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Unique determination of a time-dependent potential for wave equations from partial data

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Abstract

We consider the inverse problem of determining a time-dependent potential q, appearing in the wave equation $\partial_t^2 u - \Delta_x u + q(t,x)u = 0$ in $Q = (0,T) \times \Omega$ with $T > 0$ and Ω a $\mathcal{C}^2$ bounded domain of $\mathbb{R}^n$, $n \geqslant 2$, from partial observations of the solutions on ∂Q. More precisely, we look for observations on ∂Q that allows to recover uniquely a general time-dependent potential q without involving an important set of data. We prove global unique determination of $q \in L^\infty(Q)$ from partial observations on ∂Q. Besides being nonlinear, this problem is related to the inverse problem of determining a semilinear term appearing in a nonlinear hyperbolic equation from boundary measurements.

MSC: 35R30; 35L05

Keywords: Inverse problems; Wave equation; Time-dependent potential; Uniqueness; Carleman estimates; Partial data

1. Introduction

1.1. Statement of the problem

We fix Ω a $\mathcal{C}^2$ bounded domain of $\mathbb{R}^n$, $n \geqslant 2$, and we set $\Sigma = (0,T) \times \partial\Omega$, $Q = (0,T) \times \Omega$ with $0 < T < \infty$. We consider the wave equation

$$\partial_t^2 u - \Delta_x u + q(t,x)u = 0, \quad (t,x) \in Q, \tag{1.1}$$

where $q \in L^\infty(Q)$ is real valued. We study the inverse problem of determining q from observations of solutions of (1.1) on $\partial Q = \Sigma \cup (\{0\} \times \overline{\Omega}) \cup (\{T\} \times \overline{\Omega})$.

It is well known that for $T > \mathrm{Diam}(\Omega)$ the data

$$\mathcal{A}_q = \{(u_{|\Sigma}, \partial_\nu u_{|\Sigma}) : \ u \in L^2(Q), \ \Box u + qu = 0, \ u_{|t=0} = \partial_t u_{|t=0} = 0\} \tag{1.2}$$

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determines uniquely a time-independent potential q (e.g. [27]). Here ν denotes the outward unit normal vector to $\partial\Omega$, $\partial_\nu = \nu \cdot \nabla_x$ the normal derivative and from now on $\Box$ denotes the differential operator $\partial_t^2 - \Delta_x$. It has been even proved that partial knowledge of $\mathcal{A}_q$ determines a time-independent potential q (e.g. [9]). In contrast to time-independent potentials, we can not recover the restriction of a general time-dependent potential q to the set

$$D = \{(t,x) \in Q : \ 0 < t < \mathrm{Diam}(\Omega)/2, \ \mathrm{dist}(x, \partial\Omega) > t\}$$

from the data $\mathcal{A}_q$. Indeed, assume that $\Omega = \{x \in \mathbb{R}^n : \ |x| < R\}$, $T > R > 0$. Now let u solve

$$\Box u = 0, \ u_{|\Sigma} = f, \quad u_{|t=0} = \partial_t u_{|t=0} = 0$$

with $f \in H^1(\Sigma)$ satisfying $f_{|t=0} = 0$. Since $u_{|t=0} = \partial_t u_{|t=0} = 0$, the finite speed of propagation implies that $u_{|D} = 0$. Therefore, for any $q \in \mathcal{C}_0^\infty(D)$, we have $qu = 0$ and u solves

$$\Box u + qu = 0, \ u_{|\Sigma} = f, \quad u_{|t=0} = \partial_t u_{|t=0} = 0.$$

This last result implies that for any $q \in \mathcal{C}_0^\infty(D)$ we have $\mathcal{A}_q = \mathcal{A}_0$ where $\mathcal{A}_0$ stands for $\mathcal{A}_q$ when $q = 0$.

Facing this obstruction to uniqueness, it appears that four different approaches have been considered so far to solve this problem:

1) Considering the equation (1.1) for any time $t \in \mathbb{R}$ instead of $0 < t < T$ (e.g. [28,29]).
2) Recovering the restriction on a subset of Q of a time-dependent potential q from the data $\mathcal{A}_q$ (e.g. [26]).
3) Recovering a time-dependent potential q from the extended data C_q (e.g. [13]) given by

$$C_q = \{(u_{|\Sigma}, u_{|t=0}, \partial_t u_{|t=0}, \partial_\nu u_{|\Sigma}, u_{|t=T}, \partial_t u_{|t=T}) : \ u \in L^2(Q), \ (\partial_t^2 - \Delta_x + q)u = 0\}. \tag{1.3}$$

4) Recovering time-dependent coefficients that are analytic with respect to the t variable (e.g. [10]).

Therefore, it seems that the only results of unique global determination of a time-dependent potential q proved so far (at finite time) involve strong smoothness assumptions such as analyticity with respect to the t variable or the important set of data C_q. In the present paper we investigate some conditions that guaranty unique determination of general time-dependent potentials without involving an important set of data. More precisely, our goal is to prove unique global determination of a general time-dependent potential q from partial knowledge of the set of data C_q.

1.2. Physical and mathematical interest

Physically speaking, our inverse problem can be stated as the determination of physical properties such as the time evolving density of an inhomogeneous medium by probing it with disturbances generated on some parts of the boundary and at initial time. The data is the response of the medium to these disturbances, measured on some parts of the boundary and at the end of the experiment, and the purpose is to recover the function q which measures the property of the medium. Note also that the determination of time-dependent potentials can be associated with models where it is necessary to take into account the evolution in time of the perturbation.

We also precise that the determination of time-dependent potentials can be an important tool for the more difficult problem of determining a non-linear term appearing in a nonlinear wave equation from observations of the solutions on ∂Q. Indeed, in [15] Isakov applied such results for the determination of a semilinear term appearing in a semilinear parabolic equation from observations of the solutions on ∂Q.

1.3. Existing papers

In recent years the determination of coefficients for hyperbolic equations from boundary measurements has been growing in interest. Many authors have considered this problem with observations given by the set $\mathcal{A}_q$ defined by (1.2). In [27], Rakesh and Symes proved that $\mathcal{A}_q$ determines uniquely a time-independent potential q and [14] proved unique determination of a potential and a damping coefficient. The uniqueness by partial boundary observations has been considered in [9]. For sake of completeness we also mention that the stability issue related to this problem has been treated in [2,16,19,24,31,32]. Note that [19] extended the results of [27] to time-independent coefficients of order zero in an unbounded cylindrical domain. It has been proved that measurements on a bounded subset determine some classes of coefficients including periodic coefficients and compactly supported coefficients.

All the above mentioned results are concerned with time-independent coefficients. Several authors considered the problem of determining time-dependent coefficients for hyperbolic equations. In [30], Stefanov proved unique determination of a time-dependent potential for the wave equation from the knowledge of scattering data which is equivalent to the problem with boundary measurements. In [28], Ramm and Sjöstrand considered the determination of a time-dependent potential q from the data $(u_{|\mathbb{R}\times\partial\Omega}, \partial_\nu u_{|\mathbb{R}\times\partial\Omega})$ of forward solutions of (1.1) on the infinite time–space cylindrical domain $\mathbb{R}_t \times \Omega$ instead of Q ($t \in \mathbb{R}$ instead of $0 < t < T < \infty$). Rakesh and Ramm [26] considered this problem at finite time on Q, with $T > \mathrm{Diam}(\Omega)$, and they determined uniquely q restricted to some subset of Q from $\mathcal{A}_q$. Isakov established in [13, Theorem 4.2] unique determination of general time-dependent potentials on the whole domain Q from the extended data C_q given by (1.3). Applying a result of unique continuation borrowed from [33], Eskin [10] proved that the data $\mathcal{A}_q$ determines time-dependent coefficients analytic with respect to the time variable t. Salazar [29] extended the result of [28] to more general coefficients. Finally, [34] stated stability in the recovery of X-ray transforms of time-dependent potentials on a manifold and [3] proved log-type stability in the determination of time-dependent potentials from the data considered by [26] and [13].

We also mention that [5–7,11] examined the determination of time-dependent coefficients for parabolic and Schrödinger equations and proved stability estimate for these problems.

1.4. Main result

In order to state our main result, we first introduce some intermediate tools and notations. For all $\omega \in \mathbb{S}^{n-1} = \{y \in \mathbb{R}^n : |y| = 1\}$ we introduce the ω-shadowed and ω-illuminated faces

$$\partial\Omega_{+,\omega} = \{x \in \partial\Omega : \nu(x)\cdot\omega \geqslant 0\}, \quad \partial\Omega_{-,\omega} = \{x \in \partial\Omega : \nu(x)\cdot\omega \leqslant 0\}$$

of $\partial\Omega$. Here, for all $k \in \mathbb{N}^*$, $\cdot$ denotes the scalar product in $\mathbb{R}^k$ defined by

$$x\cdot y = x_1y_1 + \ldots + x_ky_k, \quad x = (x_1,\ldots,x_k) \in \mathbb{R}^k,\ y = (y_1,\ldots,y_k) \in \mathbb{R}^k.$$

We consider also the parts of the lateral boundary Σ given by

$$\Sigma_{+,\omega} = \{(t,x) \in \Sigma : \nu(x)\cdot\omega > 0\}, \quad \Sigma_{-,\omega} = \{(t,x) \in \Sigma : \nu(x)\cdot\omega \leqslant 0\}.$$

From now on we fix $\omega_0 \in \mathbb{S}^{n-1}$ and we consider $F = [0,T] \times F'$ (resp. $G = (0,T) \times G'$) with F' (resp. G') an open neighborhood of $\partial\Omega_{+,\omega_0}$ (resp. $\partial\Omega_{-,\omega_0}$) in $\partial\Omega$.

The main purpose of this paper is to prove the unique global determination of a time-dependent and real valued potential $q \in L^\infty(Q)$ from the data

$$C_q^* = \{(u_{|\Sigma}, \partial_t u_{|t=0}, \partial_\nu u_{|G}, u_{|t=T}) : u \in L^2(Q),\ \Box u + qu = 0,\ u_{|t=0} = 0,\ \mathrm{supp} u_{|\Sigma} \subset F\}.$$

See also Section 2 for a rigorous definition of this set. Our main result can be stated as follows.

Theorem 1.1. *Let $q_1,\ q_2 \in L^\infty(Q)$. Assume that*

$$C_{q_1}^* = C_{q_2}^*. \tag{1.4}$$

Then $q_1 = q_2$.

