

Positive solutions to Kirchhoff type equations with nonlinearity having prescribed asymptotic behavior [☆]

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Abstract

Existence and bifurcation of positive solutions to a Kirchhoff type equation

$$\begin{cases} -\left(a + b \int_{\Omega} |\nabla u|^2\right) \Delta u = \nu f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

are considered by using topological degree argument and variational method. Here f is a continuous function which is asymptotically linear at zero and is asymptotically 3-linear at infinity. The new results fill in a gap of recent research about the Kirchhoff type equation in bounded domain, and in our results the nonlinearity may be resonant near zero or infinity.

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1. Introduction

Let Ω be a bounded domain in \mathbb{R}^N , $N = 1, 2, 3$, with a smooth boundary $\partial\Omega$. We consider the following Kirchhoff type nonlocal problem with Dirichlet boundary condition

$$\begin{cases} -\left(a + b \int_{\Omega} |\nabla u|^2\right) \Delta u = \nu f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $a \geq 0$, $b \geq 0$ are real constants, $a + b > 0$ and ν is a positive parameter.

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Problem (1.1) is the stationary case of a nonlinear wave equation

$$u_{tt} - \left(a + b \int_{\Omega} |\nabla u|^2 \right) \Delta u = \nu f(x, u), \tag{1.2}$$

which was first proposed by Kirchhoff in 1883 to describe the transversal oscillations of a stretched string, where u denotes the displacement, f is the external force, b represents the initial tension, and a is related to the intrinsic properties of the string. The solvability of Kirchhoff type equation (1.2) has been well studied in general dimension by many authors, see [6,7] and the references therein. More recently, there have been many papers studying the elliptic version Kirchhoff type equations (1.1) with $\nu = 1$ by using variational method, see for example, [1,3–5,10,11, 15–17,20,21,23]. To state the conditions and conclusions in this paper, we recall some results about the following two eigenvalue problems:

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

and

$$\begin{cases} -\left(\int_{\Omega} |\nabla u|^2 \right) \Delta u = \mu u^3, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

Let $\lambda_1 > 0$ be the principal eigenvalue of the problem (1.3) and let $\varphi_1 > 0$ be its associated eigenfunction. It is known that λ_1 can be characterized by

$$\lambda_1 = \inf \left\{ \int_{\Omega} |\nabla u|^2 : u \in H_0^1(\Omega), \int_{\Omega} |u|^2 = 1 \right\}, \tag{1.5}$$

where $H_0^1(\Omega)$ is the usual Sobolev space defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\| = (\int_{\Omega} |\nabla u|^2)^{1/2}$. On the other hand define

$$\mu_1 = \inf \left\{ \|u\|^4 : u \in H_0^1(\Omega), \int_{\Omega} |u|^4 = 1 \right\}. \tag{1.6}$$

As shown in [17], there exists $\mu_1 > 0$ which is the principal eigenvalue of (1.4) and there is a corresponding eigenfunction $\phi_1 > 0$ in Ω . It is well known that λ_1 is a simple eigenvalue and any eigenfunction corresponding to other eigenvalue must be sign-changing. In Lemma 5.3, we show that if Ω is a ball in \mathbb{R}^N , then μ_1 must be a simple eigenvalue of (1.4) and any eigenfunction corresponding to another eigenvalue must be sign-changing. This appears to be the first such result for the eigenvalue problem (1.4).

We impose on f the following global conditions.

- (f₁) $f \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$, $f(x, t) \geq 0$ for any $x \in \bar{\Omega}$, $t > 0$ and $f(x, t) = 0$ for any $x \in \bar{\Omega}$, $t \leq 0$;
- (f₂) For $f_0, f_\infty < \infty$, the limits

$$\lim_{t \rightarrow 0^+} \frac{f(x, t)}{a\lambda_1 t + b\mu_1 t^3} = f_0, \quad \lim_{t \rightarrow \infty} \frac{f(x, t)}{a\lambda_1 t + b\mu_1 t^3} = f_\infty$$

exist uniformly for $x \in \bar{\Omega}$.

Recall that f is called asymptotically linear at zero and asymptotically 3-linear at infinity if (f₂) holds and $a > 0$, $b > 0$. In addition, (f₁) and (f₂) guarantee that f satisfies the subcritical growth condition for $1 \leq N \leq 3$, that is,

- (f₀) there exist $C > 0$ and $p \in [2, 2^*)$ such that

$$|f(x, t)| \leq C(1 + |t|^{p-1}), \quad (x, t) \in \bar{\Omega} \times \mathbb{R},$$

where $2^* = \infty$ for $N = 1, 2$ and $2^* = 6$ for $N = 3$.

The goal of this paper is to obtain sufficient conditions on the constants f_0, f_∞ for problem (1.1) to have positive solutions by using topological degree method and critical point theory.

In [20], the authors assumed that f is asymptotically linear near zero and is 3-superlinear at infinity which means that

$$\lim_{t \rightarrow \infty} \frac{f(x, t)}{t^3} = \infty.$$

In [16], it was assumed that f was superlinear at zero and is 3-superlinear at infinity. In [17], the situation that $a > 0, b > 0$ and f satisfies (f_2) is considered, and under the additional conditions that $f_0\lambda_1$ and $f_\infty\mu_1$ are not an eigenvalue of (1.3) and (1.4) respectively, the existence of a nontrivial solution to problem (1.1) with $v = 1$ was proved. Motivated by [17], in the present paper we are concerned with problem (1.1) under the assumptions (f_1) and (f_2) . Our main results are as follows. Note that in our results the constants $f_0\lambda_1$ or $f_\infty\mu_1$ could be an eigenvalue to (1.3) or (1.4) respectively.

