New results on Γ-limits of integral functionals

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Abstract

For $\psi \in W^{1,p}(\Omega;\mathbb{R}^m)$ and $g \in W^{-1,p}(\Omega;\mathbb{R}^d)$, $1 < p < +\infty$, we consider a sequence of integral functionals $F_k^{\psi,g} : W^{1,p}(\Omega;\mathbb{R}^m) \times L^p(\Omega;\mathbb{R}^{d \times n}) \to [0, +\infty]$ of the form

$$F_k^{\psi,g}(u,v) = \begin{cases} \int_\Omega f_k(x, \nabla u, v)\, dx & \text{if } u - \psi \in W_0^{1,p}(\Omega;\mathbb{R}^m) \text{ and div } v = g, \\ +\infty & \text{otherwise}, \end{cases}$$

where the integrands f_k satisfy growth conditions of order p, uniformly in k. We prove a Γ-compactness result for $F_k^{\psi,g}$ with respect to the weak topology of $W^{1,p}(\Omega;\mathbb{R}^m) \times L^p(\Omega;\mathbb{R}^{d \times n})$ and we show that under suitable assumptions the integrand of the Γ-limit is continuously differentiable. We also provide a result concerning the convergence of momenta for minimizers of $F_k^{\psi,g}$.
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Résumé

Pour tout $\psi \in W^{1,p}(\Omega;\mathbb{R}^m)$ et $g \in W^{-1,p}(\Omega;\mathbb{R}^d)$, $1 < p < +\infty$, nous considérons une suite de fonctionnelles intégrales $F_k^{\psi,g} : W^{1,p}(\Omega;\mathbb{R}^m) \times L^p(\Omega;\mathbb{R}^{d \times n}) \to [0, +\infty]$ définies par

$$F_k^{\psi,g}(u,v) = \begin{cases} \int_\Omega f_k(x, \nabla u, v)\, dx & \text{si } u - \psi \in W_0^{1,p}(\Omega;\mathbb{R}^m) \text{ et div } v = g, \\ +\infty & \text{sinon}, \end{cases}$$

où les intégrandes f_k satisfont des conditions de croissance d'ordre p, uniformément en k. Nous démontrons un résultat de Γ-compacité pour $F_k^{\psi,g}$ par rapport à la topologie faible sur $W^{1,p}(\Omega;\mathbb{R}^m) \times L^p(\Omega;\mathbb{R}^{d \times n})$ et nous prouvons que sous des conditions appropriées, l'intégrande de la Γ-limite est continûment différentiable. Nous montrons également un résultat de convergence des moments pour les minima de $F_k^{\psi,g}$.
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1. Introduction

In this paper we consider sequences of integral functionals $F_k : W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d \times n}) \to [0, +\infty)$ of the form

$$F_k(u, v) = \int_\Omega f_k(x, \nabla u, v) \, dx, \tag{1.1}$$

where Ω is a bounded open set in $\mathbb{R}^n$ and the integrands f_k satisfy suitable coerciveness and growth conditions of order $p \in (1, \infty)$, uniformly in k (see (2.1) below). Specifically we are interested in the asymptotic behavior of the solutions to the following minimization problems

$$\min\{F_k(u, v): u - \psi \in W_0^{1,p}(\Omega; \mathbb{R}^m), \ v \in L^p(\Omega; \mathbb{R}^{d \times n}), \ \text{div } v = g\}, \tag{1.2}$$

where $\psi \in W^{1,p}(\Omega; \mathbb{R}^m)$ and $g \in W^{-1,p}(\Omega; \mathbb{R}^d)$ are given. The relevance of this setting where the functionals are defined on pairs (u, v) satisfying the differential constraint $(\text{curl } u, \text{div } v) = (0, g)$ lies in the fact that in many applications (see e.g. the case of electromagnetism) PDEs constraints of this type arise naturally.

To take into account the boundary and divergence constraints on u and v, respectively, we introduce the functionals

$$F_k^{\psi, g}(u, v) = \begin{cases} F_k(u, v) & \text{if } u - \psi \in W_0^{1,p}(\Omega; \mathbb{R}^m) \text{ and div } v = g, \\ +\infty & \text{otherwise.} \end{cases} \tag{1.3}$$

The main result of the present paper is as follows: if f_k is a sequence of functions satisfying (2.1) and F_k are as in (1.1), there exist a subsequence of f_k, not relabeled, and a function f such that the sequence $F_k^{\psi, g}$ Γ-converges to the corresponding functional $F^{\psi, g}$, with respect to the weak topology of $W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d \times n})$. Moreover, the integrand f does not depend on ψ and g (see Theorem 2.1 and Theorem 3.3). The case of functionals independent of u, with the constraint $\text{div } v = 0$, has been studied in [2], while similar problems in the framework of $\mathcal{A}$-quasiconvexity have been studied in [9]. However, it does not seem that the techniques used in these papers can lead directly, in our case, to a limit integrand f independent of g.

We prove our main result in a nonconstructive way, following the so-called localization method of Γ-convergence. To this end, for every open set $U \subseteq \Omega$ we consider the functionals

$$F_k(u, v, U) = \int_U f_k(x, \nabla u, v) \, dx. \tag{1.4}$$

Notice that at this first stage both the boundary condition $u = \psi$ and the constraint $\text{div } v = g$ are omitted. In order to add the divergence constraint in the final step of the proof, it is convenient to introduce the following distance

$$d\big((u_1, v_1), (u_2, v_2)\big) := \|u_1 - u_2\|_{L^p(\Omega; \mathbb{R}^m)} + \|v_1 - v_2\|_{W^{-1,p}(\Omega; \mathbb{R}^{d \times n})} + \big\|\text{div}(v_1 - v_2)\big\|_{W^{-1,p}(\Omega; \mathbb{R}^d)} \tag{1.5}$$

for which $(W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d \times n}), d)$ is separable. Then, thanks to the general theory of Γ-convergence in separable metric spaces, in Section 2 we prove that there exist a subsequence of f_k, not relabeled, and a function f such that for every open set $U \subseteq \Omega$ the functionals $F_k(\cdot, \cdot, U)$ $\Gamma(d)$-converge to the functional $F(\cdot, \cdot, U)$ corresponding to f (see Theorem 2.3). In particular this gives the $\Gamma(d)$-convergence for $U = \Omega$ (see Theorem 2.1). We also prove that under suitable assumptions on f_k, for a.e. $x \in \Omega$ the integrand $f(x, \cdot, \cdot)$ of the $\Gamma(d)$-limit F is continuously differentiable (see Theorem 2.8).

By virtue of the above results, in Section 3 we deduce that the functionals $F_k^{\psi, g}$ Γ-converge to $F^{\psi, g}$ with respect to the weak topology of $W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d \times n})$. By general properties of Γ-convergence this gives the convergence of minima and minimizers.

In Section 4 we also prove a result about the convergence of momenta for minimizers. Specifically, in Corollary 4.6 we show that, if (u_k, v_k) is a minimizer of $F_k^{\psi, g}$, then there exist a subsequence of (u_k, v_k), not relabeled, and a minimum point (u, v) of $F^{\psi, g}$ such that $u_k \rightharpoonup u$ weakly in $W^{1,p}(\Omega; \mathbb{R}^m)$, $v_k \rightharpoonup v$ weakly in $L^p(\Omega; \mathbb{R}^{d \times n})$ (convergence of minimizers), and

$$\partial_\xi f_k(x, \nabla u_k, v_k) \rightharpoonup \partial_\xi f(x, \nabla u, v) \quad \text{weakly in } L^q\left(\Omega; \mathbb{R}^{m \times n}\right), \tag{1.6}$$

$$\partial_\eta f_k(x, \nabla u_k, v_k) \rightharpoonup \partial_\eta f(x, \nabla u, v) \quad \text{weakly in } L^q\left(\Omega; \mathbb{R}^{d \times n}\right) \tag{1.7}$$

(convergence of momenta). More in general we show that (1.6) and (1.7) can be obtained without assuming the minimality of (u_k, v_k) (see Theorem 4.5). Indeed, the only hypotheses we need are: $u_k \rightharpoonup u$ weakly in $W^{1,p}(\Omega; \mathbb{R}^m)$, $v_k \rightharpoonup v$ weakly in $L^p(\Omega; \mathbb{R}^{d \times n})$, and

$$\int_\Omega f_k(x, \nabla u_k, v_k)\, dx \to \int_\Omega f(x, \nabla u, v)\, dx.$$

We finally remark that in the recent paper [3], inspired by previous results contained in [10,16], it is shown that minimum problems like (1.2) naturally arise when dealing with sequences of Dirichlet problems of the type

$$\begin{cases} -\operatorname{div}(\sigma_k \nabla w_k) = h & \text{in } \Omega, \\ w_k \in H_0^1(\Omega), \end{cases}$$

with (σ_k) uniformly elliptic and non-symmetric. In this respect, the results contained in the present paper are used in [1] to provide a Γ-convergence approach to the study of H-convergence of non-symmetric linear elliptic operators (see also [3]). Namely, thanks to Proposition 2.6 and Theorem 4.5 of the present paper, in [1] we give an alternative and purely variational proof of the compactness of H-convergence, originally proved by other methods by Murat and Tartar [17,18].

2. Γ-convergence of integral functionals

In this section we prove a compactness result, with respect to Γ-convergence, for integral functionals depending on the gradient of a vector-valued function and on a matrix-valued field.

Let Ω be a bounded open set in $\mathbb{R}^n$ and let $1 < p < +\infty$. Let $f_k : \Omega \times \mathbb{R}^{m \times n} \times \mathbb{R}^{d \times n} \to [0, +\infty)$ be a sequence of Borel functions satisfying the following growth conditions of order p: there exist $a_0, a_1 > 0$ and two nonnegative functions $b_0, b_1 \in L^1(\Omega)$ such that for almost every $x \in \Omega$

$$a_0\left(|\xi|^p + |\eta|^p\right) - b_0(x) \leqslant f_k(x, \xi, \eta) \leqslant a_1\left(|\xi|^p + |\eta|^p\right) + b_1(x), \tag{2.1}$$

for every $k \in \mathbb{N}$, $\xi \in \mathbb{R}^{m \times n}$, and $\eta \in \mathbb{R}^{d \times n}$.

Consider the sequence of integral functionals $F_k : W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d \times n}) \to [0, +\infty)$ defined as follows

$$F_k(u, v) := \int_\Omega f_k(x, \nabla u, v)\, dx. \tag{2.2}$$

On $W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d \times n})$ consider the distance d defined by (1.5). Clearly we have

$$d\left((u_k, v_k), (u, v)\right) \to 0 \quad \Leftrightarrow \quad \begin{cases} u_k \to u & \text{strongly in } L^p\left(\Omega; \mathbb{R}^m\right), \\ v_k \to v & \text{strongly in } W^{-1,p}\left(\Omega; \mathbb{R}^{d \times n}\right), \\ \operatorname{div} v_k \to \operatorname{div} v & \text{strongly in } W^{-1,p}\left(\Omega; \mathbb{R}^d\right). \end{cases}$$

Notice that $(W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d \times n}), d)$ is a separable metric space.

