Asymptotic behavior for critical Patlak–Keller–Segel model and a repulsive–attractive aggregation equation

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Abstract

In this paper we study the long time asymptotic behavior for a class of diffusion–aggregation equations. Most results except the ones in Section 3.3 concern radial solutions. The main tools used in the paper are maximum principle type arguments on mass concentration of solutions, as well as energy method. For the Patlak–Keller–Segel problem with critical power $m = 2 - 2/d$, we prove that all radial solutions with critical mass would converge to a family of stationary solutions, while all radial solutions with subcritical mass converge to a self-similar dissipating solution algebraically fast. For non-radial solutions, we obtain convergence towards the self-similar dissipating solution when the mass is sufficiently small. We also apply the mass comparison method to another aggregation model with repulsive–attractive interaction, and prove that radial solutions converge to the stationary solution exponentially fast.
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1. Introduction

Recently there has been a growing interest in the study of nonlocal aggregation phenomena. The most widely studied models are the Patlak–Keller–Segel (PKS) models, which describe the cell movement driven by chemotaxis [22,28,25]. In this paper, we study the PKS equation in dimension $d \geqslant 3$ with degenerate diffusion, given by

$$
\begin{cases}
u_t = \Delta u^m + \nabla \cdot (u \nabla c), & x \in \mathbb{R}^d, \ t \geqslant 0, \\
\Delta c = u, & x \in \mathbb{R}^d, \ t \geqslant 0, \\
u(x, 0) = u_0(x) \in L^1_+ \big(\mathbb{R}^d; \big(1 + |x|^2 \big) \, dx \big) \cap L^\infty(\mathbb{R}^d), & x \in \mathbb{R}^d,
\end{cases}
\tag{1.1}
$$

here $m > 1$ models the local repulsion of cells with anti-crowding effects [30,31]. This model is a generalization of the classical parabolic–elliptic PKS model in 2D, which has been extensively studied over the years (see the review [25,7,17,10,9]).

Throughout this paper, we focus on the critical power $m = m_d := 2 - 2/d$, which produces an exact balance between the diffusion term and the aggregation term when one performs a mass-invariant scaling. To study the

well-posedness of (1.1), the following *free energy* functional (1.2) is an important quantity, where the first term is usually referred to as the *entropy* and the latter term is referred to as the *interaction energy*:

$$\mathcal{F}(u) = \int_{\mathbb{R}^d} \left(\frac{1}{m-1} u^m + \frac{1}{2} u(u * \mathcal{N}) \right) dx. \tag{1.2}$$

Here $\mathcal{N}(x) = -\frac{1}{(d-2)\sigma_d} |x|^{2-d}$ is the Newtonian potential in $d \geqslant 3$, with σ_d the surface area of the sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$. In [8, Definition 3.1], a notion of free energy solution is defined, which is a weak solution to (1.1) such that the free energy is non-increasing in time and satisfies some inequalities. Indeed, it is formally shown that (1.1) is the gradient flow for $\mathcal{F}$ with respect to the Wasserstein metric (see for example [1] and [14]).

The key observation in [8] is the sharp Hardy–Littlewood–Sobolev inequality, which bounds the interaction term in the free energy functional by the entropy term:

$$\left| \int_{\mathbb{R}^d} u(u * \mathcal{N}) \, dx \right| \leqslant C^* \|u\|_{L^1(\mathbb{R}^d)}^{2/d} \|u\|_{L^m(\mathbb{R}^d)}^m, \tag{1.3}$$

where C^* is a constant only depending on the dimension d. This inequality suggests that the behavior of the solution depends on its mass. Making use of this inequality, it is proved in [8] (and generalized by [4] for more general kernels) that there exists a critical mass M_c only depending on d, such that all the solutions with mass $M < M_c$ exist globally in time, whereas for all supercritical mass $M > M_c$ one can construct a solution which blows up in finite time. Although the global existence/blow-up results are well understood, the asymptotic behavior of the solution has not been fully investigated: this motivates our study. In this paper we study the asymptotic behavior of solutions with mass $M \leqslant M_c$, and try to answer the open questions raised in [8] and [7].

○ *Critical case*:

When $M = M_c$, it is proved in [8] that the global minimizers of the free energy functional $\mathcal{F}$ have zero free energy, and are given by the one-parameter family

$$u_R(x) = \frac{1}{R^d} u_1 \left(\frac{x}{R} \right) \tag{1.4}$$

subject to translations. Here u_1 is the unique radial classical solution to

$$\frac{m}{m-1} \Delta u_1^{m-1} + u_1 = 0 \quad \text{in } B(0, 1), \text{ with } u_1 = 0 \text{ on } \partial B(0, 1). \tag{1.5}$$

It was unknown that whether this family of stationary solutions attract some solutions.

In Section 3.1, we prove the convergence of radial solutions towards this family of stationary solution (see Theorem 3.2) using a combination of mass comparison and energy method. The mass comparison property is a version of comparison principle on the mass distribution of solutions (see Proposition 2.4), which has been introduced in [23]. Although it only works for radial solutions, it provides more delicate control than the energy method: it has been recently used in [3] to prove finite time blow-up of solutions with supercritical mass for a diffusion–aggregation model where Virial identity does not apply. We mention that the mass comparison property has been previously observed for the porous medium equation [32] and PKS models with linear diffusion [21,6].

○ *Subcritical case*:

When $0 < M < M_c$, the free energy solution exists globally in time, as long as its initial L^m-norm is finite [8,4]. Moreover it has been proved in [8] that there exists a dissipating self-similar solution, with the same scaling as the porous medium equation. However it was unknown whether this self-similar solution would attract all solutions in the intermediate asymptotics.

In Section 3.2, we prove algebraic convergence of radial solutions towards this self-similar dissipating solution, given that the initial data is bounded and compactly supported. This is done by constructing explicit barriers in the mass comparison sense.

For general (non-radial) solutions, the asymptotic behavior of solution to (1.1) is unknown for all mass sizes when $m = 2 - 2/d$. In Section 3.3, we prove that when the mass is sufficiently small, every solution to (1.1) with compactly supported initial data converges to the self-similar dissipating solution. We mention that for the PKS equation in 2D with linear diffusion, similar results for small-mass solutions are obtained by [11], and recently the small-mass assumption is removed by [15], where they prove that all solutions with subcritical mass converge to the self-similar solution with exponential rate after proper rescaling. Both of these results are based on a spectral gap method, hence cannot be generalized to (1.1) due to the nonlinear diffusion term.

When dealing with non-radial solutions with small mass, our key result is the uniqueness of stationary solution (in rescaled variables), which is proved using a maximum principle type argument (see Theorem 3.11). We point out that although it is well known that the global minimizer to the rescaled energy (3.15) is unique [8], there are few results concerning the uniqueness of stationary solutions for nonlocal PDEs, except in the following special cases: the stationary solution to a similar equation is proved to be unique by [12] in the 1D case, and stationary solution to the 2D Navier–Stokes equation (in rescaled variable) is proved to be unique by [20], where their proof is based on the fact that all stationary solutions have the same second moment, thus cannot be applied to (1.1).

In Section 4 we generalize the mass comparison methods to an aggregation equation

$$u_t - \nabla \cdot (u \nabla K * u) = 0, \tag{1.6}$$

and prove that the radial solution converges towards the stationary solution exponentially fast for a class of kernel K. Eq. (1.6) appears in various contexts as a mathematical model for biological aggregations [27,31]. In order to capture the biologically relevant features of solutions, it is desirable for the interaction kernel K to be repulsive for short-range interactions and attractive for long-range. Aggregation equations with such kinds of kernel are studied in [2,26,19, 18,24]. In this paper we adopt the kernel K proposed in [19], which has a repulsion component in the form of the Newtonian potential $\mathcal{N}$ and an attraction component satisfying the power law:

$$K(x) = \mathcal{N}(x) + \frac{1}{q}|x|^q, \tag{1.7}$$

here when $q = 0$ the second term is replaced by $\ln |x|$. We assume that the attraction part is less singular than the repulsion part at the origin, i.e. $q > 2 - d$. In addition, we assume $q \leqslant 2$, i.e. the long-range attraction does not grow more than linearly as the distance goes to infinity. Note that in [18] they mostly focus on the case $q \geqslant 2$.

The existence and uniqueness of weak solution is established in [18] for $q > 2 - d$. In addition, they proved that for any mass there is a unique radial stationary solution which is continuous in its support. While numerical results in [18] suggest that this stationary solution should be a global attractor, there is no rigorous proof. We prove that when $2 - d < q \leqslant 2$, the mass comparison argument can be easily generalized to (1.6), then we construct barriers in the mass comparison sense to prove that all radial solution converges to the stationary solution exponentially fast.

Outline of the paper. In Section 2 we state a mass comparison result for a general diffusion–aggregation equation. In Section 3 we apply it to the PKS model with critical power, and obtain some asymptotic results for radial solutions with critical and subcritical mass sizes respectively. In Section 4 we demonstrate that the mass comparison can also be applied to an aggregation model with repulsive–attractive interaction, and prove the asymptotic convergence of solution towards the stationary solution.