Theorem 1.1. *Assume that $N = 1, 2, 3, a > 0, b > 0$ and f satisfies (f_1) and (f_2) with $f_0 > 1$ and $f_\infty < 1$. Then (1.1) with $v = 1$ has a positive solution. In addition, if $a > 0, b = 0$, then (1.1) has a positive solution with $v = 1$ for all $N \geq 1$.*

Theorem 1.2. *Assume that $N = 1, 2, 3, a \geq 0, b > 0$, and f satisfies (f_1) and (f_2) with $f_0 < 1$ and $f_\infty > 1$. Then one of the following alternatives holds:*

- (i) $(1, \infty)$ is a bifurcation point of (1.1) where a bifurcation from infinity occurs, that is, there exists a sequence of positive solutions (v_n, u_n) to (1.1) such that $\lim_{n \rightarrow \infty} v_n = 1$ and $\lim_{n \rightarrow \infty} \|u_n\| = \infty$; or
- (ii) (1.1) has a positive solution with $v = 1$.

In addition, if $a > 0, b = 0$, then (1.1) has a positive solution with $v = 1$ for all $N \geq 1$.

Remark 1.3. The results above complete the study made in the recent papers in the following sense. The methods used in [17] cannot be applied to the case that $f_0\lambda_1$ or $f_\infty\mu_1$ is an eigenvalue. In addition, there have been no previous studies considering the bifurcation phenomena in Kirchhoff type equations to the best of our knowledge. In Theorem 5.2, we will show that if Ω is a ball, then the alternative (ii) in Theorem 1.2 must hold, thus the bifurcation from infinity does not occur in this case.

Remark 1.4. For the Kirchhoff equation (1.1), the usual assumption is that $a \geq 0, b \geq 0$ and $a + b > 0$ as in this paper, see [5]. When $a > 0$ and $b = 0$, we can reduce (1.1) to

$$\begin{cases} -\Delta u = g(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where $g(x, t) = (v/a)f(x, t)$. Eq. (1.7) has been extensively studied for bounded domain $\Omega \subset \mathbb{R}^N$ with $N \geq 1$. Under the assumption:

- (g) For $g_0, g_\infty < \infty$, the limits $\lim_{t \rightarrow 0^+} \frac{g(x, t)}{t} = g_0$ and $\lim_{t \rightarrow \infty} \frac{g(x, t)}{t} = g_\infty$ exist uniformly for $x \in \bar{\Omega}$,

the following results have been proved in [13]:

1. If $g_0 > \lambda_1$ and $g_\infty < \lambda_1$, then (1.7) has a positive solution;
2. If $g_0 < \lambda_1$ and $g_\infty > \lambda_1$, then (1.7) has a positive solution.

In this context, our results in this paper can be seen as the generalization of the results above to the Kirchhoff type equation (1.1).

Remark 1.5. In this paper, we prove Theorems 1.1 and 1.2 only for the case of $N = 1, 2, 3$ and $b > 0$. This is because that our proofs depend fundamentally on the subcritical growth condition (f_0) , which is essential in the definition of

L in Section 2 and the compact embedding from $H_0^1(\Omega)$ to $L^4(\Omega)$ in Section 4. We conjecture that Theorems 1.1 and 1.2 still hold for dimension $N \geq 4$. In Theorem 1.1, we provide a positive answer to the existence of solutions to (1.1) for the cases $a > 0, b > 0$ and $a > 0, b = 0$, except the situation that $a = 0$ and $b > 0$, which is an open problem.

Throughout this paper, we denote by X the Sobolev space $H_0^1(\Omega)$ with the inner product $(u, v) = \int_{\Omega} \nabla u \cdot \nabla v$ and norm $\|u\|^2 = \int_{\Omega} |\nabla u|^2$, by X^* the duality space of X , by \rightharpoonup the weak convergence in X , and by $\langle \cdot, \cdot \rangle$ the duality pairing between X^* and X . The symbols $C_1, C_2, C_\varepsilon, \dots$ denote various positive constants whose exact values are not essential to the analysis of the problem.

This paper is organized as follows. In Section 2, we recall some preliminaries and prove some lemmas. In Section 3, using topological degree argument, we give the proof of Theorem 1.1. Section 4 is dedicated to prove Theorem 1.2 by using critical point theory. Furthermore, we will give some applications of Theorem 1.2 in Section 5.

2. Preliminaries

Let $P = \{u \in X: u(x) \geq 0, \text{ a.e. } x \in \Omega\}$ be the positive cone in X and let $P^* = \{h \in X^*: \langle h, u \rangle \geq 0, u \in P\}$ be its dual cone. Define nonlinear operators $A, L, K : X \rightarrow X^*$ by

$$\langle Au, v \rangle = (a + b\|u\|^2) \int_{\Omega} \nabla u \cdot \nabla v, \quad \langle Lu, v \rangle = \int_{\Omega} f(x, u)v, \quad \langle Ku, v \rangle = \int_{\Omega} uv, \quad u, v \in X.$$

We first show the following property of the operator A .

