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A Keller–Segel type system in higher dimensions

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Abstract

We analyze an equation that is gradient flow of a functional related to Hardy–Littlewood–Sobolev inequality in whole Euclidean space $\mathbb{R}^d$, $d \geq 3$. Under the hypothesis of integrable initial data with finite second moment and energy, we show local-in-time existence for any mass of "free-energy solutions", namely weak solutions with some free energy estimates. We exhibit that the qualitative behavior of solutions is decided by a critical value. Actually, there is a critical value of a parameter in the equation below which there is a global-in-time energy solution and above which there exist blowing-up energy solutions.

MSC: 35K65; 35B45; 35J20

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1. Introduction

There has been recent interest in introducing a higher-dimensional analog of Patlak–Keller–Segel (PKS) system; see [3,10,19,22,23] and the references therein. The original model is a simplified version of the model that describes the collective motion of cells that are attracted by a self-emitted chemical substance. There are many proposed mathematical models for chemotaxis. As far as we know, the first mathematical model was introduced by Patlak in [21] and later by Keller and Segel in [15]. Further simplification has been proposed later, in which case the equations take the following form which we call the PKS system:

$$\begin{cases} \frac{\partial f}{\partial t}(t,x) = \Delta f(t,x) - \chi \nabla \cdot (f(t,x)\nabla c(t,x)), \quad t > 0, x \in \mathbb{R}^2, \\ -\Delta c(t,x) = f(t,x), \quad t > 0, x \in \mathbb{R}^2, \\ f(0,x) = f_0(x) \geq 0. \end{cases} \tag{1.1}$$

Here, $(t,x) \mapsto f(t,x)$ is the cell density, and $(t,x) \mapsto c(t,x)$ is the concentration of chemoattractant. The first equation in (1.1) takes into account that the motion of cells is driven by the steepest increase in the concentration of chemoattractant while following a Brownian motion due to external interactions. The second equation in (1.1) takes

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into account that the cells are producing the chemoattractant themselves and while this is diffusing into the environment.

$\chi > 0$ is the sensitivity of the bacteria to the chemoattractant, assumed to be a constant; which measures the nonlinearity of the system.

The existence of solutions, critical mass phenomena, blow-up of solutions, qualitative behavior of solutions for equation (1.1) and similar equations have been attracting many researchers recently. See [4,5,11] and some of the references cited therein. In fact in [3] and [10] a higher-dimensional analog of (1.1) was proposed and analyzed.

The sharp form of Hardy–Littlewood–Sobolev (HLS) inequality is due to Lieb [16]. It states that for a nonnegative measurable function f on $\mathbb{R}^d$, and all $0 < \lambda < d$,

$$\frac{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{f(x)f(y)}{|x-y|^\lambda}\,dx\,dy}{||f||_p^2} \le \frac{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{h(x)h(y)}{|x-y|^\lambda}\,dx\,dy}{||h||_p^2}, \tag{1.2}$$

where

$$h(x) := \left(\frac{1}{1+|x|^2}\right)^{\frac{2d-\lambda}{2}}, \tag{1.3}$$

and $p = \frac{2d}{2d-\lambda}$. Moreover, there is equality in (1.2) if and only if for some $x_0 \in \mathbb{R}^d$ and $s \in \mathbb{R}_+$, f is a nonzero multiple of $h(\frac{x}{s} - x_0)$. The $\lambda = d-2$ cases of the sharp HLS inequality (1.2) are particularly interesting since they express the L^p smoothing properties of $(-\Delta)^{-1}$ on $\mathbb{R}^d$. For $d \ge 3$, one has

$$\int_{\mathbb{R}^d} f(x)\left[(-\Delta)^{-1} f\right](x)\,dx = \tilde{C} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{f(x)f(y)}{|x-y|^{d-2}}\,dx\,dy. \tag{1.4}$$

We note that the integrals on the right hand side of (1.2) can be computed explicitly in terms of Γ-functions and after some calculation with constants one sees that for $\lambda = d-2$, (1.2) can be rewritten $F[f] \ge 0$ for all $f \in L^{2d/(d+2)}(\mathbb{R}^d)$, where

$$F[f] := C_{HLS}||f||^2_{\frac{2d}{d+2}} - \int_{\mathbb{R}^d} f(x)\left[(-\Delta)^{-1} f\right](x)\,dx. \tag{1.5}$$

We refer to this functional F on $L^{2d/(d+2)}(\mathbb{R}^d)$ as the *HLS functional*. Throughout the paper, we shall use $||f||_p$ to denote the usual L^p norms with respect to the Lebesgue measure:

$$||f||_p := \left(\int_{\mathbb{R}^d} |f|^p\,dx\right)^{1/p}, \quad \text{for} \quad 1 \le p < \infty.$$

1.1. The proposed model

Gradient flow approach of certain functionals results in interesting equations. This idea started with the seminal work [14] which analyzes the Fokker–Planck equation as a gradient flow with respect to the Wasserstein distance using the Boltzmann entropy. Later on Otto used this approach for the porous medium equation in [20] and this led to the very nice books [1] and [26], which can be referred to for further details of the optimal mass transportation theory. We mention here also the very interesting paper [12] which introduces the gradient flow equation of the Fisher information and this gives the quantum drift-diffusion equation. Thin-film equation can also be viewed as a gradient flow with respect to the Wasserstein distance and [6] and [17] use this approach to analyze the equation. On the other hand, the PKS system (1.1) can also be written as a gradient flow with respect to the Wasserstein distance using the log–HLS functional

$$F_{PKS}[f] := \int_{\mathbb{R}^2} f(x)\log f(x)\,dx + \frac{1}{4\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x)\log(|x-y|)f(y)\,dx\,dy. \tag{1.6}$$

