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Semilinear parabolic equation in $\mathbf{R}^N$ associated with critical Sobolev exponent

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Abstract

We consider the semilinear parabolic equation $u_t - \Delta u = |u|^{p-1}u$ on the whole space $\mathbf{R}^N$, $N \geqslant 3$, where the exponent $p = (N+2)/(N-2)$ is associated with the Sobolev imbedding $H^1(\mathbf{R}^N) \subset L^{p+1}(\mathbf{R}^N)$. First, we study the decay and blow-up of the solution by means of the potential-well and forward self-similar transformation. Then, we discuss blow-up in infinite time and classify the orbit.

MSC: 35K55

Keywords: Parabolic equation; Critical Sobolev exponent; Cauchy problem; Stable and unstable sets; Self-similarity

1. Introduction

The purpose of the present paper is to classify the asymptotic behavior of the solution to the semilinear parabolic equation

$$u_t - \Delta u = |u|^{p-1}u \quad \text{in } \mathbf{R}^N \times (0,T) \tag{1}$$

with

$$u|_{t=0} = u_0 \quad \text{in } \mathbf{R}^N, \tag{2}$$

where $N \geqslant 3$ and $p = (N+2)/(N-2)$, the critical exponent associated with the Sobolev imbedding. In the previous work [9,8,11,23], we studied the long time behavior of the solution defined on the bounded domain in connection with

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the stable and unstable sets introduced by [22,20]. We have shown that if the orbit enters in the stable set, then the solution exists globally in time [11]. If the orbit enters in the unstable set, then the solution blows up in finite time [9]. The other case is called "floating". Thus, the orbit is floating, by definition, if it never enters in the stable set nor the unstable set. If the domain is star-shaped, then the orbit floating globally in time must blow up in infinite time, while the floating orbit blowing up in finite time never exists under the additional assumption of the domain and the initial value, that is, the convexity and symmetry [23].

In contrast with these cases on the bounded domain, there is a family of stationary solutions to (1)–(2) concerning the whole space $\mathbf{R}^N$. We have the lack of the Poincaré inequality also. These differences are made clear by the forward self-similar transformation and the non-existence of the self-similar solution [5,14]. Consequently, we can classify the rate of $\|u(t)\|_\infty$ as $t \uparrow +\infty$ for the solution $u = u(\cdot, t)$ global in time.

More precisely, if $u = u(x, t)$ is the solution, then

$$v(y, s) = (1+t)^{1/(p-1)} u(x, t), \quad t = e^s - 1, \quad x = (1+t)^{1/2} y \tag{3}$$

satisfies

$$v_s + Lv = |v|^{p-1} v + \frac{1}{p-1} v \quad \text{in } \mathbf{R}^N \times (0, S) \tag{4}$$

with

$$v|_{s=0} = u_0 \quad \text{in } \mathbf{R}^N, \tag{5}$$

where $S = \log(1 + T)$ and

$$Lf = -\Delta f - \frac{1}{2} y \cdot \nabla f.$$

Since

$$Lf = -\frac{1}{K} \nabla \cdot (K \nabla f)$$

holds for

$$K(y) = e^{|y|^2/4},$$

problem (4)–(5) is associated with the Hilbert space $L^2(K)$, the set of measurable functions $f = f(y)$ defined in $\mathbf{R}^N$ such that

$$\|f\|_{2,K} = \left\{ \int_{\mathbf{R}^N} \left|f(y)\right|^2 K(y)\, dy \right\}^{1/2} < +\infty.$$

We also take

$$H^m(K) = \left\{ f \in L^2(K) \;\middle|\; D^\alpha f \in L^2(K) \text{ for any multi-index } \alpha \text{ in } |\alpha| \leqslant m \right\},$$

where $m = 1, 2, \ldots$. It is a Hilbert space provided with the norm

$$\|f\|_{H^m(K)} = \left\{ \sum_{|\alpha| \leqslant m} \left\| D^\alpha f \right\|_{2,K}^2 \right\}^{1/2}.$$