1.1. Summary of results

By constructing explicit barriers in the mass comparison sense and using energy method, we obtain the following results for radial solutions of (1.1):

Theorem 1.1. *Suppose $d \geqslant 3$ and $m = 2 - 2/d$. Let $u(x, t)$ be the free energy solution to (1.1) with mass A and initial data u_0, where u_0 is continuous, radially symmetric and compactly supported. Then the following results hold:*

- *If $0 < A < M_c$, $u(\cdot, t)$ converges to the dissipating self-similar solution u_A as $t \to \infty$, where the Wasserstein distance between $u(\cdot, t)$ and u_A decays algebraically fast as $t \to \infty$. (Corollary 3.8)*
- *If $A = M_c$ and u_0 satisfies $\nabla u_0^m \in L^2(\mathbb{R}^d)$ in addition to the assumptions above, then $u(\cdot, t) \to u_{R_0}$ in $L^\infty(\mathbb{R}^d)$ as $t \to \infty$ for some $R_0 > 0$, where u_{R_0} is a stationary solution defined in (1.4). (Theorem 3.2)*

For general (possibly non-radial) initial data, we use a maximum principle type method to prove that when the mass is sufficiently small, every compactly supported stationary solution must be radially symmetric. This leads to the following asymptotic convergence result:

Theorem 1.2. *Suppose $d \geqslant 3$ and $m = 2 - 2/d$. Let $u(x, t)$ be the free energy solution to* (1.1) *with mass $0 < A < M_c/2$ being sufficiently small, and the initial data u_0 is continuous and compactly supported. Then we have*

$$\lim_{t \to \infty} \big\| u(\cdot, t) - u_A \big\|_p = 0 \quad \text{for all } 1 \leqslant p \leqslant \infty,$$

where u_A is the self-similar dissipating solution defined in (3.17). (*Corollary* 3.13)

In Section 4 we generalize the mass comparison methods to the repulsive–attractive aggregation equation (4.1), and obtain the following asymptotic convergence result for $2 - d < q \leqslant 2$:

Theorem 1.3. *Assume $2 - d < q \leqslant 2$. Let u be a free energy solution to* (1.6) *with initial data u_0 and mass A, where $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ is non-negative, radially symmetric and compactly supported. In addition, we assume that u_0 is strictly positive in a neighborhood of 0. Let u_s be the unique stationary solution with mass A, as given by Proposition* 4.2. *Then as $t \to \infty$, $u(\cdot, t)$ converges to u_s exponentially fast in Wasserstein distance.* (*Theorem* 4.5)

2. Mass comparison for radial solutions

In this section we consider the following type of diffusion–aggregation equation

$$u_t = c_1 \Delta u^m + \nabla \cdot \big(u \nabla \big(u * (c_2 \mathcal{N} + c_3 \mathcal{K}) + V \big) \big), \tag{2.1}$$

where $\mathcal{N}(x) = -\frac{1}{(d-2)\sigma_d} |x|^{2-d}$ is the Newtonian potential in $d \geqslant 3$, with σ_d the surface area of the sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$. We make the following assumptions on the kernel $\mathcal{K}$, the potential V and the coefficients:

  (**C**) $c_1, c_3 \geqslant 0$, and $c_2 \in \mathbb{R}$ can be of any sign.
(**K1**) $\mathcal{K}$ is radially symmetric.
(**K2**) $\Delta \mathcal{K} \in L^1(\mathbb{R}^d)$, $\Delta \mathcal{K} \geqslant 0$ and is radially decreasing.
(**V1**) $V \in C^2(\mathbb{R}^d)$ is radially symmetric.

For a radially symmetric function $u(x, t)$, we define its mass function $M(r, t; u)$ by

$$M(r, t; u) := \int_{B(0,r)} u(x, t) \, dt, \tag{2.2}$$

and we may write it as $M(r, t)$ if the dependence on the function u is clear. The following lemma describes the PDE satisfied by the mass function.

Lemma 2.1 (*Evolution of mass function*). *Let $u(x, t)$ be a non-negative smooth radially symmetric solution to* (2.1). *Let $M(r, t) = M(r, t; u)$ be as defined in* (2.2). *Then $M(r, t)$ satisfies*

$$\frac{\partial M}{\partial t} = c_1 \sigma_d r^{d-1} \partial_r \left(\frac{\partial_r M}{\sigma_d r^{d-1}} \right)^m + \partial_r M \frac{c_2 M + c_3 \tilde{M}}{\sigma_d r^{d-1}} + \partial_r M \partial_r V, \tag{2.3}$$

where

$$\tilde{M}(r, t; u) := \int_{B(0,r)} u * \Delta \mathcal{K} \, dx. \tag{2.4}$$

Proof. Due to divergence theorem and radial symmetry of u, we have

$$\frac{\partial M}{\partial t} = \sigma_d r^{d-1} \big[c_1 \partial_r u^m + u \big(\partial_r \big(u * (c_2 \mathcal{N} + c_3 \mathcal{K}) \big) + \partial_r V \big) \big]. \tag{2.5}$$

Note that radial symmetry of u also gives

$$u(r) = \frac{\partial_r M}{\sigma_d r^{d-1}}. \tag{2.6}$$

It remains to write $\partial_r(u * \mathcal{N})$ and $\partial_r(u * \mathcal{K})$ in terms of M. For $\partial_r(u * \mathcal{N})$, divergence theorem gives

$$\partial_r(u * \mathcal{N}) = \frac{\int_{B(0,r)} \Delta u * \mathcal{N} \, dx}{\sigma_d r^{d-1}} = \frac{M(r,t)}{\sigma_d r^{d-1}}. \tag{2.7}$$

We can similarly obtain

$$\partial_r(u * \mathcal{K}) = \frac{\int_{B(0,r)} u * \Delta \mathcal{K} \, dx}{\sigma_d r^{d-1}} = \frac{\tilde{M}(r,t)}{\sigma_d r^{d-1}}, \tag{2.8}$$

where $\tilde{M}(r,t;u) := \int_{B(0,r)} u * \Delta \mathcal{K} \, dx$. Plug (2.6), (2.7) and (2.8) into Eq. (2.5), and then we can obtain (2.3). □

Definition 2.2. Let u_1 and u_2 be two non-negative radially symmetric functions in $L^1(\mathbb{R}^d)$. We say u_1 is *less concentrated than* u_2, or $u_1 \prec u_2$, if

$$\int\limits_{B(0,r)} u_1(x) \, dx \leqslant \int\limits_{B(0,r)} u_2(x) \, dx \quad \text{for all } r \geqslant 0.$$

Definition 2.3. Let $u_1(x,t)$ be a non-negative, radially symmetric function in $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, which is C^1 in its positive set. We say u_1 is a *supersolution of* (2.1) *in the mass comparison sense* if $M_1(r,t) := M(r,t;u_1)$ is a supersolution of (2.3), i.e. $M_1(r,t)$ and $\tilde{M}_1(r,t) := \tilde{M}(r,t;u_1)$ satisfy

$$\frac{\partial M_1}{\partial t} \geqslant c_1 \sigma_d r^{d-1} \partial_r \left(\frac{\partial_r M_1}{\sigma_d r^{d-1}} \right)^m + \partial_r M_1 \frac{c_2 M_1 + c_3 \tilde{M}_1}{\sigma_d r^{d-1}} + \partial_r M_1 \partial_r V, \tag{2.9}$$

in the positive set of u_1, where $\tilde{M}(r,t;u_1)$ is as defined in (2.4).

Similarly we can define a *subsolution* of (2.1) in the mass comparison sense.

Proposition 2.4 (*Mass comparison*). *Suppose $m > 1$, and $c_1, c_2, c_3, V, \mathcal{K}$ satisfy the assumptions* (**C**), (**K1**), (**K2**), (**V1**). *Let $u_1(x,t)$ be a supersolution and $u_2(x,t)$ be a subsolution of* (2.1) *in the mass comparison sense for $t \in [0, T]$. Further assume that both u_i are bounded, and u_i's preserve their mass over time, i.e., $\int u_1(\cdot, t) \, dx$ and $\int u_2(\cdot, t) \, dx$ stay constant for all $0 \leqslant t \leqslant T$. Then their mass functions are ordered for all times: i.e., if $u_1(x,0) \succ u_2(x,0)$, then we have $u_1(x,t) \succ u_2(x,t)$ for all $t \in [0, T]$.*

Proof. Let $M_i(r,t)$ be the mass function for u_i, where $i = 1, 2$. We claim that $M_1(r,t) \geqslant M_2(r,t)$ for all $r \geqslant 0$ and $t \in [0, T]$, which proves the proposition.

For the boundary conditions of M_i, note that

$$\begin{cases} M_1(0,t) = M_2(0,t) = 0 & \text{for all } t \in [0, T], \\ \lim\limits_{r \to \infty} \left(M_1(r,t) - M_2(r,t) \right) = \int\limits_{\mathbb{R}^d} \left(u_1(x,0) - u_2(x,0) \right) dx \geqslant 0 & \text{for all } t \in [0, T]. \end{cases}$$

As for initial data, we have $M_1(r,0) \geqslant M_2(r,0)$ for all $r \geqslant 0$.

For given $\lambda > 0$, we define

$$w(r,t) := \left(M_2(r,t) - M_1(r,t) \right) e^{-\lambda t},$$

where λ is a large constant to be determined later. Suppose the claim is false, then w attains a positive maximum at some point (r_1, t_1) in the domain $(0, \infty) \times (0, T]$. Moreover, since the mass of both u_1 and u_2 are preserved over time and thus are ordered, we know that (r_1, t_1) must lie inside the positive set for both u_1 and u_2, where M_i's are $C_{x,t}^{2,1}$.

Since w attains a maximum at (r_1, t_1), the following inequalities hold at (r_1, t_1):

$$w_t \geqslant 0 \quad \Rightarrow \quad \partial_t(M_2 - M_1) \geqslant \lambda(M_2 - M_1), \tag{2.10}$$

$$w_r = 0 \quad \Rightarrow \quad \partial_r M_1 = \partial_r M_2 > 0, \tag{2.11}$$

$$w_{rr} \leqslant 0 \quad \Rightarrow \quad \partial_{rr} M_1 \geqslant \partial_{rr} M_2. \tag{2.12}$$

Now we will analyze the terms on the right-hand side of (2.9) one by one. For the first term, (2.11) and (2.12) imply that

$$c_1 \partial_r \left(\frac{\partial_r M_2}{\sigma_d r^{d-1}} \right)^m - c_1 \partial_r \left(\frac{\partial_r M_1}{\sigma_d r^{d-1}} \right)^m \leqslant 0 \quad \text{at } (r_1, t_1). \tag{2.13}$$

For the term coming from Newtonian potential, we have

$$\partial_r M_1 \frac{c_2(M_2 - M_1)}{\sigma_d r^{d-1}} = c_2 u_1(r_1, t_1)(M_2 - M_1) \leqslant u_{\max}|c_2|(M_2 - M_1), \tag{2.14}$$

where $u_{\max} := \max\{\sup_{\mathbb{R}^d \times [0,T]} u_1, \sup_{\mathbb{R}^d \times [0,T]} u_2\}$ is finite by assumption on u_1 and u_2.