Note that our uniqueness result is stated for bounded potentials with, roughly speaking, half of the data (1.3) considered in [13, Theorem 4.2] which seems to be, with [3], the only result of unique global determination of general time-dependent coefficients for the wave equation, at finite time, in the mathematical literature. More precisely, we consider $u \in L^2(Q)$ solutions of $(\partial_t^2 - \Delta + q)u = 0$, in Q, with initial condition $u_{|t=0} = 0$ and Dirichlet boundary condition $u_{|\Sigma}$ supported on F (which, roughly speaking, corresponds to half of the boundary). Moreover, we exclude the data $\partial_t u_{|t=T}$ and we consider the Neumann data $\partial_\nu u$ only on G (which, roughly speaking, corresponds to the other half of the boundary). We also mention that in contrast to [10], we do not use results of unique continuation where the analyticity of the coefficients with respect to t is required. To our best knowledge condition (1.4) is the weakest condition that guaranties global uniqueness of general time-dependent potentials.

Let us also mention that, according to the obstruction to uniqueness given by domain of dependence arguments (see Subsection 1.1), there is no hope to remove all the information on $\{t=0\}$ and $\{t=T\}$ for the global recovery of general time-dependent coefficients. Thus, for our problem the data $\partial_t u_{|t=0}$ and $u_{|t=T}$, of solutions u of (1.1), can not be removed.

The main tools in our analysis are geometric optics (GO in short) solutions and Carleman estimates. Following an approach used for elliptic equations (e.g. [4,8,18,25]) and for determination of time-independent potentials by [2], we construct two kind of GO solutions: GO solutions lying in $H^1(Q)$ without condition on ∂Q (see Section 3) and GO solutions associated with (1.1) that vanish on parts of ∂Q (see Section 5). With these solutions and some Carleman estimates with linear weight (see Section 4), we prove Theorem 1.1.

Let us observe that in the present paper we consider the recovery of a time-dependent potential q in the flat case for operators whose principal part is characterized by constant coefficients. This allows us to conclude through an argument using the Fourier transform of the potential q. The same problem stated on a manifold with boundaries should be carry out differently. In some recent work [20] studied this problem on simple Riemannian manifolds, but the recovery of a time-dependent potential q on more general Riemannian manifolds, that may not be simple, is still an open problem.

1.5. Outline

This paper is organized as follows. In Section 2 we give a suitable definition of the set of data C_q^* and we define the associated boundary operator. In Section 3, using some results of [5] and [12], we build suitable GO solutions associated with (1.1) without conditions on ∂Q. In Section 4, we establish a Carleman estimate for the wave equation with linear weight. In Section 5, we use the Carleman estimate introduced in Section 4 to build GO solutions associated with (1.1) that vanish on parts of ∂Q. More precisely, we build GO u which are solutions of (1.1) with $u_{|t=0}=0$ and $\mathrm{supp} u_{|\Sigma} \subset F$. In Section 6 we combine all the results of the previous sections in order to prove Theorem 1.1. We prove also some auxiliary results in the appendix.

2. Preliminary results

The goal of this section is to give a suitable definition to the set of data C_q^* and to introduce some properties of the solutions of (1.1) for any $q \in L^\infty(Q)$. We first introduce the space

$$J = \{u \in L^2(Q) :\ (\partial_t^2 - \Delta_x)u = 0\}$$

and topologize it as a closed subset of $L^2(Q)$. We work with the space

$$H_\Box(Q) = \{u \in L^2(Q) :\ \Box u = (\partial_t^2 - \Delta_x)u \in L^2(Q)\},$$

with the norm

$$\|u\|^2_{H_\Box(Q)} = \|u\|^2_{L^2(Q)} + \left\|(\partial_t^2 - \Delta_x)u\right\|^2_{L^2(Q)}.$$

Repeating some arguments of [22, Chapter 2, Theorem 6.4] we prove in the appendix (see Theorem A.1) that $H_\Box(Q)$ is embedded continuously into the closure of $\mathcal{C}^\infty(\overline{Q})$ in the space

$$K_\Box(Q) = \{u \in H^{-1}(0,T; L^2(\Omega)) :\ \Box u = (\partial_t^2 - \Delta_x)u \in L^2(Q)\}$$

topologized by the norm

$$\|u\|^2_{K_\Box(Q)} = \|u\|^2_{H^{-1}(0,T;L^2(\Omega))} + \left\|(\partial_t^2 - \Delta_x)u\right\|^2_{L^2(Q)}.$$

Then, following [22, Chapter 2, Theorem 6.5], we prove in the appendix that the maps

$$\tau_0 w = (w_{|\Sigma}, w_{|t=0}, \partial_t w_{|t=0}), \quad \tau_1 w = (\partial_\nu w_{|\Sigma}, w_{|t=T}, \partial_t w_{|t=T}), \quad w \in \mathcal{C}^\infty(\overline{Q}),$$

can be extended continuously to $\tau_0 : H_\Box(Q) \to H^{-3}(0,T; H^{-\frac{1}{2}}(\partial\Omega)) \times H^{-2}(\Omega) \times H^{-4}(\Omega)$, $\tau_1 : H_\Box(Q) \to H^{-3}(0,T; H^{-\frac{3}{2}}(\partial\Omega)) \times H^{-2}(\Omega) \times H^{-4}(\Omega)$ (see Proposition A.1). Here for all $w \in \mathcal{C}^\infty(\overline{Q})$ we set

$$\tau_0 w = (\tau_{0,1}w, \tau_{0,2}w, \tau_{0,3}w), \quad \tau_1 w = (\tau_{1,1}w, \tau_{1,2}w, \tau_{1,3}w),$$

where

$$\tau_{0,1}w = w_{|\Sigma},\ \tau_{0,2}w = w_{|t=0},\ \tau_{0,3}w = \partial_t w_{|t=0},\ \tau_{1,1}w = \partial_\nu w_{|\Sigma},\ \tau_{1,2}w = w_{|t=T},\ \tau_{1,3}w = \partial_t w_{|t=T}.$$

Therefore, we can introduce

$$\mathcal{H}(\partial Q) = \{\tau_0 u :\ u \in H_\Box(Q)\} \subset H^{-3}(0,T; H^{-\frac{1}{2}}(\partial\Omega)) \times H^{-2}(\Omega) \times H^{-4}(\Omega).$$

Following [4] and [25], in order to define an appropriate topology on $\mathcal{H}(\partial Q)$ we consider the restriction of τ_0 to the space J.

Proposition 2.1. *The restriction of τ_0 to J, that maps J onto $\mathcal{H}(\partial Q)$, is one to one and onto.*

Proof. Let $v_1, v_2 \in J$ with $\tau_0 v_1 = \tau_0 v_2$. Then, in light of the theory introduced in [22, Chapter 3, Section 8], there exists $F \in H_\Box(Q)$ such that, for $j = 1, 2$, we have $v_j = F + w_j$ with $w_j \in \mathcal{C}^1([0,T]; L^2(\Omega)) \cap \mathcal{C}([0,T]; H_0^1(\Omega))$ solving

$$\begin{cases} \partial_t^2 w_j - \Delta_x w_j = -\Box F, & (t,x) \in Q, \\ w_{j\,|t=0} = \partial_t w_{j\,|t=0} = 0, & \\ w_{j\,|\Sigma} = 0. & \end{cases}$$

Then, the uniqueness of solutions of this initial boundary value problem (IBVP in short) implies that $v_1 = v_2$. Thus, the restriction of τ_0 to J is one to one. Now let $(g, v_0, v_1) \in \mathcal{H}(\partial Q)$. There exists $S \in H_\Box(Q)$ such that $\tau_0 S = (g, v_0, v_1)$. Consider the initial boundary value problem

$$\begin{cases} \partial_t^2 v - \Delta_x v = -\Box S, & (t,x) \in Q, \\ v_{|t=0} = \partial_t v_{|t=0} = 0, & \\ v_{|\Sigma} = 0. & \end{cases}$$

Since $-\Box S \in L^2(Q)$, we deduce that this IBVP admits a unique solution $v \in \mathcal{C}^1([0,T]; L^2(\Omega)) \cap \mathcal{C}([0,T]; H_0^1(\Omega))$. Then, $u = v + S \in L^2(Q)$ satisfies $(\partial_t^2 - \Delta_x)u = 0$ and $\tau_0 u = \tau_0 v + \tau_0 S = (g, v_0, v_1)$. Thus, τ_0 is onto. $\Box$

From now on, we set $\mathcal{P}_0$ the inverse of $\tau_0 : J \to \mathcal{H}(\partial Q)$ and define the norm of $\mathcal{H}(\partial Q)$ by

$$\|(g, v_0, v_1)\|_{\mathcal{H}(\partial Q)} = \|\mathcal{P}_0(g, v_0, v_1)\|_{L^2(Q)}, \quad (g, v_0, v_1) \in \mathcal{H}(\partial Q).$$

In the same way, we introduce the space $\mathcal{H}_F(\partial Q)$ defined by

$$\mathcal{H}_F(\partial Q) = \{(\tau_{0,1}h, \tau_{0,3}h) :\ h \in H_\Box(Q),\ \tau_{0,2}h = 0,\ \mathrm{supp}\tau_{0,1}h \subset F\}$$

with the associated norm given by

$$\|(g, v_1)\|_{\mathcal{H}_F(\partial Q)} = \|(g, 0, v_1)\|_{\mathcal{H}(\partial Q)}, \quad (g, v_1) \in \mathcal{H}_F(\partial Q).$$

One can easily check that the space $\mathcal{H}_F(\partial Q)$ is embedded continuously into $\mathcal{H}(\partial Q)$. Let us consider the IBVP

$$\begin{cases} \partial_t^2 u - \Delta_x u + q(t,x)u = 0, & \text{in } Q, \\ u(0,\cdot) = 0, \quad \partial_t u(0,\cdot) = v_1, & \text{in } \Omega, \\ u = g, & \text{on } \Sigma. \end{cases} \tag{2.1}$$

We are now in position to state existence and uniqueness of solutions of this IBVP for $(g, v_1) \in \mathcal{H}_F(\partial Q)$.

Proposition 2.2. *Let $(g, v_1) \in \mathcal{H}_F(\partial Q)$ and $q \in L^\infty(Q)$. Then, the IBVP (2.1) admits a unique weak solution $u \in L^2(Q)$ satisfying*

$$\|u\|_{L^2(Q)} \leqslant C \, \|(g, v_1)\|_{\mathcal{H}_F(\partial Q)} \tag{2.2}$$

and the boundary operator $B_q : (g, v_1) \mapsto (\tau_{1,1}u_{|G}, \tau_{1,2}u)$ is a bounded operator from $\mathcal{H}_F(\partial Q)$ to $H^{-3}(0, T; H^{-\frac{3}{2}}(G')) \times H^{-2}(\Omega)$.

