligned} \tag{66}$$

notice that we have used the fact that the boundary term in the integration is zero since $\partial\mathcal{C} \subset Z_u$. Remarkably, using the test function Ψ defined in (65), we are able to integrate on the boundary $\partial\mathcal{C}$ which could be not regular. We estimate the first term on the left-hand side of (66), denoting $h_\varepsilon(t) = \frac{J_\varepsilon(t)}{t}$. So we have

$$\left|\int_{\mathcal{C}} e^{-u}\left(|\nabla u|^{p-2}\nabla u, \nabla\left(\frac{J_\varepsilon(|\nabla u|)}{|\nabla u|}\right)\right)\phi_\rho\, dx\right| \leqslant C\int_{\mathcal{C}} |\nabla u|^{p-1}\left|h'_\varepsilon\big(|\nabla u|\big)\right|\left|\nabla\big(|\nabla u|\big)\right|\phi_\rho\, dx$$
$$\leqslant C\int_{\mathcal{C}} |\nabla u|^{p-2}\big(|\nabla u| h'_\varepsilon\big(|\nabla u|\big)\big)\left\|D^2u\right\|\phi_\rho\, dx. \tag{67}$$

We show now the following

Claim. *One has*

(i) $|\nabla u|^{p-2}\|D^2u\|\phi_\rho \in L^1(\mathcal{C})\ \forall\rho > 0$;
(ii) $|\nabla u| h'_\varepsilon(|\nabla u|) \to 0$ *a.e. in* $\mathcal{C}$ *as* $\varepsilon \to 0$ *and* $|\nabla u| h'_\varepsilon(|\nabla u|) \leqslant C$ *with* C *not depending on* ε.

Let us prove (i). By Hölder's inequality it follows

$$\int_{\mathcal{C}} |\nabla u|^{p-2}\left\|D^2u\right\|\phi_\rho\, dx \leqslant C(\mathcal{C})\left(\int_{\mathcal{C}} |\nabla u|^{2(p-2)}\left\|D^2u\right\|^2\phi_\rho^2\, dx\right)^{\frac{1}{2}}$$
$$\leqslant C\left(\int_{\mathcal{C}} |\nabla u|^{p-2-\beta}\left\|D^2u\right\|^2\phi_\rho^2|\nabla u|^{p-2+\beta}\, dx\right)^{\frac{1}{2}}$$
$$\leqslant C\|\nabla u\|_{L^\infty}^{(p-2+\beta)/2}\left(\int_{\mathcal{C}\setminus B_\rho} |\nabla u|^{p-2-\beta}\left\|D^2u\right\|^2 dx\right)^{\frac{1}{2}} \leqslant C, \tag{68}$$

where we have used Theorem 4 and the fact that $\phi_\rho^2|\nabla u|^{p-2+\beta}$ is bounded since β can be any value with $0 \leqslant \beta < 1$.

Let us prove (ii). Exploiting the definition (64), by straightforward calculation we obtain

$$h'_\varepsilon(t) = \begin{cases} 0 & \text{if } t \geqslant 2\varepsilon, \\ \frac{2\varepsilon}{t^2} & \text{if } \varepsilon \leqslant t \leqslant 2\varepsilon, \\ 0 & \text{if } 0 \leqslant t \leqslant \varepsilon, \end{cases}$$

and then we have $|\nabla u| h'_\varepsilon(|\nabla u|) \to 0$ a.e. for $\varepsilon \to 0$ in $\mathcal{C}$ and $|\nabla u| h'_\varepsilon(|\nabla u|) \leqslant 2$.

Then, by Claim (using dominated convergence) and Eq. (67) we have

$$\int_{\mathcal{C}} e^{-u}\left(|\nabla u|^{p-2}\nabla u, \nabla\left(\frac{J_\varepsilon(|\nabla u|)}{|\nabla u|}\right)\right)\phi_\rho\, dx \to 0 \quad \text{as } \varepsilon \to 0,\ \forall\rho > 0.$$

Exploiting (64) and passing to the limit in (66), by the dominated convergence theorem, it follows

$$\int_{\mathcal{C}} e^{-u}\big(|\nabla u|^{p-2}\nabla u, \nabla\phi_\rho\big)\, dx = \vartheta\int_{\mathcal{C}} \frac{u^q}{|x|^p}e^{-u}\phi_\rho\, dx + \int_{\mathcal{C}} f e^{-u}\phi_\rho\, dx \quad \forall\rho > 0.$$

Then

$$\int_{B_{2\rho}\setminus B_\rho} e^{-u}\big(|\nabla u|^{p-2}\nabla u, \nabla\phi_\rho\big)\, dx = \vartheta\int_{\mathcal{C}} \frac{u^q}{|x|^p}e^{-u}\phi_\rho\, dx + \int_{\mathcal{C}} f e^{-u}\phi_\rho\, dx. \tag{69}$$