This L is realized as a self-adjoint operator in $L^2(K)$ associated with the bilinear form

$$\mathcal{A}_K(u, v) = \int_{\mathbf{R}^N} \nabla u(y) \cdot \nabla v(y) K(y)\, dy$$

defined for $u, v \in H^1(K)$ through the relation

$$\mathcal{A}_K(u, v) = (Lu, v)_K, \quad u \in D(L) \subset H^1(K),\ v \in H^1(K),$$

where

$$(u, v)_K = \int_{\mathbf{R}^N} u(y)v(y)K(y)\,dy,$$

see [13], for the general theory of the bilinear form. The domain $D(L)$ of this operator is the set of $v \in L^2(K)$ satisfying $Lv \in L^2(K)$, and we have $D(L) = H^2(K)$, see Lemma 2.1 of [14]. It holds also that L is positive self-adjoint and has the compact inverse, and in particular, the set of normalized eigenfunctions of L forms a complete ortho-normal system in $L^2(K)$. The first eigenvalue λ_1 of L is given by $\lambda_1 = N/2$, and hence the Poincaré inequality

$$\lambda_1 \|v\|_{2,K}^2 \leqslant \|\nabla v\|_{2,K}^2, \quad v \in H^1(K), \tag{6}$$

is valid, see Proposition 2.3 of [5].

We have

$$\lambda_1 = \frac{N}{2} > \lambda \equiv \frac{1}{p-1} = \frac{N-2}{4}$$

and therefore, the operator

$$A = L - \frac{1}{p-1}$$

in $L^2(K)$ is also positive self-adjoint with the domain $D(A) = H^2(K)$. The fractional powers A^α and the semigroup $\{e^{-tA}\}_{t\geqslant 0}$ are thus defined in $L^2(K)$, where $\alpha \in [0, 1]$. These structures guarantee the well-posedness of (4)–(5) locally in time. Later, we shall show the following fact.

Proposition 1.1. *Each $u_0 \in H^1(K)$ admits $T > 0$ such that* (4)–(5) *has a unique solution $v \in C([0, T]; H^1(K))$ satisfying the following*:

1. $v \in C((0, T]; D(A^\nu))$.
2. $v(s) = e^{-sA}u_0 + \int_0^t e^{-(s-\sigma)A}|v(\sigma)|^{p-1}v(\sigma)\,d\sigma$.
3. $\lim_{s\downarrow 0} s^{\nu-\frac{1}{2}}\|A^\nu v(s)\|_{2,K} = 0$,

where

$$\nu \begin{cases} = N/(N+2) & (N \geqslant 5), \\ \in (2/3, 1) & (N = 4), \\ = 4/5 & (N = 3). \end{cases}$$

We call $v = v(\cdot, s)$ the $H^1(K)$-solution, or simply the *solution* to (4)–(5). Proposition 1.1 assures local in time unique existence of the solution $u = u(\cdot, t)$ to (1)–(2) for $u_0 \in H^1(K)$. Also, this $v = v(\cdot, s)$ coincides with the solution discussed in [14]. Henceforth, $T_m(K) \in (0, +\infty]$ denotes the supremum of T such that the solution $v = v(\cdot, s)$ to (4)–(5) exists for $s \in [0, T]$. The local existence time T assured by Proposition 1.1, however, is not estimated from below by $\|u_0\|_{H^1(K)}$. Consequently, we cannot conclude

$$\limsup_{s\uparrow T_m(K)} \|v(s)\|_{H^1(K)} = +\infty$$

from $T_m(K) < +\infty$ (cf. Theorem 1.10(ii) of [14]), similarly to the case of the bounded domain, see [9]. Henceforth, we put $T = T_m(K) \in (0, +\infty]$ for simplicity.

In this paper, we show that the orbit made from the above mentioned solution is classified in the following way. We emphasize that the solution $u = u(\cdot, t)$ may be sign-changing.

Theorem 1. *If $u = u(\cdot, t)$ is the solution to* (1)–(2) *with $u_0 \in H^1(K)$ and $p = \frac{N+2}{N-2}$, $N \geqslant 3$, then we have the following alternatives.*

1. $T=+\infty$ *and* $\limsup_{t\uparrow+\infty} t^{N/2}\|u(t)\|_\infty<+\infty$.
2. $T=+\infty$ *and* $\lim_{t\uparrow+\infty} t^{(N-2)/4}\|u(t)\|_\infty=+\infty$.
3. $T<+\infty$ *and* $\lim_{t\uparrow T}\|u(t)\|_\infty=+\infty$.

If $u_0=u_0(x)\geqslant 0$, $u_0\not\equiv 0$, *the first case is refined as* $T=+\infty$ *with*

$$\|u(t)\|_\infty \sim t^{-N/2} \quad \text{as } t\uparrow+\infty. \tag{7}$$

Here are some comments.