Proof. We split u into two terms $u = v + \mathcal{P}_0(g, 0, v_1)$ where v solves

$$\begin{cases} \partial_t^2 v - \Delta_x v + qv = -q\mathcal{P}_0(g, 0, v_1), & (t, x) \in Q, \\ v_{|t=0} = \partial_t v_{|t=0} = 0, \\ v_{|\Sigma} = 0. \end{cases} \tag{2.3}$$

Since $\mathcal{P}_0(g, 0, v_1) \in L^2(Q)$, the IBVP (2.3) admits a unique solution $v \in \mathcal{C}^1([0, T]; L^2(\Omega)) \cap \mathcal{C}([0, T]; H_0^1(\Omega))$ (e.g. [22, Chapter 3, Section 8]) satisfying

$$\|v\|_{\mathcal{C}^1([0,T];L^2(\Omega))} + \|v\|_{\mathcal{C}([0,T];H_0^1(\Omega))} \leqslant C \, \|-q\mathcal{P}_0(g, 0, v_1)\|_{L^2(Q)} \leqslant C \, \|q\|_{L^\infty(Q)} \, \|\mathcal{P}_0(g, 0, v_1)\|_{L^2(Q)} . \tag{2.4}$$

Therefore, $u = v + \mathcal{P}_0(g, 0, v_1)$ is the unique solution of (2.1) and estimate (2.4) implies (2.2). Now let us show the last part of the proposition. For this purpose fix $(g, v_1) \in \mathcal{H}_F(\partial Q)$ and consider u the solution of (2.1). Note first that $u \in L^2(Q)$ and $(\partial_t^2 - \Delta_x)u = -qu \in L^2(Q)$. Thus, $u \in H_\Box(Q)$ and $\tau_{1,1}u \in H^{-3}(0, T; H^{-\frac{3}{2}}(\partial\Omega))$, $\tau_{1,2}u \in H^{-2}(\Omega)$ with

$$\left\|\tau_{1,1}u\right\|^2 + \left\|\tau_{1,2}u\right\|^2 \leqslant C^2 \, \|u\|_{H_\Box(Q)}^2 = C^2(\|u\|_{L^2(Q)}^2 + \|qu\|_{L^2(Q)}^2) \leqslant C^2(1 + \|q\|_{L^\infty(Q)}^2) \, \|u\|_{L^2(Q)}^2 .$$

Combining this with (2.2) we deduce that B_q is a bounded operator from $\mathcal{H}_F(\partial Q)$ to $H^{-3}(0, T; H^{-\frac{3}{2}}(G')) \times H^{-2}(\Omega)$. $\Box$

From now on we consider the set C_q^* to be the graph of the boundary operator B_q given by

$$C_q^* = \{(g, v_1, B_q(g, v_1)) : \ (g, v_1) \in \mathcal{H}_F(\partial Q)\}.$$

3. Geometric optics solutions without boundary conditions

In this section we build geometric optics solutions $u \in H^1(Q)$ associated with the equation

$$\partial_t^2 u - \Delta_x u + q(t, x)u = 0 \quad \text{on } Q. \tag{3.1}$$

More precisely, for $\lambda > 1$, $\omega \in \mathbb{S}^{n-1} = \{y \in \mathbb{R}^n : \ |y| = 1\}$ and $\xi \in \mathbb{R}^{1+n}$ satisfying $\xi \cdot (1, -\omega) = 0$, we consider solutions of the form

$$u(t, x) = e^{-\lambda(t+x\cdot\omega)}(e^{-i\xi\cdot(t,x)} + w(t, x)), \quad (t, x) \in Q. \tag{3.2}$$

Here $w \in H^1(Q)$ fulfills

$$\|w\|_{L^2(Q)} \leqslant \frac{C}{\lambda}$$

with $C > 0$ independent of λ. For this purpose, for all $s \in \mathbb{R}$ and all $\omega \in \mathbb{S}^{n-1}$, we consider the operators $P_{s,\omega}$ defined by $P_{s,\omega} = e^{-s(t+x\cdot\omega)}\Box e^{s(t+x\cdot\omega)}$. One can check that

$$P_{s,\omega} = p_{s,\omega}(D_t, D_x) = \Box + 2s(\partial_t - \omega \cdot \nabla_x)$$

with $D_t = -i\partial_t$, $D_x = -i\nabla_x$ and $p_{s,\omega}(\mu, \eta) = -\mu^2 + |\eta|^2 + 2si(\mu - \omega \cdot \eta)$, $\mu \in \mathbb{R}$, $\eta \in \mathbb{R}^n$. Applying some results of [5] and [12] about solutions of PDEs with constant coefficients we obtain the following.

Lemma 3.1. *For every $\lambda > 1$ and $\omega \in \mathbb{S}^{n-1}$ there exists a bounded operator $E_{\lambda,\omega}: L^2(Q) \to L^2(Q)$ such that:*

$$P_{-\lambda,\omega}E_{\lambda,\omega}f = f, \quad f \in L^2(Q), \tag{3.3}$$

$$\|E_{\lambda,\omega}\|_{\mathcal{B}(L^2(Q))} \leqslant C\lambda^{-1}, \tag{3.4}$$

$$E_{\lambda,\omega} \in \mathcal{B}(L^2(Q); H^1(Q)) \quad \text{and} \quad \|E_{\lambda,\omega}\|_{\mathcal{B}(L^2(Q);H^1(Q))} \leqslant C \tag{3.5}$$

with $C >$ depending only on T and Ω.

Proof. In light of [5, Theorem 2.3] (see also [12, Theorem 10.3.7]), there exists a bounded operator $E_{\lambda,\omega}: L^2(Q) \to L^2(Q)$, defined from a fundamental solution associated with $P_{-\lambda,\omega}$ (see Section 10.3 of [12]), such that (3.3) is fulfilled. In addition, fixing

$$\tilde{p}_{-\lambda,\omega}(\mu,\eta) := \left(\sum_{k\in\mathbb{N}}\sum_{\alpha\in\mathbb{N}^n} |\partial_\mu^k\partial_\eta^\alpha p_{-\lambda,\omega}(\mu,\eta)|^2\right)^{\frac{1}{2}}, \quad \mu \in \mathbb{R},\ \eta \in \mathbb{R}^n,$$

for all differential operator $S(D_t, D_x)$ with $\frac{S(\mu,\eta)}{\tilde{p}_{-\lambda,\omega}(\mu,\eta)}$ a bounded function, we have $S(D_t, D_x)E_{\lambda,\omega} \in \mathcal{B}(L^2(Q))$ and there exists a constant C depending only on Ω, T such that

$$\|S(D_t, D_x)E_{\lambda,\omega}\|_{\mathcal{B}(L^2(Q))} \leqslant C \sup_{(\mu,\eta)\in\mathbb{R}^{1+n}} \frac{|S(\mu,\eta)|}{\tilde{p}_{-\lambda,\omega}(\mu,\eta)}. \tag{3.6}$$

Note that $\tilde{p}_{-\lambda,\omega}(\mu,\eta) \geqslant \left|\Im\partial_\mu p_{-\lambda,\omega}(\mu,\eta)\right| = 2\lambda$. Therefore, (3.6) implies

$$\|E_{\lambda,\omega}\|_{\mathcal{B}(L^2(Q))} \leqslant C \sup_{(\mu,\eta)\in\mathbb{R}^{1+n}} \frac{1}{\tilde{p}_{-\lambda,\omega}(\mu,\eta)} \leqslant C\lambda^{-1}$$

and (3.4) is fulfilled. In a same way, we have $\tilde{p}_{-\lambda,\omega}(\mu,\eta) \geqslant \left|\Re\partial_\mu p_{-\lambda,\omega}(\mu,\eta)\right| = 2|\mu|$ and $\tilde{p}_{-\lambda,\omega}(\mu,\eta) \geqslant \left|\Re\partial_{\eta_j} p_{-\lambda,\omega}(\mu,\eta)\right| = 2|\eta_j|$, $j = 1,\dots,n$ and $\eta = (\eta_1,\dots,\eta_n)$. Therefore, in view of [5, Theorem 2.3], we have $E_{\lambda,\omega} \in \mathcal{B}(L^2(Q); H^1(Q))$ with

$$\|E_{\lambda,\omega}\|_{\mathcal{B}(L^2(Q);H^1(Q))} \leqslant C \sup_{(\mu,\eta)\in\mathbb{R}^{1+n}} \frac{|\mu| + |\eta_1| + \ldots + |\eta_n|}{\tilde{p}_{-\lambda,\omega}(\mu,\eta)} + C\lambda^{-1} \leqslant C(n+2)$$

and (3.5) is proved. □

Applying this result, we can build geometric optics solutions of the form (3.2).

Proposition 3.1. *Let $q \in L^\infty(Q)$, $\omega \in \mathbb{S}^{n-1}$. Then, there exists $\lambda_0 > 1$ such that for $\lambda \geqslant \lambda_0$ the equation (3.1) admits a solution $u \in H^1(Q)$ of the form (3.2) with*

$$\|w\|_{H^k(Q)} \leqslant C\lambda^{k-1}, \quad k = 0, 1, \tag{3.7}$$

where C and λ_0 depend on Ω, ξ, T, $M \geqslant \|q\|_{L^\infty(Q)}$.

