Nonlinear equations for fractional Laplacians, I: Regularity, maximum principles, and Hamiltonian estimates ☆

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Abstract

This is the first of two articles dealing with the equation $(-\Delta)^s v = f(v)$ in $\mathbb{R}^n$, with $s \in (0, 1)$, where $(-\Delta)^s$ stands for the fractional Laplacian — the infinitesimal generator of a Lévy process. This equation can be realized as a local linear degenerate elliptic equation in $\mathbb{R}^{n+1}_+$ together with a nonlinear Neumann boundary condition on $\partial \mathbb{R}^{n+1}_+ = \mathbb{R}^n$.

In this first article, we establish necessary conditions on the nonlinearity f to admit certain type of solutions, with special interest in bounded increasing solutions in all of $\mathbb{R}$. These necessary conditions (which will be proven in a follow-up paper to be also sufficient for the existence of a bounded increasing solution) are derived from an equality and an estimate involving a Hamiltonian — in the spirit of a result of Modica for the Laplacian. Our proofs are uniform as $s \uparrow 1$, establishing in the limit the corresponding known results for the Laplacian.

In addition, we study regularity issues, as well as maximum and Harnack principles associated to the equation.

1. Introduction

This paper is devoted to the study of the nonlinear problem

$$(-\Delta)^s v = f(v) \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

where $s \in (0, 1)$ and

$$(-\Delta)^s v(x) = C_{n,s} \, \text{P.V.} \int_{\mathbb{R}^n} \frac{v(x) - v(\bar{x})}{|x - \bar{x}|^{n+2s}} \, d\bar{x}. \tag{1.2}$$

Here P.V. stands for the Cauchy principal value, and $C_{n,s}$ is a positive constant depending only on n and s — whose value is given in Remark 3.11 below. The above integral is well defined if, for instance, v is bounded (which ensures the integrability at infinity) and v is $C^2_{\text{loc}}(\mathbb{R}^n)$ (which ensures the integrability at $\bar{x} = x$ in the principal value sense).

The following is an essential tool in our treatment of the nonlinear equation (1.1). As explained in more detail in Section 3 below, up to an explicit multiplicative constant (given in Remark 3.11) in front of the nonlinearity f, problem (1.1) can be realized in a local manner through the nonlinear boundary value problem

$$\begin{cases} \text{div}(y^a \nabla u) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ (1+a)\dfrac{\partial u}{\partial \nu^a} = f(u) & \text{on } \partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n, \end{cases} \tag{1.3}$$

where $n \geqslant 1$, $\mathbb{R}^{n+1}_+ = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : y > 0\}$ is the upper half space, $\partial\mathbb{R}^{n+1}_+ = \{y = 0\}$, $u = u(x, y)$ is real valued, and

$$\frac{\partial u}{\partial \nu^a} = -\lim_{y \downarrow 0} y^a \partial_y u$$

is the conormal exterior derivative of u. Points in $\mathbb{R}^n$ are denoted by $x = (x_1, \ldots, x_n)$. The parameter a belongs to $(-1, 1)$ and is related to the power s of the fractional Laplacian $(-\Delta)^s$ by the formula

$$a = 1 - 2s \in (-1, 1)$$

— a relation that we assume throughout the paper.

Indeed, Caffarelli and Silvestre [7] have proved the following formula relating the fractional Laplacian $(-\Delta)^s$ to the Dirichlet to Neumann operator for (1.3):

$$(-\Delta)^s \{u(\cdot, 0)\} = d_s \frac{\partial u}{\partial \nu^a} \quad \text{in } \mathbb{R}^n = \partial\mathbb{R}^{n+1}_+, \tag{1.4}$$

where d_s is a positive constant depending only on s (see Remark 3.11 below for its value).

The aim of this paper — and of the forthcoming one [4] — is to study two types of bounded solutions of (1.1):

(a) Solutions $v = v(x)$ of (1.1) which are monotone increasing, say from -1 to 1, in one of the x-variables. These solutions are named *layer solutions* and constitute our main interest.
(b) Radial solutions $v = v(|x|)$ of (1.1) tending, say, to 0 as $|x| \to \pm\infty$.

In the second part [4] of this work, we will be concerned with the existence, uniqueness, symmetry and variational properties, as well as the asymptotic behavior of layer solutions. These questions will be related, of course, to structural assumptions made on the nonlinearity f.

In this first article, we establish necessary conditions on the nonlinearity f to admit a layer solution in $\mathbb{R}$ (i.e., in dimension one), and also to admit radial solutions in $\mathbb{R}^n$ with limit at infinity. In the case of layer solutions, our necessary conditions (2.7) and (2.8) below will be proven in [4] to be also sufficient for the existence of a layer solution. Conditions (2.7) and (2.8) express that f is of bistable balanced type, and thus the potential G (with $G' = -f$) is of double-well type. Classical examples are the Allen–Cahn equation $-\Delta v = v - v^3$ and the Peierls–Nabarro problem $(-\Delta)^{1/2} v = \pi^{-1} \sin(\pi v)$.

Our necessary conditions are derived from a new equality and a new estimate involving the Hamiltonian for (1.3) — in the spirit of a celebrated inequality of Modica [17] for the Laplacian. For $s = 1/2$ the Hamiltonian equality and estimate were proven by Solà-Morales and the first author in [5]. Also for $s = 1/2$ but in the non-stationary case, Caffarelli, Mellet, and the second author [6] establish a corresponding Hamiltonian equality. It is a crucial tool in their proof of existence of traveling waves for the homogeneous heat equation in a half-plane together with a nonlinear Neumann boundary condition of combustion type. In Subsection 1.1 below we explain the Hamiltonian structure of problem (1.3).

Let us also recall that Modica proved that every bounded solution of $\Delta v - G'(v) = 0$ in the whole $\mathbb{R}^n$ satisfies $(1/2)|\nabla v|^2 \leqslant G(v)$ pointwise in all $\mathbb{R}^n$, assuming only that $G \geqslant 0$ in $\mathbb{R}$. Using the Hamiltonian structure, here we prove an analogue of such an estimate in dimension one in the case of nonlocal operators given by the fractional Laplacians. This is done via a careful study of the local boundary value problem (1.3).

In addition, in this first paper we also study regularity issues, as well as maximum, Liouville, and Harnack principles associated to the fractional Laplacian in $\mathbb{R}^n$. These tools will be needed in this paper and in its follow-up.

Our work extends to all fractions $s \in (0, 1)$ the results of one of the authors and J. Solà-Morales [5] for the case $s = 1/2$. [5] studies bounded (specially layer) solutions of

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \dfrac{\partial u}{\partial v} = f(u) & \text{on } \partial \mathbb{R}^{n+1}_+, \end{cases}$$

which corresponds to the case $a = 0$ in (1.3). It is well known that the Dirichlet to Neumann operator associated to the previous problem is precisely $(-\Delta)^{1/2}$. Therefore, layer solutions of the previous equation are actually heteroclinic connections (between -1 and 1) of

$$(-\Delta)^{1/2} v = f(v) \quad \text{in } \mathbb{R}^n,$$

where v is the trace of u on the boundary $\{y = 0\}$. The goal of our papers is to generalize this study to any fractional power $s \in (0, 1)$ of the Laplacian. We will make a great use of the tools developed in [5]. However, some new difficulties arise due to the degeneracy of the operator in (1.3).

The first equation in (1.3) is a linear degenerate elliptic equation with weight y^a. Since $a \in (-1, 1)$, the weight y^a belongs to the Muckenhoupt class A_2; see [18]. More precisely, a nonnegative function w defined in $\mathbb{R}^N$ is said to be A_2 if, for some constant C,

$$\sup_B \left(\frac{1}{|B|} \int_B w \right) \left(\frac{1}{|B|} \int_B w^{-1} \right) \leqslant C$$

for all balls B in $\mathbb{R}^N$. It is easy to verify that $|y|^a \in A_2(\mathbb{R}^{n+1})$ for $a \in (-1, 1)$. As we explain in Section 3, this fact allows to use the regularity results of Fabes, Jerison, Kenig, and Serapioni [11,10] concerning divergence form equations with A_2 weights. Another important property of the weight y^a is that it depends only on the extension variable y and not on the tangential variables x. Thus, (1.3) is invariant under translations in x — as it is Eq. (1.1). In [4], this will allow us, for instance, to use for (1.3) the sliding method of Berestycki and Nirenberg.

The study of elliptic equations involving fractional powers of the Laplacian appears to be important in many physical situations in which one has to consider long-range or anomalous diffusions. From a probabilistic point of view, the fractional Laplacian is the infinitesimal generator of a Lévy process — see the book of Bertoin [3], for instance.

A lot of interest is currently devoted to the study of nonlinear equations involving fractional powers of the Laplacian. This type of operator seems to have a prevalent role in physical situations such as combustion (see [8]), and in dislocations in mechanical systems (see [14]) or in crystals (see [13,24]). Layer solutions are a central object in equations modeling phase transitions, for instance in the Peierls–Nabarro equation $(-\Delta)^{1/2} v = \pi^{-1} \sin(\pi v)$ for dislocations in crystals. In addition, these operators arise in modeling diffusion and transport in a highly heterogeneous medium; they may be used in the description of the diffusion of a liquid in some heterogeneous media, or as an effective diffusion in a limiting advection–diffusion equation with a random velocity field; see for instance [16].

1.1. Formal Hamiltonian structure

As in [5], the quantity appearing in our main results — see for instance (2.9) below — arises naturally when one looks at problem (1.3) for $n = 1$ as a formal Hamiltonian system in infinite dimensions. Here the time variable is $\tau = x$, the position q is the function $u(x, \cdot) = u(\tau, \cdot)$ in the halfline $\{y \geqslant 0\}$, and the momentum is $p = q' = u_x(\tau, \cdot)$.

From the action — that is, in PDE terminology the energy functional (2.1) below, which we already have at hand — we see that the Lagrangian is $L(q, p) = (1/2) \|p\|^2_{2,a} + W(q)$, where

$$W(q) = \frac{1}{2} \|\partial_y q\|^2_{2,a} + \frac{1}{1+a} G\big(q(0)\big),$$

$\|w\|^2_{2,a} = \int_0^{+\infty} y^a |w(y)|^2 \, dy$, and $G' = -f$.

The Legendre transform of L with respect to p gives the Hamiltonian

$$H(q, p) = \int\limits_{0}^{+\infty} \frac{t^a}{2} \{u_x^2(x, t) - u_y^2(x, t)\} dt - \frac{1}{1+a} G(u(x, 0))$$

$$= \frac{1}{2} \|p\|_{2,a}^2 - W(q).$$

One can easily check that its associated Hamiltonian system

$$\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} p \\ W'(q) \end{pmatrix}$$

is formally problem (1.3).

Thus, our equation admits a Hamiltonian structure in an infinite dimensional phase space. However, in this paper we do not address the question of setting it as a true well posed semigroup. Note that a lot of challenging issues usually arise with infinite dimensional Hamiltonian systems — see for instance [9].

1.2. Outline of the article

The paper is organized as follows. Section 2 contains the statements of our main results. In Section 3 we explain the relation between problems (1.1) and (1.3), and we present the Poisson kernel and a regularity result for (1.3). Section 4 contains results on the operator L_a appearing in (1.3); we establish Schauder estimates, a Harnack inequality, a Hopf principle, maximum principles, and a Liouville theorem. Section 5 is concerned with the proof of the Hamiltonian equality and estimates. In Section 6 we prove our results on layers as the fraction s tends to 1. Finally, in Section 7 we collect the proofs of our main results, Theorems 2.2, 2.3, 2.4 and 2.5, using the results established in previous sections.

2. Main results: Hamiltonian identity and necessary conditions on the nonlinearity for existence

Throughout the paper we assume that $s \in (0, 1)$ and that the nonlinearity satisfies

$$f \in C^{1,\gamma}(\mathbb{R}) \quad \text{for some } \gamma > \max(0, 1 - 2s).$$

We will denote G the associated potential, i.e.,

$$G' = -f$$

— which is defined up to an additive constant. We recall that the problem under study is (1.3), i.e.,

$$\begin{cases} \operatorname{div}(y^a \nabla u) = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ (1 + a) \dfrac{\partial u}{\partial \nu^a} = f(u) & \text{on } \partial \mathbb{R}_+^{n+1}, \end{cases}$$

with $a = 1 - 2s$. In the sequel we will denote

$$L_a w = \operatorname{div}(y^a \nabla w).$$

We use the notation

$$B_R^+ = \{(x, y) \in \mathbb{R}^{n+1}: y > 0, \ |(x, y)| < R\},$$
$$\Gamma_R^0 = \{(x, 0) \in \partial \mathbb{R}_+^{n+1}: |x| < R\}, \quad \text{and}$$
$$\Gamma_R^+ = \{(x, y) \in \mathbb{R}^{n+1}: y \geqslant 0, \ |(x, y)| = R\}.$$

It is easy to see that (1.3) has a variational structure, corresponding to the energy functional

$$E_{B_R^+}(u) = \int\limits_{B_R^+} \frac{1}{2} y^a |\nabla u|^2 \, dx \, dy + \int\limits_{\Gamma_R^0} \frac{1}{1+a} G(u) \, dx. \tag{2.1}$$

We are concerned with the following types of solutions. The first class (layer solutions) consists of solutions which are increasing and have limits at infinity in one Euclidean variable in the space $\mathbb{R}^n$ of x-variables. In the following definition, and for future convenience, after a rotation we may assume that such variable is the x_1-variable.

Definition 2.1. We say that v is a *layer solution* of (1.1) if v is a solution of (1.1) satisfying

$$v_{x_1} > 0 \quad \text{in } \mathbb{R}^n, \quad \text{and}$$
$$\lim_{x_1 \to \pm\infty} v(x) = \pm 1 \quad \text{for every } (x_2, \dots, x_n) \in \mathbb{R}^{n-1}. \tag{2.2}$$

We say that u is a *layer solution* of (1.3) if it is a solution of (1.3),

$$u_{x_1} > 0 \quad \text{on } \partial\mathbb{R}_+^{n+1}, \quad \text{and} \tag{2.3}$$
$$\lim_{x_1 \to \pm\infty} u(x, 0) = \pm 1 \quad \text{for every } (x_2, \dots, x_n) \in \mathbb{R}^{n-1}. \tag{2.4}$$

It is important to emphasize that, for $n \geq 2$, the limits in (2.2) and (2.4) are taken for $(x_2, \dots, x_n)$ fixed, and are not assumed to be uniform in $(x_2, \dots, x_n) \in \mathbb{R}^{n-1}$.