The following compactness theorem is the main result of this section.

Theorem 2.1 (*Γ-compactness of integral functionals*). *Let F_k be the sequence of functionals defined in* (2.2) *with f_k satisfying* (2.1). *Then, there exist a subsequence F_{k_j} and a Borel function $f : \Omega \times \mathbb{R}^{m \times n} \times \mathbb{R}^{d \times n} \to [0, +\infty)$ such that the functionals F_{k_j} $\Gamma(d)$-converge to the functional $F : W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d \times n}) \to [0, +\infty)$ defined as*

$$F(u, v) := \int_\Omega f(x, \nabla u, v)\, dx, \tag{2.3}$$

with f satisfying

$$a_0\big(|\xi|^p + |\eta|^p\big) - b_0(x) \leqslant f(x, \xi, \eta) \leqslant a_1\big(|\xi|^p + |\eta|^p\big) + b_1(x), \tag{2.4}$$

$$\big|f(x, \xi_1, \eta_1) - f(x, \xi_2, \eta_2)\big| \leqslant a_2\big(|\xi_1 - \xi_2| + |\eta_1 - \eta_2|\big)\big(|\xi_1| + |\xi_2| + |\eta_1| + |\eta_2| + b_2(x)\big)^{p-1} \tag{2.5}$$

for almost every $x \in \Omega$, $\xi, \xi_1, \xi_2 \in \mathbb{R}^{m \times n}$, $\eta, \eta_1, \eta_2 \in \mathbb{R}^{d \times n}$, *where* $a_2 \in \mathbb{R}^+$ *and* $b_2 \in L^p(\Omega)^+$ *depend only on* a_0, a_1, b_0, b_1.

We prove Theorem 2.1 in a nonconstructive way, following the so-called localization method of Γ-convergence, for which we refer the reader to [12, Chapters 14–20]. Loosely speaking, this method consists of two main steps. In the first one, based on compactness arguments, we prove the existence of a Γ-converging (sub)sequence. While in the second one we recover enough information on the structure of the Γ-limit as to obtain a representation in an integral form.

As a preliminary step, we localize the sequence F_k by introducing an explicit dependence on the set of integration. With a little abuse of notation, we consider the functionals

$$F_k(u, v, B) := \int_B f_k(x, \nabla u, v)\, dx, \tag{2.6}$$

defined for every $u \in W^{1,p}(\Omega; \mathbb{R}^m)$, $v \in L^p(\Omega; \mathbb{R}^{d \times n})$, and for every Borel set $B \subseteq \Omega$, so that $F_k(u, v, \Omega) = F_k(u, v)$.

Let $\mathcal{A}(\Omega)$ be the set of all open subset of Ω. The following compactness theorem shows that there exists a subsequence whose Γ-limit $F(u, v, U)$ is a measure with respect to U. This is a preliminary step to obtain the integral representation (2.8) below.

Proposition 2.2 (Γ-*compactness*). *Let* $F_k : W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d \times n}) \times \mathcal{A}(\Omega) \to [0, +\infty)$ *be the sequence of functionals defined by* (2.6)*, with* f_k *satisfying the growth conditions* (2.1)*. Then, there exist a subsequence* F_{k_j} *and a local functional* $F : W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d \times n}) \times \mathcal{A}(\Omega) \to [0, +\infty)$ *such that*

$$F_{k_j}(\cdot, \cdot, U) \quad \Gamma(d)\text{-converges to } F(\cdot, \cdot, U)$$

for every $U \in \mathcal{A}(\Omega)$*. Moreover, for all* $(u, v) \in W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d \times n})$ *the set function* $F(u, v, \cdot)$ *is the restriction to* $\mathcal{A}(\Omega)$ *of a nonnegative Borel measure defined on* Ω*. Finally,*

$$F(u + s, v, U) = F(u, v, U) \tag{2.7}$$

for every $(u, v) \in W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d \times n})$, $s \in \mathbb{R}^m$, *and* $U \in \mathcal{A}(\Omega)$*.*

Proof. Arguing as in [12, Theorem 19.1] we can prove that the sequence F_k satisfies the fundamental estimate introduced in [12, Definition 18.2]. Since $(W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d \times n}), d)$ is a separable metric space, the result can be obtained by adapting the arguments of [12, Section 18]. The functional F is local by [12, Proposition 16.15]. The final statement of the proposition is trivial. $\square$

Using Proposition 2.2, we now provide an integral representation formula for the $\Gamma(d)$-limit F. The proof relies on standard arguments that we repeat and adapt to our context for the reader's convenience. Note that Theorem 2.1 follows from Theorem 2.3 below by taking $U = \Omega$.

Theorem 2.3 (Γ-*compactness of local integral functionals*). *Let* $f_k : \Omega \times \mathbb{R}^{m \times n} \times \mathbb{R}^{d \times n} \to [0, +\infty)$ *be Borel functions satisfying the growth assumptions* (2.1)*, and let* F_k *be as in* (2.6)*. Then, there exist a subsequence* F_{k_j} *and a Borel function* $f : \Omega \times \mathbb{R}^{m \times n} \times \mathbb{R}^{d \times n} \to [0, +\infty)$*, satisfying* (2.4) *and* (2.5)*, such that for every* $U \in \mathcal{A}(\Omega)$ *the functionals* $F_{k_j}(\cdot, \cdot, U)$ $\Gamma(d)$-*converge to the functional* $F(\cdot, \cdot, U) : W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d \times n}) \to [0, +\infty)$ *defined as*

$$F(u, v, U) := \int_U f(x, \nabla u, v)\, dx. \tag{2.8}$$

Proof. Proposition 2.2 ensures the existence of a subsequence $F_{k_j}(\cdot, \cdot, U)$ that $\Gamma(d)$-converges to a functional $F(\cdot, \cdot, U)$, for every $U \in \mathcal{A}(\Omega)$. Hence, it remains to deduce the integral representation formula (2.8). This is done in several steps.

Step 1. Definition of f.

Fix $(\xi, \eta) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{d \times n}$ and define $u_\xi(x) = \xi x$; by Proposition 2.2 $F(u_\xi, \eta, \cdot)$ can be extended to a Borel measure on Ω which, by (2.1), is absolutely continuous with respect to the Lebesgue measure. For every $x \in \Omega$ we define

$$f(x, \xi, \eta) := \limsup_{\rho \to 0^+} \frac{F(u_\xi, \eta, B_\rho(x))}{|B_\rho(x)|}, \tag{2.9}$$

where $B_\rho(x)$ is the n-dimensional ball of radius $\rho > 0$, centered at x. Then f is a Borel function and by the Lebesgue Differentiation Theorem we have

$$F(u_\xi, \eta, U) = \int_U f(x, \xi, \eta) \, dx, \tag{2.10}$$

for every $U \in \mathcal{A}(\Omega)$. By (2.1) it follows that (2.4) holds at every Lebesgue point common to b_0 and b_1.

Step 2. Integral representation on piecewise affine and piecewise constant functions.

Since F is local and $F(u, v, \cdot)$ is a measure, from (2.7) and (2.10) we obtain that

$$F(u, v, U) = \int_U f(x, \nabla u, v) \, dx$$

when u is piecewise affine and v is piecewise constant (we assume that the boundaries of the sets where u is affine and v is constant have zero Lebesgue measure). We refer to the proof of [12, Theorem 20.1] for the details.

Step 3. Convexity properties of f.

The proof of the rank-1-convexity in ξ is standard (see, e.g., [8, Theorem 9.1]). Let us prove that f is rank-$(n-1)$ convex in η, i.e., for every $t \in (0, 1)$

$$f(x, \xi, t\eta_1 + (1-t)\eta_2) \leqslant t f(x, \xi, \eta_1) + (1-t) f(x, \xi, \eta_2) \tag{2.11}$$

for every $x \in \Omega$, for every $\xi \in \mathbb{R}^{m \times n}$, and for every $\eta_1, \eta_2 \in \mathbb{R}^{d \times n}$ with $\text{rank}(\eta_1 - \eta_2) \leqslant n - 1$. By (2.9) it suffices to show that, if $B_\rho(x) \subset \Omega$, then for all $t \in (0, 1)$

$$F(u_\xi, t\eta_1 + (1-t)\eta_2, B_\rho(x)) \leqslant t F(u_\xi, \eta_1, B_\rho(x)) + (1-t) F(u_\xi, \eta_2, B_\rho(x))$$

for every $\xi \in \mathbb{R}^{m \times n}$ and for every $\eta_1, \eta_2 \in \mathbb{R}^{d \times n}$ with $\text{rank}(\eta_1 - \eta_2) \leqslant n - 1$. This inequality can be obtained as in [2, Theorem 4.2, Step 3]. Indeed, as a consequence of the rank property of η_1, η_2 we deduce the existence of a unit vector $\nu \in \mathbb{S}^{n-1}$ such that $(\eta_1 - \eta_2)\nu = 0$. Then, if we define $v : \mathbb{R}^n \mapsto \{\eta_1, \eta_2\}$ as $v(y) = \eta_1$ if $y \in U_1^\nu$ and $v(y) = \eta_2$ if $y \in U_2^\nu$, with

$$U_1^\nu := \{y \in \mathbb{R}^n \colon h \leqslant y \cdot \nu < h + t, \ h \in \mathbb{Z}\},$$
$$U_2^\nu := \{y \in \mathbb{R}^n \colon h + t \leqslant y \cdot \nu < h + 1, \ h \in \mathbb{Z}\},$$

we clearly have $\text{div} \, v = 0$. These sets represent a lamination of $\mathbb{R}^n$ in the direction orthogonal to ν, with volume fraction t and $1 - t$. If we define $v_h(y) := v(hy)$ for $y \in \mathbb{R}^n$, it is easy to show that

$$v_h \rightharpoonup t\eta_1 + (1-t)\eta_2 \quad \text{weakly* in } L^\infty(\Omega; \mathbb{R}^{d \times n}), \text{ as } h \to \infty.$$

Hence, in particular v_h converges strongly to v in $W^{-1,p}(\Omega; \mathbb{R}^{d \times n})$ and $\text{div} \, v_h = 0 = \text{div} \, v$.