Proof. We start by recalling that, for every $(t,x) \in Q$, we have

$$\begin{aligned}\Box e^{-\lambda(t+x\cdot\omega)}e^{-i\xi\cdot(t,x)} &= e^{-\lambda(t+x\cdot\omega)}\left(\Box e^{-i\xi\cdot(t,x)} + 2i\lambda\xi\cdot(1,-\omega)e^{-i\xi\cdot(t,x)}\right)\\ &= e^{-\lambda(t+x\cdot\omega)}\Box e^{-i\xi\cdot(t,x)}.\end{aligned}$$

Thus, w should be a solution of

$$\partial_t^2 w - \Delta_x w - 2\lambda(\partial_t - \omega\cdot\nabla_x)w = -\left((\Box + q)e^{-i\xi\cdot(t,x)} + qw\right). \tag{3.8}$$

Therefore, according to Lemma 3.1, we can define w as a solution of the equation

$$w = -E_{\lambda,\omega}\left((\Box + q)e^{-i\xi\cdot(t,x)} + qw\right), \quad w \in L^2(Q)$$

with $E_{\lambda,\omega} \in \mathcal{B}(L^2(Q))$ given by Lemma 3.1. For this purpose, we will use a standard fixed point argument associated with the map

$$\mathcal{G}: \quad L^2(Q) \to L^2(Q),$$
$$h \mapsto -E_{\lambda,\omega}\left[(\Box + q)e^{-i\xi\cdot(t,x)} + qh\right].$$

Indeed, in view of (3.4), fixing $M_1 > 0$, there exists $\lambda_0 > 1$ such that for $\lambda \geqslant \lambda_0$ the map $\mathcal{G}$ admits a unique fixed point w in $\{u \in L^2(Q) : \|u\|_{L^2(Q)} \leqslant M_1\}$. In addition, condition (3.4)–(3.5) imply that $w \in H^1(Q)$ fulfills (3.7). This completes the proof. □

4. Carleman estimates

This section is devoted to the proof of Carleman estimates similar to [2] and [4]. More precisely, we fix $\omega \in \mathbb{S}^{n-1}$ and we consider the following estimates.

Theorem 4.1. *Let $q \in L^\infty(Q)$ and $u \in \mathcal{C}^2(\overline{Q})$. If u satisfies the condition*

$$u_{|\Sigma} = 0, \quad u_{|t=0} = \partial_t u_{|t=0} = 0 \tag{4.1}$$

then there exists $\lambda_1 > 1$ depending only on Ω, T and $M \geqslant \|q\|_{L^\infty(Q)}$ such that the estimate

$$\begin{aligned}&\lambda \int_\Omega e^{-2\lambda(T+\omega\cdot x)} |\partial_t u(T,x)|^2 dx + \lambda \int_{\Sigma_{+,\omega}} e^{-2\lambda(t+\omega\cdot x)} |\partial_\nu u|^2 |\omega\cdot\nu(x)| d\sigma(x)dt + \lambda^2 \int_Q e^{-2\lambda(t+\omega\cdot x)} |u|^2 dxdt \\ &\leqslant C\left(\int_Q e^{-2\lambda(t+\omega\cdot x)} \left|(\partial_t^2 - \Delta_x + q)u\right|^2 dxdt + \lambda^3 \int_\Omega e^{-2\lambda(T+\omega\cdot x)} |u(T,x)|^2 dx\right) \\ &\quad + C\left(\lambda \int_\Omega e^{-2\lambda(T+\omega\cdot x)} |\nabla_x u(T,x)|^2 dx + \lambda \int_{\Sigma_{-,\omega}} e^{-2\lambda(t+\omega\cdot x)} |\partial_\nu u|^2 |\omega\cdot\nu(x)| d\sigma(x)dt\right)\end{aligned} \tag{4.2}$$

holds true for $\lambda \geqslant \lambda_1$ with C depending only on Ω, T and $M \geqslant \|q\|_{L^\infty(Q)}$. If u satisfies the condition

$$u_{|\Sigma} = 0, \quad u_{|t=T} = \partial_t u_{|t=T} = 0 \tag{4.3}$$

then the estimate

$$\begin{aligned}&\lambda \int_\Omega e^{2\lambda\omega\cdot x} \left|\partial_t u_{|t=0}\right|^2 dx + \lambda \int_{\Sigma_{-,\omega}} e^{2\lambda(t+\omega\cdot x)} |\partial_\nu u|^2 |\omega\cdot\nu(x)| d\sigma(x)dt + \lambda^2 \int_Q e^{2\lambda(t+\omega\cdot x)} |u|^2 dxdt \\ &\leqslant C\left(\int_Q e^{2\lambda(t+\omega\cdot x)} \left|(\partial_t^2 - \Delta_x + q)u\right|^2 dxdt + \lambda^3 \int_\Omega e^{2\lambda\omega\cdot x} |u(0,x)|^2 dx + \lambda \int_\Omega e^{2\lambda\omega\cdot x} |\nabla_x u(0,x)|^2 dx\right) \\ &\quad + C\lambda \int_{\Sigma_{+,\omega}} e^{2\lambda(t+\omega\cdot x)} |\partial_\nu u|^2 |\omega\cdot\nu(x)| d\sigma(x)dt\end{aligned} \tag{4.4}$$

holds true for $\lambda \geqslant \lambda_1$.

In order to prove these estimates, we fix $u \in \mathcal{C}^2(\overline{Q})$ satisfying (4.1) (resp. (4.3)) and we set $v = e^{-\lambda(t+\omega\cdot x)}u$ (resp. $v = e^{\lambda(t+\omega\cdot x)}u$) in such a way that

$$e^{-\lambda(t+\omega\cdot x)}\Box u = P_{\lambda,\omega} v, \quad \left(\text{resp. } e^{\lambda(t+\omega\cdot x)}\Box u = P_{-\lambda,\omega} v\right). \tag{4.5}$$

Then, we consider the following estimates associated with the weighted operators $P_{\pm\lambda,\omega}$.

Lemma 4.1. *Let $v \in \mathcal{C}^2(\overline{Q})$ and $\lambda > 1$. If v satisfies the condition*

$$v_{|\Sigma} = 0, \quad v_{|t=0} = \partial_t v_{|t=0} = 0 \tag{4.6}$$

then the estimate

$$\begin{aligned}&\lambda \int_\Omega |\partial_t v(T,x)|^2 dx + 2\lambda \int_{\Sigma_{+,\omega}} |\partial_\nu v|^2 \omega\cdot\nu(x) d\sigma(x)dt + c\lambda^2 \int_Q |v|^2 dxdt \\ &\leqslant \int_Q \left|P_{\lambda,\omega} v\right|^2 dxdt + 14\lambda \int_\Omega |\nabla_x v(T,x)|^2 dx + 2\lambda \int_{\Sigma_{-,\omega}} |\partial_\nu v|^2 |\omega\cdot\nu(x)| d\sigma(x)dt\end{aligned} \tag{4.7}$$

holds true for $c > 0$ depending only on Ω and T. If v satisfies the condition

$$v_{|\Sigma} = 0, \quad v_{|t=T} = \partial_t v_{|t=T} = 0 \tag{4.8}$$

then the estimate

$$\begin{aligned} &\lambda \int_\Omega |\partial_t v(0,x)|^2 \, dx + 2\lambda \int_{\Sigma_{-,\omega}} |\partial_\nu v|^2 \, |\omega \cdot \nu(x)| \, d\sigma(x) dt + c\lambda^2 \int_Q |v|^2 \, dx dt \\ &\leqslant \int_Q \left| P_{-\lambda,\omega} v \right|^2 dx dt + 14\lambda \int_\Omega |\nabla_x v(0,x)|^2 \, dx + 2\lambda \int_{\Sigma_{+,\omega}} |\partial_\nu v|^2 \, \omega \cdot \nu(x) d\sigma(x) dt \end{aligned} \tag{4.9}$$

holds true.

Proof. We start with (4.7). For this purpose we fix $v \in \mathcal{C}^2(\overline{Q})$ satisfying (4.6) and we consider

$$I_{\lambda,\omega} = \int_Q |P_{\lambda,\omega} v|^2 dt dx.$$

Without lost of generality we assume that v is real valued. Repeating some arguments of [2] (see the formula 2 lines before (2.4) in page 1225 of [2] and formula (2.5) in page 1226 of [2]) we obtain the following

$$\begin{aligned} I_{\lambda,\omega} \geqslant \int_Q |\Box v|^2 dt dx + c\lambda^2 \int_Q |v|^2 \, dx dt + 2\lambda \int_\Sigma |\partial_\nu v|^2 \, \omega \cdot \nu(x) d\sigma(x) dt \\ +2\lambda \int_\Omega |\partial_t v(T,x)|^2 \, dx + 2\lambda \int_\Omega |\nabla_x v(T,x)|^2 \, dx - 4\lambda \int_\Omega (\partial_t v(T,x))(\omega \cdot \nabla_x v(T,x)) dx. \end{aligned}$$

On the other hand, an application of the Cauchy–Schwarz inequality yields

$$4\lambda \left| \int_\Omega (\partial_t v(T,x))(\omega \cdot \nabla_x v(T,x)) dx \right| \leqslant \frac{\lambda}{4} \int_\Omega |\partial_t v(T,x)|^2 \, dx + 16\lambda \int_\Omega |\nabla_x v(T,x)|^2 \, dx$$

and we deduce that

$$\begin{aligned} &I_{\lambda,\omega} + 14\lambda \int_\Omega |\nabla_x v(T,x)|^2 \, dx \\ &\geqslant \int_Q |\Box v|^2 dt dx + c\lambda^2 \int_Q |v|^2 \, dx dt + 2\lambda \int_\Sigma |\partial_\nu v|^2 \, \omega \cdot \nu(x) d\sigma(x) dt + \lambda \int_\Omega |\partial_t v(T,x)|^2 \, dx. \end{aligned}$$

From this last estimate we deduce easily (4.7). Now let us consider (4.9). For this purpose note that for v satisfying (4.8), w defined by $w(t,x) = v(T-t,x)$ satisfies (4.6). Thus, applying (4.7) to w with ω replaced by $-\omega$ we obtain (4.9). □

In light of Lemma 4.1, we are now in position to prove Theorem 4.1.