Letting $\rho \to 0$ in (69), by Hölder's inequality we estimate the left-hand side as

$$\left|\int_{B_{2\rho}\setminus B_\rho} e^{-u}\big(|\nabla u|^{p-2}\nabla u, \nabla\phi_\rho\big)\, dx\right| \leqslant C\left(\int_{B_{2\rho}\setminus B_\rho} |\nabla u|^p\right)^{\frac{p-1}{p}}\left(\int_{B_{2\rho}\setminus B_\rho} |\nabla\phi_\rho|^p\right)^{\frac{1}{p}} \leqslant C\left(\frac{\rho^N}{\rho^p}\right)^{\frac{1}{p}} \to 0,$$

where we have used that $|\nabla \phi_\rho| \leqslant \frac{C}{\rho}$ and $p < N$. On the other hand, for $\rho \to 0$, the right-hand side of (69), by dominated convergence theorem, becomes

$$\vartheta \int_{\mathcal{C}} \frac{u^q}{|x|^p} e^{-u}\,dx + \int_{\mathcal{C}} f e^{-u} > 0,$$

which is a contradiction.

If Ω is smooth, since the right-hand side of (63) is positive, by Hopf's Lemma (see [19]), a neighborhood of the boundary belongs to a component $\mathcal{C}$ of $\Omega \setminus Z_u$. By what we have just proved above, a second component $\mathcal{C}'$ cannot be contained compactly in Ω. Thus $\Omega \setminus Z_u$ is connected. □

3.3. Comparison principles to problem ($\mathcal{P}$)

We shall prove the following

Proposition 2 *(Weak comparison principle). Let $\lambda < 0$ and $\tilde{\Omega}$ be a bounded domain such that $\tilde{\Omega} \Subset \Omega^\nu_\lambda$. Assume that $u \in C^1(\overline{\Omega} \setminus \{0\})$ is a solution to ($\mathcal{P}$) such that $u \leqslant u^\nu_\lambda$ on $\partial\tilde{\Omega}$. Then there exists a positive constant $\delta = \delta(\lambda, \operatorname{dist}(\tilde{\Omega}, \partial\Omega))$ such that if we assume $|\tilde{\Omega}| \leqslant \delta$, then it holds*

$$u \leqslant u^\nu_\lambda \quad \text{in } \tilde{\Omega}.$$

Proof. We have (in the weak sense)

$$-\Delta_p u + |\nabla u|^p = \vartheta \frac{u^q}{|x|^p} + f \quad \text{in } \Omega, \tag{70}$$

$$-\Delta_p u^\nu_\lambda + \left|\nabla u^\nu_\lambda\right|^p = \vartheta \frac{(u^\nu_\lambda)^q}{|x^\nu_\lambda|^p} + f^\nu_\lambda \quad \text{in } \Omega, \tag{71}$$

where $f^\nu_\lambda(x) = f(x^\nu_\lambda)$.

Let us set $\phi^\nu_{\rho,\lambda}(x) = \phi_\rho(x^\nu_\lambda)$, with $\phi_\rho(\cdot)$ as in (62). By contradiction, we assume the statement false and we consider

(i) $e^{-u}(u - u^\nu_\lambda)^+(\phi^\nu_{\rho,\lambda})^2 \chi_{\tilde{\Omega}} \in W^{1,p}_0(\tilde{\Omega})$, as a test function in (70);
(ii) $e^{-u^\nu_\lambda}(u - u^\nu_\lambda)^+(\phi^\nu_{\rho,\lambda})^2 \chi_{\tilde{\Omega}} \in W^{1,p}_0(\tilde{\Omega})$, as a test function in (71).

Notice that, by Lemma 4 we have that $\lim_{|x|\to 0} u(x) = +\infty$. This, together with the fact that $u \in L^\infty(\tilde{\Omega})$, implies that (see Eq. (49))

$$0^\nu_\lambda = R^\nu_\lambda(0) \notin \operatorname{supp}\big(u - u^\nu_\lambda\big)^+. \tag{72}$$