We next claim

$$\partial_r M_1 \frac{c_3(\tilde{M}_2 - \tilde{M}_1)(r_1, t_1)}{\sigma_d r^{d-1}} \leqslant u_{\max} c_3 \|\Delta \mathcal{K}\|_1 (M_2 - M_1)(r_1, t_1). \tag{2.15}$$

To prove the claim, note that $\tilde{M}_2 - \tilde{M}_1$ can be rewritten as

$$\tilde{M}_2(r_1, t_1) - \tilde{M}_1(r_1, t_1) = \int_{\mathbb{R}^d} \left((u_2 - u_1) * \Delta \mathcal{K} \right) \chi_{B(0, r_1)} \, dx$$

$$= \int_{\mathbb{R}^d} (u_2 - u_1)(\chi_{B(0, r_1)} * \Delta \mathcal{K}) \, dx. \tag{2.16}$$

Note that $\Delta \mathcal{K} \geqslant 0$ is radially decreasing due to assumption (**K2**), thus $\chi_{B(0, r_1)} * \Delta \mathcal{K}$ is non-negative, radially decreasing and has maximum less than or equal to $\|\Delta \mathcal{K}\|_1$. Therefore we can use a sum of bump function to approximate $\chi_{B(0, r_1)} * \Delta \mathcal{K}$, where the sum of the height is less than $\|\Delta \mathcal{K}\|_1$. Hence

$$\tilde{M}_2(r_1, t_1) - \tilde{M}_1(r_1, t_1) \leqslant \|\Delta \mathcal{K}\|_1 \sup_x (M_2 - M_1)(x, t_1) = \|\Delta \mathcal{K}\|_1 (M_2 - M_1)(r_1, t_1),$$

which proves the claim (2.15). Finally, for the last term in (2.9), as a result of (2.11) we have

$$\partial_r M_2 \partial V - \partial_r M_1 \partial V = 0. \tag{2.17}$$

Now we subtract (2.9) with the corresponding equation for the subsolution. Due to the inequalities (2.13), (2.14), (2.15) and (2.17), we obtain that

$$\partial_t(M_2 - M_1) \leqslant u_{\max}\left(|c_2| + c_3\|\Delta K\|_1\right)(M_2 - M_1).$$

Hence if we choose $\lambda > u_{\max}(|c_2| + c_3\|\Delta K\|_1)$ in the beginning of the proof, the inequality above will contradict (2.10). $\quad \square$

Remark 2.5. If both u_1 and u_2 are supported in some compact set $B(0, R)$ for $0 \leqslant t \leqslant T$, then the assumption (**K2**) can be replaced by (**K2′**) as follows:

(**K2′**) $\Delta K \in L^1_{loc}(\mathbb{R}^d)$, $\Delta K \geqslant 0$ and is radially decreasing.

The proof under condition (K2′) is almost the same, except that there is some change in (2.15). Note that in this case (2.16) still holds, and we can bound the maximum of $\chi_{B(0, r_1)} * \Delta \mathcal{K}$ by $\int_{B(0, R)} \Delta \mathcal{K} \, dx$. This yields an inequality similar to (2.15), with the $\|\Delta \mathcal{K}\|_1$ in the right-hand side replaced by $\int_{B(0, R)} \Delta \mathcal{K} \, dx$.

3. Application to critical Patlak–Keller–Segel models

3.1. Convergence towards stationary solution for critical mass

In this subsection, we prove that every radial solution with mass M_c and continuous, compactly supported initial data will be eventually attracted to some stationary solution within the family (1.4).

If the initial data is bounded above and has the critical mass M_c, then it is proved in [8] that the free energy solution to (1.1) exists globally in time. In the next lemma we prove the solution has a global (in time) L^∞ bound. In addition, if the initial data is radially symmetric and compactly supported, then the support of the solution would stay uniformly bounded in time.

Lemma 3.1. *Suppose $d \geqslant 3$ and $m = 2 - 2/d$. Consider the problem* (1.1) *with initial data $u_0 \in L^1_+(\mathbb{R}^d; (1+|x|^2)\, dx) \cap L^\infty(\mathbb{R}^d)$, where u_0 is continuous and has critical mass M_c. Then the L^∞-norm of the weak solution $u(x, t)$ is globally bounded, i.e. there exists $K_1 > 0$ depending on $\|u_0\|_\infty$ and d, such that $\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^d)} \leqslant K_1$ for all $t \geqslant 0$.*

If u_0 is radially symmetric and compactly supported in addition to the assumptions above, then there exists some $R_2 > 0$, such that $\{u(\cdot, t) > 0\} \subseteq B(0, R_2)$ for all $t \geqslant 0$, where R_2 depend on d and u_0.

Proof. In order to bound the L^∞-norm of $u(\cdot, t)$, we first consider Eq. (1.1) with symmetrized initial data, which is described below. Let $\bar{u}(\cdot, t)$ be the solution to (1.1) with initial data u_0^*, where $u_0^*(\cdot)$ is the radial decreasing rearrangement of u_0. Here the *radial decreasing rearrangement* of a non-negative function f is defined as

$$f^*(x) := \int_0^\infty \chi_{\{f > t\}^*}(x)\, dt. \tag{3.1}$$

Since $\bar{u}$ has a radially symmetric initial data and has mass M_c, due to [8], we readily obtain that $\bar{u}$ exists globally in time, and $\bar{u}$ is radially symmetric for all $t \geqslant 0$. We first prove that there is a global L^∞ bound for $\bar{u}$.

Since $\|\bar{u}(\cdot, 0)\|_\infty = \|u_0\|_\infty < \infty$, we can choose $R_1 > 0$ depending on $\|u_0\|_\infty$, where R_1 is sufficiently small such that $u_0^* \prec u_{R_1}$, where u_{R_1} is as defined in (1.4). Then the mass comparison result in Proposition 2.4 yields that

$$\bar{u}(\cdot, t) \prec u_{R_1} \quad \text{for all } t \geqslant 0. \tag{3.2}$$

Now we go back to the original solution u, and compare u with $\bar{u}$. It is proved in Theorem 6.3 of [23] that

$$u^*(\cdot, t) \prec \bar{u}(\cdot, t) \quad \text{for all } t \geqslant 0.$$

Combining the above two inequalities together, we readily obtain that

$$u^*(\cdot, t) \prec u_{R_1} \quad \text{for all } t \geqslant 0,$$

which implies that $u^*(0, t) \leqslant u_{R_1}(0)$ for $t \geqslant 0$. Note that $u^*(\cdot, t)$ is radially decreasing for all $t \geqslant 0$ by definition, and u_{R_1} is radially decreasing due to [8]. Hence the above inequality implies that

$$\big\| u(\cdot, t) \big\|_\infty = \big\| u^*(\cdot, t) \big\|_\infty \leqslant \| u_{R_1} \|_\infty = R_1^{-d} u_1(0) \quad \text{for } t \geqslant 0, \tag{3.3}$$

thus u has a global L^∞ bound $R_1^{-d} u_1(0)$, where u_1 is as defined in (1.4).

Next we hope to show that if u_0 is radially symmetric and compactly supported in addition to the conditions above, the support of $u(\cdot, t)$ will stay in some compact set for all time. We first prove it for the case where $u_0(0) > 0$. Due to the continuity of u_0, we have u_0 is uniformly positive in a neighborhood of 0. This enables us to choose $R_2 > 0$ sufficiently large such that $u_0 \succ u_{R_2}$, where u_{R_2} is as defined in (1.4). Proposition 2.4 again gives us $u(\cdot, t) \succ u_{R_2}$ for all $t \geqslant 0$, which implies that

$$\operatorname{supp} u(\cdot, t) \subseteq \operatorname{supp} u_{R_2} = B(0, R_2) \quad \text{for all } t \geqslant 0.$$

If $u_0(0) = 0$, we claim that after some finite time t_1, $u(0, t_1)$ becomes positive, and $u(\cdot, t_1)$ has a compact support, where t_1 depends on d and u_0. Then we can take t_1 as the starting time and argue as in the case $u_0(0) > 0$.

Now we will prove the claim. This is done by performing the mass comparison between u and w, where w is the solution to the porous medium equation

$$w_t = \Delta w^m \quad \text{in } \mathbb{R}^d \times [0, \infty), \tag{3.4}$$

with initial data $w(\cdot, 0) = u_0$. It can be readily checked that u is a supersolution of (3.4) in the mass comparison sense, hence Proposition 2.4 implies that $u(\cdot, t) \succ w(\cdot, t)$ for all $t \geqslant 0$, which yields

$$u(0, t) \geqslant w(0, t) \quad \text{and} \quad \operatorname{supp} u(\cdot, t) \subseteq \operatorname{supp} w(\cdot, t) \quad \text{for all } t \geqslant 0. \tag{3.5}$$

For the porous medium equation (3.4), it is well known that the solution will eventually converge to the self-similar Barenblatt profile (see [32] for example), which has a positive density at 0 for all $t \geqslant 0$. Hence there exists some $t_1 \geqslant 0$ such that $w(0, t_1) > 0$. In addition, $w(\cdot, t_1)$ has a compact support, due to the finite speed of propagation property of porous medium equation [32]. Therefore (3.5) yields that $u(0, t_1) \geqslant w(0, t_1) > 0$ and $u(\cdot, t_1)$ is compactly supported, which prove the claim. □

Next we prove that under the conditions in Lemma 3.1, every radial solution converges to some stationary solution in the family (1.4) as $t \to \infty$. To do this we investigate the free energy functional (1.2), and make use of the following result proved in [8] and [4]: Let u be a free energy solution (for definition, see [8, Definition 3.1]) to (1.1), then it satisfies the following energy dissipation inequality for almost every t during its existence:

$$\mathcal{F}(u(t)) + \int_0^t \int_{\mathbb{R}^d} u \left| \frac{m}{m-1} \nabla u^{m-1} + \nabla \mathcal{N} * u \right|^2 dx \, dt \leqslant \mathcal{F}(u_0). \tag{3.6}$$

Theorem 3.2. *Suppose $d \geqslant 3$ and $m = 2 - 2/d$. Let $u(x, t)$ be the free energy solution to (1.1) with critical mass M_c and nonnegative initial data u_0, where u_0 is continuous, radially symmetric and compactly supported, and satisfies $\nabla u_0^m \in L^2(\mathbb{R}^d)$. Then there exists $R_0 > 0$ depending on u_0 and d, such that $u(\cdot, t) \to u_{R_0}$ in $L^\infty(\mathbb{R}^d)$ as $t \to \infty$, where u_{R_0} is as defined in (1.4).*

Proof. Due to Lemma 3.1, we obtain the existence of a free energy solution globally in time, which has a global L^∞ bound. In addition, by treating $u * \mathcal{N}$ as an *a priori* potential in (1.1) and applying the continuity result in [16], we obtain that $u(x, t)$ is uniformly continuous in space and time in $[\tau, \infty)$ for all $\tau > 0$.