Proof of Theorem 4.1. Let us first consider the case $q = 0$. Note that for u satisfying (4.1), $v = e^{-\lambda(t+\omega\cdot x)} u$ satisfies (4.6). Moreover, we have (4.5) and (4.1) implies $\partial_\nu v_{|\Sigma} = e^{-\lambda(t+\omega\cdot x)} \partial_\nu u_{|\Sigma}$. Finally, using the fact that

$$\partial_t u = \partial_t (e^{\lambda(t+\omega\cdot x)} v) = \lambda u + e^{\lambda(t+\omega\cdot x)} \partial_t v, \quad \nabla_x v = e^{-\lambda(t+\omega\cdot x)} (\nabla_x u - \lambda u \omega),$$

we obtain

$$\begin{aligned} \int_\Omega e^{-2\lambda(T+\omega\cdot x)} |\partial_t u(T,x)|^2 \, dx \leqslant 2 \int_\Omega |\partial_t v(T,x)|^2 \, dx + 2\lambda^2 \int_\Omega e^{-2\lambda(T+\omega\cdot x)} |u(T,x)|^2 \, dx, \\ \int_\Omega |\nabla_x v(T,x)|^2 \, dx \leqslant 2\lambda^2 \int_\Omega e^{-2\lambda(T+\omega\cdot x)} |u(T,x)|^2 \, dx + 2 \int_\Omega e^{-2\lambda(T+\omega\cdot x)} |\nabla_x u(T,x)|^2 \, dx. \end{aligned}$$

Thus, applying the Carleman estimate (4.7) to v, we deduce (4.2). For $q \neq 0$, we have

$$\left| \partial_t^2 u - \Delta_x u \right|^2 = \left| \partial_t^2 u - \Delta_x u + qu - qu \right|^2 \leqslant 2 \left| (\partial_t^2 - \Delta_x + q) u \right|^2 + 2 \, \|q\|_{L^\infty(Q)}^2 \, |u|^2$$

and hence if we choose $\lambda_1 > 2C\|q\|^2_{L^\infty(Q)}$, replacing C by

$$C_1 = \frac{C\lambda_1^2}{\lambda_1^2 - 2C\|q\|^2_{L^\infty(Q)}},$$

we deduce (4.2) from the same estimate when $q = 0$. Using similar arguments, we prove (4.4). □

Remark 4.1. Note that, by density, estimate (4.2) can be extended to any function $u \in \mathcal{C}^1([0,T]; L^2(\Omega)) \cap \mathcal{C}([0,T]; H^1(\Omega))$ satisfying (4.6), $(\partial_t^2 - \Delta_x)u \in L^2(Q)$ and $\partial_\nu u \in L^2(\Sigma)$.

5. Geometric optics solutions vanishing on parts of the boundary

In this section we fix $q \in L^\infty(Q)$. From now on, for all $y \in \mathbb{S}^{n-1}$ and all $r > 0$, we set

$$\partial\Omega_{+,r,y} = \{x \in \partial\Omega : \nu(x)\cdot y > r\}, \quad \partial\Omega_{-,r,y} = \{x \in \partial\Omega : \nu(x)\cdot y \leqslant r\}$$

and $\Sigma_{\pm,r,y} = (0,T)\times\partial\Omega_{\pm,r,y}$. Here and in the remaining of this text we always assume, without mentioning it, that y and r are chosen in such way that $\partial\Omega_{\pm,r,\pm y}$ contain a non-empty relatively open subset of $\partial\Omega$. Without lost of generality we assume that there exists $0 < \varepsilon < 1$ such that for all $\omega \in \{y \in \mathbb{S}^{n-1} : |y - \omega_0| \leqslant \varepsilon\}$ we have $\partial\Omega_{-,\varepsilon,-\omega} \subset F'$. The goal of this section is to use the Carleman estimate (4.4) in order to build solutions $u \in H_\Box(Q)$ to

$$\begin{cases} (\partial_t^2 - \Delta_x + q(t,x))u = 0 & \text{in } Q,\\ u_{|t=0} = 0, & \\ u = 0, & \text{on } \Sigma_{+,\varepsilon/2,-\omega}, \end{cases} \tag{5.1}$$

of the form

$$u(t,x) = e^{\lambda(t+\omega\cdot x)}\left(1 + z(t,x)\right), \quad (t,x)\in Q. \tag{5.2}$$

Here $\omega \in \{y \in \mathbb{S}^{n-1} : |y-\omega_0| \leqslant \varepsilon\}$, $z \in e^{-\lambda(t+\omega\cdot x)}H_\Box(Q)$ fulfills: $z(0,x) = -1$, $x\in\Omega$, $z = -1$ on $\Sigma_{+,\varepsilon/2,-\omega}$ and

$$\|z\|_{L^2(Q)} \leqslant C\lambda^{-\frac{1}{2}} \tag{5.3}$$

with C depending on F', Ω, T and any $M \geqslant \|q\|_{L^\infty(Q)}$. Since $\Sigma\setminus F \subset \Sigma\setminus\Sigma_{-,\varepsilon,-\omega} = \Sigma_{+,\varepsilon,-\omega}$ and since $\Sigma_{+,\varepsilon/2,-\omega}$ is a neighborhood of $\Sigma_{+,\varepsilon,-\omega}$ in Σ, it is clear that condition (5.1) implies $(\tau_{0,1}u, \tau_{0,3}u) \in \mathcal{H}_F(\partial Q)$ (recall that for $v \in \mathcal{C}^\infty(\overline{Q})$, $\tau_{0,1}v = v_{|\Sigma}$, $\tau_{0,3}v = \partial_t v_{|t=0}$).

The main result of this section can be stated as follows.

Theorem 5.1. *Let $q \in L^\infty(Q)$, $\omega \in \{y \in \mathbb{S}^{n-1} : |y-\omega_0| \leqslant \varepsilon\}$. For all $\lambda \geqslant \lambda_1$, with λ_1 the constant of Theorem 4.1, there exists a solution $u \in H_\Box(Q)$ of (5.1) of the form (5.2) with z satisfying (5.3).*

In order to prove existence of such solutions of (5.1) we need some preliminary tools and an intermediate result.

5.1. *Weighted spaces*

In this subsection we give the definition of some weighted spaces. We set $s \in \mathbb{R}$, we fix $\omega \in \{y \in \mathbb{S}^{n-1} : |y - \omega_0| \leqslant \varepsilon\}$ and we denote by γ the function defined on $\partial\Omega$ by

$$\gamma(x) = |\omega\cdot\nu(x)|, \quad x \in \partial\Omega.$$

We introduce the spaces $L_s(Q)$, $L_s(\Omega)$, and for all non-negative measurable function h on $\partial\Omega$ the spaces $L_{s,h,\pm}$ defined respectively by

$$L_s(Q) = e^{-s(t+\omega\cdot x)}L^2(Q), \quad L_s(\Omega) = e^{-s\omega\cdot x}L^2(\Omega), \quad L_{s,h,\pm} = \{f : e^{s(t+\omega\cdot x)}h^{\frac{1}{2}}(x)f \in L^2(\Sigma_{\pm,\omega})\}$$

with the associated norm

$$\|u\|_s = \left(\int_Q e^{2s(t+\omega\cdot x)} |u|^2 \, dxdt\right)^{\frac{1}{2}}, \quad u \in L_s(Q),$$

$$\|u\|_{s,0} = \left(\int_\Omega e^{2s\omega\cdot x} |u|^2 \, dx\right)^{\frac{1}{2}}, \quad u \in L_s(\Omega),$$

$$\|u\|_{s,h,\pm} = \left(\int_{\Sigma_{\pm,\omega}} e^{2s(t+\omega\cdot x)} h(x) |u|^2 \, d\sigma(x)dt\right)^{\frac{1}{2}}, \quad u \in L_{s,h,\pm}.$$

5.2. *Intermediate result*

We set the space

$$\mathcal{D} = \{v \in \mathcal{C}^2(\overline{Q}): \ v_{|\Sigma} = 0, \ v_{|t=T} = \partial_t v_{|t=T} = v_{|t=0} = 0\}$$

and, in view of Theorem 4.1, applying the Carleman estimate (4.4) to any $f \in \mathcal{D}$ we obtain

$$\lambda \|f\|_\lambda + \lambda^{\frac{1}{2}} \left\|\partial_t f_{|t=0}\right\|_{\lambda,0} + \lambda^{\frac{1}{2}} \|\partial_\nu f\|_{\lambda,\gamma,-} \leqslant C(\left\|(\partial_t^2 - \Delta_x + q)f\right\|_\lambda + \|\partial_\nu f\|_{\lambda,\lambda\gamma,+}), \quad \lambda \geqslant \lambda_1. \tag{5.4}$$

We introduce also the space

$$\mathcal{M} = \{((\partial_t^2 - \Delta_x + q)v, \partial_\nu v_{|\Sigma_{+,\omega}}): \ v \in \mathcal{D}\}$$

and think of $\mathcal{M}$ as a subspace of $L_\lambda(Q) \times L_{\lambda,\lambda\gamma,+}$. We consider the following intermediate result.

Lemma 5.1. *Given $\lambda \geqslant \lambda_1$, with λ_1 the constant of Theorem 4.1, and*

$$v \in L_{-\lambda}(Q), \quad v_- \in L_{-\lambda,\gamma^{-1},-}, \quad v_0 \in L_{-\lambda}(\Omega),$$

there exists $u \in L_{-\lambda}(Q)$ such that:

1) $(\partial_t^2 - \Delta_x + q)u = v$ *in* Q,
2) $u_{|\Sigma_{-,\omega}} = v_-$, $u_{|t=0} = v_0$,
3) $\|u\|_{-\lambda} \leqslant C\left(\lambda^{-1}\|v\|_{-\lambda} + \lambda^{-\frac{1}{2}}\|v_-\|_{-\lambda,\gamma^{-1},-} + \lambda^{-\frac{1}{2}}\|v_0\|_{-\lambda,0}\right)$ *with C depending on $\Omega, T, M \geqslant \|q\|_{L^\infty(Q)}$.*

Proof. In view of (5.4), we can define the linear function S on $\mathcal{M}$ by

$$S[((\Box + q)f, \partial_\nu f_{|\Sigma_{+,\omega}})] = \langle f, v\rangle_{L^2(Q)} - \langle \partial_\nu f, v_-\rangle_{L^2(\Sigma_{-,\omega})} - \left\langle \partial_t f_{|t=0}, v_0\right\rangle_{L^2(\Omega)}, \quad f \in \mathcal{D}.$$