We will also study solutions v of (1.1) which are radially symmetric (not necessarily decreasing) and such that

$$\lim_{|x| \to \infty} v(|x|) = 0. \tag{2.5}$$

We can now state our main results. The next theorem provides a necessary condition — (2.7) and (2.8) — on the nonlinearity f to admit a layer solution in $\mathbb{R}$. In our subsequent paper [4], this necessary condition will be proven to be also sufficient for the existence of a layer solution. It is interesting to point out that conditions (2.7) and (2.8) are independent of the fraction $s \in (0, 1)$, and that they are also the necessary and sufficient conditions for the existence of a layer solution to the local equation $-v'' = f(v)$ in all of $\mathbb{R}$.

The theorem also states that families of layer solutions indexed by $s \in (0, 1)$ converge as s goes to 1 to a layer solution of the equation $-v'' = f(v)$ in $\mathbb{R}$.

Theorem 2.2. (i) *Let $s \in (0, 1)$ and f any $C^{1,\gamma}(\mathbb{R})$ function, for some $\gamma > \max(0, 1 - 2s)$. Assume that there exists a layer solution v of*

$$(-\partial_{xx})^s v = f(v) \quad \text{in } \mathbb{R}, \tag{2.6}$$

that is, v is a solution of (2.6) satisfying

$$v' > 0 \quad \text{in } \mathbb{R} \quad \text{and} \quad \lim_{x \to \pm\infty} v(x) = \pm 1.$$

Then, we have

$$G'(1) = G'(-1) = 0 \tag{2.7}$$

and

$$G > G(1) = G(-1) \quad \text{in } (-1, 1). \tag{2.8}$$

(ii) *Let f be any $C^{1,\gamma}(\mathbb{R})$ function with $\gamma \in (0, 1)$. Assume that $\{v_s\}$, with $s = s_k \in (0, 1)$ and $s_k \uparrow 1$, is a sequence of layer solutions of*

$$(-\partial_{xx})^s v_s = f(v_s) \quad \text{in } \mathbb{R},$$

such that $v_s(0) = 0$. Then, there exits a function $\bar{v}$ such that

$$\lim_{s = s_k \uparrow 1} v_s = \bar{v}$$

in the uniform C^2 convergence on every compact set of $\mathbb{R}$. Furthermore, the function $\bar{v}$ is the layer solution of

$$-\bar{v}'' = f(\bar{v}) \quad \text{in } \mathbb{R}$$

with $\bar{v}(0) = 0$.

Conditions (2.7) and (2.8) express that is G is of double-well type and f is of bistable balanced type. Note that the statement $G(1) = G(-1)$ is equivalent to

$$\int_{-1}^{1} f = 0.$$

Theorem 2.2 is actually a consequence of the following Hamiltonian equality and estimate, which are of independent interest. We have introduced the Hamiltonian associated to problem (1.3) in Subsection 1.1 above. The following Hamiltonian identity for layer solutions in $\mathbb{R}$ states the conservation of the Hamiltonian in "time" — recall that x plays the role of time variable. Instead, the Hamiltonian inequality below is the analogue in dimension 1 of the classical Modica estimate for bounded solutions of semilinear equations $\Delta v - G'(v) = 0$ in $\mathbb{R}^n$, which states that the kinetic energy is bounded at every point by the potential energy, i.e., $(1/2)|\nabla v|^2 \leqslant G(v)$ everywhere in $\mathbb{R}^n$, whenever $G \geqslant 0$ in $\mathbb{R}$.

Notice that our Modica-type estimate is stated for $n = 1$. It is still an open problem for $n \geqslant 2$.

The theorem also provides an asymptotic result as s goes to 1, in which we recover the classical Hamiltonian identity.

Theorem 2.3. (i) *Let* $a \in (-1, 1)$ *and* f *any* $C^{1,\gamma}(\mathbb{R})$ *function, for some* $\gamma > \max(0, a)$. *Let* $n = 1$ *and* u *be a layer solution of* (1.3). *Then, for every* $x \in \mathbb{R}$ *we have* $\int_0^{+\infty} t^a |\nabla u(x, t)|^2 dt < \infty$ *and the Hamiltonian equality*

$$(1 + a) \int_0^{+\infty} \frac{1}{2} t^a \{u_x^2(x, t) - u_y^2(x, t)\} dt = G(u(x, 0)) - G(1). \tag{2.9}$$

Furthermore, for all $y \geqslant 0$ *and* $x \in \mathbb{R}$ *we have*

$$(1 + a) \int_0^{y} \frac{1}{2} t^a \{u_x^2(x, t) - u_y^2(x, t)\} dt < G(u(x, 0)) - G(1). \tag{2.10}$$

(ii) *Let* f *be any* $C^{1,\gamma}(\mathbb{R})$ *function with* $\gamma \in (0, 1)$, *$n = 1$ and* $\{u_a\}$, *with* $a = a_k \in (-1, 1)$ *and* $a_k \downarrow -1$ *be a sequence of layer solutions of* (1.3) (*with* u *replaced by* u_a *for each* a) *such that* $u_a(0, 0) = 0$. *Then,* $\lim_{a=a_k\downarrow-1} u_a(\cdot, 0) = \bar{v}$ *in the uniform* C^2 *convergence on every compact set of* $\mathbb{R}$, *where* $\bar{v}$ *is the layer solution of* $-\bar{v}'' = f(\bar{v})$ *in* $\mathbb{R}$ *with* $\bar{v}(0) = 0$. *In addition, for every* $x \in \mathbb{R}$ *we have*

$$\lim_{a\downarrow-1} (1 + a) \int_0^{+\infty} \frac{1}{2} t^a (u_a)_x^2(x, t) dt = \frac{1}{2} \bar{v}'(x)^2 = G(\bar{v}(x)) - G(1)$$

and

$$\lim_{a\downarrow-1} (1 + a) \int_0^{+\infty} \frac{1}{2} t^a (u_a)_y^2(x, t) dt = 0.$$

We emphasize once again that the previous estimate (2.10) is pointwise and nonlocal.

The asymptotic result when $a \to -1$ (i.e., $s \to 1$) of part (ii) in the previous theorem allows to recover from (2.9) the standard conservation of the Hamiltonian for the Laplacian. This will be presented in Section 6 below.

In the case of radial solutions with limit at infinity, we establish the following result. Here the dimension n is arbitrary.

Theorem 2.4. *Let* $s \in (0, 1)$ *and* f *any* $C^{1,\gamma}(\mathbb{R})$ *function, for some* $\gamma > \max(0, 1 - 2s)$.

Let $n > 1$ *and* $v = v(x) = v(|x|)$ *be a nonconstant radial solution of*

$$(-\Delta)^s v = f(v) \quad in \ \mathbb{R}^n$$

satisfying

$$\lim_{|x| \to +\infty} v(|x|) = 0.$$

Then, we have

$$f(0) = 0 = G'(0) \quad and \quad G(0) > G(v(0)).$$

If, in addition, v is decreasing in |x|, then

$$f'(0) = -G''(0) \leqslant 0.$$

The statement $G(0) > G(v(0))$ is equivalent to

$$\int_0^{v(0)} f > 0.$$

As in the case of layer solutions, Theorem 2.4 relies on the following statement about the Hamiltonian. In this next theorem we do not assume $u(\cdot, 0)$ to have a limit at infinity.

Theorem 2.5. *Let $a \in (-1, 1)$ and f any $C^{1,\gamma}(\mathbb{R})$ function, for some $\gamma > \max(0, a)$.*
Let $n \geqslant 1$ and u be a bounded solution of (1.3) which is radial in x, i.e., $u(x, y) = u(|x|, y)$.
Then, $\int_0^{+\infty} t^a |\nabla u(r, t)|^2 \, dt < \infty$ for every $r \geqslant 0$, and the quantity

$$(1 + a) \int_0^{+\infty} \frac{t^a}{2} \{u_r^2(r, t) - u_y^2(r, t)\} \, dt - G(u(r, 0)) \tag{2.11}$$

is nonincreasing in $r \geqslant 0$.

3. Local realization of the fractional Laplacian and results on degenerate elliptic equations

This section is concerned with the relation between the local problem (1.3) and the nonlocal problem (1.1). We collect also several results on degenerate elliptic equations with A_2 weights.

We first introduce the spaces

$$H^s(\mathbb{R}^n) = \{v \in L^2(\mathbb{R}^n) \colon |\xi|^s (\mathcal{F}v)(\xi) \in L^2(\mathbb{R}^n)\},$$

where $s \in (0, 1)$ and $\mathcal{F}$ denotes Fourier transform. For $\Omega \subset \mathbb{R}_+^{n+1}$ a Lipschitz domain (bounded or unbounded) and $a \in (-1, 1)$, we denote

$$H^1(\Omega, y^a) = \{u \in L^2(\Omega, y^a \, dx \, dy) \colon |\nabla u| \in L^2(\Omega, y^a \, dx \, dy)\}.$$

3.1. Local realization of the fractional Laplacian

The fractional Laplacian can be defined in various ways, which we review now. It can be defined using Fourier transform by

$$\mathcal{F}((-\Delta)^s v) = |\xi|^{2s} \mathcal{F}(v),$$

for $v \in H^s(\mathbb{R}^n)$. It can also be defined through the kernel representation (see the book by Landkof [15])

$$(-\Delta)^s v(x) = C_{n,s} \, \text{P.V.} \int_{\mathbb{R}^n} \frac{v(x) - v(\bar{x})}{|x - \bar{x}|^{n+2s}} \, d\bar{x}, \tag{3.1}$$

for instance for $v \in \mathcal{S}(\mathbb{R}^n)$, the Schwartz space of rapidly decaying functions. One can also define the fractional Laplacian acting on spaces of functions with weaker regularity. Indeed, following [21], one defines the space $\mathcal{S}_s(\mathbb{R}^n)$

of C^∞ functions v such that for every $k \geqslant 0$, the quantity $(1 + |x|^{n+2s})D^k v(x)$ is bounded. We denote $\mathcal{S}'_s(\mathbb{R}^n)$ its dual. It is then possible to check that $(-\Delta)^s$ maps $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}_s(\mathbb{R}^n)$. By duality, this allows to define the fractional Laplacian for functions in the space

$$\mathcal{L}_s(\mathbb{R}^n) := \left\{ v \in L^1_{\mathrm{loc}}(\mathbb{R}^n): \int_{\mathbb{R}^n} \frac{|v(x)|}{(1+|x|)^{n+2s}}\, dx < \infty \right\}$$
$$= L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap \mathcal{S}'_s(\mathbb{R}^n).$$

For $v \in \mathcal{L}_s(\mathbb{R}^n) \cap C^2_{\mathrm{loc}}(\mathbb{R}^n)$, the integral in (3.1) is well defined. This is clear for $|\bar{x}|$ large. For the Cauchy principal value to be well defined (as $\bar{x} \to x$), it suffices to assume that v is $C^2_{\mathrm{loc}}(\mathbb{R}^n)$. In particular, expression (3.1) defines the operator on the type of solutions that we consider, since they will always be bounded in $\mathbb{R}^n$ and locally C^2. See [21,15] for more comments on the various definitions of the fractional Laplacian and their agreement. We refer the reader to the book by Landkof [15] where an extensive study of integro-differential operators with Martin–Riesz kernels, i.e., kernels of the type (up to a normalizing constant) $1/|z|^{n+2s}$ is presented.

It is well known that one can see the operator $(-\Delta)^{1/2}$ by considering it as the Dirichlet to Neumann operator associated to the harmonic extension in the halfspace, paying the price to add a new variable. In [7], Caffarelli and Silvestre proved that such a kind of realization is also possible for any power of the Laplacian between 0 and 1, as follows.

Given $s \in (0, 1)$, let $a = 1 - 2s \in (-1, 1)$. By a result of Nekvinda [20], it is known that the space $H^s(\mathbb{R}^n)$ coincides with the trace on $\partial \mathbb{R}^{n+1}_+$ of $H^1(\mathbb{R}^{n+1}_+, y^a)$. In particular, every $v \in H^s(\mathbb{R}^n)$ is the trace of a function $u \in L^2_{\mathrm{loc}}(\mathbb{R}^{n+1}_+, y^a)$ such that $\nabla u \in L^2(\mathbb{R}^{n+1}_+, y^a)$. In addition, the function u which minimizes

$$\min \left\{ \int_{\mathbb{R}^{n+1}_+} y^a |\nabla u|^2 \, dx \, dy: \; u|_{\partial \mathbb{R}^{n+1}_+} = v \right\} \tag{3.2}$$

solves the Dirichlet problem

$$\begin{cases} L_a u := \mathrm{div}(y^a \nabla u) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ u = v & \text{on } \partial \mathbb{R}^{n+1}_+. \end{cases} \tag{3.3}$$

By standard elliptic regularity, u is smooth in $\mathbb{R}^{n+1}_+$. It turns out that $-y^a u_y(\cdot, y)$ converges in $H^{-s}(\mathbb{R}^n)$ to a distribution $h \in H^{-s}(\mathbb{R}^n)$ as $y \downarrow 0$. That is, u weakly solves

$$\begin{cases} \mathrm{div}(y^a \nabla u) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ -y^a \partial_y u = h & \text{on } \partial \mathbb{R}^{n+1}_+. \end{cases} \tag{3.4}$$

Consider the Dirichlet to Neumann operator

$$\Gamma_a : H^s(\mathbb{R}^n) \to H^{-s}(\mathbb{R}^n), \quad v \mapsto \Gamma_a(v) = h := -\lim_{y \to 0^+} y^a \partial_y u,$$

where u is the solution of (3.3). Then, we have:

Theorem 3.1. *(See [7].) For every $v \in H^s(\mathbb{R}^n)$,*

$$(-\Delta)^s v = d_s \Gamma_a(v) = -d_s \lim_{y \to 0^+} y^a \partial_y u$$

holds in the distributional sense, where $a = 1 - 2s$ and d_s is a positive constant depending only on s — whose value is given in Remark 3.11.