Moreover, setting $U_{1,h}^\nu := (1/h)U_1^\nu$ and $U_{2,h}^\nu := (1/h)U_2^\nu$, we have

$$\chi_{U_{1,h}^\nu} \rightharpoonup t \quad \text{and} \quad \chi_{U_{2,h}^\nu} \rightharpoonup 1 - t \quad \text{weakly* in } L^\infty(\Omega), \text{ as } h \to \infty.$$

By Step 2 and by the lower semicontinuity of F with respect to the metric d, we have

$$
\begin{aligned}
F\big(u_\xi, t\eta_1 + (1-t)\eta_2, B_\rho(x)\big) &\leqslant \liminf_{h\to\infty} F\big(u_\xi, v_h, B_\rho(x)\big) \\
&= \liminf_{h\to\infty}\bigg(\int_{U_{1,h}^v \cap B_\rho(x)} f(y, \xi, \eta_1)\, dy + \int_{U_{2,h}^v \cap B_\rho(x)} f(y, \xi, \eta_2)\, dy \bigg) \\
&= t \int_{B_\rho(x)} f(y, \xi, \eta_1)\, dy + (1-t) \int_{B_\rho(x)} f(y, \xi, \eta_2)\, dy \\
&= t F\big(u_\xi, \eta_1, B_\rho(x)\big) + (1-t) F\big(u_\xi, \eta_2, B_\rho(x)\big),
\end{aligned}
$$

for every $\xi \in \mathbb{R}^{m\times n}$. Finally, taking the limsup in ρ yields (2.11).

The rank-1 convexity in ξ together with the rank-$(n-1)$ convexity in η ensures that f is separately convex in each component; *i.e.*, if $\xi = (\xi)_{ij}$ and $\eta = (\eta)_{\ell j}$, f is convex in ξ_{ij} for every $i = 1, \ldots, m$ and $j = 1, \ldots, n$, and it is convex in $\eta_{\ell j}$ for every $\ell = 1, \ldots, d$ and $j = 1, \ldots, n$. Therefore, the growth condition (2.4) together with the separate convexity yields that $f(x, \cdot, \cdot)$ is locally Lipschitz (see, e.g., [11, Lemma 2.2]). More precisely, there exist $a_2 > 0$ and a nonnegative function $b_2 \in L^p(\Omega)$ such that (2.5) holds.

Step 4. Integral representation.

By (2.4) and (2.5) for every $U \in \mathcal{A}(\Omega)$ the functional

$$
(u, v) \mapsto \int_U f(x, \nabla u, v)\, dx \tag{2.12}
$$

is continuous with respect to the strong convergence of $W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d\times n})$.

Let $(u, v) \in W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d\times n})$ and let $U \in \mathcal{A}(\Omega)$ with $U \subset\subset \Omega$. We can find a sequence of functions $u_h \in W^{1,p}(\Omega; \mathbb{R}^m)$ strongly converging to u in $W^{1,p}(\Omega; \mathbb{R}^m)$ with piecewise affine restrictions to U and a sequence of piecewise constant functions v_h strongly converging to v in $L^p(\Omega; \mathbb{R}^{d\times n})$. Note that (u_h, v_h) converge to (u, v) with respect to the distance d. Since F is lower semicontinuous and (2.12) is continuous, by Step 2 we get

$$
F(u, v, U) \leqslant \liminf_{h\to\infty} F(u_h, v_h, U) = \lim_{h\to\infty} \int_U f(x, \nabla u_h, v_h)\, dx = \int_U f(x, \nabla u, v)\, dx.
$$

We now prove the converse inequality. Let $(u, v) \in W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d\times n})$ and let $G : W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d\times n}) \times \mathcal{A}(\Omega) \to [0, +\infty)$ be defined by

$$
G(\tilde{u}, \tilde{v}, U) := F(u + \tilde{u}, v + \tilde{v}, U).
$$

Since G satisfies the same properties as F, there exists a Carathéodory function $g : \Omega \times \mathbb{R}^{m\times n} \times \mathbb{R}^{d\times n} \to [0, +\infty)$ such that for every $(\tilde{u}, \tilde{v}) \in W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d\times n})$ and every $U \in \mathcal{A}(\Omega)$, with $U \subset\subset \Omega$, we have that

$$
G(\tilde{u}, \tilde{v}, U) \leqslant \int_U g(x, \nabla\tilde{u}, \tilde{v})\, dx, \tag{2.13}
$$

with equality whenever $\tilde{u}$ and $\tilde{v}$ are piecewise affine and piecewise constant, respectively.

Let (u_h, v_h) be the approximating functions considered above; then

$$
\begin{aligned}
\int_U g(x, 0, 0)\, dx = G(0, 0, U) = F(u, v, U) &\leqslant \int_U f(x, \nabla u, v)\, dx \\
&= \lim_{h\to\infty} \int_U f(x, \nabla u_h, v_h)\, dx = \lim_{h\to\infty} F(u_h, v_h, U) = \lim_{h\to\infty} G(u_h - u, v_h - v, U) \\
&\leqslant \lim_{h\to\infty} \int_U g\big(x, \nabla(u_h - u), v_h - v\big)\, dx = \int_U g(x, 0, 0)\, dx.
\end{aligned}
$$

Hence all inequalities are actually equalities, and in particular

$$F(u, v, U) = \int_U f(x, \nabla u, v) \, dx, \tag{2.14}$$

for all $(u, v) \in W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d \times n})$ and $U \in \mathcal{A}(\Omega)$, with $U \subset\subset \Omega$. Finally, the above equality holds for all $U \in \mathcal{A}(\Omega)$ since F is the restriction to $\mathcal{A}(\Omega)$ of a Borel measure. $\quad\square$

Remark 2.4. The functional F is lower semicontinuous with respect to the distance d defined by (1.5), then we can apply [14, Theorem 3.6] to obtain that for a.e. $x \in \Omega$ the function $(\xi, \eta) \mapsto f(x, \xi, \eta)$ is $\mathcal{A}$-quasiconvex according to [14, Definition 3.1] where

$$\mathcal{A}(\psi, v) := (\operatorname{curl} \psi, \operatorname{div} v)$$

for every $(\psi, v) \in L^p(\Omega; \mathbb{R}^{m \times n} \times \mathbb{R}^{d \times n})$. More precisely, we say that $f(x, \cdot, \cdot)$ is (curl, div)-quasiconvex if

$$f(x, \xi, \eta) \leqslant \int_Q f\big(x, \xi + \psi(y), \eta + v(y)\big) \, dy$$

for all $(\xi, \eta) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{d \times n}$ and all $(\psi, v) \in C^\infty(\mathbb{R}^n; \mathbb{R}^{m \times n} \times \mathbb{R}^{d \times n})$, Q-periodic, such that $\operatorname{curl} \psi = 0$, $\operatorname{div} v = 0$, $\int_Q \psi \, dy = 0$, $\int_Q v \, dy = 0$. In view of (2.4) and (2.5) in the above definition we may replace $C^\infty(\mathbb{R}^n; \mathbb{R}^{m \times n} \times \mathbb{R}^{d \times n})$ by $L^p(\mathbb{R}^n; \mathbb{R}^{m \times n} \times \mathbb{R}^{d \times n})$ (see [14, Remark 3.3]).

We consider now the special case of quadratic functionals depending on the gradient of a scalar function u and on a vector field v. In Corollary 2.5 and Proposition 2.6 we choose $m = d = 1$ and $p = 2$.

Corollary 2.5. *Let $\Sigma_k \in L^\infty(\Omega; \mathbb{R}^{2n \times 2n})$ be symmetric matrices such that for a.e. $x \in \Omega$, and for every $\mathrm{w} \in \mathbb{R}^{2n}$*

$$c_0 |\mathrm{w}|^2 \leqslant \Sigma_k(x) \mathrm{w} \cdot \mathrm{w} \leqslant c_1 |\mathrm{w}|^2 \tag{2.15}$$

for some $0 < c_0 \leqslant c_1$ independent of k, where the dot denotes the scalar product. Let $Q_k : W^{1,2}(\Omega) \times L^2(\Omega; \mathbb{R}^n) \times \mathcal{A}(\Omega) \to [0, +\infty)$ be the sequence of quadratic forms defined as follows

$$Q_k(u, v, U) := \int_U \Sigma_k(x) \begin{pmatrix} \nabla u \\ v \end{pmatrix} \cdot \begin{pmatrix} \nabla u \\ v \end{pmatrix} dx. \tag{2.16}$$

Then, there exist a subsequence Q_{k_j} and a matrix $\Sigma \in L^\infty(\Omega; \mathbb{R}^{2n \times 2n})$ satisfying (2.15), such that for every $U \in \mathcal{A}(\Omega)$ the quadratic forms $Q_{k_j}(\cdot, \cdot, U)$ $\Gamma(d)$-converge to the quadratic form $Q(\cdot, \cdot, U) : W^{1,2}(\Omega) \times L^2(\Omega; \mathbb{R}^n) \to [0, +\infty)$ given by

$$Q(u, v, U) := \int_U \Sigma(x) \begin{pmatrix} \nabla u \\ v \end{pmatrix} \cdot \begin{pmatrix} \nabla u \\ v \end{pmatrix} dx. \tag{2.17}$$

Proof. By Theorem 2.3 and [12, Theorem 11.10] we have that for every $U \in \mathcal{A}(\Omega)$ the quadratic forms $Q_{k_j}(\cdot, \cdot, U)$ $\Gamma(d)$-converge to the quadratic form $Q(\cdot, \cdot, U)$ given by

$$Q(u, v, U) = \int_U f(x, \nabla u, v) \, dx.$$

By (2.9) and [12, Proposition 11.9] on the characterization of the quadratic forms, we get that for every $x \in \Omega$ also $f(x, \cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ is a quadratic form. Therefore, there exists a matrix $\Sigma \in L^\infty(\Omega; \mathbb{R}^{2n \times 2n})$ satisfying (2.15) such that

$$f(x, \xi, \eta) = \Sigma(x) \begin{pmatrix} \xi \\ \eta \end{pmatrix} \cdot \begin{pmatrix} \xi \\ \eta \end{pmatrix},$$

for every $(x, \xi, \eta) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^n$. $\quad\square$

For future applications (see, e.g., [1]) it is useful to highlight the following fact.

Proposition 2.6. *Let $\Sigma_k, \Sigma \in L^\infty(\Omega; \mathbb{R}^{2n \times 2n})$ be symmetric matrices satisfying conditions* (2.15). *Let $Q_k(u, v, U)$ and $Q(u, v, U)$ be quadratic forms defined as in* (2.16), (2.17), *and let $Q_k(u, v) := Q_k(u, v, \Omega)$, $Q(u, v) := Q(u, v, \Omega)$. If $Q_k \, \Gamma(d)$-converges to Q then for every $U \in \mathcal{A}(\Omega)$ the quadratic forms $Q_k(\cdot, \cdot, U) \, \Gamma(d)$-converge to the quadratic form $Q(\cdot, \cdot, U)$.*

Proof. Following the argument of [3, Theorem 4.6] we may prove that the matrix Σ is independent of the set U. □

We now prove that the convergence in measure of the integrands f_k together with (curl, div)-quasiconvexity implies $\Gamma(d)$-convergence of the corresponding integral functionals F_k.