Lemma 2.1. *Assume that $a > 0$. Then the operator A is a homeomorphism from X to X^* and $A^{-1}(P^*) \subset P$.*

Proof. For any $u, v \in X$, we have

$$\begin{aligned} &\langle Au - Av, u - v \rangle \\ &= \langle Au, u \rangle - \langle Av, u \rangle - \langle Au, v \rangle + \langle Av, v \rangle \\ &= (a + b\|u\|^2)\|u\|^2 - (a + b\|v\|^2) \int_{\Omega} \nabla u \cdot \nabla v - (a + b\|u\|^2) \int_{\Omega} \nabla u \cdot \nabla v + (a + b\|v\|^2)\|v\|^2 \\ &= a \left(\|u\|^2 - 2 \int_{\Omega} \nabla u \cdot \nabla v + \|v\|^2 \right) + b \left[\|u\|^4 + \|v\|^4 - (\|u\|^2 + \|v\|^2) \int_{\Omega} \nabla u \cdot \nabla v \right] \\ &\geq a\|u - v\|^2 + b \left[\|u\|^4 + \|v\|^4 - \frac{1}{2}(\|u\|^2 + \|v\|^2)^2 \right] \\ &\geq a\|u - v\|^2. \end{aligned}$$

Hence, A is a strongly monotone operator. It is easy to see that A is continuous from X to X^* . By the strong monotone operator theorem [22, Theorem 26.A, p. 557], A is a homeomorphism.

To show the second part of the lemma, we assume that $h \in P^*$. By the first part of the lemma, there exists $u \in X$ such that

$$(a + b\|u\|^2) \int_{\Omega} \nabla u \cdot \nabla v = \langle h, v \rangle, \quad v \in X. \tag{2.1}$$

Taking $v = u^- = \min\{u, 0\}$ in (2.1), we have that $(a + b\|u\|^2)\|u^-\|^2 \leq 0$. Hence $u(x) \geq 0$ almost everywhere for $x \in \Omega$, that is, $u \in P$. The proof is completed. \square

Since the assumptions (f_1) and (f_2) hold, then f satisfies (f_0) . By [18, Proposition B.10, p. 90], we know that $L : X \rightarrow X^*$ is compact. Furthermore, we can easily see that L maps P into P^* by (f_1) . Similarly, $K : X \rightarrow X^*$ is

also compact and K maps P into P^* . Because we are concerned with the existence of positive solutions of (1.1) with $v = 1$ in Theorem 1.1, we may consider the following operator equation in P

$$Au = Lu. \tag{2.2}$$

Denote by P_r for $r > 0$ the bounded open subset $\{u \in P: \|u\| < r\}$ of P . If (2.2) has no solution on ∂P_r , that is, the completely continuous operator $A^{-1}L : \bar{P}_r \subset P \rightarrow P$ has no fixed point on ∂P_r , then by [2] the index of fixed point $i(A^{-1}L, P_r, P)$ is well defined. Hence, we can use the fixed point index theory to complete the proof of Theorem 1.1. To this end, we recall some necessary results about the fixed point index in the sequel.

Theorem 2.2. (See [2].) *Let E be a real Banach space, $V \subset E$ a cone, and $U \subset V$ a bounded open subset of V . If the completely continuous operator $B : \bar{U} \rightarrow V$ has no fixed point on ∂U , then there exists an integer $i(B, U, V)$, which is regarded as the fixed point index, and the following statements hold:*

- (i) *If $B : \bar{U} \rightarrow U$ is a constant mapping, then $i(B, U, V) = 1$;*
- (ii) *Assume that U_1 and U_2 are disjoint open subsets of U and B has no fixed point in $\bar{U} \setminus (U_1 \cup U_2)$. Then*

$$i(B, U, V) = i(B, U_1, V) + i(B, U_2, V),$$
where $i(B, U_i, V) = i(B|_{\bar{U}_i}, U_i, V)$, $i = 1, 2$;
- (iii) *If $H : [0, 1] \times \bar{U} \rightarrow V$ is a completely continuous homotopy and $H(t, u) \neq u$ for any $(t, u) \in [0, 1] \times \partial U$, then $i(H(t, \cdot), U, V)$ is independent of $t \in [0, 1]$;*
- (iv) *If $i(B, U, V) \neq 0$, then B has a fixed point in U .*

Before concluding this section, we recall another theorem from [14], which will be used to prove our second theorem in this paper. The “monotonicity trick” at the core of the recalled theorem was first formulated by Struwe [19].

Theorem 2.3. *Let $(E, \|\cdot\|)$ be a Banach space and $I \subset \mathbb{R}_+$ an interval. Consider the family of C^1 functionals on E ,*

$$J_v(u) = S(u) - vT(u), \quad v \in I,$$

with $J_v(0) = 0$, $v \in I$, T nonnegative and either $S(u) \rightarrow \infty$ or $T(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$. For any $v \in I$, we set

$$\Gamma_v = \{\gamma \in C([0, 1], E) : \gamma(0) = 0, J_v(\gamma(1)) < 0\}.$$

If for every $v \in I$ the set Γ_v is nonempty and

$$c_v = \inf_{\gamma \in \Gamma_v} \max_{t \in [0, 1]} J_v(\gamma(t)) > 0,$$

then for almost every $v \in I$ there exists a sequence $\{u_n^v\} \subset E$ such that

- (i) $\{u_n^v\}$ *is bounded;*
- (ii) $J_v(u_n^v) \rightarrow c_v$ *as $n \rightarrow \infty$;*
- (iii) $J'_v(u_n^v) \rightarrow 0$ *in the dual E^* as $n \rightarrow \infty$.*

3. Proof of Theorem 1.1

To prove Theorem 1.1, we first prove two lemmas.