Motivated by some of these works and observing that in the 2-dimensional case the steady state solutions of PKS model (1.1) have infinite support, in this paper we propose to consider the formal gradient flow of the following energy functional:

$$E[f] := ||f||^2_{\frac{2d}{d+2}} - \kappa \int_{\mathbb{R}^d} f(x) \left[(-\Delta)^{-1} f\right](x)\, dx, \tag{1.7}$$

where $\kappa = \frac{\alpha}{C_{HLS}}$ with C_{HLS} being the optimal constant in the $d-2$ case of the HLS inequality (1.2).

Definition 1.1. The parameter α appearing in (1.7) will be a critical parameter in analyzing the global existence versus blow up of solutions for our proposed model.

Writing down the formal gradient flow for E, that is plugging E into the following equation

$$\frac{\partial f}{\partial t} = \mathrm{div}\left(f \nabla \left(\frac{\delta E}{\delta f}\right)\right), \tag{1.8}$$

where $\frac{\delta E}{\delta f}$ is the first variation of the energy with respect to the L^2-metric, we obtain

$$\frac{\partial f}{\partial t} = \left(\frac{d-2}{d}\right) ||f||^{\frac{4}{d+2}}_{\frac{2d}{d+2}} \left\{\Delta\left(f^{\frac{2d}{d+2}}\right) - \mathrm{div}\,(f \nabla c)\right\}, \tag{1.9}$$

where ∇c is given by

$$\nabla c = \frac{d\kappa}{(d-2)||f||^{\frac{4}{d+2}}_{\frac{2d}{d+2}}} \nabla\left(\left[(-\Delta)^{-1} f\right](x)\right). \tag{1.10}$$

The equation (1.9) is complemented by the initial data

$$f(t=0, x) = f_0(x), \tag{1.11}$$

where

$$0 \le f_0(x), \quad (1+|x|^2) f_0(x) \in L^1(\mathbb{R}^d), \quad E[f_0] < \infty. \tag{1.12}$$

Definition 1.2 *(Weak and free energy solutions).* Let f_0 be an initial condition satisfying (1.12) and $T \in (0, \infty]$.

(**D**.1) A weak solution to (1.9) on $[0, T)$ with initial condition f_0 is a non-negative function $f \in C([0, T); L^1(\mathbb{R}^d))$ such that $f \in L^\infty((0, t) \times \mathbb{R}^d)$, $f^{2d/(d+2)} \in L^2(0, t; H^1(\mathbb{R}^d))$ for each $t \in [0, T)$ and

$$\begin{aligned} \int_{\mathbb{R}^d} f_0(x)\psi(0,x)\,dx = &- \int_0^T \int_{\mathbb{R}^d} f(t,x)\partial_t \psi(t,x)\,dx\,dt \\ &+ \int_0^T \int_{\mathbb{R}^d} f(t,x) \nabla\left(2||f||^{\frac{4}{d+2}}_{\frac{2d}{d+2}} f^{\frac{d-2}{d+2}} - 2\kappa\left[(-\Delta)^{-1} f\right]\right) \cdot \nabla\psi(t,x)\,dx\,dt, \end{aligned} \tag{1.13}$$

for any test function $\psi \in D([0, T) \times \mathbb{R}^d)$.

(**D**.2) A free energy solution to (1.9) on $[0, T)$ with initial condition f_0 is a weak solution f to (1.9) on $[0, T)$ with initial condition f_0 satisfying additionally

$$E[f(t)] + \int_0^t \int_{\mathbb{R}^d} f(t,x) \left|2||f||^{\frac{4}{d+2}}_{\frac{2d}{d+2}} f^{\frac{d-2}{d+2}} - 2\kappa\left[(-\Delta)^{-1} f\right]\right|^2 dx\,dt \le E[f_0], \tag{1.14}$$

for all $t \in (0, T)$.

Note that both (1.13) and (1.14) are well-defined. In addition, it follows from (1.9) that

$$||f||_1 = \int_{\mathbb{R}^d} f(t,x)\,dx = \int_{\mathbb{R}^d} f_0(x)\,dx = ||f_0||_1 =: M \quad \text{for } t \in [0,T). \tag{1.15}$$

Indeed, to see (1.15) take a sequence of functions ψ_R in $D([0,T)\times\mathbb{R}^d)$ such that $\psi_R \to 1_{[0,t]}$ almost everywhere and $0 \le \psi_r \le 1$. Writing (1.13) with $\psi = \psi_R$, the integrability properties of f, ∇f and ∇c allow us to pass to the limit as $R \to \infty$ and deduce (1.15).