Theorem 2.7. *Let f_k be a sequence of Borel functions satisfying* (2.1) *and such that $f_k(x, \cdot, \cdot)$ is* (curl, div)-*quasiconvex for a.e. $x \in \Omega$. Assume that*

$$f_k(\cdot, \xi, \eta) \to f(\cdot, \xi, \eta) \quad \text{in measure on } \Omega \tag{2.18}$$

for every $\xi \in \mathbb{R}^{m \times n}$ and $\eta \in \mathbb{R}^{d \times n}$. Then f is a Borel function, it satisfies (2.4), *and $f(x, \cdot, \cdot)$ is* (curl, div)-*quasiconvex for a.e. $x \in \Omega$. Let F_k and F be the functionals defined by* (2.2) *and* (2.3). *Then $F_k \, \Gamma(d)$-converges to F.*

Proof. The first statement is a straightforward consequence of (2.18). Since $f_k(x, \cdot, \cdot)$ and $f(x, \cdot, \cdot)$ are (curl, div)-quasiconvex, by [14, Proposition 3.4] $f_k(x, \cdot, \eta)$ and $f(x, \cdot, \eta)$ are rank-1 convex on $\mathbb{R}^{m \times n}$ for a.e. $x \in \Omega$ and every $\eta \in \mathbb{R}^{d \times n}$, while $f_k(x, \xi, \cdot)$ and $f(x, \xi, \cdot)$ are rank-$(n-1)$ convex on $\mathbb{R}^{d \times n}$ for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^{m \times n}$. Hence, they are separately convex with respect to the scalar components of ξ and η. This property, together with the growth assumptions (2.1) and (2.4), implies that f_k and f satisfy the continuity estimates (2.5) uniformly with respect to k (see also Step 3 of the proof of Theorem 2.3).

Under our hypotheses we can prove that there exists a subsequence of f_k, not relabeled, such that $f_k(x, \xi, \eta) \to f(x, \xi, \eta)$ for a.e. $x \in \Omega$, for every $\xi \in \mathbb{Q}^{m \times n}$, $\eta \in \mathbb{Q}^{d \times n}$ where $\mathbb{Q}$ denotes the set of rational numbers. By (2.5) and a diagonal argument the same property holds for a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^{m \times n}$, $\eta \in \mathbb{R}^{d \times n}$. Taking into account the growth conditions (2.1), this implies that $F_k(u, v) \to F(u, v)$ for every $(u, v) \in W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d \times n})$. Therefore to achieve the Γ-convergence result it is enough to prove the Γ-liminf inequality. To this end let $(u, v) \in W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d \times n})$ and let $(u_k, v_k) \in W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d \times n})$ be a sequence such that $(u_k, v_k) \to (u, v)$ in the distance d and such that $F_k(u_k, v_k)$ converges to a finite number. We want to prove that

$$F(u, v) \leqslant \lim_{k \to \infty} F_k(u_k, v_k). \tag{2.19}$$

From the convergence in the distance d and the boundedness of $F_k(u_k, v_k)$ we deduce that $u_k \rightharpoonup u$ weakly in $W^{1,p}(\Omega; \mathbb{R}^m)$ and $v_k \rightharpoonup v$ weakly in $L^p(\Omega; \mathbb{R}^{d \times n})$. Up to subsequence we may assume that $(\nabla u_k, v_k)$ generates a Young measure $\nu : \Omega \to \mathcal{M}(\mathbb{R}^{m \times n} \times \mathbb{R}^{d \times n})$. We refer to [14, Section 2] for the properties of Young measures we use in this proof. For a general treatment of this subject we refer to [5,7,18].

We now prove that

$$\int_\Omega \left(\int_{\mathbb{R}^{m \times n} \times \mathbb{R}^{d \times n}} f(x, \xi, \eta) \, d\nu_x(\xi, \eta) \right) dx \leqslant \lim_{k \to \infty} \int_\Omega f_k(x, \nabla u_k, v_k) \, dx. \tag{2.20}$$

Arguing as in [12, Theorem 5.14] for every $\varepsilon > 0$ we find a measurable set $A_\varepsilon \subset \Omega$, with $|A_\varepsilon| < \varepsilon$, such that for every $M > 0$ we have

$$f_k(x, \xi, \eta) \to f(x, \xi, \eta) \quad \text{uniformly on } (\Omega \setminus A_\varepsilon) \times B_M, \tag{2.21}$$

where B_M is the ball in $\mathbb{R}^{m \times n} \times \mathbb{R}^{d \times n}$ with center $(0, 0)$ and radius M. We can easily construct a sequence φ_M of Carathéodory functions such that

$$\varphi_M(x, \xi, \eta) \to f(x, \xi, \eta) \quad \text{as } M \to +\infty \text{ for a.e. } x \in \Omega \text{ and for every } (\xi, \eta) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{d \times n}, \tag{2.22a}$$

$$\varphi_M(x, \xi, \eta) = 0 \quad \text{for a.e. } x \in \Omega \text{ and for every } (\xi, \eta) \notin B_M, \tag{2.22b}$$

$$0 \leqslant \varphi_M(x, \xi, \eta) \leqslant \left(f(x, \xi, \eta) - \frac{1}{M} \right)^+ \quad \text{for a.e. } x \in \Omega \text{ and for every } (\xi, \eta) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{d \times n}, \tag{2.22c}$$

where $(t)^+$ denotes the positive part of $t \in \mathbb{R}$. By (2.21) and (2.22) there exists $k_M \in \mathbb{N}$ such that for every $k \geqslant k_M$ we have $\varphi_M(x, \xi, \eta) \leqslant f_k(x, \xi, \eta)$ for a.e. $x \in \Omega \setminus A_\varepsilon$ and for every $(\xi, \eta) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{d \times n}$. Integrating this inequality we obtain

$$\int\limits_{\Omega \setminus A_\varepsilon} \varphi_M(x, \nabla u_k, v_k) \, dx \leqslant \int\limits_\Omega f_k(x, \nabla u_k, v_k) \, dx \tag{2.23}$$

for every $k \geqslant k_M$. By the Fundamental Theorem on Young Measures (see e.g. [14, Theorem 2.2])

$$\int\limits_{\Omega \setminus A_\varepsilon} \left(\int\limits_{\mathbb{R}^{m \times n} \times \mathbb{R}^{d \times n}} \varphi_M(x, \xi, \eta) \, d\nu_x(\xi, \eta) \right) dx \leqslant \liminf_{k \to \infty} \int\limits_{\Omega \setminus A_\varepsilon} \varphi_M(x, \nabla u_k, v_k) \, dx.$$

Taking into account (2.23) and passing to the limit first as $M \to \infty$ and then as $\varepsilon \to 0$, thanks to (2.22a) we obtain (2.20).

To prove (2.19) it remains to show that

$$f\bigl(x, \nabla u(x), v(x)\bigr) \leqslant \int\limits_{\mathbb{R}^{m \times n} \times \mathbb{R}^{d \times n}} f(x, \xi, \eta) \, d\nu_x(\xi, \eta)$$

for a.e. $x \in \Omega$. Since $f(x, \cdot, \cdot)$ is (curl, div)-quasiconvex, this can be done arguing as in the proof of [14, Theorem 3.7]. This concludes the proof of the Γ-liminf inequality and therefore shows that the selected subsequence of F_k $\Gamma(d)$-converges to F. Finally, since the Γ-limit F does not depend on the subsequence, we may conclude thanks to the Urysohn property of Γ-convergence [12, Proposition 8.3]. $\square$

We now prove that under suitable assumptions the integrand f of the $\Gamma(d)$-limit F is continuously differentiable with respect to (ξ, η). Similar results have been proved in [15] for the convex case and in [4,6] for quasiconvex envelopes.

Theorem 2.8. *Let $f_k : \Omega \times \mathbb{R}^{m \times n} \times \mathbb{R}^{d \times n} \to [0, +\infty)$ be Borel functions satisfying* (2.1). *Assume that for every k, for a.e. $x \in \Omega$, and for every $\xi_1, \xi_2 \in \mathbb{R}^{m \times n}$, $\eta_1, \eta_2 \in \mathbb{R}^{d \times n}$*

$$(\xi, \eta) \mapsto f_k(x, \xi, \eta) \quad \text{belongs to } C^1\bigl(\mathbb{R}^{m \times n} \times \mathbb{R}^{d \times n}\bigr), \tag{2.24a}$$

$$\bigl| \partial_\xi f_k(x, \xi_1, \eta) - \partial_\xi f_k(x, \xi_2, \eta) \bigr| \leqslant a |\xi_1 - \xi_2|^\alpha \bigl(|\xi_1| + |\xi_2| + |\eta| + b(x) \bigr)^{p-1-\alpha}, \tag{2.24b}$$

$$\bigl| \partial_\eta f_k(x, \xi, \eta_1) - \partial_\eta f_k(x, \xi, \eta_2) \bigr| \leqslant a |\eta_1 - \eta_2|^\alpha \bigl(|\xi| + |\eta_1| + |\eta_2| + b(x) \bigr)^{p-1-\alpha}, \tag{2.24c}$$

where $a > 0$, $b \in L^p(\Omega)^+$, and $0 < \alpha < \min\{1, (p-1)\}$ are independent of k.

Let $f : \Omega \times \mathbb{R}^{m \times n} \times \mathbb{R}^{d \times n} \to [0, +\infty)$ be a Borel functions satisfying (2.4), (2.5), *and let F_k and F be the functionals defined by* (2.6) *and* (2.8). *Assume that for every $U \in \mathcal{A}(\Omega)$*

$$F_k(\cdot, \cdot, U) \quad \Gamma(d)\text{-converges to } F(\cdot, \cdot, U). \tag{2.25}$$

Then,

$$(\xi, \eta) \mapsto f(x, \xi, \eta) \quad \text{belongs to } C^1\bigl(\mathbb{R}^{m \times n} \times \mathbb{R}^{d \times n}\bigr) \tag{2.26}$$

for a.e. $x \in \Omega$.