Then, using (5.4), for all $f \in \mathcal{D}$, we obtain

$$\begin{aligned}
&\left|S[((\Box + q)f, \partial_\nu f_{|\Sigma_{+,\omega}})]\right| \\
&\leqslant \|f\|_\lambda \|v\|_{-\lambda} + \|\partial_\nu f\|_{\lambda,\gamma,-}\|v_-\|_{-\lambda,\gamma^{-1},-} + \left\|\partial_t f_{|t=0}\right\|_{\lambda,0}\|v_0\|_{-\lambda,0} \\
&\leqslant \lambda^{-1}\|v\|_{-\lambda}\left(\lambda\|f\|_\lambda\right) + \lambda^{-\frac{1}{2}}\|v_-\|_{-\lambda,\gamma^{-1},-}\left(\lambda^{\frac{1}{2}}\|\partial_\nu f\|_{\lambda,\gamma,-}\right) + \lambda^{-\frac{1}{2}}\|v_0\|_{-\lambda,0}\left(\lambda^{\frac{1}{2}}\left\|\partial_t f_{|t=0}\right\|_{\lambda,0}\right) \\
&\leqslant C\left(\lambda^{-1}\|v\|_{-\lambda} + \lambda^{-\frac{1}{2}}\|v_-\|_{-\lambda,\gamma^{-1},-} + \lambda^{-\frac{1}{2}}\|v_0\|_{-\lambda,0}\right)\left(\|(\Box + q)f\|_\lambda + \|\partial_\nu f\|_{\lambda,\lambda\gamma,+}\right) \\
&\leqslant 2C\left(\lambda^{-1}\|v\|_{-\lambda} + \lambda^{-\frac{1}{2}}\|v_-\|_{-\lambda,\gamma^{-1},-} + \lambda^{-\frac{1}{2}}\|v_0\|_{-\lambda,0}\right)\left\|((\Box + q)f, \partial_\nu f_{|\Sigma_{+,\omega}})\right\|_{L_\lambda(Q)\times L_{\lambda,\lambda\gamma,+}}
\end{aligned}$$

with C the constant of (5.4). Applying the Hahn Banach theorem we deduce that S can be extended to a continuous linear form, also denoted by S, on $L_\lambda(Q) \times L_{\lambda,\lambda\gamma,+}$ satisfying

$$\|S\| \leqslant C\left(\lambda^{-1}\|v\|_{-\lambda}+\lambda^{-\frac{1}{2}}\|v_-\|_{-\lambda,\gamma^{-1},-}+\lambda^{-\frac{1}{2}}\|v_0\|_{-\lambda,0}\right). \tag{5.5}$$

Thus, there exists

$$(u,u_+)\in L_{-\lambda}(Q)\times L_{-\lambda,(\lambda\gamma)^{-1},+}$$

such that for all $f\in\mathcal{D}$ we have

$$S[((\Box+q)f,\partial_\nu f_{|\Sigma_{+,\omega}})]=\langle(\Box+q)f,u\rangle_{L^2(Q)}-\langle\partial_\nu f,u_+\rangle_{L^2(\Sigma_{+,\omega})}.$$

Therefore, for all $f\in\mathcal{D}$ we have

$$\begin{aligned}&\langle(\Box+q)f,u\rangle_{L^2(Q)}-\langle\partial_\nu f,u_+\rangle_{L^2(\Sigma_{+,\omega})}\\&=\langle f,v\rangle_{L^2(Q)}-\langle\partial_\nu f,v_-\rangle_{L^2(\Sigma_{-,\omega})}-\left\langle\partial_t f_{|t=0},v_0\right\rangle_{L^2(\Omega)}.\end{aligned} \tag{5.6}$$

Note first that, since $L_{\pm\lambda}(Q)$ is embedded continuously into $L^2(Q)$, we have $u\in L^2(Q)$. Therefore, taking $f\in\mathcal{C}_0^\infty(Q)$ shows 1). For condition 2), using the fact that $L_{\pm\lambda}(Q)$ is embedded continuously into $L^2(Q)$ we deduce that $u\in H_\Box(Q)$. Thus, we can define the trace $u_{|\Sigma}$, $u_{|t=0}$ and allowing $f\in\mathcal{D}$ to be arbitrary shows that $u_{|\Sigma_{-,\omega}}=v_-$, $u_{|t=0}=v_0$ and $u_{|\Sigma_{+,\omega}}=-u_+$. Here we use the fact that $\Sigma_{+,\omega}\cap\Sigma_{-,\omega}=\emptyset$. Finally, condition 3) follows from the fact that

$$\|u\|_{-\lambda}\leqslant\|S\|\leqslant C\left(\lambda^{-1}\|v\|_{-\lambda}+\lambda^{-\frac{1}{2}}\|v_-\|_{-\lambda,\gamma^{-1},-}+\lambda^{-\frac{1}{2}}\|v_0\|_{-\lambda,0}\right).\quad\Box$$

Armed with this lemma we are now in position to prove Theorem 5.1.

5.3. *Proof of Theorem 5.1*

Note first that z must satisfy

$$\begin{cases}z\in L^2(Q)\\(\partial_t^2-\Delta_x+q)(e^{\lambda(t+\omega\cdot x)}z)=-qe^{\lambda(t+\omega\cdot x)}\ \text{ in } Q\\z(0,x)=-1,\quad x\in\Omega,\\z=-1\quad\text{on }\Sigma_{+,\varepsilon/2,-\omega}.\end{cases} \tag{5.7}$$

Let $\psi\in\mathcal{C}_0^\infty(\mathbb{R}^n)$ be such that supp$\psi\cap\partial\Omega\subset\{x\in\partial\Omega:\ \omega\cdot\nu(x)<-\varepsilon/3\}$ and $\psi=1$ on $\{x\in\partial\Omega:\ \omega\cdot\nu(x)<-\varepsilon/2\}=\partial\Omega_{+,\varepsilon/2,-\omega}$. Choose $v_-(t,x)=-e^{\lambda(t+\omega\cdot x)}\psi(x)$, $(t,x)\in\Sigma_{-,\omega}$. Since $v_-(t,x)=0$ for $t\in(0,T)$, $x\in\{x\in\partial\Omega:\ \omega\cdot\nu(x)\geqslant-\varepsilon/3\}$ we have $v_-\in L_{-\lambda,\gamma^{-1},-}$. Fix also $v(t,x)=-qe^{\lambda(t+\omega\cdot x)}$ and $v_0(x)=-e^{\lambda\omega\cdot x}$, $(t,x)\in Q$. From Lemma 5.1, we deduce that there exists $w\in H_\Box(Q)$ such that

$$\begin{cases}(\partial_t^2-\Delta_x+q)w=v(t,x)=-qe^{\lambda(t+\omega\cdot x)} & \text{in } Q,\\w(0,x)=v_0(x)=-e^{\lambda x\cdot\omega}, & x\in\Omega,\\w(t,x)=v_-(t,x)=-e^{\lambda(t+\omega\cdot x)}\psi(x), & (t,x)\in\Sigma_{-,\omega}.\end{cases}$$

Then, for $z=e^{-\lambda(t+\omega\cdot x)}w$ condition (5.7) will be fulfilled. Moreover, condition 3) of Lemma 5.1 implies

$$\begin{aligned}\|z\|_{L^2(Q)}=\|w\|_{-\lambda}&\leqslant C\left(\lambda^{-1}\|v\|_{-\lambda}+\lambda^{-\frac{1}{2}}\|v_-\|_{-\lambda,\gamma^{-1},-}+\lambda^{-\frac{1}{2}}\|v_0\|_{-\lambda,0}\right)\\&\leqslant C\left(\lambda^{-1}\|q\|_{L^2(Q)}+\lambda^{-\frac{1}{2}}\left\|\psi\gamma^{-1/2}\right\|_{L^2(\Sigma_{-,\omega})}+\lambda^{-\frac{1}{2}}\|1\|_{L^2(\Omega)}\right)\leqslant C\lambda^{-\frac{1}{2}}\end{aligned}$$

with C depending only on Ω, T and $\|q\|_{L^\infty(Q)}$. Therefore, estimate (5.3) holds. Using the fact that $e^{\lambda(t+\omega\cdot x)}z=w\in H_\Box(Q)$, we deduce that u defined by (5.2) is lying in $H_\Box(Q)$ and is a solution of (5.1). This completes the proof of Theorem 5.1.

6. Uniqueness result

This section is devoted to the proof of Theorem 1.1. From now on we set $q = q_2 - q_1$ on Q and we assume that $q = 0$ on $\mathbb{R}^{1+n} \setminus Q$. Without lost of generality we assume that for all $\omega \in \{y \in \mathbb{S}^{n-1} : |y - \omega_0| \leqslant \varepsilon\}$ we have $\partial\Omega_{-,\varepsilon,\omega} \subset G'$ with $\varepsilon > 0$ introduced in the beginning of the previous section. Let $\lambda > \max(\lambda_1, \lambda_0)$ and fix $\omega \in \{y \in \mathbb{S}^{n-1} : |y - \omega_0| \leqslant \varepsilon\}$. According to Proposition 3.1, we can introduce

$$u_1(t,x) = e^{-\lambda(t+\omega\cdot x)}\left(e^{-i\xi\cdot(t,x)} + w(t,x)\right),\ (t,x) \in Q,$$

where $u_1 \in H^1(Q)$ satisfies $\partial_t^2 u_1 - \Delta_x u_1 + q_1 u_1 = 0$, $\xi \cdot (1, -\omega) = 0$ and w satisfies (3.7). Moreover, in view of Theorem 5.1, we consider $u_2 \in H_\Box(Q)$ a solution of (5.1) with $q = q_2$ of the form

$$u_2(t,x) = e^{\lambda(t+\omega\cdot x)}\left(1 + z(t,x)\right), \quad (t,x) \in Q$$

with z satisfying (5.3), such that $\mathrm{supp}\tau_{0,1}u_2 \subset F$ and $\tau_{0,2}u_2 = 0$ (we recall that $\tau_{0,j}$, $j = 1, 2$, are the extensions on $H_\Box(Q)$ of the operators defined by $\tau_{0,1}v = v_{|\Sigma}$ and $\tau_{0,2}v = v_{|t=0}$, $v \in \mathcal{C}^\infty(\overline{Q})$). In view of Proposition 2.2, there exists a unique solution $w_1 \in H_\Box(Q)$ of

$$\begin{cases} \partial_t^2 w_1 - \Delta_x w_1 + q_1 w_1 = 0 & \text{in } Q, \\ \tau_0 w_1 = \tau_0 u_2. \end{cases} \tag{6.1}$$