Lemma 3.1. *Assume that $a > 0$, $b > 0$, (f₁) and (f₂) hold. Then*

$$L'_+(0) = af_0\lambda_1 K \tag{3.1}$$

and

$$\lim_{\|u\| \rightarrow \infty, u \in P} \frac{Lu - bf_\infty\mu_1 Ku^3}{\|u\|^3} = 0, \tag{3.2}$$

where $L'_+(0)$ is the right derivative of L at 0, see [8, p. 225].

Proof. The proof of the lemma is similar to that in [12,13]. For the sake of completeness, we give a proof here. Since (f₁) and (f₂) hold, then for any ε > 0, there exists C_ε > 0 such that

$$|f(x, t) - af_0\lambda_1 t| \leq \varepsilon t + C_\varepsilon t^3, \quad x \in \bar{\Omega}, \quad t \geq 0.$$

For any $u \in P \setminus \{0\}$, set $w = u/\|u\|$. Then we have by Hölder’s inequality and Sobolev’s inequality that

$$\begin{aligned} \sup_{\|v\| \leq 1} \left| \left\langle \frac{Lu - af_0\lambda_1 Ku}{\|u\|}, v \right\rangle \right| &\leq \sup_{\|v\| \leq 1} \int_{\Omega} \frac{|f(x, u) - af_0\lambda_1 u|}{\|u\|} |v| \\ &\leq \sup_{\|v\| \leq 1} \int_{\Omega} [\varepsilon w |v| + C_\varepsilon \|u\|^2 w^3 |v|] \\ &\leq \varepsilon C_1 + C_2 C_\varepsilon \|u\|^2, \end{aligned}$$

where C₁, C₂ > 0 are constants independent of ε. Therefore,

$$\lim_{\|u\| \rightarrow 0, u \in P} \frac{Lu - af_0\lambda_1 Ku}{\|u\|} = 0,$$

and (3.1) holds.

Since (f₁) and (f₂) hold, for any ε > 0, there exists C_ε > 0 such that

$$|f(x, t) - bf_\infty\mu_1 t^3| \leq C_\varepsilon + \varepsilon t^3, \quad x \in \bar{\Omega}, \quad t \geq 0.$$

For any $u \in P \setminus \{0\}$, set $w = u/\|u\|$. Then we have by Hölder’s inequality and Sobolev’s inequality that

$$\begin{aligned} \sup_{\|v\| \leq 1} \left| \left\langle \frac{Lu - bf_\infty\mu_1 Ku^3}{\|u\|^3}, v \right\rangle \right| &\leq \sup_{\|v\| \leq 1} \int_{\Omega} \frac{|f(x, u) - bf_\infty\mu_1 u^3|}{\|u\|^3} |v| \\ &\leq \sup_{\|v\| \leq 1} \int_{\Omega} [C_\varepsilon \|u\|^{-3} |v| + \varepsilon w^3 |v|] \\ &\leq C_3 C_\varepsilon \|u\|^{-3} + \varepsilon C_2, \end{aligned}$$

where C₃ > 0 is a constant. Therefore, (3.2) holds. The proof is completed. □

Lemma 3.2. Assume that a > 0, b > 0 and f₀ > 1. Then for r ∈ (0, r₁) where r₁ = √{a(f₀ - 1)/b},

$$i(A^{-1}(af_0\lambda_1 K), P_r, P) = 0.$$

Proof. Given 0 ≤ h ∈ C[∞]₀(Ω) with h ≠ 0, define a completely continuous homotopy H : [0, 1] × X → X* by

$$H(t, u) = af_0\lambda_1 Ku + tKh, \quad (t, u) \in [0, 1] \times P.$$

We first claim that the operator equation

$$Au = H(t, u)$$

has no solutions on [0, 1] × ∂P_r for r ∈ (0, r₁). Suppose this is not true. Then there exist t₁ ∈ [0, 1] and u₁ ∈ P with 0 < \|u₁\| < r₁ such that

$$Au_1 = H(t_1, u_1).$$

Thus for any v ∈ X, we have

$$(a + b\|u_1\|^2) \int_{\Omega} \nabla u_1 \cdot \nabla v = af_0\lambda_1 \int_{\Omega} u_1 v + t_1 \int_{\Omega} h v.$$

That is, u₁ is a weak solution of the following problem

$$\begin{cases} -\left(a + b \int_{\Omega} |\nabla u|^2\right) \Delta u = af_0\lambda_1 u + t_1 h, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

By the elliptic regularity theory and the strong maximum principle, we know that $u_1 \in C^2(\Omega) \cap C_0^1(\bar{\Omega})$ and $u_1 > 0$ in Ω . Hence, u_1 satisfies the following equation

$$-\Delta u = \lambda_1 u + \left(\frac{af_0\lambda_1}{a + b\|u\|^2} - \lambda_1\right)u + \frac{t_1}{a + b\|u\|^2}h, \quad \text{in } \Omega. \tag{3.3}$$