Then, if we subtract (in the weak formulation) (70) and (71), we get

$$\begin{aligned}
&\int_{\tilde{\Omega}} e^{-u^\nu_\lambda}\big(|\nabla u|^{p-2}\nabla u - \left|\nabla u^\nu_\lambda\right|^{p-2}\nabla u^\nu_\lambda, \nabla\big(u - u^\nu_\lambda\big)^+\big)\big(\phi^\nu_{\rho,\lambda}\big)^2\,dx \\
&\quad \leqslant \int_{\tilde{\Omega}} \big|\big(e^{-u} - e^{-u^\nu_\lambda}\big)\big(|\nabla u|^{p-2}\nabla u, \nabla\big(u - u^\nu_\lambda\big)^+\big)\big|\big(\phi^\nu_{\rho,\lambda}\big)^2 \\
&\qquad + C\int_{\tilde{\Omega}} \big|\big(e^{-u}|\nabla u|^{p-2}\nabla u - e^{-u^\nu_\lambda}\left|\nabla u^\nu_\lambda\right|^{p-2}\nabla u^\nu_\lambda, \nabla\phi^\nu_{\rho,\lambda}\big)\big|\big(u - u^\nu_\lambda\big)^+\phi^\nu_{\rho,\lambda}\,dx \\
&\qquad + \vartheta\int_{\tilde{\Omega}} e^{-u}\frac{u^q}{|x|^p}\big(u - u^\nu_\lambda\big)^+\big(\phi^\nu_{\rho,\lambda}\big)^2\,dx - \vartheta\int_{\tilde{\Omega}} e^{-u^\nu_\lambda}\frac{(u^\nu_\lambda)^q}{|x^\nu_\lambda|^p}\big(u - u^\nu_\lambda\big)^+\big(\phi^\nu_{\rho,\lambda}\big)^2\,dx \\
&\qquad + \int_{\tilde{\Omega}} e^{-u}f(x)\big(u - u^\nu_\lambda\big)^+\big(\phi^\nu_{\rho,\lambda}\big)^2\,dx - \int_{\tilde{\Omega}} e^{-u^\nu_\lambda}f\big(x^\nu_\lambda\big)\big(u - u^\nu_\lambda\big)^+\big(\phi^\nu_{\rho,\lambda}\big)^2\,dx.
\end{aligned} \tag{73}$$

The term in the third line of (73) can be estimated by Hölder's inequality and since $p < N$,

$$
\begin{aligned}
&C \int_{\tilde{\Omega}} \left| \left(e^{-u} |\nabla u|^{p-2} \nabla u - e^{-u_\lambda^\nu} \left|\nabla u_\lambda^\nu\right|^{p-2} \nabla u_\lambda^\nu, \nabla \phi_{\rho,\lambda}^\nu \right) \right| \left(u - u_\lambda^\nu\right)^+ \phi_{\rho,\lambda}^\nu \, dx \\
&\quad \leqslant C\left(\|u\|_{L^\infty(\Omega_\lambda^\nu)}\right) \int_{\tilde{\Omega}} \left| |\nabla u|^{p-1} + \left|\nabla u_\lambda^\nu\right|^{p-1} \right| \left|\nabla \phi_{\rho,\lambda}^\nu\right| \phi_{\rho,\lambda}^\nu \, dx \\
&\quad \leqslant C\left(\|u\|_{L^\infty(\Omega_\lambda^\nu)}\right) \left(\int_{\tilde{\Omega}} \left(|\nabla u|^p + \left|\nabla u_\lambda^\nu\right|^p \right) dx \right)^{\frac{p-1}{p}} \left(\int_{B_{2\rho} \setminus B_\rho} \left|\nabla \phi_{\rho,\lambda}^\nu\right|^p dx \right)^{\frac{1}{p}} \to 0 \quad \text{as } \rho \to 0. \qquad (74)
\end{aligned}
$$

Notice that we are considering the set $\tilde{\Omega} \cap \{u \geqslant u_\lambda\}$ and therefore $|x| \geqslant |x_\lambda^\nu|$. Using (74), Eq. (73) becomes

$$
\begin{aligned}
&\int_{\tilde{\Omega}} e^{-u_\lambda^\nu} \left(|\nabla u|^{p-2} \nabla u - \left|\nabla u_\lambda^\nu\right|^{p-2} \nabla u_\lambda^\nu, \nabla \left(u - u_\lambda^\nu\right)^+ \right) \left(\phi_{\rho,\lambda}^\nu\right)^2 dx \\
&\quad \leqslant \int_{\tilde{\Omega}} \left| \left(e^{-u} - e^{-u_\lambda^\nu}\right) \left(|\nabla u|^{p-2} \nabla u, \nabla \left(u - u_\lambda^\nu\right)^+ \right) \right| \left(\phi_{\rho,\lambda}^\nu\right)^2 dx \\
&\qquad + \vartheta \int_{\tilde{\Omega}} e^{-u} \left(\frac{u^q - (u_\lambda^\nu)^q}{|x|^p} \right) \left(u - u_\lambda^\nu\right)^+ \left(\phi_{\rho,\lambda}^\nu\right)^2 dx \\
&\qquad + \int_{\tilde{\Omega}} e^{-u} \left(f(x) - f\left(x_\lambda^\nu\right) \right) \left(u - u_\lambda^\nu\right)^+ \left(\phi_{\rho,\lambda}^\nu\right)^2 dx + o(1).
\end{aligned}
$$