Proof. Given $x \in \Omega$ and $0 < \rho < \text{dist}(x, \partial\Omega)$ we define

$$F_k^{x,\rho}(u, v) := \int_{B_1} f_k^{x,\rho}\big(y, \nabla u(y), v(y)\big)\, dy, \qquad F^{x,\rho}(u, v) := \int_{B_1} f^{x,\rho}\big(y, \nabla u(y), v(y)\big)\, dy,$$

where B_1 is the open ball in $\mathbb{R}^n$ with center 0 and radius 1 and

$$f_k^{x,\rho}(y, \xi, \eta) := f_k(x + \rho y, \xi, \eta), \qquad f^{x,\rho}(y, \xi, \eta) := f(x + \rho y, \xi, \eta).$$

By (2.25) and by a change of variables we obtain that $F_k^{x,\rho}$ $\Gamma(d)$-converges to $F^{x,\rho}$, where now the distance d is defined using B_1 instead of Ω. Thanks to the continuity assumptions (2.5) there exists a negligible set $N \subset \Omega$ such that every $x \in \Omega \setminus N$ is a Lebesgue point of $f(\cdot, \xi, \eta)$ for every $(\xi, \eta) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{d \times n}$, as well a Lebesgue point of the function b_0 and b_1 in (2.4). It follows that for every $x \in \Omega \setminus N$, $f^{x,\rho}(\cdot, \xi, \eta) \to f(x, \xi, \eta)$ in measure on B_1 as $\rho \to 0$. Moreover, there exist a sequence $\rho_i \to 0$, possibly depending on x, and two functions b_0^x and b_1^x such that

$$b_0(x + \rho_i y) \leqslant b_0^x(y), \qquad b_1(x + \rho_i y) \leqslant b_1^x(y)$$

for every i and for a.e. $y \in B_1$. By Remark 2.4 for a.e. $x \in \Omega$ the functions $f^{x,\rho_i}(y, \cdot, \cdot)$ are (curl, div)-quasiconvex for a.e. $y \in B_1$. Therefore for every $x \in \Omega \setminus N$ we can apply Theorem 2.7 and obtain that

$$F^{x,\rho_i} \quad \Gamma(d)\text{-converges to } F^x$$

as $\rho_i \to 0$, where

$$F^x(u, v) := \int_{B_1} f\big(x, \nabla u(y), v(y)\big)\, dy.$$

By diagonal argument we find a sequence k_i such that $F_{k_i}^{x,\rho_i}$ $\Gamma(d)$-converges to F^x.

We can now apply the arguments of the proof of [15, Proposition 2.5] using Lemma 4.4 in place of [15, Lemma 2.4] and we obtain the existence of the partial derivatives of f with respect to the components of (ξ, η). The continuity with respect to (ξ, η) of these partial derivatives is a consequence of the convexity in each components which, in its turn, follows from the (curl, div)-quasiconvexity. $\quad\square$

3. Boundary condition and divergence constraint

In this section we study functionals with a Dirichlet boundary condition $u = \psi$ and divergence constraint $\text{div } v = g$. We begin with the case of boundary conditions.

Theorem 3.1 (*Γ-convergence with boundary data*). *Let F_k be the sequence of functionals defined in (2.2) with f_k satisfying (2.1). Assume that F_k $\Gamma(d)$-converges to F satisfying (2.3)–(2.5). Let $\psi \in W^{1,p}(\Omega; \mathbb{R}^m)$ and let $F_k^\psi : W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d \times n}) \to [0, +\infty]$ be the functionals defined as*

$$F_k^\psi(u, v) := \begin{cases} F_k(u, v) & \text{if } u - \psi \in W_0^{1,p}(\Omega; \mathbb{R}^m), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.1}$$

Then, the functionals F_k^ψ $\Gamma(d)$-converge to

$$F^\psi(u, v) := \begin{cases} F(u, v) & \text{if } u - \psi \in W_0^{1,p}(\Omega; \mathbb{R}^m), \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. Clearly, we have to deal only with the u variable. Then, the proof exactly follows that of [12, Theorem 21.1]. Indeed, the Γ-liminf inequality is a straightforward consequence of (2.1), of the Poincaré inequality, and of the fact that $W_0^{1,p}(\Omega; \mathbb{R}^m) + \psi$ is closed in the weak topology of $W^{1,p}(\Omega; \mathbb{R}^m)$. To prove the Γ-limsup inequality we have to exhibit a recovery sequence such that $u_k - \psi \in W_0^{1,p}(\Omega; \mathbb{R}^m)$, for every k. To this end, we suitably modify a recovery sequence for F_k as in [12, Theorem 21.1]. Then the fundamental estimate allows us to show that the error introduced in the energy goes to zero as $k \to +\infty$. $\quad\square$

Remark 3.2. Let $f_0 : \Omega \times \mathbb{R}^{m \times n} \times \mathbb{R}^{d \times n} \to [0, +\infty)$ be a Borel function satisfying the growth assumption (2.1) and let F_0 be defined as in (2.2). Applying Theorem 2.1 with $F_k = F_0$ for every k and using [12, Proposition 3.6], we find that there exists a Borel function $\bar{f}_0$ satisfying (2.4) and (2.5) such that the d-lower semicontinuous envelope $\overline{F}_0$ of F_0 can be represented as

$$\overline{F}_0(u, v) := \int_\Omega \bar{f}_0(x, \nabla u, v)\, dx. \tag{3.2}$$

Given $\psi \in W^{1,p}(\Omega; \mathbb{R}^m)$, we can consider the functional F_0^ψ defined in (3.1). Then, applying Theorem 3.1 with $F_k = F_0$ for every k we obtain that the d-lower semicontinuous envelope of F_0^ψ coincides with

$$\overline{F}_0^\psi(u, v) := \begin{cases} \overline{F}_0(u, v) & \text{if } u - \psi \in W_0^{1,p}(\Omega; \mathbb{R}^m), \\ +\infty & \text{otherwise.} \end{cases}$$

In the following theorem we consider the case of prescribed divergence.

Theorem 3.3 (*Γ-convergence with prescribed divergence*). *Let F_k be the sequence of functionals defined in* (2.2) *with f_k satisfying* (2.1). *Assume that F_k $\Gamma(d)$-converges to F satisfying* (2.3)–(2.5) *and that $f_k(x, \cdot, \cdot)$ is continuous on $\mathbb{R}^{m \times n} \times \mathbb{R}^{d \times n}$ for every k and for a.e. $x \in \Omega$. Let $\psi \in W^{1,p}(\Omega; \mathbb{R}^m)$, let $g \in W^{-1,p}(\Omega; \mathbb{R}^d)$, and let $F_k^{\psi,g} : W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d \times n}) \to [0, +\infty]$ be the functionals defined by*

$$F_k^{\psi,g}(u, v) := \begin{cases} F_k(u, v) & \text{if } u - \psi \in W_0^{1,p}(\Omega; \mathbb{R}^m) \text{ and } \operatorname{div} v = g, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.3}$$

Then, the functionals $F_k^{\psi,g}$ Γ-converge to

$$F^{\psi,g}(u, v) := \begin{cases} F(u, v) & \text{if } u - \psi \in W_0^{1,p}(\Omega; \mathbb{R}^m) \text{ and } \operatorname{div} v = g, \\ +\infty & \text{otherwise,} \end{cases} \tag{3.4}$$

with respect to the weak topology of $W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d \times n})$.

Proof. *Γ-liminf.* Let $(u_k, v_k) \in W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d \times n})$ be such that $u_k - \psi \in W_0^{1,p}(\Omega; \mathbb{R}^m)$ and $\operatorname{div} v_k = g$, and suppose that $u_k \rightharpoonup u$ weakly in $W^{1,p}(\Omega; \mathbb{R}^m)$ and $v_k \rightharpoonup v$ weakly in $L^p(\Omega; \mathbb{R}^{d \times n})$. Then, $u_k \to u$ strongly in $L^p(\Omega; \mathbb{R}^m)$, $v_k \to v$ strongly in $W^{-1,p}(\Omega; \mathbb{R}^{d \times n})$, and $\operatorname{div} v_k \to \operatorname{div} v$ strongly in $W^{-1,p}(\Omega; \mathbb{R}^d)$. Therefore, $(u_k, v_k) \to (u, v)$ with respect to the distance d and the Γ-liminf inequality follows immediately from Theorem 3.1.

Γ-limsup. By Theorem 3.1, Remark 3.2, and [12, Proposition 6.11] the sequence $\overline{F}_k^\psi$ $\Gamma(d)$-converges to F^ψ. Let $(u, v) \in W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d \times n})$ be such that $u - \psi \in W_0^{1,p}(\Omega; \mathbb{R}^m)$ and $\operatorname{div} v = g$; by the $\Gamma(d)$-limsup inequality there exists a sequence $(u_k, v_k) \in W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d \times n})$ such that $u_k - \psi \in W_0^{1,p}(\Omega; \mathbb{R}^m)$,

$$u_k \to u \quad \text{in } L^p(\Omega; \mathbb{R}^m), \qquad v_k \to v \quad \text{in } W^{-1,p}(\Omega; \mathbb{R}^{d \times n}), \qquad \operatorname{div} v_k \to g \quad \text{in } W^{-1,p}(\Omega; \mathbb{R}^d), \tag{3.5}$$

$$\lim_{k \to \infty} \overline{F}_k^\psi(u_k, v_k) = F^\psi(u, v) = F^{\psi,g}(u, v) < +\infty. \tag{3.6}$$

Now we need to modify v_k in order to get a new recovery sequence w_k for $\overline{F}_k^\psi$ satisfying the divergence constraint $\operatorname{div} w_k = g$. To this end, for every $k \in \mathbb{N}$ let φ_k be the solution to the following Dirichlet problem

$$\begin{cases} -\Delta_q \varphi_k = g - \operatorname{div} v_k, \\ \varphi_k \in W_0^{1,q}(\Omega; \mathbb{R}^d), \end{cases} \tag{3.7}$$

where $\Delta_q \varphi_k := \operatorname{div}(|\nabla \varphi_k|^{q-2} \nabla \varphi_k)$ and $1/p + 1/q = 1$. We define $\hat{v}_k := |\nabla \varphi_k|^{q-2} \nabla \varphi_k$ with the convention $\hat{v}_k = 0$ if $q < 2$ and $|\nabla \varphi_k| = 0$. Then using (3.7) we obtain

$$\int\limits_\Omega |\hat{v}_k|^p \, dx = \int\limits_\Omega |\nabla \varphi_k|^q \, dx \leqslant \|g - \operatorname{div} v_k\|_{W^{-1,p}(\Omega;\mathbb{R}^d)} \|\nabla \varphi_k\|_{L^q(\Omega;\mathbb{R}^{d\times n})},$$

hence,

$$\|\hat{v}_k\|_{L^p(\Omega;\mathbb{R}^{d\times n})} \leqslant \|g - \operatorname{div} v_k\|_{W^{-1,p}(\Omega;\mathbb{R}^d)}. \tag{3.8}$$