Our preliminary goal is to find a time sequence $\{t_n\}$ which increases to infinity, such that $u(t_n)$ uniformly converges to some stationary solution as $n \to \infty$. Note that $\mathcal{F}(u(\cdot, t))$ is non-increasing for almost every t due to (3.6), and is bounded below as $t \to \infty$. This enables us to find a time sequence $\{t_n\}$ which increases to infinity, such that

$$\lim_{n \to \infty} \int_{\mathbb{R}^d} u(t_n) \left| \frac{m}{m-1} \nabla u(t_n)^{m-1} + \nabla \mathcal{N} * u(t_n) \right|^2 dx = 0. \tag{3.7}$$

We will slightly abuse the notation and denote $u(t_n)$ by u_n. Note that $\{u_n\}$ is uniformly bounded and equicontinuous, hence Arzelà–Ascoli theorem enables us to find a subsequence of $\{u_n\}$, such that

$$u_n \to u_\infty \quad \text{uniformly in } n, \tag{3.8}$$

where u_∞ is some radially symmetric and continuous function. Moreover, Lemma 3.1 implies that the support of $\{u_n\}$ all stays in some fixed compact set, hence we have u_∞ is compactly supported as well, and it has mass M_c. We will prove that u_∞ is indeed a stationary solution later.

We first claim that $\{\nabla u_n^m\}$ are uniformly bounded in $L^2(\mathbb{R}^d)$. To prove the claim, note that

$$\int_{\mathbb{R}^d} \left| \nabla u_n^m + u_n \nabla \mathcal{N} * u_n \right|^2 dx \leqslant \|u_n\|_\infty \int_{\mathbb{R}^d} u_n \left| \frac{m}{m-1} \nabla u_n^{m-1} + \nabla \mathcal{N} * u_n \right|^2 dx,$$

where the right-hand side is uniformly bounded for all n. In addition, since $\{u_n\}$ are uniformly bounded and are all supported in some $B(0, R)$, we know $\int_{\mathbb{R}^d} u_n |\nabla \mathcal{N} * u_n|^2 dx$ is also uniformly bounded for all n. Therefore triangle inequality yields the uniform boundedness of $\int_{\mathbb{R}^d} |\nabla u_n^m|^2 dx$, which proves the claim.

The uniform boundedness of $\{\nabla u_n^m\}$ in $L^2(\mathbb{R}^d)$ implies that $\{\nabla u_\infty^m\}$ is in $L^2(\mathbb{R}^d)$ as well. Moreover, we can find a subsequence of $\{u_n\}$ (which we again denote by $\{u_n\}$ for the simplicity of notation), such that

$$\nabla u_n^m \rightharpoonup \nabla u_\infty^m \quad \text{as } n \to \infty \text{ weakly in } L^1\big(\mathbb{R}_L^d : \mathbb{R}^d\big). \tag{3.9}$$

Using (3.7) and the two convergence properties (3.8) and (3.9), we can proceed in the same way as Lemma 10 in [13] and prove that u_∞ satisfies

$$\int_{\mathbb{R}^d} u_\infty \left| \frac{m}{m-1} \nabla u_\infty^{m-1} + \nabla \mathcal{N} * u_\infty \right|^2 dx = 0. \tag{3.10}$$

Next we claim that u_∞ is a radial stationary solution to (1.1). First note that u_∞ is continuous, compactly supported (since the support of $u(\cdot, t)$ is uniformly bounded in t) and radially symmetric. Moreover, due to (3.10), $\frac{m}{m-1}\partial_r u_\infty^{m-1} = -\partial_r(\mathcal{N} * u_\infty) = -\frac{M(r)}{\sigma_d r^{d-1}} \leqslant 0$ holds almost everywhere in the support of u_∞, where $M(r) := \int_{B(0,r)} u_\infty(x)\, dx$ is the mass function for u_∞. This implies that $u_\infty(x)$ must be radially decreasing, hence it only has one positive component. In other words, the support of u_∞ must coincide with $B(0, R_0)$ for some R_0. (3.10) then implies $\frac{m}{m-1}u_\infty^{m-1} + \mathcal{N} * u_\infty = C$ holds in $B(0, R_0)$ for some constant C. Due to the fact that $\mathcal{N} * u_\infty$ is once more differentiable than u_∞, one can iteratively show that u_∞ is indeed C^∞ in $B(0, R_0)$, hence it is a classical solution to the following equation:

$$\begin{cases} \dfrac{m}{m-1}\Delta u_\infty^{m-1} + u_\infty = 0 & \text{in } B(0, R_0), \\ u_\infty = 0 & \text{on } \partial B(0, R_0), \end{cases}$$

hence u_∞ is indeed in the family (1.4).

Next we will prove that $u(\cdot, t) \to u_\infty$ uniformly in $L^\infty(\mathbb{R}^d)$ as $t \to \infty$. In order to prove this, we make use of the monotonicity of the second moment of $u(\cdot, t)$ in time. By combining the following Virial identity

$$\frac{d}{dt}\int_{\mathbb{R}^d} |x|^2 u(x, t)\, dx = 2(d-2)\mathcal{F}\big[u(t)\big] \quad \text{for all } t \tag{3.11}$$

with the fact that the minimizer of $\mathcal{F}$ has free energy 0, it is shown in [8] that

$$M_2\big[u(\cdot, t)\big] := \int_{\mathbb{R}^d} |x|^2 u(x, t)\, dt \text{ is non-decreasing in } t. \tag{3.12}$$

This implies that any subsequence of $u(\cdot, t)$ can converge to only one limit: if not, then we can find another sequence $\{t_n'\}$ increasing to infinity, such that $u(t_n')$ converges to another stationary solution u_∞' uniformly as $n \to \infty$, where u_∞' is also in the family (1.4). Since $u(t_n')$ are uniformly bounded and uniformly compactly supported, we have $M_2[u(t_n')] \to M_2[u_\infty']$. On the other hand for the time sequence $\{t_n\}$ we have $M_2[u(t_n)] \to M_2[u_\infty]$, hence (3.12) implies that u_∞ and u_∞' must have the same second moment. Since both u_∞ and u_∞' are within the family (1.4), they can have the same second moment only if they are the same stationary solution. $\quad\square$

Remark 3.3. Since the proof is done by extracting a subsequence of time, we are unable to obtain the rate of the convergence. We also point out that the above proof is for radial solution only; for general initial data the difficulty lies in the fact that we are unable to bound the solution in some compact set uniform in time.

3.2. Convergence towards self-similar solution for subcritical mass, radial case

In this subsection, we prove that every radial solution with subcritical mass and compactly supported initial data would converge to some self-similar solution which is dissipating with the same scaling as the solution of the porous medium equation.

Let u be the free energy solution to (1.1), with mass $A \in (0, M_c)$. Following [32] and [8], we rescale u according to the scaling from porous medium equation:

$$\mu(\lambda, \tau) = (t+1)u(x, t); \quad \lambda = x(t+1)^{-1/d}; \quad \tau = \ln(t+1). \tag{3.13}$$

Then $\mu(\lambda, 0) = u(x, 0)$, and $\mu(\lambda, \tau)$ solves the following rescaled equation

$$\mu_\tau = \Delta \mu^m + \nabla \cdot \left(\mu \nabla \frac{|\lambda|^2}{2d} \right) + \nabla \cdot \left(\mu \nabla (\mu * \mathcal{N}) \right). \tag{3.14}$$

It is pointed out in Theorem 5.2 of [8] that the free energy associated to the rescaled problem (3.14) is

$$\mathcal{G}\big(\mu(\cdot, t)\big) := \int\limits_{\mathbb{R}^d} \left(\frac{m}{m-1} \mu^m + \frac{1}{2} \mu (\mathcal{N} * \mu) + \frac{|\lambda|^2 \mu}{2d} \right) d\lambda, \tag{3.15}$$

and for any mass $A \in (0, M_c)$, there is a unique minimizer μ_A of $\mathcal{G}$ in $\mathcal{Z}_A$ subject to translation, where $\mathcal{Z}_A := \{ h \in L^1(\mathbb{R}^d) \cap L^m(\mathbb{R}^d) : \|h\|_1 = M \text{ and } \int_{\mathbb{R}^d} |x|^2 h(x) \, dx \leqslant \infty \}$. In addition, μ_A is continuous, radially decreasing and has a compact support, and μ_A satisfies

$$\frac{m}{m-1} \frac{\partial}{\partial r} \mu_A^{m-1} + \frac{r}{d} + \frac{M(r; \mu_A)}{\sigma_d r^{d-1}} = 0 \tag{3.16}$$

in its positive set, where the mass function M is as defined in (2.2).

Since μ_A is a stationary solution of (3.14), if we go back to the original scaling, μ_A gives a self-similar solution of (1.1):

$$u_A(x, t) = (t + 1)^{-1} \mu_A \left(\frac{x}{(t + 1)^{1/d}} \right). \tag{3.17}$$

It is then asked in [8] and [7] that whether this self-similar solution attracts all global solutions.

We will first prove that all radial solutions to the rescaled equation (3.14) converge to μ_A. The following lemma constructs a family of explicit solutions to (3.14), which all converge to μ_A exponentially fast as $\tau \to \infty$.

Lemma 3.4 *(A family of explicit solutions). Suppose $d \geqslant 3$ and $m = 2 - 2/d$. For $0 < A < M_c$, we denote by μ_A the stationary solution of* (3.14). *Let $\bar{\mu}$ be defined as*

$$\bar{\mu}(\lambda, \tau) := \frac{1}{R^d(\tau)} \mu_A \left(\frac{\lambda}{R(\tau)} \right), \tag{3.18}$$

where $R(\tau)$ solves the ODE

$$\begin{cases} \dot{R}(\tau) = \frac{1}{d} \left(\frac{1}{R^d} - 1 \right) R, \\ R(0) = R_0, \end{cases} \tag{3.19}$$

where $R_0 > 0$ is a constant. Then for any $R_0 > 0$, $\bar{\mu}(\lambda, \tau)$ is a weak solution to (3.14).