By (hp.2) of Section 3.2 and taking into account that for $\lambda < 0$ one has $|x| \geqslant C$ in Ω_λ^ν for some positive constant C, one has

$$
\begin{aligned}
&\int_{\tilde{\Omega}} e^{-u_\lambda^\nu} \left(|\nabla u|^{p-2} \nabla u - \left|\nabla u_\lambda^\nu\right|^{p-2} \nabla u_\lambda^\nu, \nabla \left(u - u_\lambda^\nu\right)^+ \right) \left(\phi_{\rho,\lambda}^\nu\right)^2 dx \\
&\quad \leqslant \int_{\tilde{\Omega}} \left| \left(e^{-u} - e^{-u_\lambda^\nu}\right) \left(|\nabla u|^{p-2} \nabla u, \nabla \left(u - u_\lambda^\nu\right)^+ \right) \right| \left(\phi_{\rho,\lambda}^\nu\right)^2 dx \\
&\qquad + C_3 \int_{\tilde{\Omega}} \left[\left(u - u_\lambda^\nu\right)^+ \right]^2 \left(\phi_{\rho,\lambda}^\nu\right)^2 dx + o(1), \qquad (75)
\end{aligned}
$$

with $C_3 = C_3(\lambda, \vartheta, \|u\|_{L^\infty(\Omega_\lambda^\nu)}, \operatorname{dist}(\tilde{\Omega}, \partial\Omega))$. We note that, in the last inequality, we have used the fact that the term $u^q - (u_\lambda^\nu)^q$ is locally Lipschitz continuous in $(0, +\infty)$ and that, by strong maximum principle (see [19]), the solution u is strictly positive in $\tilde{\Omega}$.

Thus, since the term $(e^{-u} - e^{-u_\lambda^\nu})$ is locally Lipschitz continuous, from (75) we get

$$
\begin{aligned}
&C_1 \int_{\tilde{\Omega}} \left(|\nabla u| + \left|\nabla u_\lambda^\nu\right| \right)^{p-2} \left|\nabla \left(u - u_\lambda^\nu\right)^+\right|^2 \left(\phi_{\rho,\lambda}^\nu\right)^2 dx \\
&\quad \leqslant C_2 \int_{\tilde{\Omega}} |\nabla u|^{p-1} \left|\nabla \left(u - u_\lambda^\nu\right)^+\right| \left(u - u_\lambda^\nu\right)^+ \left(\phi_{\rho,\lambda}^\nu\right)^2 dx + C_3 \int_{\tilde{\Omega}} \left[\left(u - u_\lambda^\nu\right)^+ \right]^2 \left(\phi_{\rho,\lambda}^\nu\right)^2 dx + o(1), \qquad (76)
\end{aligned}
$$

with $C_1 = C_1(p, \|u\|_{L^\infty(\Omega_\lambda^\nu)})$ and $C_2 = C_2(\|u\|_{L^\infty(\Omega_\lambda^\nu)})$ positive constants.

Let us now consider

Case: $p \geqslant 2$. Let us evaluate the terms on the right-hand side of the inequality (76). Exploiting the weighted Young's inequality we get

$$
\begin{aligned}
&C_2 \int_{\tilde{\Omega}} |\nabla u|^{p-1} \big|\nabla\big(u-u_\lambda^\nu\big)^+\big| \big(u-u_\lambda^\nu\big)^+ \big(\phi_{\rho,\lambda}^\nu\big)^2\,dx \\
&\leqslant \varepsilon C_2 \int_{\tilde{\Omega}} |\nabla u|^{p-2} \big|\nabla\big(u-u_\lambda^\nu\big)^+\big|^2 \big(\phi_{\rho,\lambda}^\nu\big)^2\,dx + \frac{C_2}{\varepsilon} \int_{\tilde{\Omega}} |\nabla u|^p \big[\big(u-u_\lambda^\nu\big)^+\big]^2 \big(\phi_{\rho,\lambda}^\nu\big)^2\,dx \\
&\leqslant \varepsilon C_2 \int_{\tilde{\Omega}} \big(|\nabla u| + \big|\nabla u_\lambda^\nu\big|\big)^{p-2} \big|\nabla\big(u-u_\lambda^\nu\big)^+\big|^2 \big(\phi_{\rho,\lambda}^\nu\big)^2\,dx + \tilde{C}_2 \int_{\tilde{\Omega}} \big[\big(u-u_\lambda^\nu\big)^+\big]^2 \big(\phi_{\rho,\lambda}^\nu\big)^2\,dx,
\end{aligned}
\tag{77}
$$

with $\tilde{C}_2 = \tilde{C}_2(\varepsilon, \|u\|_{L^\infty(\Omega_\lambda^\nu)}, \|\nabla u\|_{L^\infty(\Omega_\lambda^\nu)})$ a positive constant. Since $p > 2$, we used $|\nabla u|^{p-2} \leqslant (|\nabla u| + |\nabla u_\lambda^\nu|)^{p-2}$. Thus, choosing ε sufficiently small such that $C_1 - \varepsilon C_2 \geqslant \tilde{C}_1 > 0$, using (77), Eq. (76) becomes