Therefore, $\hat{v}_k \in L^p(\Omega;\mathbb{R}^{d\times n})$, $\operatorname{div} \hat{v}_k = \operatorname{div} v_k - g$, and by (3.5) and (3.8) we have that

$$\hat{v}_k \to 0 \quad \text{strongly in } L^p\big(\Omega;\mathbb{R}^{d\times n}\big). \tag{3.9}$$

Thus, $w_k := v_k - \hat{v}_k$ converges to v weakly in $L^p(\Omega;\mathbb{R}^{d\times n})$ and $\operatorname{div} w_k = g$. By Remark 3.2 the integrands corresponding to $\overline{F}_k^\psi$ satisfy the local Lipschitz condition (2.5) with a_2 and b_2 independent of k. Therefore, (3.6) and (3.9) yield

$$\lim_{k\to\infty} \overline{F}_k^\psi(u_k, w_k) = \lim_{k\to\infty} \overline{F}_k^\psi(u_k, v_k) = F^{\psi,g}(u, v). \tag{3.10}$$

Starting from the sequence (u_k, w_k) we want to construct a recovery sequence for the Γ-limit of F_k^ψ, still satisfying the divergence constraint. To this end we fix k and we consider a sequence (u_k^j, w_k^j) d-converging to (u_k, w_k) such that $u_k^j - \psi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ and

$$\lim_{j\to\infty} F_k^\psi\big(u_k^j, w_k^j\big) = \overline{F}_k^\psi(u_k, w_k). \tag{3.11}$$

Reasoning as above, we may modify w_k^j in order to get a new sequence z_k^j converging to w_k weakly in $L^p(\Omega;\mathbb{R}^{d\times n})$ and with $\operatorname{div} z_k^j = g$. Indeed, it is enough to take $z_k^j = w_k^j - |\nabla \varphi_k^j|^{q-2} \nabla \varphi_k^j$, where φ_k^j is solution to the Dirichlet problem (3.7), with v_k replaced by w_k^j. Since f_k is a Carathéodory function, (u_k^j, w_k^j) is bounded in $L^p(\Omega;\mathbb{R}^m \times \mathbb{R}^{d\times n})$, and $(z_k^j - w_k^j) \to 0$ strongly in $L^p(\Omega;\mathbb{R}^{d\times n})$, by [13, Lemma 4.9] we have that

$$\lim_{j\to\infty} \big(F_k\big(u_k^j, w_k^j\big) - F_k\big(u_k^j, z_k^j\big)\big) = 0.$$

Therefore, by (3.11), for every fixed k we get

$$\lim_{j\to\infty} F_k^\psi\big(u_k^j, z_k^j\big) = \overline{F}_k^\psi(u_k, w_k).$$

Hence, there exists j_k such that $d((u_k^{j_k}, z_k^{j_k}), (u_k, w_k)) < 1/k$ and

$$\big|F_k^\psi\big(u_k^{j_k}, z_k^{j_k}\big) - \overline{F}_k^\psi(u_k, w_k)\big| < 1/k. \tag{3.12}$$

We may assume that the sequence j_k is increasing and we set $\bar{u}_k := u_k^{j_k}$ and $\bar{z}_k := z_k^{j_k}$. Then $(\bar{u}_k, \bar{z}_k)$ d-converges to (u, v), $\operatorname{div} \bar{z}_k = g$, $\bar{u}_k - \psi \in W_0^{1,p}(\Omega;\mathbb{R}^m)$, and by (3.10) and (3.12)

$$\lim_{k\to\infty} F_k^{\psi,g}(\bar{u}_k, \bar{z}_k) = \lim_{k\to\infty} \overline{F}_k^\psi(u_k, w_k) = F^{\psi,g}(u, v) < +\infty.$$

Therefore, the growth condition (2.1) yields

$$\bar{u}_k \rightharpoonup u \quad \text{weakly in } W^{1,p}\big(\Omega;\mathbb{R}^m\big) \quad \text{and} \quad \bar{v}_k \rightharpoonup v \quad \text{weakly in } L^p\big(\Omega;\mathbb{R}^{d\times n}\big).$$

This concludes the proof of the Γ-limsup inequality with respect to the weak topology of $W^{1,p}(\Omega;\mathbb{R}^m) \times L^p(\Omega;\mathbb{R}^{d\times n})$. $\quad\square$

4. Convergence of momenta

In this section we prove the convergence of the momenta for minimizers of the functionals $F_k^{\psi,g}$ as in (3.3). This result is a generalization of [13, Lemma 4.11] to the case of sequences of functionals.

Here we assume that the functions f_k satisfy a local Lipschitz property, uniformly in k. More precisely, we suppose that there exist $a_2 > 0$ and a nonnegative function $b_2 \in L^p(\Omega)$ such that for almost every $x \in \Omega$

$$\left| f_k(x, \xi_1, \eta_1) - f_k(x, \xi_2, \eta_2) \right| \leqslant a_2 \left(|\xi_1 - \xi_2| + |\eta_1 - \eta_2| \right) \left(|\xi_1| + |\xi_2| + |\eta_1| + |\eta_2| + b_2(x) \right)^{p-1} \tag{4.1}$$

for every $k \in \mathbb{N}$, $x \in \Omega$, $\xi_1, \xi_2 \in \mathbb{R}^{m \times n}$, and $\eta_1, \eta_2 \in \mathbb{R}^{d \times n}$.

Remark 4.1. If the functionals $F_k^{\psi,g}$ are lower semicontinuous with respect to the weak topology of $W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d \times n})$, then by [14, Theorem 3.6] the functions $(\xi, \eta) \mapsto f_k(x, \xi, \eta)$ are (curl, div)-quasiconvex for a.e. $x \in \Omega$. As already observed in the proof of Theorem 2.7 we have in particular that the functions f_k satisfy (4.1) with a_2 and b_2 depending only on a_0, a_1, b_0, b_1.

We need the following preliminary Γ-convergence result.

Theorem 4.2 (*Γ-convergence of the perturbed functionals*). *Let F_k be the sequence of functionals defined in (2.6) with f_k satisfying (2.1) and (4.1). Assume that for every $U \in \mathcal{A}(\Omega)$ the functionals $F_k(\cdot, \cdot, U)$ $\Gamma(d)$-converge to a functional $F(\cdot, \cdot, U)$ given by (2.8), with f satisfying (2.4) and (2.5). Let $\Phi \in L^p(\Omega; \mathbb{R}^{m \times n})$, $w \in L^p(\Omega; \mathbb{R}^{d \times n})$, and let $G_k^{\Phi,w} : W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d \times n}) \times \mathcal{A}(\Omega) \to [0, +\infty)$ be the functional defined as*

$$G_k^{\Phi,w}(u, v, U) := \int_U f_k(x, \nabla u + \Phi, v + w) \, dx. \tag{4.2}$$

Then, for every $U \in \mathcal{A}(\Omega)$ the functionals $G_k^{\Phi,w}(\cdot, \cdot, U)$ $\Gamma(d)$-converge to

$$G^{\Phi,w}(u, v, U) := \int_U f(x, \nabla u + \Phi, v + w) \, dx.$$

Proof. For every $x \in \Omega$, $\xi \in \mathbb{R}^{m \times n}$, $\eta \in \mathbb{R}^{d \times n}$, and $k \in \mathbb{N}$ set

$$g_k^{\Phi,w}(x, \xi, \eta) := f_k(x, \xi + \Phi, \eta + w). \tag{4.3}$$

Notice that, by definition, $g_k^{\Phi,w}$ satisfies conditions of type (2.1) and (4.1) with b_0, b_1, b_2 replaced respectively by $\alpha_0(b_0 + |\Phi|^p + |w|^p)$, $\alpha_0(b_1 + |\Phi|^p + |w|^p)$ and $\alpha_0(b_2 + |\Phi| + |w|)$, for a suitable constant $\alpha_0 > 0$ depending only on a_0, a_1, a_2 and p. Therefore, by Theorem 2.3, for every $U \in \mathcal{A}(\Omega)$ the functionals $G_k^{\Phi,w}(\cdot, \cdot, U)$ $\Gamma(d)$-converge, up to subsequences (not relabeled), to a functional of the form

$$G^{\Phi,w}(u, v, U) = \int_U g^{\Phi,w}(x, \nabla u, v) \, dx,$$

for some Borel function $g^{\Phi,w} : \Omega \times \mathbb{R}^{m \times n} \times \mathbb{R}^{d \times n} \to [0, +\infty)$ satisfying (2.4) and (2.5) with the same values of b_0, b_1 considered above and a possibly different b_2. We want to prove that

$$G^{\Phi,w}(u, v, U) = \int_U f(x, \nabla u + \Phi, v + w) \, dx, \tag{4.4}$$

for every u, v, U. We divide the proof into four main steps.

Step 1. $\Phi = \xi = constant$.   Let $\xi \in \mathbb{R}^{m \times n}$ and let $u_\xi := \xi x$. By definition, we have $G_k^{\xi,w}(u, v, U) = F_k(u + u_\xi, v + w, U)$, hence $G^{\xi,w}(u, v, U) = F(u + u_\xi, v + w, U)$ so that

$$G^{\xi,w}(u, v, U) = \int_U f(x, \nabla u + \xi, v + w)\, dx$$

for every $(u, v, U) \in W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d \times n}) \times \mathcal{A}(\Omega)$.

Step 2. Φ piecewise constant.   Let $\xi^1, \ldots, \xi^N \in \mathbb{R}^{m \times n}$ and consider the piecewise constant function

$$\Phi(x) := \sum_{i=1}^N \chi_{U_i}(x)\xi^i,$$

with U_i pairwise disjoint open sets such that $|\Omega \setminus \bigcup_{i=1}^N U_i| = 0$.