Since $\|u_1\| = r < r_1$, we have

$$\left(\frac{af_0\lambda_1}{a + b\|u_1\|^2} - \lambda_1\right)u_1 + \frac{t_1}{a + b\|u_1\|^2}h > 0, \quad \text{in } \Omega.$$

This is impossible since (3.3) has no positive solution. Notice that this fact holds for the problem

$$\begin{cases} -(a + b\|u\|^2)\Delta u = af_0\lambda_1 u + h, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Indeed, the above problem has no solutions in P_r for $r \in (0, r_1)$. Consequently, Theorem 2.2 (iii) and (i) imply that

$$i(A^{-1}(af_0\lambda_1 K), P_r, P) = i(A^{-1}H(0, \cdot), P_r, P) = i(A^{-1}H(1, \cdot), P_r, P) = 0, \quad r \in (0, r_1),$$

which completes the proof. \square

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. In order to prove Theorem 1.1 with $a > 0, b > 0$, we only need to show that the fixed point indices $i(A^{-1}L, P_r, P)$ take different values for small r and for large r by Theorem 2.2 (ii) and (iv).

Firstly, we prove that

$$i(A^{-1}L, P_r, P) = 1 \quad \text{for large } r.$$

To this end, we define a completely continuous homotopy function $H : [0, 1] \times X \rightarrow X^*$ by

$$H(t, u) = tLu, \quad (t, u) \in [0, 1] \times P.$$

We claim that there exists $R_0 > 0$ such that the operator equation

$$Au = H(t, u) \tag{3.4}$$

has no solutions on $[0, 1] \times \partial P_r$ for $r > R_0$. We prove by contradiction. Suppose that there exists a sequence $\{(t_n, u_n)\} \subset [0, 1] \times P$ such that

$$t_n \rightarrow t_0 \in [0, 1], \quad \|u_n\| \rightarrow \infty,$$

and (t_n, u_n) satisfies (3.4), that is,

$$(a + b\|u_n\|^2) \int_{\Omega} \nabla u_n \cdot \nabla v = t_n \int_{\Omega} f(x, u)v, \quad v \in X.$$

Let $w_n = u_n / \|u_n\|$ for any n . Then we have, for any $v \in X$,

$$\frac{a + b\|u_n\|^2}{\|u_n\|^2} \int_{\Omega} \nabla w_n \cdot \nabla v = t_n b f_{\infty} \mu_1 \int_{\Omega} w_n^3 v + t_n \int_{\Omega} \frac{f(x, u_n) - b f_{\infty} \mu_1 u_n^3}{\|u_n\|^3} v. \tag{3.5}$$

Since $\{w_n\}$ is bounded in P , we may assume, by passing to a subsequence if necessary, that $w_n \rightharpoonup w_0 \in P$. Taking $v = w_n$ in (3.5) and letting $n \rightarrow \infty$, we have from Lemma 3.1 that

$$1 = t_0 f_{\infty} \mu_1 \int_{\Omega} w_0^4.$$

Hence,

$$1 \leq f_\infty \mu_1 \int_{\Omega} w_0^4 \leq f_\infty,$$

which contradicts to $f_\infty < 1$. Taking $r > R_0$, we have

$$i(A^{-1}L, P_r, P) = i(A^{-1}H(1, \cdot), P_r, P) = i(A^{-1}H(0, \cdot), P_r, P) = 1.$$

Secondly, we complete the proof of Theorem 1.1 by showing that

$$i(A^{-1}L, P_r, P) = 0 \quad \text{for small } r.$$

Define now another completely continuous homotopy function

$$\tilde{H}(t, u) = (1-t)Lu + taf_0\lambda_1 Ku, \quad (t, u) \in [0, 1] \times P.$$

We show that there exists $r_0 > 0$ such that the operator equation

$$Au = \tilde{H}(t, u) \tag{3.6}$$

has no solutions on $[0, 1] \times \partial P_r$ for $r \in (0, r_0)$. Again we prove it by contradiction argument. Suppose that there exists a sequence $\{(t_n, u_n)\} \subset [0, 1] \times P$ such that

$$t_n \rightarrow t_0, \quad \|u_n\| \rightarrow 0,$$

and (t_n, u_n) satisfies (3.6). Let $w_n = u_n/\|u_n\|$ for any n . Then we have, for any $v \in X$,

$$\begin{aligned} \int_{\Omega} \nabla w_n \cdot \nabla v &= \frac{1-t_n}{a+b\|u_n\|^2} \int_{\Omega} \frac{f(x, u_n)}{\|u_n\|} v + \frac{t_n af_0 \lambda_1}{a+b\|u_n\|^2} \int_{\Omega} w_n v \\ &= \frac{1-t_n}{a+b\|u_n\|^2} \int_{\Omega} \frac{f(x, u_n) - af_0 \lambda_1 u_n}{\|u_n\|} v + \frac{af_0 \lambda_1}{a+b\|u_n\|^2} \int_{\Omega} w_n v. \end{aligned} \tag{3.7}$$