In other words, given $h \in H^{-s}(\mathbb{R}^n)$, a function $v \in H^s(\mathbb{R}^n)$ solves the equation $(-\Delta)^s v = d_s h$ in $\mathbb{R}^n$ if and only if its extension $u \in H^1(\mathbb{R}^{n+1}_+, y^a)$ solves (3.4). By duality, the same relation can be stated when $v \in \mathcal{L}_s(\mathbb{R}^n)$ — as it is the case of the solutions considered in this paper.

3.2. Degenerate elliptic equations with A_2 weights

According to the previous result, we must study the operator $L_a = \text{div}(y^a \nabla)$ in $\mathbb{R}_+^{n+1}$, whose weight y^a belongs to the class A_2 since $a \in (-1, 1)$. Through a reflection method, it will be useful to consider the equation in domains $\Omega \subset \mathbb{R}^{n+1}$ not necessarily contained in $\mathbb{R}_+^{n+1}$. In such case, we extend the weight y^a by $|y|^a$ for $y < 0$. That is, we define

$$L_a u := \text{div}\big(|y|^a \nabla u\big) \quad \text{in } \Omega \subset \mathbb{R}^{n+1}. \tag{3.5}$$

In a series of papers [11,10], Fabes, Jerison, Kenig, and Serapioni developed a systematic theory for this class of operators: existence of weak solutions, Sobolev embeddings, Poincaré inequality, Harnack inequality, local solvability in Hölder spaces, and estimates on the Green's function.

In particular, as a consequence of a Poincaré inequality related to A_2 weights, they established an existence result (via the Lax–Milgram theorem) and a Hölder continuity result. The following three results for L_a as in (3.5), with $a \in (-1, 1)$ follow from results of [11], stated there for general A_2 weights. More precisely, they follow respectively from Theorem 2.2, Theorems 2.3.12 and 2.3.15 (and Remark 1 preceding it), and Lemma 2.3.5 of [11].

Theorem 3.2 (Solvability in Sobolev spaces). (See [11].) *Let $\Omega \subset \mathbb{R}^{n+1}$ be a smooth bounded domain, $h = (h_1, \ldots, h_{n+1})$ satisfy $|h|/|y|^a \in L^2(\Omega, |y|^a)$, and $g \in H^1(\Omega, |y|^a)$. Then, there exists a unique solution $u \in H^1(\Omega, |y|^a)$ of $L_a u = -\text{div}\, h$ in Ω with $u - g \in H_0^1(\Omega, |y|^a)$.*

Theorem 3.3 (Hölder local regularity). (See [11].) *Let $\Omega \subset \mathbb{R}^{n+1}$ be a smooth bounded domain and u a solution of $L_a u = -\text{div}\, h$ in Ω, where $|h|/|y|^a \in L^{2(n+1)}(\Omega, |y|^a)$. Then, u is Hölder continuous in Ω with a Hölder exponent depending only on n and a.*

Theorem 3.4 (Harnack inequality). (See [11].) *Let u be a positive solution of $L_a u = 0$ in $B_{4R}(x_0) \subset \mathbb{R}^{n+1}$. Then, $\sup_{B_R(x_0)} u \leqslant C \inf_{B_R(x_0)} u$ for some constant C depending only on n and a — and in particular, independent of R.*

As a consequence, bounded solutions of $L_a u = 0$ in all of $\mathbb{R}^{n+1}$ are constant.

The last statement is proved applying the previous Harnack inequality to $u - \inf_{\mathbb{R}^{n+1}} u$ in $B_{4R}(0)$ and letting $R \to \infty$.

Corollary 3.5. *Problems* (3.3) *and* (3.4) *admit at most one solution u with u bounded and continuous in $\overline{\mathbb{R}_+^{n+1}}$ — up to an additive constant in the case of* (3.4).

Proof. The difference w of two solutions would solve the homogeneous problem. We can then perform odd reflection for problem (3.3) and even reflection for problem (3.4), to obtain a bounded solution of $L_a w = 0$ in all of $\mathbb{R}^{n+1}$. By Theorem 3.4, w is constant, which finishes the proof.

We remark that in case of the Neumann problem (3.4), the previous Liouville and uniqueness results also follow from the Harnack inequality in $\mathbb{R}_+^{n+1}$ that we prove in Lemma 4.9 below. □

The existence of a bounded solution for (3.3) and (3.4) is stated below in this same section.

3.3. A duality principle

An important property of the operator L_a is the following duality property. It relates the Neumann problem for the operator L_a with the Dirichlet problem for L_{-a}, the operator with the inverse weight.

Proposition 3.6. (See [7].) *Assume that $h \in C(\mathbb{R}^n)$, $u \in C^2(\mathbb{R}_+^{n+1})$, and $y^a \partial_y u \in C(\overline{\mathbb{R}_+^{n+1}})$. If u is a classical solution of*

$$\begin{cases} \text{div}\big(y^a \nabla u\big) = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ \dfrac{\partial u}{\partial \nu^a} = h & \text{on } \partial \mathbb{R}_+^{n+1}, \end{cases}$$

then $w = -y^a \partial_y u$ is a classical solution of

$$\begin{cases} \operatorname{div}(y^{-a} \nabla w) = 0 & in \ \mathbb{R}^{n+1}_+, \\ w = h & on \ \partial \mathbb{R}^{n+1}_+. \end{cases}$$

The previous duality property is related, in dimension two, to some generalized Cauchy–Riemann conditions that we describe. Indeed, writing $L_a u = 0$ in $\mathbb{R}^2_+$ as

$$\partial_x(y^a \partial_x u) + \partial_y(y^a \partial_y u) = 0,$$

we see that the associated conjugate function $\tilde{u}$ is such that

$$\begin{cases} y^a \partial_y u = \partial_x \tilde{u}, \\ -y^a \partial_x u = \partial_y \tilde{u}, \end{cases}$$

hence satisfying generalized Cauchy–Riemann conditions. The function $\tilde{u}$ is the a-conjugate of u. Similarly, u is the $-a$ conjugate of $\tilde{u}$. Complexifying the problem by denoting $\varphi = u + i\tilde{u}$, it is easy to see that φ satisfies

$$\bar{\partial}\varphi = \nu(y)\overline{\partial\varphi} \tag{3.6}$$

where $\bar{\partial} = \partial_x + i\partial_y$ is the standard $\bar{\partial}$-operator and

$$\nu(y) = \frac{1 + y^a}{1 - y^a}.$$

Eq. (3.6) is called conjugate Beltrami equation and has been extensively studied in the Calderon problem (see [2,22, 19] and references therein).

3.4. Fundamental solutions

Concerning the operator L_a involved in the extension or Dirichlet problem for the fractional Laplacian, one has the following representation formula through a Poisson kernel.

Proposition 3.7. *(See [7].) Given $s \in (0, 1)$, let $a = 1 - 2s \in (-1, 1)$. The function*

$$P_s(x, y) = p_{n,s} \frac{y^{2s}}{(|x|^2 + y^2)^{\frac{n+2s}{2}}} = p_{n,s} \frac{y^{1-a}}{(|x|^2 + y^2)^{\frac{n+1-a}{2}}}$$

is a solution of

$$\begin{cases} \operatorname{div}(y^a \nabla P_s) = 0 & in \ \mathbb{R}^{n+1}_+, \\ P_s = \delta_0 & on \ \partial \mathbb{R}^{n+1}_+ = \mathbb{R}^n, \end{cases} \tag{3.7}$$

where δ_0 is the delta distribution at the origin, and $p_{n,s}$ is a positive constant depending only on n and s chosen such that, for all $y > 0$,

$$\int_{\mathbb{R}^n} P_s(x, y) \, dx = 1.$$

Its value is given below in Remark 3.11 (c).

Remark 3.8. As a consequence of Corollary 3.5 above, we have that for $v \in (C \cap L^\infty)(\mathbb{R}^n)$, the convolution in the x-variables

$$u(\cdot, y) = P_s(\cdot, y) * v$$

is the unique solution of (3.3) in $(C \cap L^\infty)(\overline{\mathbb{R}^{n+1}_+})$.

As a consequence, for v a bounded $C^2_{\text{loc}}(\mathbb{R}^n)$ function, v is a solution of (1.1) if and only if

$$u(\cdot, y) = P_s(\cdot, y) * v$$

is a solution of (1.3) (with f replaced by $(1 + a)d_s^{-1} f = 2(1 - s)d_s^{-1} f$) whose trace on $\partial\mathbb{R}^{n+1}_+$ is v. Recall that d_s is the constant from (1.4).

From the previous duality principle (Proposition 3.6) and the knowledge of the fundamental solution (the Poisson kernel) of the Dirichlet problem (3.3), we can find the fundamental solution of the fractional Laplacian or, equivalently, the fundamental solution of the Neumann problem (3.4).

Proposition 3.9. *(See [7].) Given $s \in (0, 1)$, let $a = 1 - 2s \in (-1, 1)$. The function*

$$\Gamma_s(x, y) = e_{n,s}|(x, y)|^{2s-n} = e_{n,s}|(x, y)|^{1-a-n}$$

is a solution of

$$\begin{cases} \operatorname{div}(y^a \nabla \Gamma_s) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ -y^a \partial_y \Gamma_s = \delta_0 & \text{on } \partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n, \end{cases} \tag{3.8}$$

where δ_0 is the delta distribution at the origin, and $e_{n,s}$ is a positive constant depending only on n and s.

As a consequence, $\Gamma_s(x, 0) = e_{n,s}|x|^{2s-n}$ is, up to a multiplicative constant, the fundamental solution of $(-\Delta)^s$ in $\mathbb{R}^n$.

Remark 3.10. As a consequence of Corollary 3.5 above, we have that for $h \in C_c(\mathbb{R}^n)$ (h continuous with compact support), the convolution in the x-variables

$$u(\cdot, y) = \Gamma_s(\cdot, y) * h$$

is the unique (up to an additive constant) solution of (3.4) in $(C \cap L^\infty)(\overline{\mathbb{R}^{n+1}_+})$. Thus, its trace

$$v = |x|^{2s-n} * h$$

is up to a multiplicative constant, the unique (up to an additive constant) continuous and bounded solution of

$$(-\Delta)^s v = h \quad \text{in } \mathbb{R}^n.$$

Remark 3.11. (a) The normalizing constant $C_{n,s}$ in (1.2) is given by

$$C_{n,s} = \pi^{-\frac{n}{2}} 2^{2s} \frac{\Gamma(\frac{n+2s}{2})}{-\Gamma(-s)} = \pi^{-\frac{n}{2}} 2^{2s} \frac{\Gamma(\frac{n+2s}{2})}{\Gamma(1-s)} s = \pi^{-\frac{n}{2}} 2^{2s} \frac{\Gamma(\frac{n+2s}{2})}{\Gamma(2-s)} s(1-s).$$

In particular, up to positive multiplicative constants, $C_{n,s}$ behaves as s and $1 - s$ for $s \downarrow 0$ and $s \uparrow 1$, respectively.

(b) Let us also make some comments on the constant d_s in the Caffarelli–Silvestre extension problem — see Theorem 3.1. Its value is given by

$$d_s = 2^{2s-1} \frac{\Gamma(s)}{\Gamma(1-s)}.$$

The fact that this constant does not depend on n is already proved in Section 3.2 of [7]. Its precise value appears in several papers; see e.g. [22,12]. Using that $s\Gamma(s) = \Gamma(s+1)$ and $(1-s)\Gamma(1-s) = \Gamma(2-s)$, we deduce, respectively, that

$$\frac{d_s}{(2s)^{-1}} \to 1 \quad \text{as } s \downarrow 0 \quad \text{and} \quad \frac{d_s}{2(1-s)} \to 1 \quad \text{as } s \uparrow 1. \tag{3.9}$$

When u solves problem (1.3), its boundary condition $(1 + a)\partial_{v_a} u = 2(1 - s)\partial_{v_a} u = f(u)$ gives that the trace of u solves

$$(-\Delta)^s\{u(\cdot, 0)\} = \frac{d_s}{2(1-s)} f(u(\cdot, 0)).$$

(c) The value of the constant in front of the Poisson kernel is given by

$$p_{n,s} = \pi^{-\frac{n}{2}} \frac{\Gamma(\frac{n+2s}{2})}{\Gamma(s)};$$

see Remark 2.2 in [22].

4. Preliminary results on elliptic problems involving L_a: Schauder estimates, maximum principles, and a Liouville theorem

This section is devoted to the proof of several general results concerning problem (1.3). The following definition provides the concept of weak solution for (1.3). More generally, we consider the problem

$$\begin{cases} \operatorname{div}(y^a \nabla u) = 0 & \text{in } B_R^+, \\ -y^a u_y = h & \text{on } \Gamma_R^0, \end{cases} \tag{4.1}$$

where we have used the notation introduced in the beginning of Section 2.

Definition 4.1. Given $R > 0$ and a function $h \in L^1(\Gamma_R^0)$, we say that u is a weak solution of (4.1) if

$$y^a |\nabla u|^2 \in L^1(B_R^+)$$

and

$$\int_{B_R^+} y^a \nabla u \cdot \nabla \xi - \int_{\Gamma_R^0} h\xi = 0 \tag{4.2}$$

for all $\xi \in C^1(\overline{B_R^+})$ such that $\xi \equiv 0$ on Γ_R^+.

Remark 4.2. The (weak) maximum principle holds for weak solutions of (4.1). More generally, if u weakly solves

$$\begin{cases} -\operatorname{div}(y^a \nabla u) \geqslant 0 & \text{in } B_R^+, \\ -y^a u_y \geqslant 0 & \text{on } \Gamma_R^0, \\ u \geqslant 0 & \text{on } \Gamma_R^+ \end{cases} \tag{4.3}$$

(in the corresponding sense to the previous definition by taking nonnegative test functions), then $u \geqslant 0$ in B_R^+. This is proved simply multiplying the weak formulation by the negative part u^- of u.