Proof. Since $\bar{\mu}$ is a self-similar function, it can be easily verified that $\bar{\mu}$ solves the following transport equation

$$\bar{\mu}_\tau + \nabla \cdot \left(\bar{\mu} \frac{\dot{R}(\tau)}{R(\tau)} \lambda \right) = 0.$$

On the other hand, note that (3.14) can also be written as a transport equation

$$\mu_\tau = \nabla \cdot (\mu \vec{v}),$$

where

$$\vec{v} = \frac{m}{m-1} \nabla \mu^{m-1} + \frac{\lambda}{d} + \frac{M(|\lambda|, \tau; \mu)}{\sigma_d |\lambda|^{d-1}} \frac{\lambda}{|\lambda|}.$$

Therefore, to prove that $\bar{\mu}$ solves (3.14), it suffices to verify that

$$-\frac{\dot{R}(\tau)}{R(\tau)} r = \underbrace{\frac{m}{m-1} \frac{\partial}{\partial r} \bar{\mu}^{m-1}}_{T_1} + \frac{r}{d} + \underbrace{\frac{M(r, \tau; \bar{\mu})}{\sigma_d r^{d-1}}}_{T_2} \quad \text{for } 0 \leqslant r \leqslant R(\tau). \tag{3.20}$$

Since $\bar{\mu}$ is a rescaling of μ_A,

$$T_1 = \frac{1}{R^{(m-1)d+1}}\left(\frac{\partial}{\partial r}\mu_A^{m-1}\right)\left(\frac{r}{R(\tau)}\right) = \frac{1}{R^{d-1}}\left(\frac{\partial}{\partial r}\mu_A^{m-1}\right)\left(\frac{r}{R(\tau)}\right),$$

where in the last inequality we used the fact that m is the critical power, i.e. $m = 2 - 2/d$. For T_2 in (3.20), the definition of $\bar{\mu}$ gives

$$T_2 = \frac{1}{R(\tau)^{d-1}}\frac{M(\frac{r}{R(\tau)};\mu_A)}{\sigma_d(\frac{r}{R(\tau)})^{d-1}} \quad \text{for } 0 \leqslant r \leqslant R(\tau).$$

Now recall that μ_A satisfies (3.16) in its positive set, which implies

$$\begin{aligned}
\text{RHS of (3.20)} &= \frac{1}{R(\tau)^{d-1}}\left(\frac{m}{m-1}\frac{\partial}{\partial r}\mu_A^{m-1}\left(\frac{r}{R(\tau)}\right) + \frac{r}{dR(\tau)} + \frac{M(\frac{r}{R(\tau)};\mu_A)}{\sigma_d(\frac{r}{R(\tau)})^{d-1}}\right) + \frac{1}{d}\left(1 - \frac{1}{R^d(\tau)}\right)r \\
&= \frac{1}{d}\left(1 - \frac{1}{R^d(\tau)}\right)r \\
&= -\frac{\dot{R}(\tau)}{R(\tau)}r,
\end{aligned}$$

where the last equality comes from the definition of R in (3.19). This verifies that (3.20) is indeed true, which completes the proof. $\quad\square$

Next we use the family of explicit solution constructed above as barriers, and perform mass comparison between the real solution and the barriers.

Proposition 3.5. *Suppose $d \geqslant 3$ and $m = 2 - 2/d$. Let $\mu(\lambda, \tau)$ be a radially symmetric weak solution to (3.14) with mass $0 < A < M_c$, where the initial data $\mu(\cdot, 0)$ is nonnegative, continuous and compactly supported. Then as $\tau \to \infty$, the mass function of μ converges to the mass function of μ_A exponentially, i.e.*

$$\sup_r\left|M(r, \tau; \mu) - M(r, \tau; \mu_A)\right| \leqslant Ce^{-\tau},$$

where μ_A is as defined in (3.16), and C depends on d, A and $\mu(\cdot, 0)$.

Proof. Without loss of generality we assume that $\mu(0, 0) > 0$. (When $\mu(0, 0) = 0$, from the same discussion in Lemma 3.1, $\mu(0, \tau)$ will become positive after some finite time.) Then we can find R_{01} sufficiently small and R_{02} sufficiently large, such that

$$\frac{1}{R_{02}^d}\mu_A\left(\frac{\cdot}{R_{02}}\right) \prec \mu(\cdot, 0) \prec \frac{1}{R_{01}^d}\mu_A\left(\frac{\cdot}{R_{01}}\right),$$

where in the first inequality we used that $\mu(0, 0) > 0$, and in the second inequality we used $\|\mu(\cdot, 0)\|_\infty < \infty$.

Let $\mu_1(\lambda, \tau)$ and $\mu_2(\lambda, \tau)$ be defined as in (3.18), with $R(0)$ equal to R_{01} and R_{02} respectively. Then Lemma 3.4 says that both μ_1 and μ_2 are solutions to (3.14). Note that (3.14) is a special case of (2.1), hence the mass comparison result in Proposition 2.4 holds here as well, which gives

$$\mu_2(\cdot, \tau) \prec \mu(\cdot, \tau) \prec \mu_1(\cdot, \tau) \quad \text{for all } \tau \geqslant 0,$$

or in other words,

$$M(\cdot, \tau; \mu_2) \leqslant M(\cdot, \tau; \mu) \leqslant M(\cdot, \tau; \mu_1) \quad \text{for all } \tau \geqslant 0.$$

It remains to show that

$$\sup_r\left|M(r, \tau; \mu_i) - M(r; \mu_A)\right| \leqslant Ce^{-\tau} \quad \text{for } i = 1, 2.$$

Recall that both μ_i's are scalings of μ_A with scaling coefficient $R_i(\tau)$, hence

$$\left| M(r, \tau; \mu_i) - M(r; \mu_A) \right| = \left| M\left(\frac{r}{R_i(\tau)}; \mu_A \right) - M(r; \mu_A) \right|. \tag{3.21}$$

Since μ_A is bounded and compactly supported, it suffices to show that $R_i(\tau) \to 1$ exponentially as $r \to \infty$. Recall that $\dot{R}_i = \frac{1}{d}(\frac{1}{R_i^d} - 1)R_i$ with initial data R_{0i} for $i = 1, 2$, a simple calculation reveals that $|R_i(\tau) - 1| \leqslant C_i e^{-\tau}$, where C_i depends on R_{0i}. This implies that the right-hand side of (3.21) decays like $e^{-\tau}$, which completes the proof. □

Making use of the explicit barriers μ_1 and μ_2 constructed in the proof of Proposition 3.5, we get exponential convergence of μ/A towards the μ_A/A in the p-Wasserstein metric, which is defined below. Note that the Wasserstein metric is natural for this problem, since as pointed out in [1] and [14], Eq. (1.1) is formally a gradient flow of the free energy (1.2) with respect to the 2-Wasserstein metric.

Definition 3.6. Let μ_1 and μ_2 be two (Borel) probability measures on $\mathbb{R}^d$ with finite p-th moment. Then the *p-Wasserstein distance* between μ_1 and μ_2 is defined as

$$W_p(\mu_1, \mu_2) := \left(\inf_{\pi \in \mathcal{P}(\mu_1, \mu_2)} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \, \pi(dx \, dy) \right\} \right)^{\frac{1}{p}},$$

where $\mathcal{P}(\mu_1, \mu_2)$ is the set of all probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with first marginal μ_1 and second marginal μ_2.

Corollary 3.7. *Let $d \geqslant 3$, and $m = 2 - \frac{2}{d}$. Let $\mu(\lambda, \tau)$ and μ_A be as given in Proposition 3.5. Then for all $p > 1$, we have*

$$W_p\left(\frac{\mu(\cdot, \tau)}{A}, \frac{\mu_A}{A} \right) \leqslant C e^{-\tau},$$

where C depends on d and $\mu(\cdot, 0)$.

Proof. The proof is very similar to the proof of Corollary 5.8 in [23], and we will briefly sketch it below for the sake of self-consistency.

First we state a useful property of Wasserstein distance [33]: for two probability densities f_0, f_1 on $\mathbb{R}^d$, the p-Wasserstein distance between them coincides with the solution of Monge's optimal mass transportation problem. Namely,

$$W_p(f_1, f_0) = \left(\inf_{T \# f_0 = f_1} \int_{\mathbb{R}^d} f_0(x) |x - T(x)|^p \, dx \right)^{\frac{1}{p}}, \tag{3.22}$$

where T is a "push-forward map" from $\mathbb{R}^d$ to $\mathbb{R}^d$, and $T \# f_0 = f_1$ stands for "the map T transports f_0 onto f_1", in the sense that for all bounded continuous function h on $\mathbb{R}^d$,

$$\int_{\mathbb{R}^d} h(x) f_1(x) \, dx = \int_{\mathbb{R}^d} h(T(x)) f_0(x) \, dx.$$

Without loss of generality we assume the mass $A = 1$. In order to prove that $W_p(\mu(\cdot, \tau), \mu_A)$ is small, it suffices to show that there is a push-forward map $T(\cdot, \tau)$ satisfying $T(\cdot, \tau) \# \mu(\cdot, \tau) = \mu_A$, such that $|T(x, \tau) - x| \leqslant C e^{-\tau}$ holds for all x and $\tau > 0$ for some C.

The exact formula for $T(x, \tau)$ is a little bit complicated, so we will first look for a push-forward map $\tilde{T}$ that pushes the "subsolution" μ_2 to the "supersolution" μ_1, where μ_2 and μ_1 are defined in the proof of Proposition 3.5. Note that for each τ, μ_2 and μ_1 are exact scalings of each other:

$$\mu_1(\cdot, \tau) = \frac{R_2(\tau)^d}{R_1(\tau)^d} \, \mu_1\left(\frac{\cdot R_2(\tau)}{R_1(\tau)} \right),$$

where $R_1(\tau)$ and $R_2(\tau)$ are both solutions to (3.19) with initial data R_{01} and R_{02} respectively. This means that one can let $\tilde{T}(x, \tau)$ be a simple scaling $\tilde{T}(x, \tau) = x \frac{R_1(\tau)}{R_2(\tau)}$ in $B(0, R_2(\tau))$ (the definition of $\tilde{T}$ outside of $B(0, R_2(\tau))$ is not important since μ_2 is supported in $B(0, R_2(\tau))$), then we would have

$$\tilde{T}(x, \tau) \# \mu_2(\cdot, \tau) = \tilde{\mu}_1(\cdot, \tau),$$

and $|\tilde{T}(x, \tau) - x| \leqslant C R_1(\tau)/R_2(\tau) \leqslant C e^{-\tau}$ for all $x \in B(0, R_2(\tau))$, which means the Wasserstein distance between μ_1 and μ_2 is decaying exponentially fast.