Since $\{w_n\}$ is bounded in P , passing to a subsequence if necessary, we may assume that $w_n \rightharpoonup w_0 \in P$. Letting $n \rightarrow \infty$ in (3.7), by Lemma 3.1, we obtain

$$\int_{\Omega} \nabla w_0 \cdot \nabla v = f_0 \lambda_1 \int_{\Omega} w_0 v.$$

Taking $v = w_n$ in (3.7) and letting again $n \rightarrow \infty$,

$$1 = f_0 \lambda_1 \int_{\Omega} w_0^2,$$

which implies that $w_0 \neq 0$ and w_0 is a nontrivial eigenfunction of (1.3). But the assumption $f_0 > 1$ implies that w_0 must be sign-changing, which contradicts with $w_0 \in P$. Hence, it follows from Theorem 2.2 (iii) and Lemma 3.2 that

$$i(A^{-1}L, P_r, P) = i(A^{-1}\tilde{H}(0, \cdot), P_r, P) = i(A^{-1}\tilde{H}(1, \cdot), P_r, P) = i(A^{-1}(af_0\lambda_1 K), P_r, P) = 0$$

for $r \in (0, \min\{r_0, r_1\})$.

When $a > 0, b = 0$, the results in Theorem 1.1 is a direct conclusion of [13, Theorem 1]. For the detail, the reader can also see Remark 1.4. The proof is completed. \square

4. Proof of Theorem 1.2

In this section, we always assume that $b > 0$ unless specified otherwise and (f_1) and (f_2) hold with $f_0 < 1$ and $f_\infty > 1$. Hence, there exist $\varepsilon_1 > 0$ and $C_{\varepsilon_1} > 0$ such that

$$F(x, t) \geq \frac{1}{4}(b\mu_1 + \varepsilon_1)t^4 - C_{\varepsilon_1}, \quad x \in \bar{\Omega}, t \geq 0, \tag{4.1}$$

$$F(x, t) \leq \frac{1}{2}a(1 - \varepsilon_1)\lambda_1 t^2 + \frac{1}{4}b(1 - \varepsilon_1)\mu_1 t^4 + C_{\varepsilon_1}t^6, \quad x \in \bar{\Omega}, t \in \mathbb{R}, \tag{4.2}$$

where $F(x, t) = \int_0^t f(x, s) ds$.

In the following, we utilize Theorem 2.3 to complete the proof of Theorem 1.2. In the setting of Theorem 2.3 we have $E = X, I = [\delta, 1]$ with $\frac{b\mu_1}{b\mu_1 + \varepsilon_1} < \delta < 1$, and

$$S(u) = \frac{1}{2}a\|u\|^2 + \frac{1}{4}b\|u\|^4, \quad T(u) = \int_{\Omega} F(x, u),$$

$$J_v(u) = \frac{1}{2}a\|u\|^2 + \frac{1}{4}b\|u\|^4 - v \int_{\Omega} F(x, u), \quad u \in X, v \in I.$$

It is easy to verify that

$$\langle J'_v(u), v \rangle = (a + b\|u\|^2) \int_{\Omega} \nabla u \cdot \nabla v - v \int_{\Omega} f(x, u)v, \quad u, v \in X, v \in I.$$

In the following, we show that J_v satisfies the conditions of Theorem 2.3 by proving several lemmas.

Lemma 4.1. $\Gamma_v \neq \emptyset$ for any $v \in I$.

Proof. Let $\phi_1 > 0$ be a μ_1 -eigenfunction mentioned in Section 1. For $t > 0$, we have by (4.1) and (1.6) that

$$J_v(t\phi_1) = \frac{1}{2}at^2\|\phi_1\|^2 + \frac{1}{4}bt^4\|\phi_1\|^4 - v \int_{\Omega} F(x, t\phi_1)$$

$$\leq \frac{1}{2}at^2\|\phi_1\|^2 + \frac{1}{4}b\mu_1 t^4 \int_{\Omega} \phi_1^4 - \frac{1}{4}(b\mu_1 + \varepsilon_1)\delta t^4 \int_{\Omega} \phi_1^4 + C_1$$

$$= \frac{1}{2}at^2\|\phi_1\|^2 - \frac{1}{4}C_2 t^4 \int_{\Omega} \phi_1^4 + C_1,$$

where $C_2 = (b\mu_1 + \varepsilon_1)\delta - b\mu_1$. Noting that $C_2 > 0$, we can choose $t_0 > 0$ large enough so that $J_v(t_0\phi_1) < 0$, where t_0 is independent of $v \in I$. The proof is completed. \square

Lemma 4.2. There exists a constant $c > 0$ such that $c_v \geq c$ for any $v \in I$.

Proof. For any $u \in X$, it follows from (4.2), (1.5) and (1.6) that

$$J_v(u) \geq \frac{1}{2}a\|u\|^2 + \frac{1}{4}b\|u\|^4 - \frac{1}{2}a(1 - \varepsilon_1)\lambda_1 \int_{\Omega} |u|^2 - \frac{1}{4}b\mu_1(1 - \varepsilon_1) \int_{\Omega} |u|^4 - C_{\varepsilon_1} \int_{\Omega} |u|^6$$

$$\geq \frac{1}{2}a\varepsilon_1\|u\|^2 + \frac{1}{4}b\varepsilon_1\|u\|^4 - C_{\varepsilon_1} \int_{\Omega} |u|^6.$$