In addition, one has the strong maximum principle: either $u \equiv 0$ or $u > 0$ in $B_R^+ \cup \Gamma_R^0$. That u cannot vanish at an interior point follows from the classical strong maximum principle for strictly elliptic operators. That u cannot vanish at a point in Γ_R^0 follows from the Hopf principle that we establish below (see Proposition 4.11) or by the strong maximum principle of [11].

Note that the same weak and strong maximum principles (and proofs) hold in other bounded domains of $\mathbb{R}_+^{n+1}$ different than B_R^+. It also holds for the Dirichlet problem in B_R^+, i.e., replacing the Neumann condition in (4.3) by $u \geqslant 0$ on Γ_R^0.

4.1. Schauder estimates

In the following, we prove several estimates for solutions of (1.1) and (1.3), as well as for solutions of the Neumann problem (4.1) in B_R^+.

Remark 4.3. Note that the function $u(x, y) = y^{1-a}$ solves $L_a u = 0$. Therefore, one cannot expect the Hölder regularity in y to be higher than $C^{\min(1,1-a)}$ up to the boundary $\{y = 0\}$. Thus, for $s < 1/2$ (i.e. $a > 0$), there are solutions of $L_a u = 0$ vanishing on $\{y = 0\}$ which are not C^1 up to $\{y = 0\}$.

The next lemmas provide several regularity results. We start with estimates on the nonlocal equation. Here we use results from [21] and the equivalent notions of bounded solutions described there.

Lemma 4.4. *Let f be a $C^{1,\gamma}(\mathbb{R})$ function with $\gamma > \max(0, 1 - 2s)$. Then, any bounded solution of*

$$(-\Delta)^s v = f(v) \quad in \ \mathbb{R}^n$$

is $C^{2,\beta}(\mathbb{R}^n)$ for some $0 < \beta < 1$ depending only on s and γ.

Furthermore, given $s_0 > 1/2$ there exists $0 < \beta < 1$ depending only on n, s_0, and γ — and hence independent of s — such that for every $s > s_0$,

$$\|v\|_{C^{2,\beta}(\mathbb{R}^n)} \leqslant C$$

for some constant C depending only on n, s_0, $\|f\|_{C^{1,\gamma}}$, and $\|v\|_{L^\infty(\mathbb{R}^n)}$ — and hence independent of $s \in (s_0, 1)$.

In addition, the function defined by $u = P_s * v$ (where P_s is the Poisson kernel in Proposition 3.7) satisfies for every $s > s_0$,

$$\|u\|_{C^\beta(\overline{\mathbb{R}^{n+1}_+})} + \|\nabla_x u\|_{C^\beta(\overline{\mathbb{R}^{n+1}_+})} + \|D^2_x u\|_{C^\beta(\overline{\mathbb{R}^{n+1}_+})} \leqslant C$$

for some constant C independent of $s \in (s_0, 1)$, indeed depending only on the same quantities as the previous one.

Proof. Since v is bounded, $f(v)$ is also bounded. Applying Proposition 2.9 in [21], we have

- If $s \leqslant 1/2$, then for any $\alpha < 2s$, $v \in C^{0,\alpha}(\mathbb{R}^n)$.
- If $s > 1/2$, then for any $\alpha < 2s - 1$, $v \in C^{1,\alpha}(\mathbb{R}^n)$.

This implies in particular that $f(v)$ is $C^\alpha(\mathbb{R}^n)$. Applying now Proposition 2.8 in [21], we have

- If $\alpha + 2s \leqslant 1$, then $v \in C^{0,\alpha+2s}(\mathbb{R}^n)$.
- If $\alpha + 2s > 1$, then $v \in C^{1,\alpha+2s-1}(\mathbb{R}^n)$.

Therefore, iterating the procedure a finite number of times, one gets that $v \in C^{1,\sigma}(\mathbb{R}^n)$ for some $\sigma \in (0, 1)$ depending only on s. Indeed, if $\alpha + 2s > 1$, then one can take $\sigma = \alpha + 2s - 1$. On the other hand if $\alpha + 2s \leqslant 1$, we have that $f(v)$ is $C^{0,\alpha+2s}$. As a consequence, one gets that v is $C^{0,\alpha+4s}$. Hence iterating a finite number of times, we will end up with $\alpha + k2s > 1$ for some integer k. This gives the $C^{1,\sigma}$ regularity.

We now differentiate the equation to obtain

$$(-\Delta)^s v_{x_i} = f'(v) v_{x_i} \quad in \ \mathbb{R}^n$$

for $i = 1, \ldots, n$, with v_{x_i} and $f'(v)$ belonging to $C^{0,\sigma}(\mathbb{R}^n)$ provided we take $\sigma < \gamma$. Therefore, applying Proposition 2.8 of [21] we obtain that $v_{x_i} \in C^{0,\sigma+2s}(\mathbb{R}^n)$. We iterate this procedure a finite number of times (as long as the Hölder exponent is smaller than γ). Now, since by assumption $\gamma + 2s > 1$, we finally arrive at $v_{x_i} \in C^{1,\beta}(\mathbb{R}^n)$, and thus $v \in C^{2,\beta}(\mathbb{R}^n)$ for some $\beta > 0$ depending only on s and γ.

For the second point of the lemma, we write the nonlocal equation as

$$-\Delta v = (-\Delta)^{1-s} f(v) \quad in \ \mathbb{R}^n.$$

A careful look at the proof of Proposition 2.5 in [21] shows the following. Given $s_0 > 1/2$, if $0 < 2 - 2s_0 < \alpha < 1$, the operator $(-\Delta)^{1-s}$ maps $C^\alpha(\mathbb{R}^n)$ into $C^{\alpha-2+2s}(\mathbb{R}^n)$ continuously with a constant independent of s. Here we use that the constant $C_{n,s}$ in (1.2) is uniformly bounded as $s \uparrow 1$ — see Remark 3.11. As a consequence, applying $C^{2,\beta}$ estimates for Poisson equation, we deduce that $v \in C^{2,\alpha-2+2s_0}(\mathbb{R}^n) = C^{2,\beta}(\mathbb{R}^n)$ and a $C^{2,\beta}$ estimate with a constant independent of s — indeed depending on the quantities in the statement of the lemma.

We now come to the last point of the lemma. Let $v \in (L^\infty \cap C^\beta)(\mathbb{R}^n)$ for some $\beta \in (0, \min(1, 2s_0)]$ — here we allow $s_0 \in (0, 1)$. We claim that there exists a constant C depending on n and s_0, independent of $s > s_0$, such that the function u defined by

$$u(\cdot, y) = P_s(\cdot, y) * v$$

is $(L^\infty \cap C^\beta)(\mathbb{R}^{n+1}_+)$ with the estimate

$$\|u\|_{C^\beta(\overline{\mathbb{R}^{n+1}_+})} \leqslant C\|v\|_{C^\beta(\mathbb{R}^n)}.$$

Applying this fact to v, v_{x_i} and $v_{x_i x_j}$, we conclude the statement of the lemma. To prove the claim, the Poisson kernel P_s writes

$$P_s(x, y) = \frac{1}{y^n} H_s\left(\frac{x}{y}\right)$$

where

$$H_s(\xi) = p_{n,s} \frac{1}{(1 + |\xi|^2)^{\frac{n+2s}{2}}}.$$

The constant $p_{n,s}$ is such that $\int_{\mathbb{R}^n} H_s(\xi)\, d\xi = 1$ and therefore it is bounded uniformly in s, for $s > s_0$ — alternatively, use the formula in Remark 3.11 (c). We have

$$u(x, y) = \int_{\mathbb{R}^n} v(x - y\xi) H_s(\xi)\, d\xi$$

and then

$$\left| u(x_1, y_1) - u(x_2, y_2) \right| = \left| \int_{\mathbb{R}^n} \{v(x_1 - y_1\xi) - v(x_2 - y_2\xi)\} H_s(\xi)\, d\xi \right|$$

$$\leqslant C\left\{ |x_1 - x_2|^\beta + |y_1 - y_2|^\beta \int_{\mathbb{R}^n} H_s(\xi)|\xi|^\beta\, d\xi \right\} \|v\|_{C^\beta(\mathbb{R}^n)}.$$

Thus we deduce the desired result taking $\beta < 2s_0 < 2s$.   □

The next lemma provides estimates for solutions of the Neumann problem in a half-ball.

Lemma 4.5. *Let $a \in (-1, 1)$ and $R > 0$. Let $\varphi \in C^\sigma(\Gamma^0_{2R})$ for some $\sigma \in (0, 1)$ and $u \in L^\infty(B^+_{2R}) \cap H^1(B^+_{2R}, y^a)$ be a weak solution of*

$$\begin{cases} L_a u = 0 & \text{in } B^+_{2R} \subset \mathbb{R}^{n+1}_+, \\ \dfrac{\partial u}{\partial \nu^a} = \varphi & \text{on } \Gamma^0_{2R}. \end{cases}$$

Then, there exists $\beta \in (0, 1)$ depending only on n, a, and σ, such that $u \in C^{0,\beta}(\overline{B^+_R})$ and $y^a u_y \in C^{0,\beta}(\overline{B^+_R})$.

Furthermore, there exist constants C^1_R and C^2_R depending only on n, a, R, $\|u\|_{L^\infty(B^+_{2R})}$ and also on $\|\varphi\|_{L^\infty(\Gamma^0_{2R})}$ (for C^1_R) and $\|\varphi\|_{C^\sigma(\Gamma^0_{2R})}$ (for C^2_R), such that

$$\|u\|_{C^{0,\beta}(\overline{B^+_R})} \leqslant C^1_R$$

and

$$\|y^a u_y\|_{C^{0,\beta}(\overline{B^+_R})} \leqslant C^2_R.$$

Proof. Multiply φ by a cut-off function identically 1 in $\Gamma^0_{3R/2}$ and call it $\bar\varphi$. Thus $\bar\varphi$ is $C^\sigma_c(\mathbb{R}^n)$ and $\bar\varphi \equiv \varphi$ in $\Gamma^0_{3R/2}$. Let $\bar u$ be the solution of

$$\begin{cases} L_a \bar u = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \dfrac{\partial \bar u}{\partial \nu^a} = \bar\varphi & \text{on } \partial\mathbb{R}^{n+1}_+ \end{cases}$$

given in Remark 3.10. We have that $\bar u$ is continuous and bounded.

Then, the function $\bar{u}|_{\partial\mathbb{R}^{n+1}_+}$ solves

$$(-\Delta)^s \bar{u} = d_s \bar{\varphi} \quad \text{in } \mathbb{R}^n$$

and by Proposition 2.9 in [21], we have that $\bar{u} \in C^{0,\beta}(\mathbb{R}^n)$ for some $\beta \in (0,1)$ depending only on n and $s = \frac{1-a}{2}$.

We consider now $w = u - \bar{u}$. Then, w is a bounded function and a weak solution of

$$\begin{cases} L_a w = 0 & \text{in } B^+_{3R/2} \subset \mathbb{R}^{n+1}_+, \\ \dfrac{\partial w}{\partial \nu^a} = 0 & \text{on } \Gamma^0_{3R/2}. \end{cases}$$

Reflecting evenly the function w with respect to $\{y = 0\}$, the reflected function $\tilde{w}$ satisfies in the weak sense the problem

$$\text{div}\big(|y|^a \nabla \tilde{w}\big) = 0 \quad \text{in } B_{3R/2} \subset \mathbb{R}^{n+1}.$$

This fact is verified with a classical computation using test functions and integration by parts. Now, since the weight $|y|^a$ is A_2 and the function $\tilde{w}$ is bounded, the regularity theory in [11] (see Theorem 3.3 above) ensures that $\tilde{w}$ is $C^{0,\beta}$ for some $\beta \in (0,1)$ depending only on n and s.

Putting these two results together ensures that $u \in C^{0,\beta}(\overline{B^+_R})$ for some $\beta \in (0,1)$ depending on n and s. Furthermore, we get the estimate

$$\|u\|_{C^{0,\beta}(\overline{B^+_R})} \leqslant C^1_R$$

for some constant C^1_R as in the statement of the lemma (depending on $\|\varphi\|_{L^\infty}$ and not on $\|\varphi\|_{C^\sigma}$).

On the other hand, by Proposition 3.6, $\psi(x,y) := -y^a u_y(x,y)$ satisfies

$$\begin{cases} L_{-a}\psi = 0 & \text{in } B^+_{2R}, \\ \psi = \varphi & \text{on } \Gamma^0_{2R}. \end{cases}$$

We introduce

$$\tilde{\psi}(x,y) = P_{\bar{s}}(\cdot, y) * \bar{\varphi},$$

where $\bar{s}$ is such that $1 - 2\bar{s} = -a$. Recall that $\bar{\varphi} \in C^\sigma(\mathbb{R}^n)$ has compact support. Thus, $\tilde{\psi}$ is bounded and $C^\beta(\overline{\mathbb{R}^{n+1}_+})$ if $\beta \leqslant \min(\sigma, 2\bar{s})$ (for this, recall the argument on convolutions at the end of the proof of Lemma 4.4).

Now, we have that the odd reflection of the function $\overline{\psi} = \psi - \tilde{\psi}$ satisfies in weak sense

$$\text{div}\big(|y|^{-a} \nabla \overline{\psi}\big) = 0 \quad \text{in } B_{3R/2} \subset \mathbb{R}^{n+1}.$$

Hence by the results in [11] (see Theorem 3.3 above), $\overline{\psi}$ is C^β for some $\beta \in (0,1)$, depending only on n and s (perhaps different than the previous β). This and the above fact on $\tilde{\psi}$ give the desired result and estimates for $\psi = -y^a u_y = \overline{\psi} + \tilde{\psi}$. $\quad\square$

4.2. Gradient estimates and integrability at infinity

The following two results concern bounds for solutions of problem (1.3).

Proposition 4.6. *Let f be a $C^{1,\gamma}(\mathbb{R})$ function with $\gamma > \max(0, 1 - 2s)$ and $u \in L^\infty(\mathbb{R}^{n+1}_+)$ a weak solution of problem* (1.3).