Recently, there have been various studies on problems involving functionals of the form

$$H[f] := \int_{\mathbb{R}^d} U(f(x))\,dx + \int_{\mathbb{R}^d} V(x) f(x)\,dx + \iint_{\mathbb{R}^d\times\mathbb{R}^d} W(x-y) f(x) f(y)\,dx\,dy,$$

with the basic assumption $U : \mathbb{R}_+ \to \mathbb{R}$ is a density of internal energy, $V : \mathbb{R}^d \to \mathbb{R}$ is a convex smooth confinement potential and $W : \mathbb{R}^d \to \mathbb{R}$ is a symmetric convex smooth interaction potential [1,3,8,9,18]. The internal energy U should satisfy the following condition, due to McCann [18]

$$\lambda \mapsto \lambda^d U(\lambda^{-d}), \quad \text{is convex non-increasing on} \quad \mathbb{R}_+.$$

The local part of the free energy functional E is not exactly in this form. Nevertheless, we consider a formal gradient flow of the energy functional E. The free energy functional E plays a central role for the equation (1.9) since it can be formally considered as a gradient flow of the energy functional E with respect to the Wasserstein distance like for the porous medium equation [20]. See also [1] and [26].

Although we introduce the formal gradient flow for the non-displacement convex functional E, we analyze the resulting equation using purely PDE techniques in this introductory paper. Use of the optimal gradient flow techniques to analyze the equation for further properties, and the critical case $\alpha = 1$, will require the use of some functionals that are displacement convex and some new ideas and this will be done in a follow-up paper [7], see [2] for such a study for the PKS system (1.1).

In this paper we recover the critical value to be $\alpha = 1$ for the blow-up scenarios unlike the usual PKS model (1.1) where the mass is the critical value for such a phenomena. The reason for this is that in the log–HLS inequality the mass appears as a parameter but in the HLS inequality it does not.

Similar arguments as in this paper have been used in the literature before. In particular, proofs of existence via virial identity for a different equation have been provided in [13] and [24]. The technique for obtaining the rate of blow up has been considered in [25]. Our contribution in this paper is to provide a model for higher dimensional analog of two dimensional Patlak–Keller–Segel model. It is still an active field of research to provide alternative models that share the same and similar properties as the two dimensional Patlak–Keller–Segel model. Our paper can be regarded as one of these efforts.

2. Existence criterion

Following the ideas of previous papers, in order to show the existence of free-energy solutions we regularize (1.9) and consider

$$\begin{cases} \frac{\partial f_\varepsilon}{\partial t}(t,x) = \underbrace{(\frac{d-2}{d})||f_\varepsilon||_{2d/(d+2)}^{4/(d+2)}}_{=:C_1} \operatorname{div}\Big\{\nabla(\psi_\varepsilon \circ f_\varepsilon) - f_\varepsilon \nabla c_\varepsilon\Big\}, & t>0, x\in\mathbb{R}^d, \\ c_\varepsilon(t,x) = -\frac{\kappa}{C_1}\mathcal{K} * f_\varepsilon, \quad t>0, x\in\mathbb{R}^d, \\ f_\varepsilon(0,x) = f_0^\varepsilon(x) \ge 0, \quad x\in\mathbb{R}^d \end{cases} \tag{2.1}$$

where $\psi_\varepsilon : [0,\infty) \to \mathbb{R}$ is given by $\psi_\varepsilon(s) := (s+\varepsilon)^{2d/(d+2)} - \varepsilon^{2d/(d+2)}$. Here f_0^ε is the convolution of f_0 with a sequence of mollifiers and $||f_0^\varepsilon||_1 = ||f_0|| = M$. As the solution of the Poisson equation $-\Delta c_\varepsilon = \frac{\kappa}{C_1} f_\varepsilon$ is given up to a harmonic function, we choose the one given by $c_\varepsilon(t,x) = -\frac{\kappa}{C_1}\mathcal{K} * f_\varepsilon$ with

$$\mathcal{K}(x) = c_d \frac{1}{|x|^{d-2}} \quad \text{and} \quad c_d := \frac{1}{(d-2)\sigma_d}$$

where $\sigma_d := 2\pi^{d/2}/\gamma(d/2)$ is the surface area of the sphere $\mathcal{S}^{d-1}$ in $\mathbb{R}^d$.

Such regularized problems have been considered in the literature. See [3,23] and some of the references therein. Following these references we can safely deduce that the regularized problem (2.1) has global in time smooth solutions and this approximation is convergent.

Precisely, as in Section 4 of [23], see also the remarks in [3], if

$$\sup_{0<t<T} ||f_\varepsilon||_{L^\infty} \le \tilde{K}, \tag{2.2}$$

where $\tilde{K}$ is independent of ε, then there is a subsequence $\varepsilon_n \to 0$ such that

$$\begin{aligned}
&f_{\varepsilon_n} \to f \quad \text{strongly in} \quad C([0,T]; L^p_{\text{loc}}) \quad \text{and almost everywhere in } (0,T)\times\mathbb{R}^d,\\
&\nabla(f_\varepsilon^{2d/(d+2)}) \rightharpoonup \nabla(f^{2d/(d+2)}) \quad \text{weakly-}* \quad \text{in} \quad L^\infty(0,T; L^2(\mathbb{R}^d)),\\
&c_{\varepsilon_n} \to c(t) \quad \text{strongly in} \quad L^r_{\text{loc}}(\mathbb{R}^d) \quad \text{and almost everywhere in} \quad (0,T),\\
&\nabla c_{\varepsilon_n} \to \nabla c(t) \quad \text{strongly in} \quad L^r_{\text{loc}}(\mathbb{R}^d) \quad \text{and almost everywhere in} \quad (0,T),
\end{aligned} \tag{2.3}$$

for any $p \in (1,\infty)$ and $r \in (1,\infty]$, and f is a weak solution of (1.9) on $[0,T)$ with $c = -\frac{\kappa}{C_1}\mathcal{K} * f$; free-energy, similarly to Proposition 6.1 in [3], satisfies $E[f(t)] \le E[f_0]$ for almost every $t \ge 0$. In fact, f is a free energy solution of (1.9).