By Sobolev’s embedding theorem, we conclude that there exist $\rho > 0$ and $c > 0$ such that $J_\nu(u) > 0$ for $\|u\| \in (0, \rho]$ and

$$J_\nu(u) \geq c, \quad \|u\| = \rho.$$

Fix $\nu \in I$ and $\gamma \in \Gamma_\nu$. By the definition of Γ_ν , we have that $\|\gamma(1)\| > \rho$. Hence, there exists $t_\gamma \in (0, 1)$ such that $\|\gamma(t_\gamma)\| = \rho$. So,

$$c_\nu = \inf_{\gamma \in \Gamma_\nu} \max_{t \in [0,1]} J_\nu(\gamma(t)) \geq \inf_{\gamma \in \Gamma_\nu} J_\nu(\gamma(t_\gamma)) \geq c.$$

The proof is completed. \square

Lemma 4.3. *For any $\nu \in I$, if $\{u_n\}$ is bounded and $J'_\nu(u_n) \rightarrow 0$ in X^* , then $\{u_n\}$ admits a convergent subsequence.*

Proof. Given $\nu \in I$, assume that $\{u_n\}$ is bounded, $J'_\nu(u_n) \rightarrow 0$ in X^* . By extracting a subsequence, we may suppose that there exists $u \in X$ such that

$$\begin{aligned} u_n &\rightharpoonup u, && \text{in } X, \\ u_n &\rightarrow u, && \text{in } L^s(\Omega), \quad s \in [1, 2^*), \\ u_n(x) &\rightarrow u(x), && \text{a.e. } x \in \Omega. \end{aligned}$$

It follows from (f₁) and (f₂) that there exist $C_1, C_2 > 0$ such that

$$f(x, t) \leq C_1|t| + C_2|t|^3, \quad x \in \bar{\Omega}, \quad t \in \mathbb{R}. \tag{4.3}$$

Hence, by Hölder’s inequality and Sobolev’s embedding theorem, we have

$$\begin{aligned} \left| \int_{\Omega} f(x, u_n)(u_n - u) \right| &\leq C_1 \int_{\Omega} |u_n| |u_n - u| + C_2 \int_{\Omega} |u_n|^3 |u_n - u| \\ &\leq C_3 \left(\int_{\Omega} |u_n - u|^2 \right)^{1/2} + C_4 \left(\int_{\Omega} |u_n - u|^4 \right)^{1/4} \rightarrow 0. \end{aligned}$$

Noting that

$$(a + b\|u_n\|^2) \int_{\Omega} \nabla u_n \cdot \nabla (u_n - u) = (a + b\|u_n\|^2) [(u_n, u_n) - (u_n, u)],$$

we know that

$$o(1) = \langle J'_\nu(u_n), u_n - u \rangle + \nu \int_{\Omega} f(x, u_n)(u_n - u) = (a + b\|u_n\|^2) [(u_n, u_n) - (u_n, u)].$$

Hence, $(u_n, u_n) \rightarrow (u, u)$. This together with $u_n \rightharpoonup u$ shows that $u_n \rightarrow u$ in X . The proof is completed. \square

Lemma 4.4. *There exist a sequence $\{\nu_n\} \subset I$ with $\nu_n \rightarrow 1^-$ as $n \rightarrow \infty$ and $\{u_{\nu_n}\} \subset X$ such that*

$$J_{\nu_n}(u_{\nu_n}) = c_{\nu_n}, \quad J'_{\nu_n}(u_{\nu_n}) = 0.$$

Proof. We only need to show that for almost every $\nu \in I$ there exists $u^\nu \in X$ such that $J_\nu(u^\nu) = c_\nu$ and $J'_\nu(u^\nu) = 0$. By Theorem 2.3, for almost each $\nu \in I$, there exists a bounded sequence $\{u_n^\nu\} \subset X$ such that

$$J_\nu(u_n^\nu) \rightarrow c_\nu, \quad J'_\nu(u_n^\nu) \rightarrow 0.$$

By Lemma 4.3, we may assume that $u_n^\nu \rightarrow u^\nu$ in X . Then the continuity of J_ν and J'_ν imply that $J_\nu(u^\nu) = c_\nu$ and $J'_\nu(u^\nu) = 0$. The proof is completed. \square

In the sequel, we give the proof of Theorem 1.2.

Proof of Theorem 1.2. First we consider the case of $a \geq 0, b > 0$. By Lemma 4.4, there exists a sequence $\{v_n\} \subset I$ with $v_n \rightarrow 1^-$ and $\{u_{v_n}\} \subset X$ such that

$$J_{v_n}(u_{v_n}) = c_{v_n}, \quad J'_{v_n}(u_{v_n}) = 0. \tag{4.4}$$

Since $c_{v_n} \geq c > 0$ by Lemma 4.2, by standard regularity theory, we know that u_{v_n} is a positive solution to (1.1) with $v = v_n$. To prove the theorem, we assume the first alternative does not hold. Then the sequence $\{u_{v_n}\}$ above is bounded in X . Since $v_n \rightarrow 1^-$, we can show that

$$J'_1(u_{v_n}) \rightarrow 0, \quad \text{in } X^*.$$

In fact, for any $v \in X$, it follows from (4.3), Hölder’s inequality and Sobolev’s embedding theorem that

$$\left| \int_{\Omega} f(x, u_{v_n})v \right| \leq C_1 \int_{\Omega} |u_{v_n}| |v| + C_2 \int_{\Omega} |u_{v_n}|^3 |v| \leq C_3 \|v\|.$$