Then, $\nabla_x u$ and $y^a \partial_y u$ belong to $L^\infty(\mathbb{R}^{n+1}_+)$. In addition, given $s_0 > 1/2$, there exists a constant C_1 depending only on n, s_0, $\|f\|_{C^{1,\gamma}}$ and $\|u\|_{L^\infty(\mathbb{R}^{n+1}_+)}$, such that for every $s > s_0$, we have

$$\|\nabla_x u\|_{L^\infty(\mathbb{R}^{n+1}_+)} + (1+a)\big\|y^a \partial_y u\big\|_{L^\infty(\mathbb{R}^{n+1}_+)} \leqslant C_1. \tag{4.4}$$

Furthermore, we have

$$\left|\nabla u(x, y)\right| \leqslant \frac{C_2}{y} \quad \text{for } y > 0, \tag{4.5}$$

where the constant C_2 is uniformly bounded for $a \in (-1, 1)$.

As a consequence of (4.4)*, we have*

$$y^a |\nabla u|^2 \in L^1_{\text{loc}}\left(\overline{\mathbb{R}^{n+1}_+}\right).$$

Proof. The bound $\|\nabla_x u\|_{L^\infty(\mathbb{R}^{n+1}_+)}$ in (4.4) follows from Lemma 4.4 and the fact that

$$\frac{d_s}{2(1-s)} = \frac{d_s}{1+a} \to 1 \quad \text{as } s \to 1;$$

see Remark 3.11. The bound on $(1+a)\|y^a \partial_y u\|_{L^\infty(\mathbb{R}^{n+1}_+)}$ in (4.4) follows from duality (see Proposition 3.6) and the boundary condition

$$-(1+a)y^a \partial_y u|_{y=0} = f(u) \in L^\infty(\mathbb{R}^n).$$

The last bound (4.5) follows from rescaling the equation $L_a u = 0$ in $B_{y_0/2}(x_0, y_0)$ to the same equation for $\bar{u}(x', y') = u(x_0 + y_0 x', y_0 y')$ in $B_{1/2}(0, 1)$. Then we use that the operator L_a is uniformly elliptic and has Lipschitz coefficients $(y')^a$ with constants independent of $a \in (-1, 1)$ — since $1/2 < y' < 3/2$ in this ball. $\quad \square$

The following result is concerned with solutions of (1.3) with limits in one Euclidean variable, or in the radial variable, at infinity.

Remark 4.7. If u is a layer solution of (1.3), then not only $u_{x_1}(x, 0) > 0$ but $u_{x_1}(x, y) > 0$ for every $y \geqslant 0$. Indeed, since $L_a u_{x_1} = 0$ in $\mathbb{R}^{n+1}_+$ and u_{x_1} is bounded and continuous in $\overline{\mathbb{R}^{n+1}_+}$ (by Lemma 4.4), Remark 3.8 gives that $u_{x_1}(\cdot, y)$ is the convolution of $P_s(\cdot, y)$ with $u_{x_1}(\cdot, 0) > 0$. Hence the result follows.

Lemma 4.8. (i) *Let u be a bounded solution of* (1.3) *such that*

$$\lim_{x_1 \to \pm\infty} u(x, 0) = L^\pm \tag{4.6}$$

for every $(x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$ and some constants $L^\pm$. Then,

$$f(L^+) = f(L^-) = 0 \tag{4.7}$$

and

$$\lim_{x_1 \to \pm\infty} u(x, y) = L^\pm \tag{4.8}$$

for every $(x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$ and $y \geqslant 0$. Moreover, for every fixed $R > 0$ and $(x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$, we have

$$\left\|u - L^\pm\right\|_{L^\infty(B_R^+(x,0))} \to 0 \quad \text{as } x_1 \to \pm\infty, \tag{4.9}$$

$$\left\|\nabla_x u\right\|_{L^\infty(B_R^+(x,0))} \to 0 \quad \text{as } x_1 \to \pm\infty, \tag{4.10}$$

and

$$\left\|y^a u_y\right\|_{L^\infty(B_R^+(x,0))} \to 0 \quad \text{as } x_1 \to \pm\infty. \tag{4.11}$$

(ii) *Let u be a radial solution of* (1.3) *such that*

$$\lim_{|x| \to \infty} u(|x|, 0) = 0. \tag{4.12}$$

Then,

$$f(0) = 0. \tag{4.13}$$

Moreover, for every fixed $R > 0$, we have

$$\|u\|_{L^\infty(B_R^+(x,0))} + \|\nabla_x u\|_{L^\infty(B_R^+(x,0))} + \|y^a u_y\|_{L^\infty(B_R^+(x,0))} \to 0 \quad as \ |x| \to \infty. \tag{4.14}$$

Proof. As in Lemma 2.4 in [5] for the half-Laplacian, the lemma follows easily by a compactness argument and the invariance of the problem under translations in x_1. Indeed, in both cases (i) and (ii) of the lemma, one considers the family of translated (or slided) solutions in the x_1-variable, $u^t(x, y) = u(x_1 + t, x_2, \ldots, x_n, y)$. In the radial case (ii) we proceed like this in each Euclidean variable, not only the x_1.

By the Hölder estimates of Lemma 4.5, the translated solutions converge locally uniformly and up to subsequences, to a solution of the same problem (1.3). By assumption (4.6) or (4.12), such limit is identically constant. From this, (4.7), (4.13), (4.8), (4.9) and (4.11) follow immediately. Finally, the C^β estimate for $\nabla_x u$ of Lemma 4.4 leads to (4.10) or (4.14). □

4.3. A Harnack inequality

The following Harnack inequality for linear Neumann problems will be useful in the study of stable solutions of (1.1). It has also been proved in [23] for the more general equation $\partial_{v^a}\varphi + d(x)\varphi = b(x)$.

Lemma 4.9. *Let $\varphi \in H^1(B_{4R}^+, y^a)$ be a nonnegative weak solution of*

$$\begin{cases} L_a\varphi = 0 & in \ B_{4R}^+ \subset \mathbb{R}_+^{n+1}, \\ \dfrac{\partial\varphi}{\partial v^a} + d(x)\varphi = 0 & on \ \Gamma_{4R}^0, \end{cases}$$

where d is a bounded function in Γ_{4R}^0. Then,

$$\sup_{B_R^+} \varphi \leqslant C_R \inf_{B_R^+} \varphi, \tag{4.15}$$

for some constant C_R depending only on n, a, and $R^{1-a}\|d\|_{L^\infty(\Gamma_{4R}^0)}$.

Proof. By scaling, one can assume $R = 1$. We introduce the new function

$$\Psi^A(x, y) = e^{Ay^{1-a}}\varphi(x, y).$$

The function Ψ^A satisfies

$$\begin{cases} \operatorname{div}(y^a\nabla(e^{-Ay^{1-a}}\Psi^A)) = 0 & in \ B_4^+, \\ \dfrac{\partial\Psi^A}{\partial v^a} = -(A(1-a) + d(x))\Psi^A & on \ \Gamma_4^0. \end{cases}$$

Therefore, choosing

$$A = \frac{\|d\|_{L^\infty(\Gamma_4^0)}}{1-a},$$

we have $\partial_{v^a}\Psi^A \leqslant 0$ on Γ_4^0. We consider the even extension of Ψ^A across Γ_4^0, defined by

$$\tilde{\Psi}^A(x, y) = \Psi^A(x, -y) \quad for \ (x, y) \in B_4, \ y \leqslant 0.$$

Since $\partial_{v^a}\Psi^A \leqslant 0$ on Γ_4^0, $\tilde{\Psi}^A$ satisfies

$$-\operatorname{div}(|y|^a\nabla(e^{-A|y|^{1-a}}\tilde{\Psi}^A)) \leqslant 0 \quad in \ B_4$$

in the weak sense.

Next, taking $-A$ we obtain $\partial_{y^a} \Psi^{-A} \geqslant 0$ on Γ_4^0, and arguing as before we deduce that $\tilde\Psi^{-A}$ satisfies

$$- \operatorname{div}\big(|y|^a \nabla\big(e^{A|y|^{1-a}} \tilde\Psi^{-A}\big)\big) \geqslant 0 \quad \text{in } B_4$$

in the weak sense.

Denote by $\mathcal{L}_A$ the operator

$$\mathcal{L}_A w := \operatorname{div}\big(|y|^a \nabla\big(e^{-A|y|^{1-a}} w\big)\big).$$

We introduce now the solutions $h^{\pm A}$ of

$$\begin{cases} \mathcal{L}_{\pm A} h^{\pm A} = 0 & \text{in } B_4, \\ h^{\pm A} = \tilde\Psi^{\pm A} & \text{on } \partial B_4. \end{cases}$$

These solutions are obtained from the solutions of the Dirichlet problem for L_a given by Theorem 3.2, after multiplying $h^{\pm A}$ by $e^{\pm A|y|^{1-a}}$. By the weak maximum principle and the previous considerations, we have that

$$\tilde\Psi^A \leqslant h^A \quad \text{and} \quad h^{-A} \leqslant \tilde\Psi^{-A} \quad \text{in } B_4. \tag{4.16}$$

On the other hand, since $\Psi^A / \Psi^{-A} = e^{2Ay^{1-a}} \leqslant e^{2A4^{1-a}} \leqslant e^{32A}$ in B_4^+, we have that $\tilde\Psi^A \leqslant e^{32A}\tilde\Psi^{-A}$ on ∂B_4. Next, since $\mathcal{L}_A h^A = 0 = \mathcal{L}_{-A} h^{-A} = \mathcal{L}_A(e^{2A|y|^{1-a}} h^{-A})$ and on the boundary ∂B_4, $h^A = \tilde\Psi^A \leqslant e^{32A}\tilde\Psi^{-A} \leqslant e^{32A} e^{2A|y|^{1-a}} \tilde\Psi^{-A} = e^{32A}\{e^{2A|y|^{1-a}} h^{-A}\}$, the weak maximum principle for the operator $\mathcal{L}_A$ leads to

$$h^A \leqslant e^{32A}\big\{e^{2A|y|^{1-a}} h^{-A}\big\} \leqslant e^{64A} h^{-A} \quad \text{in } B_4. \tag{4.17}$$

Next, note that $L_a(e^{-A|y|^{1-a}} h^A) = 0$ in B_4. According to the Harnack inequality of Fabes, Kenig, and Serapioni, Lemma 2.3.5 of [11] (Theorem 3.4 above), we deduce that

$$\sup_{B_1}\big(e^{-A|y|^{1-a}} h^A\big) \leqslant C \inf_{B_1}\big(e^{-A|y|^{1-a}} h^A\big)$$

for some constant C depending only on n and a. Thus,

$$\sup_{B_1^+} h^A \leqslant C e^A \inf_{B_1^+} h^A. \tag{4.18}$$

Using (4.16), (4.17), and (4.18), we deduce

$$\varphi \leqslant \tilde\Psi^A \leqslant h^A \leqslant e^{64A} h^{-A} \leqslant e^{64A}\tilde\Psi^{-A} \leqslant e^{64A}\varphi \quad \text{in } B_1^+. \tag{4.19}$$

Finally, (4.19) and (4.18) lead immediately to the desired result. $\quad\square$

4.4. A Liouville theorem

We prove the following theorem, which will be useful in [4] to prove a symmetry result for stable solutions of (1.3) in $\mathbb{R}_+^2$. This is a generalization to degenerate elliptic equations of the Liouville theorem given in [5]. This type of result had been already used in [1] to prove the De Giorgi conjecture for reactions in the interior in three dimensional spaces.

Theorem 4.10. *Let $\varphi \in L^\infty_{\mathrm{loc}}(\overline{\mathbb{R}_+^{n+1}})$ be a positive function. Suppose that $\sigma \in H^1_{\mathrm{loc}}(\overline{\mathbb{R}_+^{n+1}}, y^a)$ is such that*

$$\begin{cases} -\sigma \operatorname{div}(y^a \varphi^2 \nabla\sigma) \leqslant 0 & \text{in } \mathbb{R}_+^{n+1}, \\ -\sigma y^a \partial_y \sigma \leqslant 0 & \text{on } \partial\mathbb{R}_+^{n+1} \end{cases} \tag{4.20}$$

in the weak sense. Assume that for every $R > 1$,

$$\int\limits_{B_R^+} y^a (\sigma\varphi)^2 \, dx \, dy \leqslant C R^2 \tag{4.21}$$

for some constant C independent of R.

Then σ is constant.

Proof. We adapt the proof given in [5]. Let ζ be a C^∞ function on $[0, +\infty)$ such that $0 \leqslant \zeta \leqslant 1$ and

$$\zeta = \begin{cases} 1 & \text{for } 0 \leqslant t \leqslant 1, \\ 0 & \text{for } t \geqslant 2. \end{cases}$$

For $R > 1$ and $(x, y) \in \mathbb{R}_+^{n+1}$, let $\zeta_R(x, y) = \zeta(r/R)$, where $r = |(x, y)|$.