Theorem 2.1. *Under the assumption (1.12) on initial data and (2.2) on the approximating sequence, there exists a free energy solution of (1.9) on* $[0,T)$.

Theorem 2.1 can be proved by following [22] and the proof of Proposition 2.1 in [3].

Lemma 2.2. *For any $\eta > 0$ there exists a $\tau_\eta > 0$ depending only on d, M and η such that if*

$$\sup_{\varepsilon\in(0,1)} ||f_\varepsilon(t^*)||_{\frac{2d}{(d+2)}} \le \eta$$

for some $t^* \in [0,\infty)$, then

- *the family $(f_\varepsilon)_\varepsilon$ is bounded in $L^\infty(t^*, t^*+\tau_\eta; L^{2d/(d+2)}(\mathbb{R}^d))$.*
- *Moreover, if $(f_\varepsilon(t^*))_\varepsilon$ is also bounded in $L^p(\mathbb{R}^d)$ for some $p \in (\frac{2d}{d+2}, \infty]$, then $(f_\varepsilon)_\varepsilon$ is bounded in $L^\infty(t^*, t^*+\tau_\eta; L^p(\mathbb{R}^d))$.*

Proof. *Step 1: $L^{\frac{2d}{d+2}}$-estimate:* An easy calculation yields that

$$\begin{aligned}
\frac{d}{dt}||f_\varepsilon||^{2d/d+2}_{\frac{2d}{d+2}} &= \frac{2d}{(d+2)}\int f_\varepsilon^{\frac{d-2}{d+2}} \nabla\cdot(f_\varepsilon \nabla\tilde{c}_\varepsilon)\,dx\\
&- 2\frac{(d-2)^2}{(d+2)^2}||f_\varepsilon||^{4/d+2}_{\frac{2d}{d+2}} \int f_\varepsilon^{-4/d+2}(\frac{2d}{d+2})(f_\varepsilon+\varepsilon)^{(d-2)/(d+2)}|\nabla f_\varepsilon|^2\,dx\\
&\le -\frac{16d(d-2)^2}{(d+2)(3d+2)^2}||f_\varepsilon||^{4/d+2}_{\frac{2d}{d+2}} \int \left|\nabla\left(f_\varepsilon^{\frac{3d-2}{2(d+2)}}\right)\right|^2 dx\\
&+\kappa\frac{(d-2)}{(d+2)}\int f_\varepsilon^{\frac{3d+2}{d+2}}\,dx,
\end{aligned} \tag{2.4}$$

where $\Delta\tilde{c}_\varepsilon = -\kappa f_\varepsilon$ and recall that $\kappa = \alpha/C_{HLS}$. Hence, we obtain

$$\begin{aligned}
\frac{d}{dt}||f_\varepsilon||^{2d/d+2}_{\frac{2d}{d+2}} &\le -16\frac{d(d-2)^2}{(d+2)(3d+2)^2}||f_\varepsilon||^{4/d+2}_{\frac{2d}{d+2}}\int \left|\nabla(f_\varepsilon^{(3d-2)/2(d+2)})\right|^2 dx\\
&+\kappa\left(\frac{d-2}{d+2}\right)||f_\varepsilon||^{(3d+2)/(d+2)}_{(3d+2)/(d+2)}.
\end{aligned} \tag{2.5}$$

We have the following Gagliardo–Nirenberg inequality:

$$||u||_p \le K_{\text{opt}} ||\nabla u||_2^{\theta} ||u||_q^{1-\theta}, \quad 1 < q < p < \frac{2d}{d-2}, \tag{2.6}$$

where

$$\theta = \frac{2d(p-q)}{p(2d - q(d-2))} \in (0,1). \tag{2.7}$$

We apply the Gagliardo–Nirenberg inequality (2.6) with

$$u = f_\varepsilon^{(3d-2)/2(d+2)}, \quad p = \frac{2(3d+2)}{3d-2} \quad \text{and} \quad q = \frac{4d}{3d-2},$$

and then the Young inequality to deduce

$$||f_\varepsilon||_{(3d+2)/(d+2)}^{(3d+2)/(d+2)} \le \delta ||f_\varepsilon||_{\frac{2d}{d+2}}^{4/d+2} ||\nabla (f_\varepsilon^{(3d-2)/2(d+2)})||_2^2 + C_\delta ||f_\varepsilon||_{2d/(d+2)}^{4d^2/(d-2)(d+2)}, \tag{2.8}$$

with $\delta > 0$ is small.