Furthermore, (4.4) implies that

$$\langle J'_1(u_{v_n}), v \rangle + (1 - v_n) \int_{\Omega} f(x, u_{v_n})v = \langle J'_{v_n}(u_{v_n}), v \rangle = 0, \quad v \in X.$$

Hence, $J'_1(u_{v_n}) \rightarrow 0$ in X^* . By Lemma 4.3, $\{u_{v_n}\}$ has a convergent subsequence. Without loss of generality, we may assume that $u_{v_n} \rightarrow u$. According to Lemma 4.2, (4.4) and noting that

$$\left| \int_{\Omega} F(x, u_{v_n}) \right| \leq C_4,$$

we have

$$J_1(u) = \lim_{n \rightarrow \infty} J_1(u_{v_n}) = \lim_{n \rightarrow \infty} J_{v_n}(u_{v_n}) \geq c > 0,$$

and

$$J'_1(u) = \lim_{n \rightarrow \infty} J'_1(u_{v_n}) = 0.$$

The standard process shows that u is a positive solution to (1.1) with $v = 1$, and the second alternative holds.

For the case of $a > 0, b = 0$, by Remark 1.4, we know easily that (1.1) has a positive solution with $v = 1$ for all $N \geq 1$. The proof is completed. \square

5. Applications of Theorem 1.2

In this section, we show that Theorem 1.2 is a useful result in some applications. Here, we assume that $a \geq 0, b > 0$ and $N = 1, 2, 3$.

Theorem 5.1. Assume that f satisfies (f_1) and (f_2) with $f_0 < 1$ and $f_{\infty} > 1$. If in addition $f_{\infty}\mu_1$ is not an eigenvalue of (1.4), then problem (1.1) has a positive solution with $v = 1$.

The result in this theorem holds by the mountain pass theorem and [23, Lemma 2.2]. Here we give an alternative proof of Theorem 5.1 by using Theorem 1.2.

Proof of Theorem 5.1. From Theorem 1.2, we only need to exclude that the bifurcation from infinity occurs. Suppose by contradiction that there exists a sequence of positive solutions $\{(v_n, u_n)\}$ to (1.1) such that $\lim_{n \rightarrow \infty} v_n = 1$ and $\lim_{n \rightarrow \infty} \|u_n\| = \infty$. Let $w_n = u_n / \|u_n\|$. Similar to (3.5), we have

$$\frac{a + b\|u_n\|^2}{\|u_n\|^2} \int_{\Omega} \nabla w_n \cdot \nabla v = v_n b f_{\infty} \mu_1 \int_{\Omega} w_n^3 v + v_n \int_{\Omega} \frac{f(x, u_n) - b f_{\infty} \mu_1 u_n^3}{\|u_n\|^3} v, \quad v \in X. \tag{5.1}$$

Since $\{w_n\}$ is bounded in X , we may assume that $w_n \rightharpoonup w_0 \in P \subset X$. Passing to limit $n \rightarrow \infty$ in (5.1), we obtain that

$$\int_{\Omega} \nabla w_0 \cdot \nabla v = f_{\infty} \mu_1 \int_{\Omega} w_0^3 v, \quad v \in X, \quad (5.2)$$

and $w_0 \neq 0$. Hence, $f_{\infty} \mu_1$ is an eigenvalue of (1.4), which contradicts with the assumption. The proof is completed. \square

Finally we prove that the bifurcation from infinity in Theorem 1.2 cannot occur if the domain Ω is a ball.

Theorem 5.2. *Assume that Ω is a ball in \mathbb{R}^N and f satisfies (f_1) and (f_2) with $f_0 < 1$ and $f_{\infty} > 1$. Then problem (1.1) has a positive solution with $v = 1$.*

Proof. Similar to the proof of Theorem 5.1, we have (5.2) and $\|w_0\| = 1$. Hence, w_0 is a positive solution to (1.4) with $\mu = f_{\infty} \mu_1$. By Lemma 5.3 below, $f_{\infty} = 1$, which is a contradiction to $f_{\infty} > 1$. The proof is completed. \square

The following result on the simplicity of principal eigenvalue of (1.4) is of independent interest.

Lemma 5.3. *Assume that Ω is a ball in \mathbb{R}^N . Then the principal eigenvalue μ_1 of (1.4) is simple and every eigenfunction corresponding to an eigenvalue $\mu > \mu_1$ of (1.4) must be sign-changing.*

Proof. Assume that Ω is a ball. Then the following equation

$$\begin{cases} -\Delta u = u^3, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases} \quad (5.3)$$

has a unique solution, see [9].

Without loss of generality, suppose that $u \geq 0$ and $v \geq 0$ are both eigenfunctions corresponding to μ_1 . Then $\frac{\sqrt{\mu_1}}{\|u\|} u$ and $\frac{\sqrt{\mu_1}}{\|v\|} v$ are both solutions to (5.3). Hence $u = \frac{\|u\|}{\|v\|} v$, which shows that μ_1 is simple. To prove the second part of the lemma, we assume by contradiction that $v \geq 0$ is an eigenfunction to $\mu > \mu_1$. Then

$$\frac{\sqrt{\mu}}{\|v\|} v = \frac{\sqrt{\mu_1}}{\|\phi_1\|} \phi_1,$$

where $\phi_1 > 0$ is a μ_1 -eigenfunction mentioned in Section 1. Therefore, $\mu = \mu_1$, which is a contradiction. The proof is completed. \square

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