Multiplying (4.20) by ζ_R^2 and integrating by parts in $\mathbb{R}_+^{n+1}$, we obtain

$$\int\limits_{\mathbb{R}_+^{n+1}} y^a \zeta_R^2 \varphi^2 |\nabla \sigma|^2 \, dx \, dy \leqslant -2 \int\limits_{\mathbb{R}_+^{n+1}} y^a \zeta_R \varphi^2 \sigma \nabla \zeta_R \nabla \sigma \, dx \, dy$$

$$\leqslant 2 \left[\int\limits_{\mathbb{R}_+^{n+1} \cap \{R < r < 2R\}} y^a \zeta_R^2 \varphi^2 |\nabla \sigma|^2 \, dx \, dy \right]^{1/2} \cdot$$

$$\cdot \left[\int\limits_{\mathbb{R}_+^{n+1}} y^a \varphi^2 \sigma^2 |\nabla \zeta_R|^2 \, dx \, dy \right]^{1/2}$$

$$\leqslant C \left[\int\limits_{\mathbb{R}_+^{n+1} \cap \{R < r < 2R\}} y^a \zeta_R^2 \varphi^2 |\nabla \sigma|^2 \, dx \, dy \right]^{1/2} \cdot$$

$$\cdot \left[\frac{1}{R^2} \int\limits_{B_{2R}^+} y^a (\varphi \sigma)^2 \, dx \, dy \right]^{1/2},$$

for some constant C independent of R. Using hypothesis (4.21), we infer that

$$\int\limits_{\mathbb{R}_+^{n+1}} y^a \zeta_R^2 \varphi^2 |\nabla \sigma|^2 \, dx \, dy \leqslant C \left[\int\limits_{\mathbb{R}_+^{n+1} \cap \{R < r < 2R\}} y^a \zeta_R^2 \varphi^2 |\nabla \sigma|^2 \, dx \, dy \right]^{1/2}, \tag{4.22}$$

again with C independent of R. Hence, $\int_{\mathbb{R}_+^{n+1}} y^a \zeta_R^2 \varphi^2 |\nabla \sigma|^2 \, dx \, dy \leqslant C$ and, letting $R \to \infty$, we deduce $\int_{\mathbb{R}_+^{n+1}} y^a \varphi^2 |\nabla \sigma|^2 \, dx \, dy \leqslant C$. It follows that the right hand side of (4.22) tends to zero as $R \to \infty$, and therefore $\int_{\mathbb{R}_+^{n+1}} y^a \varphi^2 |\nabla \sigma|^2 \, dx \, dy = 0$. We conclude that σ is constant. $\square$

4.5. A Hopf principle

The following proposition provides a Hopf boundary lemma in our context.

Proposition 4.11. *Let $a \in (-1, 1)$ and consider the cylinder $\mathcal{C}_{R,1} = \Gamma_R^0 \times (0, 1) \subset \mathbb{R}_+^{n+1}$ where Γ_R^0 is the ball of center 0 and radius R in $\mathbb{R}^n$. Let $u \in C(\overline{\mathcal{C}_{R,1}}) \cap H^1(\mathcal{C}_{R,1}, y^a)$ satisfy*

$$\begin{cases} L_a u \leqslant 0 & \text{in } \mathcal{C}_{R,1}, \\ u > 0 & \text{in } \mathcal{C}_{R,1}, \\ u(0, 0) = 0. \end{cases}$$

Then,

$$\limsup_{y \to 0^+} -y^a \frac{u(0, y)}{y} < 0.$$

In addition, if $y^a u_y \in C(\overline{\mathcal{C}_{R,1}})$, then

$$\partial_{y^a} u(0, 0) < 0.$$

Proof. Consider the function on $\mathcal{C}_{R,1}$ defined by

$$w_A(x, y) = y^{-a}(y + Ay^2)\varphi(x),$$

where A is a constant to be chosen later and $\varphi = \varphi(x)$ is the first eigenfunction of $-\Delta_x$ in $\Gamma_{R/2}^0$ with Dirichlet boundary conditions, i.e.,

$$\begin{cases} -\Delta_x\varphi = \lambda_1\varphi & \text{in } \Gamma_{R/2}^0, \\ \varphi = 0 & \text{on } \partial\Gamma_{R/2}^0. \end{cases}$$

Notice that $\lambda_1 > 0$ and that we can choose $\varphi > 0$ in $\Gamma_{R/2}^0$ with $\|\varphi\|_{L^\infty} = 1$. The function w_A satisfies

$$\begin{cases} L_a w_A = \varphi(x)\{A(2 - a) - \lambda_1(y + Ay^2)\} & \text{in } \mathcal{C}_{R/2,1}, \\ w_A \geqslant 0 & \text{in } \overline{\mathcal{C}_{R/2,1}}, \\ w_A = 0 & \text{on } \partial\Gamma_{R/2}^0 \times [0, 1). \end{cases}$$

Therefore, choosing A large enough, we have in $\mathcal{C}_{R/2,1}$

$$L_a w_A \geqslant 0.$$

Hence, for $\varepsilon > 0$,

$$L_a(u - \varepsilon w_A) \leqslant 0 \quad \text{in } \mathcal{C}_{R/2,1}$$

and $u - \varepsilon w_A = u \geqslant 0$ on $\partial\Gamma_{R/2}^0 \times [0, 1)$. Moreover, taking $\varepsilon > 0$ small enough, we have on $\Gamma_{R/2}^0 \times \{y = 1/2\}$

$$u \geqslant \varepsilon w_A,$$

since u is continuous and positive on the closure of this set. Notice furthermore that $w_A = 0$ on $\Gamma_R^0 \times \{y = 0\}$. Thus, we have

$$\begin{cases} L_a(u - \varepsilon w_A) \leqslant 0 & \text{in } \mathcal{C}_{R/2,1/2}, \\ u - \varepsilon w_A \geqslant 0 & \text{on } \partial\mathcal{C}_{R/2,1/2}. \end{cases}$$

The weak maximum principle then implies that in $\overline{\mathcal{C}_{R/2,1/2}}$

$$u - \varepsilon w_A \geqslant 0.$$

Consequently, this leads to

$$\limsup_{y\to 0^+} -y^a\frac{u(0, y)}{y} \leqslant \varepsilon\limsup_{y\to 0^+} -y^a\frac{w_A(0, y)}{y} = -\varepsilon\varphi(0) < 0,$$

as claimed in the proposition.

Assume, in addition, $y^a u_y \in C(\overline{\mathcal{C}}_{R,1})$. Let $y_0 \leqslant 1/2$. Since $(u - \varepsilon w_A)(0, \cdot) \geqslant 0$ in $[0, y_0]$ and $(u - \varepsilon w_A)(0, 0) = 0$, we have $(u_y - \varepsilon(w_A)_y)(0, y_1) \geqslant 0$ for some $y_1 \in (0, y_0)$. Repeating this argument for a sequence of $y_0's$ tending to 0, we conclude that $-y^a u_y \leqslant -\varepsilon y^a(w_A)_y$ at a sequence of points $(0, y_j)$ with $y_j \downarrow 0$. Since we assume $y^a u_y$ continuous up to $\{y = 0\}$ and $-\varepsilon(y^a(w_A)_y)(0, y_j) \to -\varepsilon\varphi(0)$, we conclude that $\partial_{\nu^a} u(0, 0) < 0$. $\quad\square$

Corollary 4.12. *Let $a \in (-1, 1)$ and $\varepsilon > 0$. Let d be a Hölder continuous function in Γ_ε^0 and $u \in L^\infty(B_\varepsilon^+) \cap H^1(B_\varepsilon^+, y^a)$ be a weak solution of*

$$\begin{cases} L_a u = 0 & \text{in } B_\varepsilon^+, \\ u \geqslant 0 & \text{in } B_\varepsilon^+, \\ \partial_{\nu^a} u + d(x)u = 0 & \text{on } \Gamma_\varepsilon^0. \end{cases}$$

Then, $u > 0$ in $B_\varepsilon^+ \cup \Gamma_\varepsilon^0$ unless $u \equiv 0$ in B_ε^+.

Proof. We apply Lemma 4.5 to obtain that u (for this, see the proof of the lemma) and $y^a u_y$ are C^α up to the boundary. Hence the equation

$$\partial_{\nu^a} u + d(x) u = 0 \tag{4.23}$$

is satisfied pointwise on Γ_ε^0. If u is not identically 0 in B_ε^+ then $u > 0$ in B_ε^+ by the strong maximum principle for the operator L_a. Now, if $u(x_0, 0) = 0$ at some point $(x_0, 0) \in \Gamma_\varepsilon^0$, then a rescaled version of Proposition 4.11 gives $\partial_{\nu^a} u(x_0, 0) < 0$. This contradicts (4.23). □

4.6. A maximum principle

Here we present a maximum principle related to the operator L_a and to the fractional Laplacian. We will need it in our subsequent article to prove monotonicity properties for solutions in $\mathbb{R}$ with limits, as well as the uniqueness (up to translations) of layer solutions in $\mathbb{R}$. Recall that Section 3 already contained some Liouville and maximum principles for these operators.

Lemma 4.13. *Let* $u \in (C \cap L^\infty)(\overline{\mathbb{R}_+^{n+1}})$ *with* $y^a u_y \in C(\overline{\mathbb{R}_+^{n+1}})$ *satisfy*

$$\begin{cases} L_a u = 0 & in \ \mathbb{R}_+^{n+1}, \\ \dfrac{\partial u}{\partial \nu^a} + d(x) u \geqslant 0 & on \ \partial \mathbb{R}_+^{n+1}, \end{cases} \tag{4.24}$$

where d *is a bounded function, and also*

$$u(x, 0) \to 0 \quad as \ |x| \to \infty. \tag{4.25}$$

Assume that there exists a closed nonempty set $H \subset \mathbb{R}^n$ *such that* $u(x, 0) > 0$ *for* $x \in H$, *and such that* d *is continuous and nonnegative in* $\overline{\mathbb{R}^n \setminus H}$.

Then, $u > 0$ *in* $\overline{\mathbb{R}_+^{n+1}}$.

Proof. By Remark 3.8 applied to $v(x) = u(x, 0) - \inf_{\mathbb{R}^n} u(\cdot, 0) \geqslant 0$, we see that $u - \inf_{\mathbb{R}^n} u(\cdot, 0) \geqslant 0$ in $\mathbb{R}_+^{n+1}$. Thus, $\inf_{\mathbb{R}_+^{n+1}} u = \inf_{\mathbb{R}^n} u(\cdot, 0)$.

Arguing by contradiction, assume that there exists a point (x_0, y_0) in $\overline{\mathbb{R}_+^{n+1}}$ such that $u(x_0, y_0) \leqslant 0$. Then, in case $\inf_{\mathbb{R}_+^{n+1}} u = 0$, the minimum of u is achieved at (x_0, y_0). In case $\inf_{\mathbb{R}_+^{n+1}} u = \inf_{\mathbb{R}^n} u(\cdot, 0) < 0$, using that $u(x, 0) \to 0$ as $|x| \to +\infty$, there exists a point $(x_1, 0)$ at which the minimum of u is achieved. In both cases we conclude that the nonpositive minimum of u is achieved at a point (x_2, y_2).

By the strong maximum principle, we cannot have $y_2 > 0$, since u is not identically constant (recall $u(\cdot, 0) > 0$ in $H \neq \emptyset$). Thus $y_2 = 0$. According to the Hopf lemma, Proposition 4.11, and since $y^a u_y \in C(\overline{\mathbb{R}_+^{n+1}})$, we have

$$\frac{\partial u}{\partial \nu^a}(x_2, 0) < 0.$$

Since $u(x_2, 0) \leqslant 0$ then $x_2 \notin H$, and thus we have $d(x_2) \geqslant 0$. Now, using the boundary condition in (4.24) at $x = x_2$, we reach a contradiction. □

Remark 4.14. Lemma 4.13 can be stated in an equivalent way using the equation

$$(-\Delta)^s v + d(x) v \geqslant 0 \quad in \ \mathbb{R}^n$$

and assuming the same conditions on v as those for $u(\cdot, 0)$ in the previous lemma. In addition, an alternative proof of the lemma can be given using the integral expression (1.2) for $(-\Delta)^s v(x_2)$, that will be negative at a point of minimum (since v is not identically constant in the proof).

5. Hamiltonian estimates

This section is devoted to establish the main facts needed to prove Theorems 2.3 and 2.5. We start with an easy lemma that will be needed later in several occasions.

Lemma 5.1. *Let* $u \in L^\infty(\mathbb{R}^{n+1}_+)$ *be a bounded solution of* (1.3). *Then, for all* $x \in \mathbb{R}^n$, *we have* $\int_0^{+\infty} t^a |\nabla u(x,t)|^2 \, dt < \infty$. *In addition, the integral can be differentiated with respect to* $x \in \mathbb{R}^n$ *under the integral sign. Furthermore,*

$$\lim_{M \to +\infty} \int_M^{+\infty} t^a |\nabla u(x,t)|^2 \, dt = 0 \tag{5.1}$$

uniformly in $x \in \mathbb{R}^n$.

If in addition, u is either a layer solution in $\mathbb{R}$ (here $n = 1$) or u is a radial solution in $\mathbb{R}^n$ for which $\lim_{|x| \to \infty} u(|x|, 0)$ *exists, then*

$$\lim_{|x| \to \infty} \int_0^{+\infty} t^a |\nabla u(x,t)|^2 \, dt = 0. \tag{5.2}$$

Proof. The first two statements and (5.1) follow directly from the gradient bounds in Proposition 4.6. The statement (5.2) is a consequence of (5.1) and of Lemma 4.8. □

5.1. Hamiltonian equality and estimate for layer solutions

This subsection contains two lemmas. The first one establishes that the Hamiltonian is conserved for layer solutions in dimension one.

Lemma 5.2. *Let $n = 1$ and assume that u is a layer solution of* (1.3). *Then, for all $x \in \mathbb{R}$ we have* $\int_0^{+\infty} t^a |\nabla u(x,t)|^2 \, dt < \infty$ *and the Hamiltonian identity*

$$(1 + a) \int_0^{+\infty} \frac{t^a}{2} \{u_x^2(x,t) - u_y^2(x,t)\} \, dt = G(u(x,0)) - G(1). \tag{5.3}$$

As a consequence,

$$G(1) = G(-1). \tag{5.4}$$

Proof. The integrability of $t^a |\nabla u(x,t)|^2$ follows from Lemma 5.1. We now establish equality (5.3). It will be crucial that the weight in L_a does not depend on the tangential variable x.

Following [5], we consider the function

$$v(x) = \int_0^{+\infty} \frac{t^a}{2} \{u_x^2(x,t) - u_y^2(x,t)\} \, dt. \tag{5.5}$$

Lemma 5.1 allows us to differentiate under the integral in (5.5) to get

$$\frac{d}{dx} v(x) = \int_0^{+\infty} t^a (u_{xx} u_x - u_{xy} u_y)(x,t) \, dt.$$

Noticing that

$$L_a u = \partial_y (y^a u_y) + y^a u_{xx} = 0$$

and after an integration by parts (which is justified by Lemma 5.1) we have

$$\frac{d}{dx} v(x) = \lim_{y \to 0^+} y^a u_y(x, y) u_x(x, y) = \frac{1}{1 + a} \frac{d}{dx} G\big(u(x, 0)\big).$$

The function $(1 + a)v(x) - \{G(u(x, 0)) - G(1)\}$ is then constant in x. Letting $x \to +\infty$ and using Lemma 5.1, we have that this constant is actually zero. Letting now $x \to -\infty$ and using Lemma 5.1, we deduce $G(1) = G(-1)$. $\quad\square$

We have obtained that a necessary condition for the existence of a layer solution in $\mathbb{R}$ is that $G(1) = G(-1)$. The other necessary condition will follow from the following result — our Modica-type estimate for layer solutions in dimension 1 (Theorem 2.3).