Plugging this back into (2.5) we obtain

$$\begin{aligned} \frac{d}{dt} ||f_\varepsilon||_{\frac{2d}{d+2}}^{2d/d+2} \le & \left[\kappa (\frac{d-2}{d+2}) \delta - 16 \frac{d(d-2)^2}{(d+2)(3d+2)^2} \right] ||f_\varepsilon||_{\frac{2d}{d+2}}^{4/d+2} \\ & \times \int_{\mathbb{R}^d} \left| \nabla (f_\varepsilon^{(3d-2)/(2(d+2))}) \right|^2 dx \\ & + C_\delta \kappa (\frac{d-2}{d+2}) ||f_\varepsilon||_{\frac{2d}{d+2}}^{\frac{4d^2}{d^2-4}}. \end{aligned} \tag{2.9}$$

We now choose δ so that the expression in the bracket in (2.9) is non-positive, and let us denote this non-positive number by $-C(\delta)$. Thus, we have that

$$\begin{aligned} \frac{d}{dt} ||f_\varepsilon||_{\frac{2d}{d+2}}^{2d/(d+2)} + C(\delta) ||f_\varepsilon||_{\frac{2d}{d+2}}^{4/(d+2)} \int_{\mathbb{R}^d} \left| \nabla (f_\varepsilon^{(3d-2)/(2(d+2))}) \right|^2 dx \\ \le C_\delta \kappa (\frac{d-2}{d+2}) ||f_\varepsilon||_{\frac{2d}{d+2}}^{4d^2/(d-2)(d+2)}. \end{aligned} \tag{2.10}$$

From this we obtain for any $t_2 \ge t_1 \ge 0$,

$$||f_\varepsilon(t_2)||_{\frac{2d}{d+2}}^{2d/(d+2)} \le \left[||f_\varepsilon(t_1)||_{\frac{2d}{d+2}}^{-2d/(d-2)} - C(t_2 - t_1) \right]^{-(\frac{d-2}{d+2})}. \tag{2.11}$$

Taking $t_1 = t^*$ in (2.11), we deduce that

$$||f_\varepsilon(t)||_{\frac{2d}{d+2}}^{2d/(d+2)} \le \left[\eta^{-2d/(d+2)} - C(t - t^*) \right]^{-(\frac{d-2}{d+2})} \tag{2.12}$$

for $t \in [t^*, t^* + 2\tau_\eta)$ with $\tau_\eta = \frac{1}{2C} \eta^{-(2d/(d-2))}$. Consequently,

$$||f_\varepsilon(t)||_{\frac{2d}{d+2}}^{2d/(d+2)} \le \left(C\tau_\eta \right)^{-(\frac{d-2}{d+2})}, \tag{2.13}$$

for $t \in [t^*, t^* + \tau_\eta]$ and the proof of the first assertion is complete. We further deduce, by integration, that

$$\int_{t^*}^{t^*+\tau_\eta} ||f_\varepsilon||_{\frac{2d}{d+2}}^{2d/(d+2)} ||\nabla \left(f_\varepsilon^{3d-2/(2(d+2))} \right) ||_2^2 \, dt \le C(t^*, \eta). \tag{2.14}$$

Step 2: L^p-estimates, $p \in \left(\frac{2d}{d+2}, \infty\right)$: For $t \in [t^*, t^* + \tau_\eta]$, $K \geq 1$ and $p > \frac{2d}{d+2}$, we infer from (2.1) that

$$\begin{aligned}
&\frac{d}{dt}||(f_\varepsilon - K)_+||_p^p \\
&\leq -p(p-1)\left(\frac{d-2}{d+2}\right)||f_\varepsilon||_{\frac{2d}{d+2}}^{4/(d+2)} \int_{\mathbb{R}^d} (f_\varepsilon - K)_+^{p-2+\frac{d-2}{d+2}} |\nabla f_\varepsilon|^2\, dx \\
&+ p(p-1) \int_{\mathbb{R}^d} \left[(f_\varepsilon - K)_+^{p-1} + K(f_\varepsilon - K)_+^{p-2}\right] \nabla f_\varepsilon \cdot \nabla \tilde{c}_\varepsilon\, dx.
\end{aligned} \tag{2.15}$$

We denote the second term on the right hand side of (2.15) by B. First, by an integration by parts, we rewrite it as follows:

$$B = p(p-1)\kappa \int_{\mathbb{R}^d} \left[\frac{(f_\varepsilon - K)_+^p}{p} + K\frac{(f_\varepsilon - K)_+^{p-1}}{(p-1)}\right] f_\varepsilon\, dx,$$

from which a simple observation $f_\varepsilon \leq (f_\varepsilon - K)_+ + K$ implies

$$\begin{aligned}
B &\leq \kappa p K^2 ||(f_\varepsilon - K)_+||_{p-1}^{p-1} \\
&\quad + \kappa(2p-1)K||(f_\varepsilon - K)_+||_p^p \\
&\quad + \kappa(p-1)||(f_\varepsilon - K)_+||_{p+1}^{p+1}.
\end{aligned} \tag{2.16}$$

From Gagliardo–Nirenberg–Sobolev inequality we get

$$||w||_{p+1}^{p+1} \leq C(p)||\nabla\left(w^{\left(p+\frac{(d-2)}{(d+2)}\right)/2}\right)||_2^{\frac{2dp}{d-2}} ||w||_1^{\frac{(p+1)(d-2)-2dp}{d-2}}. \tag{2.17}$$