Finally, note that both $\mu(\cdot, \tau)$ and $\mu_A(\cdot)$ are "squeezed" between $\mu_1(\cdot, \tau)$ and $\mu_2(\cdot, \tau)$ in the mass comparison sense. This would imply $W_p(\mu(\cdot, \tau), \mu_A) \leqslant W_p(\mu_1(\cdot, \tau), \mu_2(\cdot, \tau))$, which concludes the proof. For details of the proof, we refer the readers to Corollary 5.8 in [23].   □

Rescaling back to the original space and time variables, we have

$$u(x, t) = \frac{1}{t+1} \mu\left(\frac{x}{(t+1)^{1/d}}, \ln(t+1)\right).$$

Thus Corollary 3.7 immediately yields the algebraic convergence towards the dissipating self-similar solution (3.17):

Corollary 3.8. *Let $u(x, t)$ be a radial solution to (1.1) with mass $0 < A < M_c$, where the initial data $u(\cdot, 0) \in L_+^1(\mathbb{R}^d; (1 + |x|^2)\, dx) \cap L^\infty(\mathbb{R}^d)$ is continuous and compactly supported. Let u_A be the dissipating self-similar solution with mass A defined in (3.17). Then u/A converges to u_A/A in p-Wasserstein distance algebraically fast as $t \to \infty$ for all $p > 1$. More precisely,*

$$W_p\left(\frac{u(\cdot, t)}{A}, \frac{u_A}{A}\right) \leqslant C t^{-(d-1)/d},$$

where C depends on d, A and $u(\cdot, 0)$.

3.3. Convergence towards self-similar solution for subcritical mass, non-radial case

In this subsection, we consider the rescaled equation (3.14) with general (possibly non-radial) initial data. The key result here is that when the mass $A < M_c$ is sufficiently small, the radially symmetric stationary solution μ_A as defined in (3.16) is the unique compactly supported stationary solution (in rescaled variables). Then a similar argument as in Theorem 3.2 shows that every solution to (3.14) with small mass and compactly supported initial data converges to μ_A. After scaling back to the original variables, we immediately obtain the convergence towards the self-similar solution if the mass is small.

We first prove an L^∞-regularization result, saying that if the initial mass is small, then the L^∞-norm of solution to (3.14) will become small after unit time, regardless of the L^∞-norm of the initial data. We point out that a similar L^∞-regularization result is proved in [29] for the 2D case with linear diffusion, using a De Giorgi type method.

Lemma 3.9. *Suppose $d \geqslant 3$ and $m = 2 - 2/d$. Let $\mu(\lambda, \tau)$ be a weak solution to (3.14) with mass $0 < A < M_c/2$, where the initial data $\mu_0 \in L_+^1(\mathbb{R}^d; (1 + |x|^2)\, dx) \cap L^\infty(\mathbb{R}^d)$ is continuous. Then we have*

$$\|\mu(\cdot, \tau)\|_\infty \leqslant K_A := C A^{2/d} \quad \text{for all } \tau \geqslant 1, \tag{3.23}$$

where C is some constant depending only on d.

Proof. Similar argument as the proof of Lemma 3.1 yields that

$$\mu^*(\cdot, \tau) \prec \bar{\mu}(\cdot, \tau) \quad \text{for all } \tau \geqslant 0, \tag{3.24}$$

where $\bar{\mu}(\cdot, \tau)$ is the solution to (3.14) with initial data μ_0^*. Since μ_0^* is radially symmetric and bounded above, we can find R_0 sufficiently small, such that $\mu_0^* \prec \frac{1}{R_0^d} \mu_A(\frac{\cdot}{R_0})$, where μ_A is as defined in (3.16). It then follows from Proposition 2.4 and Lemma 3.4 that

$$\bar{\mu}(\cdot, \tau) \prec \frac{1}{R(\tau)^d} \mu_A\left(\frac{\cdot}{R(\tau)}\right) \quad \text{for all } \tau \geqslant 0, \tag{3.25}$$

where $R(\tau)$ satisfies the ODE (3.19) with initial data $R(0) = R_0$. Combining (3.24) and (3.25), we obtain that

$$\|\mu(\cdot, \tau)\|_\infty = \|\mu^*(\cdot, \tau)\|_\infty \leqslant \frac{1}{R(\tau)^d} \|\mu_A\|_\infty \quad \text{for all } \tau \geqslant 0.$$

In order to bound the right-hand side of the above inequality, we first find an upper bound for $1/R(\tau)^d$. It can be readily verified that $\tilde{R}(\tau) = \min\{\frac{1}{2}\tau^{\frac{1}{d+1}}, \frac{1}{2}\}$ is a subsolution to (3.19) for any $R_0 > 0$, which implies that $R(\tau) \geqslant \tilde{R}(\tau) \geqslant \frac{1}{2}$ for all $\tau \geqslant 1$, thus $\frac{1}{R(\tau)^d} \leqslant 2^d$ for all $\tau \geqslant 1$.

Next we will estimate $\|\mu_A\|_\infty$. Note that μ_A is radially decreasing for any $0 < A < M_c$, moreover $\|\mu_A\|_\infty = \mu_A(0)$ is increasing with respect to A. Therefore we readily obtain a rough bound $\|\mu_A\|_\infty \leqslant C_1$ for all $0 < A < M_c/2$, where $C_1 = \mu_{M_c/2}(0)$ only depends on d.

Note that this rough bound of $\|\mu_A\|_\infty$ gives us an upper bound for the velocity field given by the interaction term, namely

$$\partial_r(\mu_A * \mathcal{N}) = \frac{M(r; \mu_A)}{\sigma_d r^{d-1}} \leqslant \frac{C_1 r}{d}. \tag{3.26}$$

To refine the bound for $\|\mu_A\|_\infty$, we compare μ_A with $\tilde{\mu}_A$, where $\tilde{\mu}_A$ is the radial stationary solution to the following equation

$$\mu_\tau = \Delta\mu^m + \nabla \cdot \left(\mu\nabla \frac{(1+C_1)|\lambda|^2}{2d}\right). \tag{3.27}$$

Making use of (3.26), mass comparison yields that $\mu_A \prec \tilde{\mu}_A$, which implies $\mu_A(0) \leqslant \tilde{\mu}_A(0)$. On the other hand note that (3.27) is a Fokker–Planck equation, whose stationary solution is given by

$$\tilde{\mu}_A = \left(C_A - \frac{(1+C_1)(m-1)}{2dm}|\lambda|^2\right)_+^{1/(m-1)},$$

where $C_A > 0$ is the unique constant such that $\|\tilde{\mu}_A\|_1 = A$. A simple algebraic manipulation shows that $\tilde{\mu}_A(0) \leqslant CA^{2/d}$, where $C > 0$ depends only on d, therefore we can conclude. $\square$

The next lemma shows that if the mass is small, any solution with compactly supported initial data will eventually be confined in some small disk.

Lemma 3.10. *Suppose $d \geqslant 3$ and $m = 2 - 2/d$. Then for any $R_0 > 0$, there exists some sufficiently small $A_0 > 0$, such that all weak solutions to (3.14) with continuous and compactly supported initial data and mass $0 < A < A_0$ will be eventually confined in $B(0, R_0)$.*

Proof. Let $\mu(\lambda, \tau)$ be a weak solution to (3.14) with continuous and compactly supported initial data and mass $0 < A < A_0$, where A_0 is a small constant depending on R_0 and d to be determined later.

In the proof of this lemma we take $\tau = 1$ to be the starting time, in order to take advantage of the estimate (3.23). Our goal is to show that if the support of $\mu(\cdot, 1)$ is contained in some disk $B(0, R)$ where $R > R_0 - K_A$ and K_A is as defined in (3.23), then there exists some time $T > 1$ to be determined later, such that

$$\text{supp}\,\mu(\cdot, \tau) \subset B(0, R + K_A) \quad \text{for all } \tau \in [1, T], \tag{3.28}$$

moreover at time T the support can be fit into some disk smaller than $B(0, R)$, namely

$$\text{supp}\,\mu(\cdot, T) \subset B(0, R - K_A/2). \tag{3.29}$$

By taking T as the starting time and repeating this procedure, we know that eventually the support will be confined in $B(0, R_0)$.

In order to deal with non-radial solution, we shall construct barriers in the density sense instead of in mass sense. Although comparison principle in density sense does not directly hold for (3.14) due to the nonlocal term, if we treat $V(\lambda, \tau) := \mu * \mathcal{N}$ as a fixed *a priori* potential, then (3.14) becomes

$$\mu_\tau = \Delta \mu^m + \nabla \cdot \left(\mu \nabla \left(\frac{|\lambda|^2}{2d} + V(\lambda, \tau) \right) \right), \tag{3.30}$$

which is a porous medium equation with a drift, and the weak solutions to it enjoy the comparison principle due to [5]. It follows from (3.23) that the following estimates of V hold

$$\Delta V(\lambda, \tau) \leqslant \sup_{\lambda, \tau} \mu \leqslant K_A \quad \text{for } \lambda \in \mathbb{R}^d, \ \tau \geqslant 1,$$

and

$$\left| \nabla V(\lambda, \tau) \right| \leqslant \sup_{\mu, \lambda} \left(\mu * \frac{1}{\sigma_d |\lambda|^{d-1}} \right) \leqslant C(d) A^{\frac{3}{d} - \frac{2}{d^2}} \quad \text{for } \lambda \in \mathbb{R}^d, \ \tau \geqslant 1.$$

Note that in both estimates above, the right-hand side will go to zero as $A \to 0$. We also point out that if $R \gg A^{\frac{3}{d} - \frac{2}{d^2}}$, then ∇V will be dominated by $\nabla \frac{|\lambda|^2}{2d}$ around $r = R$.