Then, $u = w_1 - u_2$ solves

$$\begin{cases} \partial_t^2 u - \Delta_x u + q_1 u = (q_2 - q_1)u_2 & \text{in } Q, \\ u(0,x) = \partial_t u(0,x) = 0 & \text{on } \Omega, \\ u = 0 & \text{on } \Sigma \end{cases} \tag{6.2}$$

and since $(q_2 - q_1)u_2 \in L^2(Q)$, in view of [1, Theorem A.2] (see also [21, Theorem 2.1] for $q = 0$), we deduce that $u \in \mathcal{C}^1([0,T]; L^2(\Omega)) \cap \mathcal{C}([0,T]; H_0^1(\Omega)) \cap H_\Box(Q) \subset H^1(Q) \cap H_\Box(Q)$ with $\partial_\nu u \in L^2(\Sigma)$. Using the fact that $u_1 \in H^1(Q) \cap H_\Box(Q)$, we deduce that $(\partial_t u_1, -\nabla_x u_1) \in H_{\mathrm{div}}(Q) = \{F \in L^2(Q; \mathbb{C}^{n+1}) :\ \mathrm{div}_{(t,x)} F \in L^2(Q)\}$. Therefore, in view of [17, Lemma 2.2], we can apply the Green formula to get

$$\int_Q u(\Box u_1)dxdt = -\int_Q (\partial_t u \partial_t u_1 - \nabla_x u \cdot \nabla_x u_1)dxdt + \langle (\partial_t u_1, -\nabla_x u_1)\cdot \mathbf{n}, u\rangle_{H^{-\frac{1}{2}}(\partial Q), H^{\frac{1}{2}}(\partial Q)}$$

with $\mathbf{n}$ the outward unit normal vector to ∂Q. In the same way, we find

$$\int_Q u_1(\Box u)dxdt = -\int_Q (\partial_t u \partial_t u_1 - \nabla_x u \cdot \nabla_x u_1)dxdt + \langle (\partial_t u, -\nabla_x u)\cdot \mathbf{n}, u_1\rangle_{H^{-\frac{1}{2}}(\partial Q), H^{\frac{1}{2}}(\partial Q)}.$$

From these two formulas we deduce that

$$\begin{aligned}\int_Q (q_2 - q_1)u_2 u_1 dxdt &= \int_Q u_1(\Box u + q_1 u)dxdt - \int_Q u(\Box u_1 + q_1 u_1)dxdt \\ &= \langle (\partial_t u, -\nabla_x u)\cdot \mathbf{n}, u_1\rangle_{H^{-\frac{1}{2}}(\partial Q), H^{\frac{1}{2}}(\partial Q)} - \langle (\partial_t u_1, -\nabla_x u_1)\cdot \mathbf{n}, u\rangle_{H^{-\frac{1}{2}}(\partial Q), H^{\frac{1}{2}}(\partial Q)}.\end{aligned}$$

On the other hand we have $u_{|t=0} = \partial_t u_{|t=0} = u_{|\Sigma} = 0$ and condition (1.4) implies that $u_{|t=T} = \partial_\nu u_{|G} = 0$. Combining this with the fact that $u \in \mathcal{C}^1([0,T]; L^2(\Omega))$ and $\partial_\nu u \in L^2(\Sigma)$, we obtain

$$\int_Q q u_2 u_1 dxdt = -\int_{\Sigma\setminus G} \partial_\nu u u_1 d\sigma(x)dt + \int_\Omega \partial_t u(T,x)u_1(T,x)dx. \tag{6.3}$$

Applying the Cauchy–Schwarz inequality to the first expression on the right hand side of this formula, we get

$$\left|\int_{\Sigma\setminus G} \partial_\nu u u_1 d\sigma(x)dt\right| \leqslant \int_{\Sigma_{+,\varepsilon,\omega}} \left|\partial_\nu u e^{-\lambda(t+\omega\cdot x)}(e^{-i\xi\cdot(t,x)}+w)\right| d\sigma(x)dt$$
$$\leqslant C\left(\int_{\Sigma_{+,\varepsilon,\omega}} \left|e^{-\lambda(t+\omega\cdot x)}\partial_\nu u\right|^2 d\sigma(x)dt\right)^{\frac{1}{2}}$$

for some C independent of λ. Here we have used both (3.7) and the fact that $(\Sigma\setminus G)\subset\Sigma_{+,\varepsilon,\omega}$. In the same way, we have

$$\left|\int_{\Omega} \partial_t u(T,x)u_1(T,x)dx\right| \leqslant \int_{\Omega} \left|\partial_t u(T,x)e^{-\lambda(T+\omega\cdot x)}\left(e^{-i\xi\cdot(T,x)}+w(T,x)\right)\right| dx$$
$$\leqslant C\left(\int_{\Omega} \left|e^{-\lambda(T+\omega\cdot x)}\partial_t u(T,x)\right|^2 dx\right)^{\frac{1}{2}}.$$

Combining these estimates with the Carleman estimate (4.2), the fact that $u_{|t=T}=\partial_\nu u_{|\Sigma_{-,\omega}}=0$, $\partial\Omega_{+,\varepsilon,\omega}\subset\partial\Omega_{+,\omega}$, we find

$$\left|\int_Q (q_2-q_1)u_2u_1dxdt\right|^2$$
$$\leqslant 2C\left(\int_{\Sigma_{+,\varepsilon,\omega}} \left|e^{-\lambda(t+\omega\cdot x)}\partial_\nu u\right|^2 d\sigma(x)dt + \int_{\Omega}\left|e^{-\lambda(T+\omega\cdot x)}\partial_t u(T,x)\right|^2 dx\right)$$
$$\leqslant 2\varepsilon^{-1}C\left(\int_{\Sigma_{+,\omega}} \left|e^{-\lambda(t+\omega\cdot x)}\partial_\nu u\right|^2 \omega\cdot\nu(x)d\sigma(x)dt + \int_{\Omega}\left|e^{-\lambda(T+\omega\cdot x)}\partial_t u(T,x)\right|^2 dx\right)$$
$$\leqslant \frac{\varepsilon^{-1}C}{\lambda}\left(\int_Q \left|e^{-\lambda(t+\omega\cdot x)}(\partial_t^2-\Delta_x+q_1)u\right|^2 dxdt\right)$$
$$\leqslant \frac{\varepsilon^{-1}C}{\lambda}\left(\int_Q \left|e^{-\lambda(t+\omega\cdot x)}qu_2\right|^2 dxdt\right) = \frac{\varepsilon^{-1}C}{\lambda}\left(\int_Q |q|^2(1+|z|)^2dxdt\right). \tag{6.4}$$

Here $C>0$ stands for some generic constant independent of λ. It follows that

$$\lim_{\lambda\to+\infty}\int_Q qu_2u_1dxdt = 0. \tag{6.5}$$

On the other hand, we have

$$\int_Q qu_1u_2dxdt = \int_{\mathbb{R}^{1+n}} q(t,x)e^{-i\xi\cdot(t,x)}dxdt + \int_Q Z(t,x)dxdt$$

with $Z(t,x)=q(t,x)(z(t,x)e^{-i\xi\cdot(t,x)}+w(t,x)+z(t,x)w(t,x))$. Then, in view of (3.7) and (5.3), an application of the Cauchy–Schwarz inequality yields

$$\left|\int_Q Z(t,x)dxdt\right| \leqslant C\lambda^{-\frac{1}{2}}$$

with C independent of λ. Combining this with (6.5), we deduce that for all $\omega \in \{y \in \mathbb{S}^{n-1} : |y-\omega_0| \leqslant \varepsilon\}$ and all $\xi \in \mathbb{R}^{1+n}$ orthogonal to $(1,-\omega)$, the Fourier transform $\mathcal{F}(q)$ of q satisfies

$$\mathcal{F}(q)(\xi) = (2\pi)^{-\frac{n+1}{2}} \int_{\mathbb{R}^{1+n}} q(t,x)e^{-i\xi\cdot(t,x)}dxdt = 0.$$

On the other hand, since $q \in L^\infty(\mathbb{R}^{1+n})$ is supported on $\overline{Q}$ which is compact, $\mathcal{F}(q)$ is analytic and it follows that $q = 0$ and $q_1 = q_2$. This completes the proof of Theorem 1.1.

Conflict of interest statement

There is no conflict of interest.

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Appendix A

In this appendix we prove that the space $\mathcal{C}^\infty(\overline{Q})$ is dense in $H_\square(Q)$ in some appropriate sense and we show that the maps τ_0 and τ_1 can be extended continuously on these spaces. Without lost of generality we consider only these spaces for real valued functions. The results of this section are well known, nevertheless we prove them for sake of completeness.

A.1. Density result in $H_\square(Q)$

Let us first recall the definition of $K_\square(Q)$:

$$K_\square(Q) = \{u \in H^{-1}(0,T;L^2(\Omega)) :\ \square u = (\partial_t^2 - \Delta_x)u \in L^2(Q)\}$$

with the norm

$$\|u\|^2_{K_\square(Q)} = \|u\|^2_{H^{-1}(0,T;L^2(\Omega))} + \|\square u\|^2_{L^2(Q)}.$$

The goal of this subsection is to prove the following.

Theorem A.1. *$H_\square(Q)$ is embedded continuously into the closure of $\mathcal{C}^\infty(\overline{Q})$ with respect to $K_\square(Q)$.*

Proof. Let N be a continuous linear form on $K_\square(Q)$ satisfying

$$Nf = 0, \quad f \in \mathcal{C}^\infty(\overline{Q}). \tag{A.1}$$

In order to show the required density result we will prove that this condition implies that $N_{|H_\square(Q)} = 0$.

By considering the application $u \mapsto (u, \square u)$ we can identify $K_\square(Q)$ to a subspace of $H^{-1}(0,T;L^2(\Omega)) \times L^2(Q)$. Then, applying the Hahn Banach theorem we deduce that N can be extended to a continuous linear form on $H^{-1}(0,T;L^2(\Omega)) \times L^2(Q)$. Therefore, there exist $h_1 \in H_0^1(0,T;L^2(\Omega))$, $h_2 \in L^2(Q)$ such that

$$N(u) = \langle u, h_1\rangle_{H^{-1}(0,T;L^2(\Omega)),H_0^1(0,T;L^2(\Omega))} + \langle \square u, h_2\rangle_{L^2(Q)}, \quad u \in K_\square(Q).$$

Now let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded $\mathcal{C}^\infty$ domain such that $\overline{\Omega} \subset \mathcal{O}$ and fix $Q_\varepsilon = (-\varepsilon, T+\varepsilon) \times \mathcal{O}$ with $\varepsilon > 0$. Let $\tilde{h}_j$ be the extension of h_j on $\mathbb{R}^{1+n}$ by 0 outside of Q for $j = 1, 2$. In view of (A.1) we have

$$\left\langle f, \tilde{h}_1 \right\rangle_{L^2(Q_\varepsilon)} + \left\langle (\partial_t^2 - \Delta_x) f, \tilde{h}_2 \right\rangle_{L^2(Q_\varepsilon)} = N(f_{|Q}) = 0, \quad f \in \mathcal{C}_0^\infty(Q_\varepsilon).$$