$$
\int_{\tilde{\Omega}} \big(|\nabla u| + \big|\nabla u_\lambda^\nu\big|\big)^{p-2} \big|\nabla\big(u-u_\lambda^\nu\big)^+\big|^2 \big(\phi_{\rho,\lambda}^\nu\big)^2\,dx \leqslant C \int_{\tilde{\Omega}} \big[\big(u-u_\lambda^\nu\big)^+\big]^2 \big(\phi_{\rho,\lambda}^\nu\big)^2\,dx + o(1),
\tag{78}
$$

for some positive constant $C = \frac{\tilde{C}_2 + C_3}{\tilde{C}_1}$. By weighted Poincaré's (Theorem 5), we get

$$
\begin{aligned}
&C \int_{\tilde{\Omega}} \big[\big(u-u_\lambda^\nu\big)^+\big]^2 \big(\phi_{\rho,\lambda}^\nu\big)^2\,dx \\
&\leqslant \tilde{C} C_p^2(\tilde{\Omega}) \int_{\tilde{\Omega}} |\nabla u|^{p-2} \big|\nabla(u-u_\lambda)^+\big|^2 \big(\phi_{\rho,\lambda}^\nu\big)^2\,dx \\
&\quad + C^*\big(\|u\|_{L^\infty(\Omega_\lambda^\nu)}, \|\nabla u\|_{L^\infty(\Omega_\lambda^\nu)}\big) \int_{B_{2\rho}\setminus B_\rho} \big|\nabla \phi_{\rho,\lambda}^\nu\big|^2 + o(1) \\
&\leqslant \tilde{C} C_p^2(\tilde{\Omega}) \int_{\tilde{\Omega}} \big(|\nabla u| + \big|\nabla u_\lambda^\nu\big|\big)^{p-2} \big|\nabla(u-u_\lambda)^+\big|^2 \big(\phi_{\rho,\lambda}^\nu\big)^2 + o(1),
\end{aligned}
\tag{79}
$$

where as before, since $N > p > 2$, we have $|\nabla u|^{p-2} \leqslant (|\nabla u| + |\nabla u_\lambda^\nu|)^{p-2}$ and

$$
\int_{B_{2\rho}\setminus B_\rho} \big|\nabla \phi_{\rho,\lambda}^\nu\big|^2 \to 0 \quad \text{as } \rho \to 0.
$$

Concluding, collecting the estimates (78) and (79) we get

$$
\begin{aligned}
&\int_{\tilde{\Omega}} \big(|\nabla u| + \big|\nabla u_\lambda^\nu\big|\big)^{p-2} \big|\nabla\big(u-u_\lambda^\nu\big)^+\big|^2 \big(\phi_{\rho,\lambda}^\nu\big)^2\,dx \\
&\leqslant \tilde{C} C_p^2(\tilde{\Omega}) \int_{\tilde{\Omega}} \big(|\nabla u| + \big|\nabla u_\lambda^\nu\big|\big)^{p-2} \big|\nabla(u-u_\lambda)^+\big|^2 \big(\phi_{\rho,\lambda}^\nu\big)^2 + o(1).
\end{aligned}
\tag{80}
$$

Since (see Theorem 5) $C_p(\tilde{\Omega})$ goes to zero provided the Lebesgue measure of $\tilde{\Omega}$ goes to 0, if $|\tilde{\Omega}| \leqslant \delta$, with δ (depending on λ) sufficiently small, we may assume $C_p(\tilde{\Omega})$ so small such that

$$
\tilde{C} C_p^2(\tilde{\Omega}) < 1.
$$

Thus, letting $\rho \to 0$ in (80), by the dominated convergence theorem we get the contradiction, showing that, actually, $(u - u_\lambda^\nu)^+ = 0$ and then the thesis for $p \geqslant 2$. We point out that here ($p \geqslant 2$) we do not need to assume that $|\nabla u|$ is bounded.

Let us consider now the

Case: $1 < p < 2$. From (72) we infer that $|\nabla u|, |\nabla u_\lambda^\nu| \in L^\infty(\tilde{\Omega} \cap \{u \geqslant u_\lambda^\nu\})$ and therefore we have that $(u - u_\lambda^\nu)^+ \in W^{1,2}(\tilde{\Omega} \cap \{u \geqslant u_\lambda^\nu\})$. Then the conclusion follows using the classical Poincaré inequality: in fact, since $p < 2$, the term $(|\nabla u| + |\nabla u_\lambda^\nu|)^{p-2}$ is bounded below being $|\nabla u|, |\nabla u_\lambda^\nu| \in L^\infty(\tilde{\Omega} \cap \{u \geqslant u_\lambda^\nu\})$. Then, Eq. (76) gives

$$C_1 \int\limits_{\tilde{\Omega} \cap \{u \geqslant u_\lambda^\nu\}} \left|\nabla \left(u - u_\lambda^\nu\right)^+\right|^2 \left(\phi_{\rho,\lambda}^\nu\right)^2 dx \leqslant C_2 \int\limits_{\tilde{\Omega} \cap \{u \geqslant u_\lambda^\nu\}} \left|\nabla \left(u - u_\lambda^\nu\right)^+\right| \left(u - u_\lambda^\nu\right)^+ \left(\phi_{\rho,\lambda}^\nu\right)^2 dx + C_3 \int\limits_{\tilde{\Omega} \cap \{u \geqslant u_\lambda^\nu\}} \left[\left(u - u_\lambda^\nu\right)^+\right]^2 \left(\phi_{\rho,\lambda}^\nu\right)^2 dx + o(1). \tag{81}$$