Next we will construct some explicit supersolution $\tilde{\mu}$ to (3.30). More precisely, we hope to find a continuous radially decreasing function $\tilde{\mu}$ defined in $\{r > R - K_A\} \times [1, T]$ for some T, such that $\tilde{\mu}$ satisfies the following inequality

$$\tilde{\mu}_\tau \geqslant \partial_{rr} \tilde{\mu}^m + \left(\partial_r + \frac{d-1}{r} \right) \left(\frac{\tilde{\mu} r}{d} \right) + \tilde{\mu} K_A + |\partial_r \tilde{\mu}| C(d) A^{\frac{3}{d} - \frac{2}{d^2}} \quad \text{for all } r > R - K_A, \ \tau \in [1, T], \tag{3.31}$$

while $\tilde{\mu}$ also satisfies the initial condition

$$\tilde{\mu}(r, 0) \geqslant K_A \quad \text{for all } R - K_A \leqslant r \leqslant R, \tag{3.32}$$

and the boundary condition

$$\tilde{\mu}(r, \tau) \geqslant K_A \quad \text{at } r = R - K_A \text{ for all } \tau \in [1, T]. \tag{3.33}$$

The inequalities (3.31)–(3.34) guarantee that $\tilde{\mu}$ is a supersolution to (3.30). If A is small enough such that $R > C A^{\frac{3}{d} - \frac{2}{d^2}}$ for some large constant C depending on d, one can check that

$$\tilde{\mu}(\lambda, \tau) = \left[2K_A - \tau \left(r - (R - K_A) \right) \right]_+^{1/m}$$

satisfies the inequalities (3.31)–(3.33) for $1 \leqslant \tau \leqslant 4$, hence comparison principle yields that $\mu \leqslant \tilde{\mu}$ in $\{r > R - K_A\}$ for all $\tau \in [1, 4]$.

The reason we choose $\tilde{\mu}$ as above is that its support will shrink after some time: note that its support stays in $B(0, R + K_A)$ for $\tau \in [1, 4]$, and most importantly, at $\tau = 4$, the support of $\tilde{\mu}$ can be fit into a disk smaller than $B(0, R)$, namely

$$\operatorname{supp} \tilde{\mu}(\cdot, 4) \subset B(0, R - K_A/2). \tag{3.34}$$

Since comparison property gives that $\operatorname{supp} \mu(\cdot, \tau) \subset \operatorname{supp} \tilde{\mu}(\cdot, \tau)$ for all $\tau \in [1, 4]$, we immediately obtain (3.28) and (3.29), which complete the proof. $\quad \square$

Making use of the above two lemmas, in the next theorem we show that when the mass is sufficiently small, there cannot be any non-radial stationary solutions.

Theorem 3.11. *Suppose $d \geqslant 3$ and $m = 2 - 2/d$. Then when $0 < A < M_c/2$ is sufficiently small, the compactly supported stationary solution to* (3.14) *is unique.*

Proof. Due to Corollary 3.7, we know that for any $0 < A < M_c$, there does not exist any compactly supported radial stationary solution other than μ_A. Hence it suffices to prove that when A is sufficiently small, every compactly supported stationary solution is radially symmetric.

Suppose $\nu_A(\lambda)$ is a compactly supported stationary solution to (3.14), which is not radially symmetric. Since ν_A is stationary, it satisfies

$$\frac{m}{m-1}\nu_A^{m-1} + \nu_A * \mathcal{N} + \frac{|\lambda|^2}{d} = C \quad \text{in } \overline{\{\nu_A > 0\}}, \tag{3.35}$$

where different positive components of ν_A may have different C's. Heuristically, the idea is to argue that the term $\nu_A * \mathcal{N}$ must be more "roundish" than $\frac{m}{m-1}\nu_A^{m-1}$ if ν_A is non-radial, thus get a contradiction.

We point out that (3.35) implies that ν_A is continuous in $\mathbb{R}^d$ and smooth inside its positive set. This enables us to find two points $a, b \in \mathbb{R}^d$ in the same connected component of $\overline{\{\nu_A > 0\}}$, satisfying $|a| = |b|$ and

$$\nu_A(a) - \nu_A(b) = \sup_{|x|=|y|} \big(\nu_A(x) - \nu_A(y)\big) > 0. \tag{3.36}$$

We claim that when A is sufficiently small, the following inequality holds

$$\frac{m}{m-1}\big|\nu_A^{m-1}(a) - \nu_A^{m-1}(b)\big| > \big|(\nu_A * \mathcal{N})(a) - (\nu_A * \mathcal{N})(b)\big|, \tag{3.37}$$

then (3.37) would contradict (3.35).

We start with the left-hand side of (3.37): Lemma 3.9 implies that both $\nu_A(a)$ and $\nu_A(b)$ are much smaller than 1 when A is small. Since $0 < m - 1 < 1$, it follows that

$$\frac{m}{m-1}\big|\nu_A^{m-1}(a) - \nu_A^{m-1}(b)\big| > \big|\nu_A(a) - \nu_A(b)\big|,$$

if A is sufficiently small. In order to prove (3.37), it suffices to show that

$$\big|(\nu_A * \mathcal{N})(a) - (\nu_A * \mathcal{N})(b)\big| < \big|\nu_A(a) - \nu_A(b)\big|. \tag{3.38}$$

We introduce a linear transformation $T : \mathbb{R}^d \to \mathbb{R}^d$ which is a rotation that maps a to b. Then radial symmetry of $\mathcal{N}$ yields that $(\nu_A * \mathcal{N})(b) = ((\nu_A \circ T) * \mathcal{N})(a)$.

In addition, T being a rotation implies that $|T(x)| = |x|$ for any $x = \mathbb{R}^d$, hence from the way we choose a and b, we have $|\nu_A(T(x)) - \nu_A(x)| \leqslant \nu_A(a) - \nu_A(b)$ for any $x \in \mathbb{R}^d$. Thus

$$
\begin{aligned}
\big|(\nu_A * \mathcal{N})(a) - (\nu_A * \mathcal{N})(b)\big| &= \big|(\nu_A * \mathcal{N})(a) - \big((\nu_A \circ T) * \mathcal{N}\big)(a)\big| \\
&\leqslant \int_{\mathbb{R}^d} \big|\nu_A(y) - \nu_A\big(T(y)\big)\big|\big|\mathcal{N}(a-y)\big|\, dy \\
&\leqslant \big(\nu_A(a) - \nu_A(b)\big) \int_{B(0,R)} \big|\mathcal{N}(y)\big|\, dy,
\end{aligned}
$$

where $B(0, R)$ is the smallest disk that contains the support of ν_A. Now we make use of Lemma 3.10, which shows that we can fit the support of ν_A into an arbitrarily small disk by letting A be sufficiently small. Therefore we can choose R such that $\int_{B(0,R)} |\mathcal{N}(y)|\, dy < 1/2$, then let A be sufficiently small such that $\operatorname{supp} \nu_A \subset B(0, R)$. This gives us (3.38), which leads to a contradiction and hence completes the proof. $\quad\square$

Remark 3.12. For general $0 < A < M_c$, we are unable to prove the uniqueness of the compactly supported stationary solution. The difficulty lies in the fact that for larger mass we are only able to show the support lies in a disk with radius $O(1)$. Hence instead of (3.38), we can only obtain $|(\nu_A * \mathcal{N})(a) - (\nu_A * \mathcal{N})(b)| < C|\nu_A(a) - \nu_A(b)|$, where C might be a large constant, which stops us from getting a contradiction.

Once we obtain the uniqueness of compactly supported stationary solution for small mass, the following corollary shows that all solution with compactly supported initial data must converge to this unique stationary solution as $\tau \to \infty$.

Corollary 3.13. *Suppose $d \geqslant 3$ and $m = 2 - 2/d$. Let $\mu(\lambda, \tau)$ be a weak solution to (3.14) with mass $0 < A < M_c/2$ being sufficiently small, where the initial data $\mu(\cdot, 0)$ is nonnegative, continuous and compactly supported. Then as $\tau \to \infty$, we have*

$$\big\| \mu(\cdot, \tau) - \mu_A(\cdot) \big\|_\infty \to 0, \tag{3.39}$$

where μ_A is as defined in (3.16).

Proof. The proof is similar as the proof of Theorem 3.2, and actually it is simpler here since there is a unique stationary solution, instead of a family of stationary solution in the case of Theorem 3.2.

When the initial data $\mu(\cdot, 0)$ is bounded and compactly supported, Lemma 3.9 and Lemma 3.10 show that $\mu(\cdot, \tau)$ would be uniformly bounded and stay in some fixed compact set for all $\tau \geqslant 1$. In addition, the continuity result in [16] indicates that $\mu(\lambda, \tau)$ is uniformly continuous in space and time in $\mathbb{R}^d \times [1, \infty)$.

As a result, for any time sequence τ_n that increases to infinity, using the same argument as in the proof of Theorem 3.2, we can extract a subsequence τ_{n_k} such that $\mu(\cdot, \tau_{n_k})$ uniformly converges to some continuous function μ_∞, where μ_∞ is a compactly supported solution that satisfies (3.35). Theorem 3.11 ensures that μ_∞ must coincide with μ_A when A is sufficiently small, yielding that $\mu(\cdot, \tau)$ indeed converges to μ_A uniformly as $\tau \to \infty$. $\quad\square$

Remark 3.14. Since $\mu(\cdot, 0)$ is confined in some compact set for all time, (3.39) implies that $\|\mu(\cdot, \tau) - \mu_A(\cdot)\|_p \to 0$ as $\tau \to \infty$ for all $p \geqslant 1$. Now if we scale back to the original variables, it immediately follows that $\|u(\cdot, t) - u_A(\cdot)\|_p \to 0$ as $t \to \infty$ for all $p \geqslant 1$, where u_A is the dissipating self-similar solution as defined in (3.17). However the rate of convergence here is unknown, since the proof is done by extracting a subsequence of time.

4. Application to aggregation models with repulsive–attractive interactions

In this section we consider the following integro-differential equation

$$u_t = \nabla \cdot (u \nabla K * u), \tag{4.1}$$

where the interaction kernel K has a repulsion component in the form of the Newtonian potential $\mathcal{N}(x) = -\frac{1}{(d-2)\sigma_d |x|^{d-2}}$ and an attraction component satisfying the power law, namely

$$K(x) = \mathcal{N}(x) + \frac{1}{q}|x|^q, \tag{4.2}$$

where $2 - d < q \leqslant 2$, and when $q = 0$ the second term is replaced by $\ln |x|$.

The global existence of weak solution is established in [18] for $q > 2 - d$. Next we show that mass comparison holds for (4.1) between weak solutions.

Proposition 4.1. *Let $u_1(x, t)$, $u_2(x, t)$ be two radially symmetric weak solution to (4.1), which are compactly supported for all $t \geqslant 0$. If $u_1(\cdot, 0) \prec u_2(\cdot, 0)$, then $u_1(\cdot, t) \prec u_2(\cdot, t)$ for all $t \geqslant 0$.*

Proof. Note that $\Delta \frac{1}{q}|x|^q = (q + d - 2)|x|^{q-2}$, which is locally integrable in $\mathbb{R}^d$, nonnegative and radially decreasing when $2 - d < q \leqslant 2$. Therefore the conditions (**C**), (**K1**), (**K2'**), (**V1**) are met, (where (**K2'**) is as defined in Remark 2.5), and (4.1) becomes a special case of the general equation (2.1) in Section 2. Due to the discussion in Remark 2.5, we can apply mass comparison to the compactly supported solutions.