Using the Hölder's inequality we get

$$\begin{aligned}
\kappa p K^2 ||(f_\varepsilon - K)_+||_{p-1}^{p-1} &\leq \kappa(p-1)K^2||(f_\varepsilon - K)_+||_p^p \\
&\quad + \kappa K^2 |\{x : f_\varepsilon(t,x) \geq K\}|.
\end{aligned} \tag{2.18}$$

We recall that

$$\begin{aligned}
&\frac{d}{dt}||(f_\varepsilon - K)_+||_p^p = \\
&\underbrace{-\left(\frac{d-2}{d+2}\right)\frac{4p(p-1)}{p+(d-2)/(d+2)}||f_\varepsilon||_{\frac{2d}{d+2}}^{4/(d+2)}}_{=:\ \text{-Coeff}} \int_{\mathbb{R}^d} |\nabla\left[(f_\varepsilon - K)_+^{\frac{p+(d-2)/(d+2)}{2}}\right]|^2\, dx + B.
\end{aligned} \tag{2.19}$$

By Step 1, we may choose $K = K_*$ large enough so that $||(f_\varepsilon - K)_+||_1$ is sufficiently small for all $t \in [t^*, t^* + \tau_\eta]$ and $\varepsilon \in (0, 1)$. Here we also use the inequality in (2.17) with $w = (f_\varepsilon - K)_+$, the inequality in (2.18) and the Young's inequality to deduce that

$$\begin{aligned}
B &\leq (\text{Coeff})||\nabla\left((f_\varepsilon - K_*)_+^{\frac{p+(d-2)/(d+2)}{2}}\right)||_2^2 \\
&\quad + C(p, t^*, \eta)\left[1 + ||(f_\varepsilon - K_*)||_p^p\right].
\end{aligned} \tag{2.20}$$

Thus, we conclude that

$$\frac{d}{dt}||(f_\varepsilon - K_*)_+||_p^p \leq C(p, t^*, \eta)\left[1 + ||(f_\varepsilon - K_*)||_p^p\right], \tag{2.21}$$

and from this we deduce that

$$||(f_\varepsilon - K)_+||_p^p \leq C(p, t^*, \eta), \quad \text{for } t \in [t^*, t^* + \tau_\eta] \text{ and } \varepsilon \in (0, 1). \tag{2.22}$$

Now, since $p > \frac{2d}{d+2}$ we have

$$||f_\varepsilon||_p^p \le C(p)\left[||(f_\varepsilon - K_*)_+||_p^p + K_*^{p-2d/(d+2)}||f_\varepsilon||_{\frac{2d}{d+2}}^{2d/(d+2)}\right].$$

This and the previous inequality, combined with Step 1, imply that

$$||f_\varepsilon||_p \le C(p, t^*, \eta), \quad \text{for } t \in [t^*, t^* + \tau_\eta] \text{ and } \varepsilon \in (0,1). \tag{2.23}$$

Step 3: L^∞-estimate: As a direct consequence of *Step 2* with $p = d+1$ and Morrey's embedding theorem $(\nabla c_\varepsilon)_\varepsilon$ is bounded in $L^\infty((t^*, t^*+\tau_\eta) \times \mathbb{R}^d; \mathbb{R}^d)$. This in turn implies that $(f_\varepsilon)_\varepsilon$ is bounded in $L^\infty((t^*, t^*+\tau_\eta) \times \mathbb{R}^d)$. See [3] and the references therein for a similar argument. One can refer to [22] and [23] for alternative arguments. □

As a consequence of the previous lemma, we can construct a free energy solution defined on a maximal existence time.

Theorem 2.3. *Under the assumption (1.12) on the initial data there are $T_{max} \in (0, \infty]$ and a free energy solution $f(t,x)$ to (1.9) on $[0, T_{max})$ with the following alternative: either $T_{max} = \infty$ or $T_{max} < \infty$ and $||f(t)||_{\frac{2d}{d+2}} \to \infty$ as $t \to T_{max}$. Furthermore, there exists a positive constant C_0 depending only on d such that f satisfies*

$$||f(t_2)||_{\frac{2d}{d+2}}^{2d/(d+2)} \le \left[||f(t_1)||_{\frac{2d}{d+2}}^{-2d/(d-2)} - C_0(t_2 - t_1)\right]^{-(\frac{d-2}{d+2})}, \tag{2.24}$$

for $t_1 \in [0, T_{max})$ and $t_2 \in (t_1, T_{max})$.

Proof. Put $\xi_p(t) := \sup_{\varepsilon\in(0,1)} ||f_\varepsilon(t)||_p \in (0,\infty]$ for $t \ge 0$ and $p \in [\frac{2d}{d+2}, \infty]$ and

$$T_1 := \{T > 0 : \xi_{\frac{2d}{d+2}} \in L^\infty(0,T)\}.$$

Clearly, by the definition of the sequence $(f_0^\varepsilon)_\varepsilon$ and (1.12) $\xi_p(0)$ is finite for all $p \in [\frac{2d}{d+2}, \infty]$. By Lemma 2.2, there exists $t_1 > 0$ such that ξ_p is bounded on $[0, t_1]$ for all $p \in [\frac{2d}{d+2}, \infty]$. Then, (2.2) is fulfilled for $T = t_1$ and there exists a free energy solution to (1.9) on $[0, t_1)$ by Theorem 2.1 and (2.14). This ensures in particular that $T_1 \ge t_1 > 0$. We now claim that

$$\xi_\infty \in L^\infty(0,T), \quad \text{for any } T \in [0, T_1). \tag{2.25}$$