By dominated convergence theorem, (81) states as

$$C_1 \int\limits_{\tilde{\Omega} \cap \{u \geqslant u_\lambda^\nu\}} \left|\nabla \left(u - u_\lambda^\nu\right)^+\right|^2 dx \leqslant C_2 \int\limits_{\tilde{\Omega} \cap \{u \geqslant u_\lambda^\nu\}} \left|\nabla \left(u - u_\lambda^\nu\right)^+\right| \left(u - u_\lambda^\nu\right)^+ dx + C_3 \int\limits_{\tilde{\Omega} \cap \{u \geqslant u_\lambda^\nu\}} \left[\left(u - u_\lambda^\nu\right)^+\right]^2 dx + o(1)$$

and by weighted Young inequality, arguing as above (see Eq. (77)), for fixed small ε such that

$$C_1 - \varepsilon C_2 \geqslant \tilde{C}_1 > 0,$$

we have

$$\int\limits_{\tilde{\Omega} \cap \{u \geqslant u_\lambda^\nu\}} \left|\nabla \left(u - u_\lambda^\nu\right)^+\right|^2 dx \leqslant C \int\limits_{\tilde{\Omega} \cap \{u \geqslant u_\lambda^\nu\}} \left[\left(u - u_\lambda^\nu\right)^+\right]^2 dx, \tag{82}$$

with $C = \frac{C_2 + \varepsilon C_3}{\varepsilon \tilde{C}_1}$. The conclusion follows using classical Poincaré inequality in (82), i.e.

$$\int\limits_{\tilde{\Omega} \cap \{u \geqslant u_\lambda^\nu\}} \left|\nabla \left(u - u_\lambda^\nu\right)^+\right|^2 dx \leqslant C C_p^2(\tilde{\Omega}) \int\limits_{\tilde{\Omega} \cap \{u \geqslant u_\lambda^\nu\}} \left|\nabla \left(u - u_\lambda^\nu\right)^+\right|^2 dx,$$

by choosing $\delta = \delta(\lambda)$ small such that $C C_p^2(\tilde{\Omega}) < 1$ and then getting $(u - u_\lambda^\nu)^+ = 0$. □

3.4. The moving plane method

We refer to the notations and definitions of Section 3.2, Eqs. (47)–(53). To prove Theorem 2, we need first the following result:

Proposition 3. *Let $u \in C^{1,\alpha}(\overline{\Omega} \setminus \{0\})$ be a solution to problem $(\mathcal{P})$. Set*

$$\lambda_1^0(\nu) := \min\{0, \lambda_1(\nu)\},$$

where $\lambda_1(\nu)$ is defined in (53). Then, for any $a(\nu) \leqslant \lambda \leqslant \lambda_1^0(\nu)$, we have

$$u(x) \leqslant u_\lambda^\nu(x), \quad \forall x \in \Omega_\lambda^\nu. \tag{83}$$

Moreover, for any λ with $a(\nu) < \lambda < \lambda_1^0(\nu)$ we have

$$u(x) < u_\lambda^\nu(x), \quad \forall x \in \Omega_\lambda^\nu \setminus Z_{u,\lambda}, \tag{84}$$

where $Z_{u,\lambda} \equiv \{x \in \Omega_\lambda^\nu\colon \nabla u(x) = \nabla u_\lambda^\nu(x) = 0\}$. Finally

$$\frac{\partial u}{\partial \nu}(x) \geqslant 0, \quad \forall x \in \Omega_{\lambda_1(\nu)}^\nu. \tag{85}$$