Since $G^{\Phi,w}(u, v, \cdot)$ is a measure, by additivity on pairwise disjoint sets and by locality we have

$$G^{\Phi,w}(u, v, U) = \sum_{i=1}^N G^{\Phi,w}(u, v, U_i \cap U) = \sum_{i=1}^N G^{\xi^i,w}(u, v, U_i \cap U),$$

hence by Step 1

$$G^{\Phi,w}(u, v, U) = \sum_{i=1}^N \int_{U_i \cap U} f\big(x, \nabla u + \xi^i, v + w\big)\, dx = \int_U f(x, \nabla u + \Phi, v + w)\, dx.$$

Step 3. Continuity estimates.   Let $\Phi_1, \Phi_2 \in L^p(\Omega; \mathbb{R}^{m \times n})$, let $U \in \mathcal{A}(\Omega)$, and let $(u, v) \in W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d \times n})$. By the definition of $\Gamma(d)$-limit there exists (u_k, v_k) d-converging to (u, v) such that

$$G_k^{\Phi_2,w}(u_k, v_k, U) \to G^{\Phi_2,w}(u, v, U). \tag{4.5}$$

Appealing to (4.1) and to the Hölder inequality, we find that for every $k \in \mathbb{N}$

$$\begin{aligned}
&G_k^{\Phi_1,w}(u_k, v_k, U) - G_k^{\Phi_2,w}(u_k, v_k, U)\\
&\quad \leqslant \int_U \big| f_k(x, \nabla u_k + \Phi_1, v_k + w) - f_k(x, \nabla u_k + \Phi_2, v_k + w) \big|\, dx\\
&\quad \leqslant \alpha_1 \|\Phi_1 - \Phi_2\|_{L^p} \big(\|\nabla u_k\|_{L^p} + \|v_k\|_{L^p} + \|w\|_{L^p} + \|\Phi_1\|_{L^p} + \|\Phi_2\|_{L^p} + 1 \big)^{p-1}\\
&\quad \leqslant \alpha_2 \|\Phi_1 - \Phi_2\|_{L^p} \big(G_k^{\Phi_2,w}(u_k, v_k, U)^{1/p} + \|w\|_{L^p} + \|\Phi_1\|_{L^p} + \|\Phi_2\|_{L^p} + 1 \big)^{p-1},
\end{aligned}$$

for some $\alpha_1, \alpha_2 > 0$ depending only on $p, a_0, a_1, a_2, b_0, b_1, b_2$, and where all the norms above refer to the set U. Therefore, (4.5) and the Γ-liminf inequality give

$$\begin{aligned}
&G^{\Phi_1,w}(u, v, U) - G^{\Phi_2,w}(u, v, U)\\
&\quad \leqslant \alpha_2 \|\Phi_1 - \Phi_2\|_{L^p} \big(G^{\Phi_2,w}(u, v, U)^{1/p} + \|w\|_{L^p} + \|\Phi_1\|_{L^p} + \|\Phi_2\|_{L^p} + 1 \big)^{p-1}.
\end{aligned}$$

Using the upper bounds in (2.4) and exchanging the roles of Φ_1 and Φ_2 we then obtain

$$\begin{aligned}
&\big| G^{\Phi_1,w}(u, v, U) - G^{\Phi_2,w}(u, v, U) \big|\\
&\quad \leqslant \alpha_3 \|\Phi_1 - \Phi_2\|_{L^p} \big(\|\nabla u\|_{L^p} + \|v\|_{L^p} + \|w\|_{L^p} + \|\Phi_1\|_{L^p} + \|\Phi_2\|_{L^p} + 1 \big)^{p-1},
\end{aligned}$$

for some constant $\alpha_3 > 0$ depending only on $p, a_0, a_1, a_2, b_0, b_1, b_2$.

Step 4. General case. Let $\Phi \in L^p(\Omega; \mathbb{R}^{m \times n})$ and let Φ_j be a sequence of piecewise constant functions converging to Φ strongly in $L^p(\Omega; \mathbb{R}^{m \times n})$. Thanks to the continuity estimate proved in Step 3 we have

$$G^{\Phi_j, w}(u, v, U) \to G^{\Phi, w}(u, v, U),$$

for every $(u, v, U) \in W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d \times n}) \times \mathcal{A}(\Omega)$. Moreover, by Step 2 we find

$$G^{\Phi_j, w}(u, v, U) = \int_U f(x, \nabla u + \Phi_j, v + w) \, dx \to \int_U f(x, \nabla u + \Phi, v + w) \, dx,$$

where the convergence follows from (2.4) and (2.5). Then, equality (4.4) is accomplished. □

The following result turns out to be useful in the applications (see, e.g., [1]).

Proposition 4.3. *Let F_k be the sequence of functionals defined in* (2.2), *with f_k satisfying* (2.1) *and* (4.1). *Assume that F_k $\Gamma(d)$-converges to F satisfying* (2.3)–(2.5). *Let $\psi \in W^{1,p}(\Omega; \mathbb{R}^m)$, $g \in W^{-1,p}(\Omega; \mathbb{R}^d)$, $\Phi \in L^p(\Omega; \mathbb{R}^{m \times n})$, $w \in L^p(\Omega; \mathbb{R}^{d \times n})$, and let $G_k^{\psi, g, \Phi, w} : W^{1,p}(\Omega; \mathbb{R}^{m \times n}) \times L^p(\Omega; \mathbb{R}^{d \times n}) \to [0, +\infty]$ be the functional defined as*

$$G_k^{\psi, g, \Phi, w}(u, v) := \begin{cases} \int_\Omega f_k(x, \nabla u + \Phi, v + w) \, dx & \text{if } u - \psi \in W_0^{1,p}(\Omega; \mathbb{R}^m), \text{ div } v = g, \\ +\infty & \text{otherwise.} \end{cases}$$

Then, the functionals $G_k^{\psi, g, \Phi, w}$ Γ-converge to the functional

$$G^{\psi, g, \Phi, w}(u, v) := \begin{cases} \int_\Omega f(x, \nabla u + \Phi, v + w) \, dx & \text{if } u - \psi \in W_0^{1,p}(\Omega; \mathbb{R}^m), \text{ div } v = g, \\ +\infty & \text{otherwise,} \end{cases}$$

with respect to the weak topology of $W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d \times n})$.

Proof. Thanks to Theorem 4.2 it is enough to apply Theorem 3.3 with f_k replaced by the functions $g_k^{\Phi, w}$ defined in (4.3). □

The following lemma is an easy variant of [13, Lemma 4.9] (see also [15, Lemma 2.4]) and it will be used in the proof of the main result of this section. We give here the proof for the reader's convenience.

Lemma 4.4. *Let $(X, \mathcal{A}, \mu)$ be a finite measure space, let $p > 1$, let $l, s \geqslant 1$, and let $H_k : X \times \mathbb{R}^s \to \mathbb{R}^l$ be a sequence of Carathéodory functions. Let $0 \leqslant \alpha \leqslant \min\{1, (p-1)\}$; assume that there exist a constant $a > 0$ and a nonnegative function $b \in L^p(X)$, such that*

$$\left| H_k(x, \zeta_1) - H_k(x, \zeta_2) \right| \leqslant a |\zeta_1 - \zeta_2|^\alpha \left(|\zeta_1| + |\zeta_2| + b(x) \right)^{p-1-\alpha} \tag{4.6}$$

for every $k \in \mathbb{N}$, $x \in X$, $\zeta_1, \zeta_2 \in \mathbb{R}^s$. Let Φ_k and Ψ_k be two sequences in $L^p(X; \mathbb{R}^s)$ such that Φ_k is bounded in $L^p(X; \mathbb{R}^s)$ and $\Psi_k \to 0$ in $L^p(X; \mathbb{R}^s)$. Then,

$$\int_X \left(H_k(x, \Phi_k + \Psi_k) - H_k(x, \Phi_k) \right) \Phi \, d\mu \to 0,$$

for every $\Phi \in L^p(X; \mathbb{R}^l)$.

Proof. By (4.6) we have

$$\left| H_k(x, \Phi_k + \Psi_k) - H_k(x, \Phi_k) \right| |\Phi| \leqslant a |\Psi_k|^\alpha \left(|\Psi_k| + 2|\Phi_k| + b(x) \right)^{p-1-\alpha} |\Phi|,$$

for every $x \in X$. Since $\Phi \in L^p(X; \mathbb{R}^s)$, to conclude it is enough to show that

$$|\Psi_k|^\alpha \left(|\Psi_k| + 2|\Phi_k| + b(x) \right)^{p-1-\alpha} \to 0 \quad \text{in } L^q(X), \tag{4.7}$$

with $q = p/(p-1)$.

To this end, we first apply Hölder's inequality with exponents (p, q) and then with $r = (p-1)/\alpha$ and $s = (p-1)/(p-1-\alpha)$; we find

$$
\int_X |\Psi_k|^{\frac{p\alpha}{p-1}} \big(|\Psi_k| + 2|\Phi_k| + b(x)\big)^{\frac{p(p-1-\alpha)}{p-1}} \, d\mu
$$

$$
\leqslant C \bigg(\int_X |\Psi_k|^p \, d\mu \bigg)^{\frac{\alpha}{p-1}} \bigg(\int_X |\Psi_k|^p \, d\mu + \int_X |\Phi_k|^p \, d\mu + \int_X b(x)^p \, d\mu \bigg)^{\frac{p-1-\alpha}{p-1}}.
$$

Hence, the thesis follows as $\|\Psi_k\|_{L^p(X;\mathbb{R}^s)} \to 0$ and Φ_k is bounded in $L^p(X; \mathbb{R}^s)$. $\qquad\square$

Let $f: \Omega \times \mathbb{R}^{m \times n} \times \mathbb{R}^{d \times n} \to [0, +\infty)$ be a Borel function satisfying (2.4), (2.5), and (2.26). In view of (2.5) we immediately deduce that

$$
\big|\partial_\xi f(x, \xi, \eta)\big| \leqslant a_2\big(2|\xi| + 2|\eta| + b_2(x)\big)^{p-1}, \qquad \big|\partial_\eta f(x, \xi, \eta)\big| \leqslant a_2\big(2|\xi| + 2|\eta| + b_2(x)\big)^{p-1},
$$

for every $x \in \Omega$, $\xi \in \mathbb{R}^{m \times n}$, $\eta \in \mathbb{R}^{d \times n}$. Therefore the functional $\mathcal{F}: L^p(\Omega; \mathbb{R}^{m \times n}) \times L^p(\Omega; \mathbb{R}^{d \times n}) \to [0, +\infty)$ defined by

$$
\mathcal{F}(\Phi, v) := \int_\Omega f(x, \Phi, v) \, dx \tag{4.8}
$$

is C^1 and its partial differentials $\partial_\Phi \mathcal{F}: L^p(\Omega; \mathbb{R}^{m \times n}) \times L^p(\Omega; \mathbb{R}^{d \times n}) \to L^q(\Omega; \mathbb{R}^{m \times n})$ and $\partial_v \mathcal{F}: L^p(\Omega; \mathbb{R}^{m \times n}) \times L^p(\Omega; \mathbb{R}^{d \times n}) \to L^q(\Omega; \mathbb{R}^{d \times n})$ are given by

$$
\partial_\Phi \mathcal{F}(\Phi, v) = \partial_\xi f(x, \Phi, v) \quad \text{and} \quad \partial_v \mathcal{F}(\Phi, v) = \partial_\eta f(x, \Phi, v). \tag{4.9}
$$

In the next theorem we consider functions f_k satisfying (2.1), (4.1), (2.24), and the corresponding functionals $\mathcal{F}_k$ defined by (4.8).