To prove (2.25), consider $T_1^\infty := \sup\{T \in (0, T_1) : \xi_\infty \in L^\infty(0,T)\}$ and assume on the contrary that $T_1^\infty < T_1$. Then, $\xi_{\frac{2d}{d+2}} \in L^\infty(0, T_1^\infty)$ and we put $\eta = ||\xi_{\frac{2d}{d+2}}||_{L^\infty(0,T_1^\infty)}$ and $t^* = T_1^\infty - (\frac{\tau_\eta}{2})$, where τ_η is defined in Lemma 2.2. As $\xi_{\frac{2d}{d+2}}(t^*) \le \eta$ and $\xi_\infty(t^*)$ is finite we may apply Lemma 2.2 to deduce both $\xi_{\frac{2d}{d+2}}$ and ξ_∞ belong to $L^\infty(t^*, t^* + \tau_\eta)$, and the latter property contradicts the definition of T_1^∞ as $t^* + \tau_\eta = T_1^\infty + (\frac{\tau_\eta}{2})$. By (2.25), (2.2) is fulfilled for any $T \in [0, T_1)$ and the existence of the free energy solution f of (1.9) on $[0, T_1)$ follows from Theorem 2.1 and (2.14). Moreover, either $T_1 = \infty$ or $T_1 < \infty$ and $||f(t)||_{\frac{2d}{d+2}} \to \infty$ as $t \to T_1$, and the proof of Theorem 2.3 is complete with $T_{max} = T_1$. Or $T_1 < \infty$ and $\lim inf_{t\to T_1} ||f(t)||_{\frac{2d}{d+2}} < \infty$. For this case, there are $\eta > 0$ and an increasing sequence of real numbers $(s_j)_{j\ge1}$ such that $s_j \to T_1$ as $j \to \infty$ and $||f(s_j)||_{\frac{2d}{d+2}} \le \eta$. Fix $j_0 \ge 1$ such that $s_{j_0} \ge T_1 - (\frac{\tau_\eta}{2})$ with τ_η defined as in Lemma 2.2 and put $\tilde{f} = f(s_{j_0})$. According to Definition 1.2 and (2.2) $\tilde{f}_0$ fulfills (1.12) and we may proceed as above to obtain a free energy solution $\tilde{f}$ to (1.9) on $[0, T_2)$ for some $T_2 \ge \tau_\eta$. Setting $\bar{f}(t) = f(t)$ for $t \in [0, s_{j_0}]$ and $\bar{f}(t) = \tilde{f}(t - s_{j_0})$ for $t \in [s_{j_0}, s_{j_0} + T_2)$, we first note that $\bar{f}$ is a free energy solution to (1.9) on $[0, s_{j_0} + T_2)$ and a true extension of f as $s_{j_0} + T_2 \ge T_1 - (\frac{\tau_\eta}{2}) + \tau_\eta = T_1 + \frac{\tau_\eta}{2}$. We note that this construction can be iterated as long as the alternative stated in Theorem 2.3 is not fulfilled to complete the proof. (2.24) follows as in the proof of Lemma 2.2 by the regularity of weak solutions. □

Corollary 2.4 *(Lower bound on the blow-up rate). Let f be a free energy solution to (1.9) on $[0, T_{max})$ constructed in Theorem 2.3 with the initial condition satisfying (1.12). If T_{max} is finite, then*

$$||f(t)||_{\frac{2d}{d+2}}^{2d/(d+2)} \geq [C_0(T_{max}-t)]^{-(\frac{d-2}{d+2})}, \tag{2.26}$$

where C_0 is defined in Theorem 2.3.

Proof. Let $t \in (0, T_{max})$ and $t_2 \in (t, T_{max})$. By (2.24)

$$||f(t_2)||_{\frac{2d}{d+2}}^{-(\frac{2d}{d-2})} \geq ||f(t)||_{\frac{2d}{d+2}}^{-(\frac{2d}{d-2})} - C_0(t_2-t_1).$$

Letting $t_2 \to T_{max}$ we deduce that

$$0 \geq ||f(t)||_{\frac{2d}{d+2}}^{-(\frac{2d}{d-2})} - C_0(t_2-t_1).$$

This implies (2.26). □

Remark 2.5. No uniqueness result for (1.9) seems to be available. We thus note that, from now on, all the results refer to the solutions constructed in Theorem 2.3.