Proof. Let $a(\nu) < \lambda < \lambda_1^0(\nu)$ with λ sufficiently close to $a(\nu)$. By Hopf's Lemma, it follows that

$$u - u_\lambda^\nu \leqslant 0 \quad \text{in } \Omega_\lambda^\nu.$$

We define

$$\Lambda_0 = \{\lambda > a(\nu)\colon\ u \leqslant u_t \text{ in } \Omega_t^\nu \text{ for all } t \in (a(\nu), \lambda]\} \tag{86}$$

and

$$\lambda_0 = \sup \Lambda_0. \tag{87}$$

Notice that by continuity we obtain $u \leqslant u_{\lambda_0}^\nu$ in $\Omega_{\lambda_0}^\nu$. We have to show that $\lambda_0 = \lambda_1^0(\nu)$. Assume by contradiction $\lambda_0 < \lambda_1^0(\nu) \leqslant 0$ and let $A_{\lambda_0} \subset \Omega_{\lambda_0}^\nu$ be an open set such that $Z_{u,\lambda_0} \cap \Omega_{\lambda_0}^\nu \subset A_{\lambda_0} \Subset \Omega$. Such set exists by Hopf's Lemma. Notice that, since $|Z_{u,\lambda_0}| = 0$ as remarked above, we can take A_{λ_0} with measure arbitrarily small. Since we are working in $\Omega_{\lambda_0}^\nu$, we have that the weight $1/|x|^p$ is not singular there. Moreover, in a neighborhood of the reflected point of the origin 0_λ^ν, we know, by Lemma 4, that $u < u_{\lambda_0}^\nu$. Since elsewhere $1/|x_\lambda^\nu|^p$ is not singular and $u, \nabla u, u_\lambda^\nu, \nabla u_\lambda^\nu$ are bounded, we can exploit the strong comparison principle, see e.g. [19, Theorem 2.5.2], to get that

$$u < u_{\lambda_0}^\nu \quad \text{or} \quad u \equiv u_{\lambda_0}^\nu$$

in any connected component of $\Omega_{\lambda_0}^\nu \setminus Z_u$. It follows now that

- the case $u \equiv u_{\lambda_0}^\nu$ in some connected component $\bar{\mathcal{C}}$ of $\Omega_{\lambda_0} \setminus Z_{u,\lambda_0}$ is not possible, since by symmetry, it would imply the existence of *a local symmetry phenomenon* and consequently that $\Omega \setminus Z_{u,\lambda_0}$ would be not connected, in spite of what stated in Lemma 5.

Note also that, since the domain is strictly convex, by Hopf's Lemma and the Dirichlet condition (see e.g. [8]), we get that there exists a neighborhood $\mathcal{N}_{\lambda_0}$ of $\partial\Omega_{\lambda_0}^\nu \cap \partial\Omega$ where $u < u_{\lambda_0}^\nu$ in $\mathcal{N}_{\lambda_0}$.

We deduce that there exists a compact set K in $\Omega_{\lambda_0}^\nu$ such that

- $|\Omega_{\lambda_0}^\nu \setminus ((K \setminus A_{\lambda_0}) \cup \mathcal{N}_{\lambda_0})|$ is sufficiently small so that Proposition 2 applies.
- $u_{\lambda_0}^\nu - u$ is positive in $(K \setminus A_{\lambda_0}) \cup \mathcal{N}_{\lambda_0}$.

Therefore by continuity (and redefining $A_{\lambda_0+\varepsilon}$ as small as we want and $\mathcal{N}_{\lambda_0+\varepsilon}$, exploiting Hopf's Lemma) we find $\varepsilon > 0$ such that

- $|\Omega_{\lambda_0+\varepsilon}^\nu \setminus ((K \setminus A_{\lambda_0+\varepsilon}) \cup \mathcal{N}_{\lambda_0+\varepsilon})|$ is sufficiently small so that Proposition 2 applies.
- $u_{\lambda_0+\varepsilon}^\nu - u$ is positive in $(K \setminus A_{\lambda_0+\varepsilon}) \cup \mathcal{N}_{\lambda_0+\varepsilon}$.

Since now $u_{\lambda_0+\varepsilon}^\nu - u \geqslant 0$ on $\partial((K \setminus A_{\lambda_0+\varepsilon}) \cup \mathcal{N}_{\lambda_0+\varepsilon})$ it follows $u \leqslant u_{\lambda_0+\varepsilon}^\nu$ on $\partial(\Omega_{\lambda_0+\varepsilon}^\nu \setminus ((K \setminus A_{\lambda_0+\varepsilon}) \cup \mathcal{N}_{\lambda_0+\varepsilon}))$. By Proposition 2 it follows $u \leqslant u_{\lambda_0+\varepsilon}^\nu$ in $\Omega_{\lambda_0+\varepsilon}^\nu \setminus ((K \setminus A_{\lambda_0+\varepsilon}) \cup \mathcal{N}_{\lambda_0+\varepsilon})$ and consequently in $\Omega_{\lambda_0+\varepsilon}^\nu$, what contradicts the assumption $\lambda_0 < \lambda_1^0(\nu)$. Therefore, $\lambda_0 \equiv \lambda_1^0(\nu)$ and the thesis is proved.

We point out that we are exploiting Proposition 2 in the set $\Omega_{\lambda_0+\varepsilon}^\nu \setminus ((K \setminus A_{\lambda_0+\varepsilon}) \cup \mathcal{N}_{\lambda_0+\varepsilon})$ which is bounded away from the boundary $\partial\Omega$ and then the constant δ in the statement is uniformly bounded.