We are now in the position to prove the result concerning the convergence of momenta. Note that in the next theorem we do not assume that (u_k, v_k) is a minimizer. We only suppose that $(u_k, v_k) \to (u, v)$ and that $\mathcal{F}_k(\nabla u_k, v_k) \to \mathcal{F}(\nabla u, v)$. We shall see in Corollary 4.6 below that these properties are always satisfied, up to subsequence, if (u_k, v_k) is a solution of a suitable minimization problem for F_k.

Theorem 4.5. *Let F_k be the sequence of functionals defined in (2.6) with f_k satisfying (2.1), (4.1), and (2.24). Assume that for every $U \in \mathcal{A}(\Omega)$ the functionals $F_k(\cdot, \cdot, U)$ $\Gamma(d)$-converge to a functional $F(\cdot, \cdot, U)$. Then F can be represented as in (2.8), with f satisfying (2.4), (2.5), and (2.26). Let $(u_k, v_k), (u, v) \in W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d \times n})$. Assume that $(u_k, v_k) \to (u, v)$ with respect to the distance d and that $\mathcal{F}_k(\nabla u_k, v_k) \to \mathcal{F}(\nabla u, v)$. Then,*

$$
\partial_\Phi \mathcal{F}_k(\nabla u_k, v_k) \rightharpoonup \partial_\Phi \mathcal{F}(\nabla u, v) \quad \text{weakly in } L^q\big(\Omega; \mathbb{R}^{m \times n}\big), \tag{4.10a}
$$

$$
\partial_v \mathcal{F}_k(\nabla u_k, v_k) \rightharpoonup \partial_v \mathcal{F}(\nabla u, v) \quad \text{weakly in } L^q\big(\Omega; \mathbb{R}^{d \times n}\big), \tag{4.10b}
$$

where $q = p/(p-1)$.

Proof. Properties (2.4) and (2.5) are proved in Theorem 2.3, while (2.26) is proved in Theorem 2.8. The proof of (4.10) follows the lines of that of [13, Lemma 4.11]. We repeat it here for the reader's convenience. We prove only (4.10a). It is enough to show that

$$
\langle \partial_\Phi \mathcal{F}(\nabla u, v), \Psi \rangle \leqslant \liminf_{k \to \infty} \langle \partial_\Phi \mathcal{F}_k(\nabla u_k, v_k), \Psi \rangle \tag{4.11}
$$

for every $\Psi \in L^p(\Omega; \mathbb{R}^{m \times n})$. Let t_i be a sequence of positive numbers converging to 0. By virtue of Theorem 4.2, for every i we obtain

$$
\frac{\mathcal{F}(\nabla u + t_i \Psi, v) - \mathcal{F}(\nabla u, v)}{t_i} \leqslant \liminf_{k \to \infty} \frac{\mathcal{F}_k(\nabla u_k + t_i \Psi, v_k) - \mathcal{F}_k(\nabla u_k, v_k)}{t_i}.
$$

Therefore there exists an increasing sequence of integers k_i such that

$$\frac{\mathcal{F}(\nabla u + t_i \Psi, v) - \mathcal{F}(\nabla u, v)}{t_i} - \frac{1}{i} \leqslant \frac{\mathcal{F}_k(\nabla u_k + t_i \Psi, v_k) - \mathcal{F}_k(\nabla u_k, v_k)}{t_i} \tag{4.12}$$

for every $k \geqslant k_i$. Setting $\varepsilon_k := t_i$ for $k_i \leqslant k < k_{i+1}$, from (4.12) we deduce

$$\liminf_{k \to \infty} \frac{\mathcal{F}(\nabla u_k + \varepsilon_k \Psi, v) - \mathcal{F}(\nabla u, v)}{\varepsilon_k} \leqslant \liminf_{k \to \infty} \frac{\mathcal{F}_k(\nabla u_k + \varepsilon_k \Psi, v_k) - \mathcal{F}_k(\nabla u_k, v_k)}{\varepsilon_k}.$$

Since $\mathcal{F}_k$ and $\mathcal{F}$ are C^1, we have

$$\langle \partial_\Phi \mathcal{F}(\nabla u, v), \Psi \rangle = \lim_{k \to \infty} \frac{\mathcal{F}(\nabla u_k + \varepsilon_k \Psi, v) - \mathcal{F}(\nabla u, v)}{\varepsilon_k}$$

and by Mean Value Theorem there exists $\tau_k \in [0, \varepsilon_k]$ such that

$$\frac{\mathcal{F}_k(\nabla u_k + \varepsilon_k \Psi, v_k) - \mathcal{F}_k(\nabla u_k, v_k)}{\varepsilon_k} = \langle \partial_\Phi \mathcal{F}_k(\nabla u_k + \tau_k \Psi, v_k), \Psi \rangle.$$

By Lemma 4.4, with $H_k = \partial_\xi f_k$ and $\Psi_k = \tau_k \Psi$, we have

$$\liminf_{k \to \infty} \langle \partial_\Phi \mathcal{F}_k(\nabla u_k + \tau_k \Psi, v_k), \Psi \rangle = \liminf_{k \to \infty} \langle \partial_\Phi \mathcal{F}_k(\nabla u_k, v_k), \Psi \rangle$$

which concludes the proof of (4.11). Similarly we can prove (4.10b). $\quad\square$

Corollary 4.6. *Assume that F_k, f_k, F, and f satisfy the hypotheses of Theorem 4.5. Let $\psi \in W^{1,p}(\Omega; \mathbb{R}^m)$, let $g \in W^{-1,p}(\Omega; \mathbb{R}^d)$, and let $F_k^{\psi,g}$ and $F^{\psi,g}$ be defined by (3.3) and (3.4). Assume that for every k the pair (u_k, v_k) is a minimizer of $F_k^{\psi,g}$. Then, there exist a subsequence of (u_k, v_k), not relabeled, and a minimum point (u, v) of $F^{\psi,g}$ such that $u_k \rightharpoonup u$ weakly in $W^{1,p}(\Omega; \mathbb{R}^m)$, $v_k \rightharpoonup v$ weakly in $L^p(\Omega; \mathbb{R}^{d \times n})$, and (4.10) hold true.*

Proof. By Theorem 3.3 the functionals $F_k^{\psi,g}$ Γ-converge to $F^{\psi,g}$ with respect to the weak topology of $W^{1,p}(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^{d \times n})$. Let (u_k, v_k) be a minimizer of $F_k^{\psi,g}$, then in particular $u_k - \psi \in W_0^{1,p}(\Omega; \mathbb{R}^m)$ and $\operatorname{div} v_k = g$. By the coerciveness condition (2.1) there exists a subsequence of (u_k, v_k), not relabeled, such that $u_k \rightharpoonup u$ weakly in $W^{1,p}(\Omega; \mathbb{R}^m)$ and $v_k \rightharpoonup v$ weakly in $L^p(\Omega; \mathbb{R}^{d \times n})$, hence, $(u_k, v_k) \to (u, v)$ with respect to the distance d. By a general property of Γ-convergence (see, e.g., [12, Corollary 7.17]) the pair (u, v) is a minimizer of $F^{\psi,g}$ and $\mathcal{F}_k(\nabla u_k, v_k) = F_k^{\psi,g}(u_k, v_k) \to F^{\psi,g}(u, v) = \mathcal{F}(\nabla u, v)$. Then the conclusion follows by Theorem 4.5. $\quad\square$

Acknowledgements

References

[1] N. Ansini, G. Dal Maso, C.I. Zeppieri, Γ-convergence and H-convergence of linear elliptic operators, J. Math. Pures Appl. 99 (2013) 321–329.

[2] N. Ansini, A. Garroni, Γ-convergence of functionals on divergence-free fields, ESAIM Control Optim. Calc. Var. 13 (2007) 809–828.

[3] N. Ansini, C.I. Zeppieri, Asymptotic analysis of non-symmetric linear operators via Γ-convergence, SIAM J. Math. Anal. 44 (2012) 1617–1635.

[4] J.F. Babadjian, Quasistatic evolution of a brittle thin film, Calc. Var. Partial Differential Equations 26 (2006) 69–118.

[5] J.M. Ball, A version of the fundamental theorem for Young measures, in: M. Rascle, D. Serre, M. Slemrod (Eds.), PDE's and Continuum Models of Phase Transitions, in: Lecture Notes in Phys., vol. 344, Springer-Verlag, Berlin, 1989, pp. 207–215.

[6] J. Ball, B. Kirchheim, J. Kristensen, Regularity of quasiconvex envelopes, Calc. Var. Partial Differential Equations 11 (2000) 333–359.

[7] H. Berliocchi, J.M. Lasry, Intégrands normales et mesures paramétrées en calcul des variations, Bull. Soc. Math. France 101 (1973) 129–184.
[8] A. Braides, A. Defranceschi, Homogenization of Multiple Integrals, Oxford University Press, New York, 1998.
[9] A. Braides, I. Fonseca, G. Leoni, $\mathcal{A}$-quasiconvexity: relaxation and homogenization, ESAIM Control Optim. Calc. Var. 5 (2000) 539–577.
[10] A.V. Chercaev, L.V. Gibiansky, Variational principles for complex conductivity, viscoelasticity, and similar problems in media with complex moduli, J. Math. Phys. 35 (1994) 127–145.
[11] B. Dacorogna, Direct Methods in the Calculus of Variations, second ed., Appl. Math. Sci., vol. 78, Springer, New York, 2008.
[12] G. Dal Maso, An Introduction to Γ-Convergence, Birkhäuser, Boston, 1993.
[13] G. Dal Maso, G.A. Francfort, R. Toader, Quasistatic crack growth in nonlinear elasticity, Arch. Ration. Mech. Anal. 176 (2005) 165–225.
[14] I. Fonseca, S. Müller, $\mathcal{A}$-quasiconvexity, lower semicontinuity, and Young measures, SIAM J. Math. Anal. 30 (1999) 1355–1390.
[15] A. Giacomini, M. Ponsiglione, A Γ-convergence approach to stability of unilateral minimality properties in fracture mechanics and applications, Arch. Ration. Mech. Anal. 180 (2006) 399–447.
[16] G.W. Milton, On characterizing the set of possible effective tensors of composites: the variational method and the translation method, Comm. Pure Appl. Math. 43 (1990) 63–125.
[17] F. Murat, Compacité par compensation: condition necessaire et suffisante de continuité faible sous une hypothése de rang constant, Ann. Sc. Norm. Super. Pisa Cl. Sci. 8 (1981) 68–102.
[18] L. Tartar, Compensated compactness and applications to partial differential equations, in: R. Knops (Ed.), Nonlinerar Analysis and Mechanics: Heriot–Watt Symposium, in: Pitman Res. Notes Math. Ser., vol. 39, Longman, Harlow, 1979, pp. 136–212.