Lemma 5.3. *Let $n = 1$ and assume that u is a layer solution of* (1.3). *Then, for every $y \geqslant 0$ and all $x \in \mathbb{R}$, we have*

$$(1 + a) \int_0^y \frac{t^a}{2} \{u_x^2(x, t) - u_y^2(x, t)\} \, dt < G\big(u(x, 0)\big) - G(1). \tag{5.6}$$

Proof. We introduce the function

$$w(x, y) = \int_0^y \frac{t^a}{2} \{u_x^2(x, t) - u_y^2(x, t)\} \, dt,$$

which is bounded in all $\mathbb{R}_+^2$ by Lemma 5.1. We also consider the function

$$\overline{w}(x, y) = \frac{1}{1 + a} \{G\big(u(x, 0)\big) - G(1)\} - w(x, y).$$

The function $\overline{w}$ is bounded in $\mathbb{R}_+^2$ and we need to show that $\overline{w} > 0$ in $\overline{\mathbb{R}_+^2}$.

We first derive some equations for $\overline{w}$ which will be useful in the sequel. We have, for all $y > 0$,

$$\overline{w}_y(x, y) = -\frac{y^a}{2} \big(u_x^2(x, y) - u_y^2(x, y)\big). \tag{5.7}$$

Furthermore, using $L_a u = 0$ and integrating by parts as in the previous proof, one gets for all $y > 0$

$$\overline{w}_x(x, y) = y^a u_x(x, y) u_y(x, y). \tag{5.8}$$

Using the two previous equalities and the equation $L_a u = 0$, we have for all $y > 0$

$$L_a \overline{w} = -a y^{2a-1} u_x^2 \tag{5.9}$$

and

$$L_{-a} \overline{w} = -a y^{-1} u_y^2. \tag{5.10}$$

We claim that $\overline{w}$ does not achieve its infimum at a point in $\overline{\mathbb{R}_+^2}$. We assume the contrary and reach a contradiction. Let (x_0, y_0) be a point where the infimum is achieved. There are now two cases depending if y_0 is on the boundary or not. We will also use that $\overline{w}$ is not identically constant. Indeed, if it were, since $w(\cdot, 0) \equiv 0$ then

$$\text{constant} = \overline{w}(\cdot, 0) = \frac{1}{1 + a} \{G\big(u(\cdot, 0)\big) - G(1)\}.$$

Thus G is constant in $(-1, 1)$, $f \equiv 0$ in $(-1, 1)$ and u is a bounded function satisfying (1.3) with $f \equiv 0$. Hence, after an even reflection across $\{y = 0\}$, Theorem 3.4 ensures that u is a constant, a contradiction with $u_x > 0$.

Case 1: $y_0 = 0$. After a translation in x, we may assume $x_0 = 0$. Since $x_0 = 0$ is a global minimum of $(1 + a)\overline{w}(\cdot, 0) = G(u(\cdot, 0)) - G(1)$, we have

$$0 = (d/dx) G\big(u(x, 0)\big)|_{x=0} = -f\big(u(0, 0)\big) u_x(0, 0),$$

and therefore

$$0 = -f\big(u(0,0)\big) = (1+a)\lim_{y\to 0^+} y^a u_y(0,y), \tag{5.11}$$

since u is a layer solution (i.e., $u_x(x,0) > 0$). For every $(x,y) \in \overline{\mathbb{R}^2_+}$, we have by Remark 4.7,

$$u_x(x,y) > 0 \quad \text{in } \overline{\mathbb{R}^2_+}. \tag{5.12}$$

We now divide the conclusion into two subcases. Let consider first the case $a \geqslant 0$. By (5.7), we see that $y^a \overline{w}_y$ is Hölder continuous up to $y = 0$. Since $L_a \overline{w} \leqslant 0$ by (5.9) and $\overline{w}$ is not identically a constant, we have $\overline{w} > \overline{w}(0,0)$ in $\mathbb{R}^2_+$. Thus the Hopf principle (see Proposition 4.11) gives that

$$0 > -\lim_{y\to 0^+} y^a \overline{w}_y(0,y).$$

Now, using (5.7) and (5.11), we have

$$0 > -\lim_{y\to 0^+} y^a \overline{w}_y(0,y)$$

$$= \lim_{y\to 0^+} \frac{y^{2a}}{2}\big\{u_x^2(0,y) - u_y^2(0,y)\big\}$$

$$= \lim_{y\to 0^+} \frac{y^{2a}}{2} u_x^2(0,y) \geqslant 0,$$

a contradiction.

We turn now to the case $a < 0$. Since $(0,0)$ is a global minimum for $\overline{w}(x,y)$, one gets

$$0 \geqslant \liminf_{y\to 0^+} -y^{-a} \overline{w}_y(0,y)$$

$$= \liminf_{y\to 0^+} \frac{1}{2}\big(u_x^2(0,y) - u_y^2(0,y)\big) = \frac{1}{2} u_x^2(0,0) > 0,$$

a contradiction. We have used that, by Lemma 4.5 $|u_y(0,y)| \leqslant C y^{-a} \to 0$ as $y \to 0^+$.

Case 2: $y_0 > 0$. By (5.12), we have $u_x > 0$ in $\overline{\mathbb{R}^2_+}$. Using (5.8) and (5.10), we obtain

$$0 = L_{-a}\overline{w} + a y^{-1} u_y^2 = L_{-a}\overline{w} + \left(a y^{-1-a}\frac{u_y}{u_x}\right)\overline{w}_x$$

which is the same as

$$0 = \nabla \cdot \big(y^{-a}\nabla\overline{w}\big) + b(x,y)\overline{w}_x \quad \text{in } \mathbb{R}^2_+,$$

with $b(x,y) := a y^{-1-a} u_y u_x^{-1}$. But this last operator is uniformly elliptic with continuous coefficients in compact sets of $\{y > 0\}$. Thus it cannot achieve its minimum at (x_0, y_0), since $y_0 > 0$ and we have proved that $\overline{w}$ is not identically constant.

Therefore, we now know that $\overline{w}$ cannot achieve its infimum at a point in $\overline{\mathbb{R}^2_+}$. To finish the proof, assume first

$$\inf_{\mathbb{R}^2_+} \overline{w} < 0.$$

By Lemma 5.2, $\overline{w}(x,y) \to 0$ as $y \to +\infty$ locally uniformly in x. By Lemma 5.1, we have $\overline{w}(x,y) \to 0$ as $|x| \to +\infty$ uniformly in y. Therefore, the infimum of $\overline{w}$ being negative, it should be achieved at a point in $\overline{\mathbb{R}^2_+}$, a contradiction with what we have proven. Therefore,

$$\inf_{\mathbb{R}^2_+} \overline{w} \geqslant 0,$$

i.e. $\overline{w} \geqslant 0$.

Thus, if $\overline{w}$ vanished at some point in $\overline{\mathbb{R}^2_+}$, this point would achieve the infimum of $\overline{w}$, a contradiction. Hence $w > 0$ in $\overline{\mathbb{R}^2_+}$ as stated in the lemma. $\square$

5.2. The Hamiltonian for radial solutions

The next lemma deals with bounded radial solutions u of (1.3). Here we do not assume u to have a limit at infinity.

Lemma 5.4. *Let u be a bounded solution of* (1.3). *Assume that $u = u(|x|, y)$ is radially symmetric in x. Then,*

$$(1 + a) \int_0^{+\infty} \frac{t^a}{2} \{u_r^2(r, t) - u_y^2(r, t)\} \, dt - G(u(r, 0)) \tag{5.13}$$

is a nonincreasing function of r.

Proof. The function u solves

$$\begin{cases} u_{rr} + \dfrac{n-1}{r} u_r + u_{yy} + \dfrac{a}{y} u_y = 0 & \text{in } (0, +\infty) \times (0, +\infty), \\ -(1 + a) y^a u_y = f(u) & \text{on } (0, +\infty) \times \{y = 0\}. \end{cases}$$

Let

$$w(r) := \int_0^{+\infty} \frac{t^a}{2} \{u_r^2(r, t) - u_y^2(r, t)\} \, dt.$$

By Lemma 5.1, we can differentiate under the integral with respect to r and obtain

$$\frac{dw(r)}{dr} = \int_0^{+\infty} t^a \{u_r u_{rr} - u_y u_{ry}\}(r, t) \, dt.$$

Performing one integration by parts and using the equation and Lemma 5.1, we end up with

$$\frac{dw(r)}{dr} = -\frac{n-1}{r} \int_0^{+\infty} t^a u_r^2(r, t) \, dt + \frac{1}{1+a} G'(u(r, 0)) u_r(r, 0).$$

As a consequence, the function

$$(1 + a) w(r) - G(u(r, 0))$$

is nonincreasing in r, as claimed. Furthermore,

$$\frac{d}{dr} \{(1 + a) w(r) - G(u(r, 0))\} = -(1 + a) \frac{n-1}{r} \int_0^{\infty} t^a u_r^2(r, t) \, dt. \qquad \square \tag{5.14}$$

6. The limit $s \to 1$ and the classical Laplacian

In the following, we investigate the asymptotic $s \to 1$. For this, we will use crucially the previous Hamiltonian estimates. We prove the following theorem.

Theorem 6.1. *Assume that $f \in C^{1,\gamma}(\mathbb{R})$ for some $\gamma \in (0, 1)$ and that $\{v_s\}$, with $s = s_k \in (0, 1)$ and $s_k \uparrow 1$, is a sequence of layer solutions of*

$$(-\partial_{xx})^s v_s = f(v_s) \quad \text{in } \mathbb{R},$$

such that $v_s(0) = 0$. Then, there exits a function $\bar{v}$ such that

$$\lim_{s \uparrow 1} v_s = \bar{v}$$

in the uniform C^2 convergence on every compact set of $\mathbb{R}$.

Furthermore, the function $\bar{v}$ is the layer solution of

$$-\bar{v}'' = f(\bar{v}) \quad in\ \mathbb{R}$$

with $\bar{v}(0) = 0$, and satisfies the Hamiltonian equality

$$\frac{1}{2}(\bar{v}')^2 = G(\bar{v}) - G(1) \quad in\ \mathbb{R}. \tag{6.1}$$

The previous theorem is stronger than just saying that the limit when s goes to 1 is a solution of an ODE, since it states that the limit is actually a layer itself. We can see Theorem 6.1 as a stability result in the class of layer solutions of nonlocal (and local) equations.

Proof of Theorem 6.1. Let v_s be a layer solution of

$$(-\partial_{xx})^s v_s = f(v_s) \quad \text{in } \mathbb{R}$$

with $v_s(0) = 0$. Then, the extension $u_a = P_s * v_s$ of v_s satisfies

$$\begin{cases} \operatorname{div}(y^a \nabla u_a) = 0 & \text{in } \mathbb{R}^2_+, \\ (1+a)\dfrac{\partial u_a}{\partial \nu^a} = c_a f(u_a) & \text{on } \partial \mathbb{R}^2_+, \\ u_a = v_s & \text{on } \partial \mathbb{R}^2_+, \end{cases}$$

where $c_a = d_s^{-1}(1+a) = d_s^{-1} 2(1-s)$ and d_s is the constant in Theorem 3.1 and Remark 3.11. By (3.9), we know that c_a tends to 1 as a goes to -1. The weak formulation of this problem is

$$(1+a) \int_{\mathbb{R}^2_+} y^a \nabla u_a \cdot \nabla \xi - \int_{\mathbb{R}} c_a f(u_a) \xi = 0 \tag{6.2}$$

for all $\xi \in C^1(\overline{\mathbb{R}^2_+})$ compactly supported.

First notice that, by the regularity result in Lemma 4.4, which is uniform as $s \uparrow 1$, the functions u_a and $\partial_x u_a$ converge over compact sets (up to a subsequence) to a function $u_{-1} = u_{-1}(x, y)$ and its x-derivative as $a \to -1$ (which corresponds to $s \to 1$). We now choose the following test function: $\xi(x, y) = \eta_1(x)\eta_2(y)$, where $\eta_2(y) = 1$ for $0 \leqslant y < 1$ and $\eta_2(y) = 0$ for $y > 2$, whereas η_1 is any test function. We deduce

$$(1+a) \int_{\mathbb{R}^2_+} y^a \{\eta_1'(x)\eta_2(y)\partial_x u_a + \eta_1(x)\eta_2'(y)\partial_y u_a\}\, dx\, dy$$

$$- \int_{\mathbb{R}} c_a f(u_a)\eta_1(x)\, dx = 0.$$

We now pass to the limit in each term. Thanks to the uniform bounds of Lemma 4.4, we have that

$$\lim_{a \downarrow -1} c_a \int_{\mathbb{R}} f(u_a)\eta_1(x)\, dx = \int_{\mathbb{R}} f(u_{-1}(x, 0))\eta_1(x)\, dx.$$

Note that the measure $(1+a)y^a\, dy$ is a probability measure on $(0, 1)$ converging as $a \downarrow -1$ (in the weak-* sense of measures) to the Dirac measure δ_0. More precisely, given functions $w_a = w_a(y)$ continuous in $[0, \infty)$, with $|y w_a(y)| \leqslant C$ in $[0, +\infty)$ (with C uniform in a) and with w_a converging to a function w_{-1} uniformly in compact sets of $[0, +\infty)$, then

$$\lim_{a \downarrow -1} (1+a) \int_0^\infty y^a w_a(y)\, dy = w_{-1}(0).$$

Indeed, given $\varepsilon > 0$ let $\delta > 0$ such that $|w_a(x) - w_a(0)| \leqslant \varepsilon$ for all $x \in (0, \delta)$. Then, we write

$$(1 + a) \int_0^\infty y^a w_a(y) \, dy = (1 + a) \int_0^\delta y^a w_a(y) \, dy + (1 + a) \int_\delta^\infty y^a w_a(y) \, dy.$$

We have that

$$(1 + a) \int_0^\delta y^a \big(w_{-1}(0) + w_a(0) - w_{-1}(0) \big) \, dy$$

tends to $\lim_{a \downarrow -1} w_{-1}(0) \delta^{1+a} = w_{-1}(0)$ and

$$(1 + a) \int_0^\delta y^a |w_a(y) - w_a(0)| \, dy \leqslant \delta^{1+a} \varepsilon.$$

Finally,

$$(1 + a) \int_\delta^\infty y^a |w_a(y)| \, dy \leqslant (1 + a) C \int_\delta^\infty y^{a-1} \, dy \leqslant C \frac{(1 + a)}{-a} \delta^a \to 0 \tag{6.3}$$

as $a \downarrow -1$. This proves the claim above.