Without loss of generality we assume that both $u_1(\cdot, 0)$ and $u_2(\cdot, 0)$ are continuous, and for general initial data we can use approximation. Due to Theorem 2.5 of [18], we have M_i is C^1 in both space and time for all t, where $i = 1, 2$. Now we can apply Proposition 2.4 to conclude that $u_1(\cdot, t) \prec u_2(\cdot, t)$ for all time: although the proof of Proposition 2.4 requires M_i be C^2 in space, the C^2 requirement are only for the diffusion term. Since the right-hand side of (4.1) only has aggregation terms, C^1 continuity of M_i is sufficient. $\quad\square$

The following existence and uniqueness result of a stationary solution is established in [18].

Proposition 4.2. *(See [18, Theorem 3.1].) For every $q > 2 - d$ and mass $A > 0$, there exists a unique radius R_A (that depends on q and d only) and a unique steady state u_s of the aggregation model* (4.1)–(4.2) *that is supported on $B(0, R_A)$, has mass A and is continuous on its support.*

In addition, u_s is radially decreasing in $B(0, R_A)$ if $q \leqslant 2$, and is radially increasing if $q \geqslant 2$.

It is proved by [18] that in its positive set, u_s satisfies $\nabla K * u_s = 0$, which can also be written as $-u_s + \Delta(\frac{1}{q}|x|^q) * u_s = 0$, i.e.

$$-u_s + (q + d - 2)|x|^{q-2} * u_s = 0 \quad \text{in } \{u_s > 0\}. \tag{4.3}$$

In the proposition below, we construct a family of explicit subsolutions, all of which are compactly supported and converge to the stationary solution u_s exponentially fast.

Lemma 4.3 *(A family of explicit subsolutions). Suppose $d \geqslant 3$ and $2 - d < q \leqslant 2$. Let u_s be the stationary solution to* (4.1) *with mass A, as given by Proposition* 4.2. *We define the self-similar function $\bar{u}$ as*

$$\bar{u}(x, t) := \frac{1}{R^d(t)} u_s\left(\frac{x}{R(t)}\right), \tag{4.4}$$

where $R(t)$ solves the ODE

$$\begin{cases} \dot{R}(t) = C_1(1 - R^{d+q-2})R^{-d+1}, \\ R(0) = R_0, \end{cases} \tag{4.5}$$

where $R_0 > 1$, and C_1 is some fixed constant only depending on q, d and A. Then for all $t \in [0, \infty)$, $\bar{u}(x, t)$ is a subsolution to (4.1) *in the mass comparison sense.*

Proof. The proof is similar to the proof of Lemma 3.4. Since $\bar{u}$ is a self-similar function, it can be easily verified that $\bar{u}$ solves the following transport equation

$$\bar{u}_t + \nabla \cdot \left(\bar{u}\frac{\dot{R}(t)}{R(t)}x\right) = 0,$$

which implies that the mass function of $\bar{u}$ satisfies

$$M_t(r, t; \bar{u}) = \sigma_d r^{d-1} M_r(r, t; \bar{u})\left(-\frac{\dot{R}(t)}{R(t)}\right)r. \tag{4.6}$$

Our goal is to show that $\bar{u}$ is a subsolution to (4.1) in mass comparison sense, which is equivalent to the following inequality due to Lemma 2.3:

$$M_t(r, t; \bar{u}) \leqslant \sigma_d r^{d-1} M_r(r, t; \bar{u})\left[\frac{M(r, t; \bar{u})}{\sigma_d r^{d-1}} + \frac{\tilde{M}(r, t; \bar{u})}{\sigma_d r^{d-1}}\right], \tag{4.7}$$

where $\tilde{M}(r, t; \bar{u}) = \int_{B(0,r)}(q + d - 2)(|x|^{q-2} * \bar{u})(x, t)\, dx$.

We point out that M_r is nonnegative by definition, since $M_r(r, t; \bar{u}) = \sigma_d r^d \bar{u}$, where $\bar{u}$ is nonnegative. By comparing (4.6) and (4.7), it suffices to prove that the following inequality holds for $\bar{u}(r, t)$ for all $0 \leqslant r \leqslant R(t)$:

$$-\frac{\dot{R}(t)}{R(t)}r \leqslant -\frac{M(r, t; \bar{u})}{\sigma_d r^{d-1}} + \frac{\tilde{M}(r, t; \bar{u})}{\sigma_d r^{d-1}}. \tag{4.8}$$

Next we will investigate the terms on the right-hand side of (4.8). Since $\bar{u}$ is defined as a continuous scaling of u_s, for the first term on the right-hand side of (4.8), we have

$$M(r, t; \bar{u}) = M\left(\frac{r}{R(t)}; u_s\right).$$

For the second term, we obtain that for any $0 \leqslant r \leqslant R(t)$,

$$\tilde{M}(r, t; \bar{u}) = (q + d - 2) \int\limits_{B(0,r)} \int\limits_{\mathbb{R}^d} |x - y|^{q-2} \frac{1}{R(t)^d} u_s\left(\frac{y}{R(t)}\right) dy \, dx$$

$$= (q + d - 2) R(t)^{q-2} \int\limits_{B(0,r)} \int\limits_{\mathbb{R}^d} \left|\frac{x}{R(t)} - z\right|^{q-2} u_s(z) \, dz \, dx \quad (z := y/R(t))$$

$$= (q + d - 2) R(t)^{q-2} \int\limits_{B(0,r)} \left(|x|^{q-2} * u_s\right)\left(\frac{x}{R(t)}\right) dx$$

$$= R(t)^{q-2} \int\limits_{B(0,r)} u_s\left(\frac{x}{R(t)}\right) dx \quad \left(\text{by (4.3)}\right)$$

$$= R(t)^{d+q-2} M\left(\frac{r}{R(t)}; u_s\right).$$

Putting the above two equations together yields

$$\text{RHS of (4.8)} = \left(-1 + R(t)^{d+q-2}\right) \frac{M\left(\frac{r}{R(t)}; u_s\right)}{\sigma_d r^{d-1}}$$

$$\geqslant C_1 \left(-1 + R(t)^{d+q-2}\right) R(t)^{-d} r, \tag{4.9}$$

where C_1 only depends on q, d and A. Here in the last inequality we used the fact that u_s is radially decreasing, which implies that $M(r; u_s) \geqslant C_A |B(0,r)|$ for all $0 \leqslant r \leqslant R_A$, where C_A is the average density of u_s in its support $B(0, R_A)$.

Since $\dot{R} = C_1(1 - R^{d+q-2})R^{-d+1}$ by definition, the above inequality implies that (4.8) is true, which completes the proof. $\quad\square$

Remark 4.4. Similarly, we can construct a family of explicit supersolutions $\bar{u}$ in the mass comparison sense: here $\bar{u}$ is defined in (4.4), where $R(t)$ solves the ODE (4.5) with initial data $0 < R_0 < 1$, and the constant C_1 in (4.5) is replaced by some other fixed constant C_2, which also only depends on q, d and A.

Making use of the subsolutions and supersolutions we constructed in Lemma 4.3 and Remark 4.4 respectively, we next prove that all radial solutions with compactly supported initial data will converge to the unique stationary solution exponentially fast.

Theorem 4.5. *Suppose $d \geqslant 3$ and $2 - d < q \leqslant 2$. Let u be a weak solution to (4.1) with initial data u_0 and mass A, where $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ is non-negative, radially symmetric and compactly supported. In addition, we assume that u_0 is strictly positive in a neighborhood of 0. Let u_s be the unique stationary solution with mass A, as given by Proposition 4.2. Then as $t \to \infty$, $u(\cdot, t)$ converges to u_s exponentially fast in Wasserstein distance.*

Proof. The proof is similar to the proof of Proposition 3.5. Since u_0 is bounded, and is strictly positive in a neighborhood of 0, we can find R_{01} sufficiently large and R_{02} sufficiently small, such that

$$\frac{1}{R_{01}^d} u_s\left(\frac{\cdot}{R_{01}}, 0\right) \prec u_0 \prec \frac{1}{R_{02}^d} u_s\left(\frac{\cdot}{R_{02}}, 0\right).$$

Let $\bar{u}_1$ be the subsolution given by Lemma 4.3 with initial data $\frac{1}{R_{01}^d} u_s(\frac{\cdot}{R_{01}}, 0)$, and $\bar{u}_2$ be the supersolution given by Remark 4.4 with initial data $\frac{1}{R_{02}^d} u_s(\frac{\cdot}{R_{02}}, 0)$. Then mass comparison in Proposition 2.4 yields $\bar{u}_1(\cdot, t) \prec u(\cdot, t) \prec \bar{u}_2(\cdot, t)$ for all t.

Now it suffices to show that $\bar{u}_1$ and $\bar{u}_2$ both converge to u_s exponentially fast in Wasserstein distance. Note that $\bar{u}_1(\cdot, t)$ is a continuous scaling of u_s with scaling coefficient $R(t)$, where $R(t)$ satisfies the ODE (4.5) and hence converges to 1 exponentially. More precisely, we have

$$\left| R(t) - 1 \right| \leqslant C_1' e^{-C_1 (d+q-2)t},$$

where C_1 is as given in Lemma 4.3, and C_1' depends on q, d and R_{01}. Similar result holds for the supersolution $\bar{u}_2$. Then argue as in Corollary 3.7, we have for all $p > 1$ that

$$W_p\left(\frac{u_i(\cdot, t)}{A}, \frac{u_s}{A} \right) \leqslant c_1 e^{-C_i (d+q-2)t} \quad \text{for } i = 1, 2,$$

where c_i depends on R_{0i} respectively. Since u_s is squeezed between $\bar{u}_1$ and $\bar{u}_2$ in the mass comparison sense, the inequality above yields

$$W_p\left(\frac{u(\cdot, t)}{A}, \frac{u_s}{A} \right) \leqslant C' e^{-C (d+q-2)t} \quad \text{for } i = 1, 2,$$

where $C := \min(C_1, C_2)$ depends on d, q and A, while C' depends on d, q, A and $u(\cdot, 0)$. $\quad\square$

Acknowledgements

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