Thus, in the sense of distribution we have

$$\Box \tilde{h}_2 = -\tilde{h}_1 \quad \text{on } Q_\varepsilon.$$

Moreover, since $\tilde{h}_2 = 0$ on $\mathbb{R}^{1+n} \setminus \overline{Q} \supset \partial Q_\varepsilon$, we deduce that $\tilde{h}_2$ solves

$$\begin{cases} \partial_t^2 \tilde{h}_2 - \Delta_x \tilde{h}_2 = -\tilde{h}_1 & \text{in } Q_\varepsilon, \\ \tilde{h}_2(-\varepsilon, x) = \partial_t \tilde{h}_2(-\varepsilon, x) = 0, & x \in \mathcal{O}, \\ \tilde{h}_2(t,x) = 0, & (t,x) \in (-\varepsilon, T+\varepsilon) \times \partial\mathcal{O}. \end{cases}$$

But, since $h_1 \in H_0^1(0,T; L^2(\Omega))$, we have $\tilde{h}_1 \in H_0^1(-\varepsilon, T+\varepsilon; L^2(\mathcal{O}))$ and we deduce from [23, Chapter 5, Theorem 2.1] that this IBVP admits a unique solution lying in $H^2(Q_\varepsilon)$. Therefore, $\tilde{h}_2 \in H^2(Q_\varepsilon)$. Combining this with the fact that $\tilde{h}_2 = 0$ on $Q_\varepsilon \setminus Q$, we deduce that $h_2 \in H_0^2(Q)$, with $H_0^2(Q)$ the closure of $\mathcal{C}_0^\infty(Q)$ in $H^2(Q)$, and that $\Box h_2 = -h_1$ in Q. Thus, for every $u \in H_\Box(Q)$ we have

$$\langle \Box u, h_2 \rangle_{L^2(Q)} = \langle \Box u, h_2 \rangle_{H^{-2}(Q), H_0^2(Q)} = \langle u, \Box h_2 \rangle_{L^2(Q)} = -\langle u, h_1 \rangle_{L^2(Q)}.$$

Here we use the fact that $H_\Box(Q) \subset L^2(Q)$. Then, it follows that

$$N(u) = \langle u, h_1 \rangle_{L^2(Q)} - \langle u, h_1 \rangle_{L^2(Q)} = 0, \quad u \in H_\Box(Q).$$

From this last result we deduce that $H_\Box(Q)$ is contained into the closure of $\mathcal{C}^\infty(\overline{Q})$ with respect to $K_\Box(Q)$. Combining this with the fact that $H_\Box(Q)$ is embedded continuously into $K_\Box(Q)$, we deduce the required result. □

A.2. *Trace operator in $H_\Box(Q)$*

In this subsection we extend the trace maps τ_0 and τ_1 into $H_\Box(Q)$ by duality in the following way.

Proposition A.1. *The maps*

$$\tau_0 w = (\tau_{0,1} w, \tau_{0,2} w, \tau_{0,3} w) = (w_{|\Sigma}, w_{|t=0}, \partial_t w_{|t=0}), \quad w \in \mathcal{C}^\infty(\overline{Q}),$$
$$\tau_1 w = (\tau_{1,1} w, \tau_{1,2} w, \tau_{1,3} w) = (\partial_\nu w_{|\Sigma}, w_{|t=T}, \partial_t w_{|t=T}), \quad w \in \mathcal{C}^\infty(\overline{Q}),$$

can be extended continuously to $\tau_0 : H_\Box(Q) \to H^{-3}(0,T; H^{-\frac{1}{2}}(\partial\Omega)) \times H^{-2}(\Omega) \times H^{-4}(\Omega)$, $\tau_1 : H_\Box(Q) \to H^{-3}(0,T; H^{-\frac{3}{2}}(\partial\Omega)) \times H^{-2}(\Omega) \times H^{-4}(\Omega)$.

Proof. It is well known that the trace maps

$$u \mapsto (u_{|\partial\Omega}, \partial_\nu u_{|\partial\Omega})$$

can be extended continuously to a bounded operator from $H^2(\Omega)$ to $H^{\frac{3}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ which is onto. Therefore, there exists a bounded operator $R : H^{\frac{3}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to H^2(\Omega)$ such that

$$R(h_1, h_2)_{|\partial\Omega} = h_1, \quad \partial_\nu R(h_1, h_2)_{|\partial\Omega} = h_2, \quad (h_1, h_2) \in H^{\frac{3}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega).$$

Fix $g \in H_0^3(0,T; H^{\frac{1}{2}}(\partial\Omega))$ and choose $G(t,.) = R(0, g(t,.))$. One can check that $G \in H_0^3(0,T; H^2(\Omega))$ and

$$\|G\|_{H^3(0,T;H^2(\Omega))} \leqslant \|R\| \, \|g\|_{H^3(0,T;H^{\frac{1}{2}}(\partial\Omega))}. \tag{A.2}$$

Applying twice the Green formula we obtain

$$\int_\Sigma v g \, d\sigma(x) dt = \int_Q \Box v G \, dx dt - \int_Q v \Box G \, dx dt, \quad v \in \mathcal{C}^\infty(\overline{Q}).$$

But $\Box G \in H_0^1(0,T;H^2(\Omega))$, and we have

$$\left\langle \tau_{0,1}v, g\right\rangle_{H^{-3}(0,T;H^{-\frac{1}{2}}(\partial\Omega)),H_0^3(0,T;H^{\frac{1}{2}}(\partial\Omega))} = \langle \Box v, G\rangle_{L^2(Q)} - \langle v, \Box G\rangle_{H^{-1}(0,T;L^2(\Omega)),H_0^1(0,T;L^2(\Omega))}.$$

Then, using (A.2) and the Cauchy Schwarz inequality, for all $v \in \mathcal{C}^\infty(\overline{Q})$, we obtain

$$\begin{aligned}\left|\left\langle \tau_{0,1}v, g\right\rangle\right| &\leqslant \|\Box v\|_{L^2(Q)}\|G\|_{L^2(Q)} + \|v\|_{H^{-1}(0,T;L^2(\Omega))}\|\Box G\|_{H^1(0,T;L^2(\Omega))}\\ &\leqslant C\,\|v\|_{K_\Box(Q)}\|g\|_{H^3(0,T;H^{\frac{1}{2}}(\partial\Omega))}\end{aligned}$$

which, combined with the density result of Theorem A.1, implies that $\tau_{0,1} : v \mapsto v_{|\Sigma}$ extend continuously to a bounded operator from $H_\Box(Q)$ to $H^{-3}(0,T;H^{-\frac{1}{2}}(\partial\Omega))$. In a same way we prove that

$$\tau_{1,1}v = \partial_\nu v_{|\Sigma}, \quad v \in \mathcal{C}^\infty(\overline{Q})$$

extend continuously to a bounded operator from $H_\Box(Q)$ to $H^{-3}(0,T;H^{-\frac{3}{2}}(\partial\Omega))$.

Now let us consider the operators $\tau_{i,j}$, $i=0,1$, $j=2,3$. We start with

$$\tau_{0,2} : v \longmapsto v_{|t=0}, \quad v \in \mathcal{C}^\infty(\overline{Q}).$$

Let $h \in H_0^2(\Omega)$ and fix $H(t,x) = t\psi(t)h(x)$ with $\psi \in \mathcal{C}_0^\infty(-T,\frac{T}{2})$ satisfying $0 \leqslant \psi \leqslant 1$ and $\psi = 1$ on $[-\frac{T}{3},\frac{T}{3}]$. Then, using the fact that $\psi = 1$ on a neighborhood of $t=0$, we deduce that

$$H_{|\Sigma} = \partial_\nu H_{|\Sigma} = H_{|t=0} = H_{|t=T} = \partial_t H_{|t=T} = \Box H_{|t=0} = \Box H_{|t=T} = 0, \quad \partial_t H_{|t=0} = h.$$

Therefore, $\Box H \in H_0^1(0,T;L^2(\Omega))$ and repeating the above arguments, for all $v \in \mathcal{C}^\infty(\overline{Q})$, we obtain the representation

$$\left\langle \tau_{0,2}v, h\right\rangle_{H^{-2}(\Omega),H_0^2(\Omega)} = \langle \Box v, H\rangle_{L^2(Q)} - \langle v, \Box H\rangle_{H^{-1}(0,T;L^2(\Omega)),H_0^1(0,T;L^2(\Omega))}.$$

Then, we prove by density that $\tau_{0,2}$ extends continuously to $\tau_{0,2} : H_\Box(Q) \longrightarrow H^{-2}(\Omega)$.

For

$$\tau_{0,3} : v \longmapsto \partial_t v_{|t=0}, \quad v \in \mathcal{C}^\infty(\overline{Q}),$$

let $\varphi \in H_0^4(\Omega)$ and fix

$$\Phi(t,x) = \psi(t)\varphi(x) + \frac{\psi(t)t^2\Delta_x\varphi(x)}{2}.$$

Then, Φ satisfies

$$\Phi_{|\Sigma} = \partial_\nu\Phi_{|\Sigma} = \partial_t\Phi_{|t=0} = \Phi_{|t=T} = \partial_t\Phi_{|t=T} = 0, \quad \Phi_{|t=0} = \varphi.$$

Moreover, we have $\Box\Phi \in H^1(0,T;L^2(\Omega))$ with

$$(\partial_t^2 - \Delta_x)\Phi_{|t=0} = -\Delta_x\varphi + \Delta_x\varphi = 0, \quad (\partial_t^2 - \Delta_x)\Phi_{|t=T} = 0$$

and it follows that $\Box\Phi \in H_0^1(0,T;L^2(\Omega))$. Therefore, repeating the above arguments we obtain the representation

$$\left\langle \tau_{0,3}v, \varphi\right\rangle_{H^{-4}(\Omega),H_0^4(\Omega)} = \langle v, \Box\Phi\rangle_{H^{-1}(0,T;L^2(\Omega)),H_0^1(0,T;L^2(\Omega))} - \langle \Box v, \Phi\rangle_{L^2(Q)}$$

and we deduce that $\tau_{0,3}$ extends continuously to $\tau_{0,3} : H_\Box(Q) \longrightarrow H^{-4}(\Omega)$. In a same way, one can check that

$$\tau_{1,2}v = v_{|t=T},\ \tau_{1,3}v = \partial_t v_{|t=T}, \quad v \in \mathcal{C}^\infty(\overline{Q})$$

extend continuously to $\tau_{1,2} : H_\Box(Q) \longrightarrow H^{-2}(\Omega)$ and $\tau_{1,3} : H_\Box(Q) \longrightarrow H^{-4}(\Omega)$. $\Box$

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