The proof of (84) follows by the strong comparison theorem applied as above.

Finally (85) follows by the monotonicity of the solution that is implicit in the arguments above. □

We can now give the

Proof of Theorem 2. Since by hypothesis Ω is strictly convex w.r.t. the ν-direction and symmetric w.r.t. to (see Eq. (47))

$$T_0^\nu = \{x \in \mathbb{R}^N\colon\ x \cdot \nu = 0\},$$

it follows by Proposition 3, being $\lambda_1(\nu) = 0 = \lambda_1^0(\nu)$ in this case, that

$$u(x) \leqslant u_\lambda^\nu(x) \quad \text{for } x \in \Omega_0^\nu,$$

see Eq. (50). In the same way, performing the moving plane method in the direction $-\nu$ we obtain

$$u(x) \geqslant u_\lambda^\nu(x) \quad \text{for } x \in \Omega_0^\nu,$$

that is, u is symmetric and non-decreasing w.r.t. the ν-direction, since monotonicity follows by (85).

Finally, if Ω is a ball, repeating this argument along any direction, it follows that u is radially symmetric. The fact that $\frac{\partial u}{\partial r}(r) < 0$ for $r \neq 0$, follows by the Hopf's boundary lemma which works in this case since the level sets are balls and therefore fulfill the interior sphere condition. □

References

[1] P. Bènilan, L. Boccardo, T. Gallout, R. Gariepy, M. Pierre, J. Vázquez, An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations, Ann. Sc. Norm. Super. Pisa, Cl. Sci. 22 (2) (1995) 241–273.
[2] H. Berestycki, L. Nirenberg, On the method of moving planes and the sliding method, Bull. Soc. Brasil. Mat. (N.S. 22 (1) (1991) 1–37.
[3] L. Boccardo, F. Murat, Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations, Nonlinear Anal. 19 (19) (1992) 581–597.
[4] L. Boccardo, F. Murat, J.P. Puel, Résultats d'existence pour certains problèmes elliptiques quasilinéaires, Ann. Sc. Norm. Super. Pisa 11 (2) (1984) 213–235.
[5] A. Dall'Aglio, Approximated solutions of equations with L^1 data. Application to the H-convergence of quasi-linear parabolic equations, Ann. Mat. Pura Appl. 170 (4) (1996) 207–240.
[6] G. Dal Maso, F. Murat, L. Orsina, A. Prignet, Renormalized solutions of elliptic equations with general measure data, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) 28 (4) (1999) 741–808.
[7] L. Damascelli, F. Pacella, Monotonicity and symmetry of solutions of p-Laplace equations, $1 < p < 2$, via the moving plane method, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) 26 (4) (1998) 689–707.
[8] L. Damascelli, F. Pacella, Monotonicity and symmetry results for p-Laplace equations and applications, Adv. Differential Equations 5 (7–9) (2000) 1179–1200.
[9] L. Damascelli, B. Sciunzi, Regularity, monotonicity and symmetry of positive solutions of m-Laplace equations, J. Differential Equations 206 (2) (2004) 483–515.
[10] J. Dávila, I. Peral, Nonlinear elliptic problems with a singular weight on the boundary, Calc. Var. Partial Differential Equations 41 (3–4) (2011) 567–586.
[11] E. Di Benedetto, $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations, Nonlinear Anal. 7 (8) (1983) 827–850.
[12] B. Gidas, W.M. Ni, L. Nirenberg, Symmetry and related properties via the maximum principle, Comm. Math. Phys. 68 (3) (1979) 209–243.
[13] J. Heinonen, T. Kilpeläinen, O. Martio, Nonlinear Potential Theory of Degenerate Elliptic Equations, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1993.
[14] G.M. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, Nonlinear Anal. 12 (11) (1988) 1203–1219.
[15] S. Merchán, I. Peral, Remarks on the solvability of an elliptic equation with a supercritical term involving the Hardy–Leray potential, J. Math. Anal. Appl. 394 (2012) 347–359.
[16] S. Merchán, L. Montoro, Remarks on the existence of solutions to some quasilinear elliptic problems involving the Hardy–Leray potential, Ann. Mat. Pura Appl., http://dx.doi.org/10.1007/s10231-012-0293-7.
[17] L. Montoro, B. Sciunzi, M. Squassina, Asymptotic symmetry for a class of quasi-linear parabolic problems, Adv. Nonlinear Stud. 10 (4) (2010) 789–818.
[18] P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, J. Differential Equations 51 (1) (1984) 126–150.
[19] P. Pucci, J. Serrin, The Maximum Principle, Birkhäuser, Boston, 2007.
[20] J. Serrin, Local behavior of solutions of quasi-linear equations, Acta Math. 111 (1964) 247–302.
[21] J. Serrin, A symmetry problem in potential theory, Arch. Ration. Mech. Anal. 43 (4) (1971) 304–318.