Lemma 2.6 *(Virial Identity). Let $0 \leq f_0 \in L^1(\mathbb{R}^d; (1+|x|^2)\,dx) \cap L^\infty(\mathbb{R}^d)$ with $E[f_0] < \infty$. Let $f = f(t,x)$ be a free energy solution of (1.9) on $[0,T)$ with initial condition f_0 for some $T\ in(0,\infty]$, constructed in Theorem 2.3. Then,*

$$\frac{d}{dt}\int_{\mathbb{R}^d} |x|^2 f(t,x)\,dx = 2(d-2)E[f(t)], \quad t \in [0,T). \tag{2.27}$$

Proof. Here we show only the formal calculations; the passing to the limit from the approximated problem (2.1) can be done by adapting the arguments in Lemma 6.2 of [22] or Lemma 2.1 of [5] without any further complication. We have by integration by parts and symmetry that

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^d} |x|^2 f(t,x)\,dx &= \int_{\mathbb{R}^d} |x|^2 \frac{\partial}{\partial t} f(t,x)\,dx \\
&= \int_{\mathbb{R}^d} |x|^2 \left(\frac{d-2}{d}\right) ||f||_{\frac{2d}{d+2}}^{4/(d+2)} \left\{\Delta(f^{2d/(d+2)}) - \nabla\cdot(f\nabla c)\right\} dx \\
&= 2(d-2)||f||_{\frac{2d}{d+2}}^{4/(d+2)} \int_{\mathbb{R}^d} f^{\frac{2d}{d+2}}\,dx \\
&\quad - \int_{\mathbb{R}^d} |x|^2 \left(\frac{d-2}{d}\right) ||f||_{\frac{2d}{d+2}}^{4/(d+2)} \nabla\cdot(f\nabla c)\,dx \\
&= 2(d-2)||f||_{\frac{2d}{d+2}}^{4/(d+2)} \int_{\mathbb{R}^d} f^{\frac{2d}{d+2}}\,dx \\
&\quad - 2(d-2)\kappa\tilde{C} \iint_{\mathbb{R}^d\times\mathbb{R}^d} (x-y)\cdot\nabla K(x-y) f(t,x) f(t,y)\,dy\,dx \\
&= 2(d-2)E[f(t)]. \qquad \square
\end{aligned}$$

Remark 2.7. We note that, in our case, the second moment is always concave in time since its second time derivative is given by the dissipation of the free energy functional,

$$\frac{d}{dt}E[f(t)] = \int_{\mathbb{R}^d} f(t,x)\left[\frac{\delta E}{\delta f}\right]^2 dx \leq 0.$$

We also emphasize that the evolution of the second moment in our case is more complicated than the classical PKS system (1.1) corresponding to $d = 2$ case. Note that in that case the time derivative of the second moment is a constant.

Remark 2.8. The following result indicates that α is the critical value instead of mass in our case. The reason is that in our functional unlike the log–HLS functional the mass is not a parameter. Depending on the value of the α there are different situations regarding the global existence and blow-up of solutions.

Proposition 2.9 *(Blowing-up solutions). If $\alpha > 1$, then there are initial data satisfying the (1.12) with a negative free energy $E[f_0]$. Moreover, if f_0 is such an initial condition and f denotes a free energy solution to (1.9) on $[0, T_{max})$ with initial condition f_0, constructed in Theorem 2.3 then $T_{max} < \infty$ and the $L^{2d/(d+2)}$-norm of f blows up in finite time.*

Proof. The proof follows the ideas of Weinstein [27]. Take $\tilde{f}$ to be a minimizer of the functional F. This means $F[\tilde{f}] = 0$. Now,

$$E[\tilde{f}] = F[\tilde{f}] + \frac{(1-\alpha)}{C_{HLS}}\tilde{C} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\tilde{f}(x)\tilde{f}(y)}{|x-y|^{d-2}}\, dy\, dx$$

$$= \frac{(1-\alpha)}{C_{HLS}}\tilde{C} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\tilde{f}(x)\tilde{f}(y)}{|x-y|^{d-2}}\, dy\, dx.$$

We recall that the minimizers are of the form $h(\frac{x-x_0}{s})$ for all $s > 0$ and $x_0 \in \mathbb{R}^d$ with $h(x) = \left(\frac{1}{1+|x|^2}\right)^{\frac{d+2}{2}}$, and they are positive.

Let us now consider an initial data f_0 to be a cut-off of the minimizer $\tilde{f}$ at a sufficiently large radius so that f_0 has a finite second moment and satisfies $E[f_0] < 0$ with $\alpha > 1$. Denote by f the corresponding free energy solution to (1.9) on $[0, T)$. (Note that we have the existence result for such an initial data in Theorem 2.3.) We deduce from the time monotonicity of E and Lemma 2.6 that

$$\frac{d}{dt} \int_{\mathbb{R}^d} |x|^2 f(t,x)\, dx = 2(d-2)E[f(t)] \leq 2(d-2)E[f_0] < 0.$$

This implies that the second moment of $f(t,x)$ will become negative after some time and this contradicts the non-negativity of f. Therefore, T_{max} is finite and $||f||_{\frac{2d}{d+2}}$ blows-up in finite time. □

Proposition 2.10. *For $\alpha < 1$ under the assumption (1.12), there exists a free energy solution to (1.9) on $[0, \infty)$ with initial condition f_0.*

Proof. By Theorem 2.3 there exists a number T_{max} and a free energy solution to (1.9) in $[0, T_{max})$ with initial condition f_0. We have

$$\left(\frac{1-\alpha}{C_{HLS}}\right) ||f(t)||^2_{\frac{2d}{d+2}} \leq E[f(t)] \leq E[f_0] < \infty.$$

Since $\alpha < 1$ we deduce from this inequality that

$$f \text{ lies in } L^\infty(0, \min\{T, T_{max}\}; L^{2d/(d+2)}(\mathbb{R}^d)) \text{ for every } T > 0,$$

which implies that $T_{max} = \infty$ by Theorem 2.3. □

Conflict of interest statement

There is no conflict of interest.

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