We now divide the integral

$$(1 + a) \int_{\mathbb{R}_+^2} y^a \big\{ \eta_1'(x) \eta_2(y) \partial_x u_a + \eta_1(x) \eta_2'(y) \partial_y u_a \big\} \, dx \, dy$$

$$= (1 + a) \int_{\mathbb{R} \times (0, +\infty)} y^a \eta_1'(x) \eta_2(y) \partial_x u_a \, dx \, dy + (1 + a) \int_{\mathbb{R} \times (1, 2)} y^a \eta_1(x) \eta_2'(y) \partial_y u_a \, dx \, dy.$$

Thanks once again to Lemma 4.4, the observation above (with $w_a = \eta_2(\cdot) \partial_x u_a(x, \cdot)$) and the gradient bounds of Proposition 4.6, we deduce

$$\lim_{a \to -1} (1 + a) \int_{\mathbb{R} \times (0, +\infty)} y^a \eta_1'(x) \eta_2(y) \partial_x u_a \, dx \, dy$$

$$= \int_{\mathbb{R}} u_{-1}'(x, 0) \eta_1'(x) \, dx.$$

By the same lemma, $|\partial_y u_a| \leqslant C y^{-1}$ uniformly in $a \in (-1, 0)$, and thus the same computation as in (6.3) shows that

$$\lim_{a \to -1} (1 + a) \int_{\mathbb{R} \times (1, 2)} y^a \eta_1(x) \eta_2'(y) \partial_y u_a \, dx \, dy = 0.$$

Therefore, the function $\bar{v} := \bar{v}(x) = u_{-1}(x, 0)$ satisfies

$$\int_{\mathbb{R}} \bar{v}' \eta_1' - f(\bar{v}) \eta_1 = 0. \tag{6.4}$$

Hence $\bar{v}$ is a weak solution of

$$-\bar{v}'' = f(\bar{v}) \quad \text{in } \mathbb{R},$$

such that $\bar{v}(0) = 0$ and $\bar{v}' \geqslant 0$ in $\mathbb{R}$. As a consequence, the function $\bar{v}$ admits limits at $\pm\infty$,

$$\lim_{x \to \pm\infty} \bar{v}(x) = L^\pm \in [-1, 1].$$

We now prove the convergence of the Hamiltonian, which will provide in addition that the function $\bar{v}$ is actually a layer, i.e. $L^{\pm} = \pm 1$. We apply the Hamiltonian estimate (5.6) with $y = 0$ to the layer u_a for some $a \in (-1, 1)$ with G replaced by $c_a G$. We deduce

$$0 < G - G(1) \quad \text{in } (-1, 1). \tag{6.5}$$

Next, we integrate the equation satisfied by $\bar{v}$, we use the above observation now with $w_a(\cdot) = (\partial_x u_a)^2(x, \cdot)$ and we use the Hamiltonian identity (5.3) for the layer u_a to obtain for all $x \in \mathbb{R}$

$$G\big(\bar{v}(x)\big) - G\big(L^{+}\big) = \frac{1}{2}\big(\bar{v}'\big)^2(x) = \lim_{a \downarrow -1} \frac{(1+a)}{2} \int_{0}^{+\infty} y^a (\partial_x u_a)^2(x, y)\, dy$$

$$= \lim_{a \downarrow -1} \left\{ \frac{(1+a)}{2} \int_{0}^{+\infty} y^a (\partial_y u_a)^2(x, y)\, dy + c_a G\big(u_a(x, 0)\big) - c_a G(1) \right\}, \tag{6.6}$$

and thus

$$G\big(\bar{v}(x)\big) - G\big(L^{+}\big) \geqslant \lim_{a \downarrow -1} c_a \big(G\big(u_a(x, 0)\big) - G(1)\big) = G\big(\bar{v}(x)\big) - G(1). \tag{6.7}$$

Hence we have that

$$G\big(L^{+}\big) \leqslant G(1),$$

that together with (6.5) and $L^{+} \geqslant 0$ (since $\bar{v}(0) = 0$) gives $L^{+} = 1$.

In addition, we deduce that the inequality (6.7) must be an equality. Thus, the term that we have dropped to obtain the inequality must be zero, i.e.

$$\lim_{a \to -1} \frac{(1+a)}{2} \int_{0}^{+\infty} y^a (\partial_y u_a)^2(x, y)\, dy = 0. \tag{6.8}$$

In the same way, we prove that $L^{-} = -1$. Hence $\bar{v}$ is the layer solution connecting -1 to 1, with $\bar{v}(0) = 0$. The uniqueness of such $\bar{v}$ follows from the Hamiltonian equality (6.1). $\square$

7. Proof of Theorems 2.2, 2.3, 2.4, and 2.5

We prove in this section the main theorems of our paper. They will follow easily from our results in previous sections.

Proof of Theorem 2.3. Part (i) follows from Lemmas 5.2 and 5.3. Part (ii) follows from Theorem 6.1 and from (6.6) and (6.8) in its proof. $\square$

Proof of Theorem 2.2. If v is a layer solution of (2.6), its extension u is a layer solution of (1.3) in $\mathbb{R}^2_+$, up to a multiplicative constant in front of f that tends to 1 as $s \uparrow 1$. In part (i) of the theorem, (2.7) follows from (4.7) in Lemma 4.8. The equality in (2.8) is (5.4) of Lemma 5.2, while the inequality in (2.8) follows from taking $y = 0$ in the statement of Lemma 5.3. Part (ii) of the theorem follows from Theorem 6.1. $\square$

Proof of Theorem 2.5. It follows from Lemmas 5.1 and 5.4. $\square$

Proof of Theorem 2.4. If v is a radial solution in $\mathbb{R}^n$, its extension $u = u(x, y)$ belongs to $L^{\infty}(\mathbb{R}^{n+1}_+)$ and it is a solution of (1.3), up to a multiplicative positive constant in front of f. Clearly, u is a radial solution in $x \in \mathbb{R}^n$.

The relation $f(0) = 0$ is (4.13) in Lemma 4.8.

The conclusion $G(0) > G(u(0, 0))$ of the theorem follows from Lemma 5.4 and (5.14). Indeed, in (5.13) we let $r = 0$ and later $r \to \infty$, and we use (5.2) in Lemma 5.1 to obtain

$$-G\big(v(0)\big) \geqslant -(1 + a) \int\limits_0^\infty \frac{t^a}{2} u_y^2(0, t) \, dt - G\big(v(0)\big) \geqslant -G(0).$$

Thus $G(0) \geqslant G(v(0))$. But if $G(0) = G(v(0))$ then the function in (5.13) would be constant in r. Hence, by (5.14) and since $n > 1$, $u_r(r, t) \equiv 0$ for all r and t, and then u is constant, contrary to our assumption.

It only remains to prove the other statement of the theorem,

$$f'(0) = -G''(0) \leqslant 0$$

under the assumption $u_r < 0$. Without loss of generality, and to simplify notation, we may replace f by a positive multiple of f in (1.3) and hence assume that u solves

$$\begin{cases} \operatorname{div}\big(y^a \nabla u\big) = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ \dfrac{\partial u}{\partial \nu^a} = f(u) & \text{on } \partial \mathbb{R}_+^{n+1} = \mathbb{R}^n. \end{cases}$$

We differentiate both equations with respect to $r = |x|$, using that the first one reads $u_{rr} + \frac{n-1}{r} u_r + u_{yy} + \frac{a}{y} u_y = 0$ in $(0, +\infty) \times (0, +\infty)$. Let

$$\psi := -u_r > 0 \quad \text{in } \big(\mathbb{R}^n \setminus \{0\}\big) \times (0, +\infty).$$

We deduce that

$$\operatorname{div}\big(y^a \nabla \psi\big) = \frac{n-1}{|x|^2} y^a \psi \quad \text{in } \big(\mathbb{R}^n \setminus \{0\}\big) \times (0, +\infty)$$

and

$$f'(u)\psi = \frac{\partial \psi}{\partial \nu^a} \quad \text{for } y = 0.$$

For $x_0 \in \mathbb{R}^n$, let $\psi^{x_0}(x, y) := \psi(x - x_0, y)$, a positive function in $(\mathbb{R}^n \setminus \{x_0\}) \times (0, +\infty)$. Let $u^{x_0}(x, y) := u(x - x_0, y)$. We have

$$\operatorname{div}\big(y^a \nabla \psi^{x_0}\big) = \frac{n-1}{|x - x_0|^2} y^a \psi^{x_0} \quad \text{in } \big(\mathbb{R}^n \setminus \{x_0\}\big) \times (0, +\infty) \tag{7.1}$$

and

$$f'\big(u^{x_0}\big)\psi^{x_0} = \frac{\partial \psi^{x_0}}{\partial \nu^a} \quad \text{for } y = 0. \tag{7.2}$$

For $R > 0$, consider the cylinder $\mathcal{C}_R = \Gamma_R^0 \times (0, R) \subset \mathbb{R}_+^{n+1}$, where Γ_R^0 is the ball of center 0 and radius R in $\mathbb{R}^n$. Let ξ be any C^1 function in $\overline{\mathcal{C}_R}$ vanishing on $\{|x| = R\} \times [0, R)$ and on $\Gamma_R^0 \times \{y = R\}$.

For $|x_0| > R$, we multiply (7.2) by ξ^2 / ψ^{x_0} — note that $\psi^{x_0} > 0$ in $\Gamma_R^0 \times [0, R)$ —, we integrate in Γ_R^0 and use (7.1) to obtain

$$\int\limits_{\Gamma_R^0} f'\big(u^{x_0}\big)\xi^2 = \int\limits_{\mathcal{C}_R} \operatorname{div}\big(y^a \nabla \psi^{x_0}\big) \frac{\xi^2}{\psi^{x_0}} + y^a \nabla \psi^{x_0} \cdot \nabla \frac{\xi^2}{\psi^{x_0}} = \int\limits_{\mathcal{C}_R} \frac{n-1}{|x - x_0|^2} y^a \xi^2 + y^a \nabla \psi^{x_0} \cdot \nabla \frac{\xi^2}{\psi^{x_0}}$$

$$= \int\limits_{\mathcal{C}_R} \frac{n-1}{|x - x_0|^2} y^a \xi^2 + y^a \left\{ 2\xi \nabla \xi \cdot \frac{\nabla \psi^{x_0}}{\psi^{x_0}} - \xi^2 \frac{|\nabla \psi^{x_0}|^2}{(\psi^{x_0})^2} \right\}$$

$$\leqslant \int\limits_{\mathcal{C}_R} \frac{n-1}{|x - x_0|^2} y^a \xi^2 + y^a |\nabla \xi|^2,$$

where we have used Cauchy–Schwarz inequality.

Letting $|x_0| \to \infty$, we deduce

$$f'(0) \leqslant \frac{\int_{\mathcal{C}_R} y^a |\nabla \xi|^2}{\int_{\Gamma_R^0} \xi^2} \tag{7.3}$$

for any C^1 function ξ in $\overline{\mathcal{C}_R}$ vanishing on $\{|x| = R\} \times [0, R)$ and on $\Gamma_R^0 \times \{y = R\}$.

Let $\varphi_R = \varphi_R(x) > 0$ be the first eigenfunction of $-\Delta_x$ in Γ_R^0 with Dirichlet boundary conditions, i.e.,

$$\begin{cases} -\Delta_x \varphi_R = \lambda_R \varphi_R & \text{in } \Gamma_R^0, \\ \varphi_R = 0 & \text{on } \partial \Gamma_R^0, \end{cases}$$

where $\lambda_R = c(n)/R^2 > 0$ is the first Dirichlet eigenvalue of $-\Delta_x$ in the ball Γ_R^0. Let $h_R = h_R(y) \in [0, 1]$ be a smooth function with compact support in $[0, R)$ such that $h_R \equiv 1$ in $[0, R/2]$ and $|h_R'| \leqslant C/R$ for some constant C. Take

$$\xi = \xi_R = \varphi_R(x) h_R(y)$$

in (7.3). We have

$$\begin{aligned} \int_{\mathcal{C}_R} y^a |\nabla \xi_R|^2 &= \int_{\mathcal{C}_R} y^a \{|\nabla_x \varphi_R|^2 h_R^2 + \varphi_R^2 (h_R')^2\} \\ &= \left\{ \lambda_R \int_0^R y^a h_R^2 \, dy + \int_0^R y^a (h_R')^2 \, dy \right\} \int_{\Gamma_R^0} \varphi_R^2 \\ &= \left\{ CR^{-2} \int_0^R y^a h_R^2 \, dy + \int_0^R y^a (h_R')^2 \, dy \right\} \int_{\Gamma_R^0} \xi_R^2 \\ &\leqslant CR^{-2} \int_0^R y^a \, dy \int_{\Gamma_R^0} \xi_R^2 = CR^{-2} \frac{R^{2-2s}}{2 - 2s} \int_{\Gamma_R^0} \xi_R^2 \\ &= \frac{C}{2 - 2s} R^{-2s} \int_{\Gamma_R^0} \xi_R^2. \end{aligned}$$

Using this in (7.3) and letting $R \to \infty$, we conclude that $f'(0) \leqslant 0$, as claimed. $\quad\square$

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