1. There is an analogous result concerning sub-critical nonlinearities [15]. Thus, if $u_0=u_0(x)\geqslant 0$ and $1+\frac{2}{N}<p<\frac{N+2}{N-2}$, then we have the following alternatives for the solution $u=u(\cdot,t)$ to (1)–(2);
   (a) $T=+\infty$ and (7).
   (b) $T=+\infty$ and $\|u(t)\|_\infty\sim t^{-1/(p-1)}$ as $t\uparrow+\infty$.
   (c) $T<+\infty$.
   The second case of this result indicates the decay rate with that of the self-similar solution. Our Theorem 1 is, actually, associated with the non-existence of the self-similar solution for the critical Sobolev exponent [14], see Proposition 5.1 below. The second case of Theorem 1, thus, may be called "type II rate" at $t=+\infty$ because $1/(p-1)=(N-2)/4$ for $p=\frac{N+2}{N-2}$.
2. The second case of Theorem 1 arises if the orbit is floating globally in time in the rescaled variables (3). Such a case occurs actually if $u_0=u_0(x)$ is a non-trivial stationary solution. In fact, in contrast with the case of the bounded star-shaped domain provided with the Dirichlet boundary condition, problem (1)–(2) admits a family of non-trivial stationary solutions, normalized by
$$-\Delta U=U^p, \qquad 0<U\leqslant U(0)=1 \quad \text{in } \mathbf{R}^N, \tag{8}$$
   see [1,3]. As for the solution converging uniformly to 0 in infinite time, however, there remains two possibilities – the first and the second cases of Theorem 1.
3. Positive radially symmetric solutions have been studied in detail. Particularly, analogous results to Theorem 1 are obtained [21], in accordance with the threshold of the modulus of the initial value for the blow-up of the solution. Its proof, however, uses the intersection comparison principle and is different from ours. The blow-up profile is also known by [12]. Thus, $T=+\infty$ implies the existence of $\lim_{t\uparrow+\infty}\|u(t)\|_\infty=\alpha\in[0,+\infty]$ for a suitable family of radially symmetric positive initial values. If $\alpha\neq 0$, the second case of Theorem 1 arises. It holds, furthermore, that
$$u(t)=\|u(t)\|_\infty U\bigl(\|u(t)\|_\infty^{\frac{2}{N-2}}\cdot\bigr)+o(1)$$
   as $t\uparrow+\infty$ in $\dot H^1(\mathbf{R}^N)$, where
$$\dot H^1(\mathbf{R}^N)=\bigl\{v\in L^{\frac{2N}{N-2}}(\mathbf{R}^N)\bigm|\nabla v\in L^2(\mathbf{R}^N)^N\bigr\}$$
   and $U=U(y)>0$ is the normalized non-trivial stationary solution defined by (8).

Our proof of Theorem 1 is involved by the imbedding theorem concerning $H^1(K)$. Henceforth, $L^q(K)$ denotes the Banach space composed of measurable functions $f=f(y)$ defined in $\mathbf{R}^N$ such that

$$\|f\|_{q,K}=\biggl\{\int_{\mathbf{R}^N}\bigl|f(y)\bigr|^q K(y)\,dy\biggr\}^{1/q}<+\infty$$

for $q\in[1,\infty)$ and

$$\|f\|_{\infty,K}=\operatorname*{ess\,sup}_{y\in\mathbf{R}^N}\bigl|f(y)\bigr|<+\infty$$

for $q=\infty$. The space $L^\infty(K)=L^\infty(\mathbf{R}^N)$ is thus compatible to the other spaces, i.e.,

$$\lim_{q\uparrow\infty}\|f\|_{q,K}=\|f\|_{\infty,K},\quad f\in L^1(K)\cap L^\infty\big(\mathbf{R}^N\big). \tag{9}$$

Although the inclusion

$$L^p(K)\subset L^q(K)\quad(1\leqslant q<p\leqslant\infty)$$

fails, we have

$$H^1(K)\subset L^{2^*}(K)$$

for $2^*=2N/(N-2)=p+1$. More precisely, Corollary 4.20 of [5] guarantees the following fact, regarded as a Sobolev–Poincaré inequality.

Proposition 1.2. *It holds that*

$$S_0\|v\|^2_{p+1,K}+\lambda_*\|v\|^2_{2,K}\leqslant\|\nabla v\|^2_{2,K},\quad v\in H^1(K), \tag{10}$$

where $\lambda_*=\max(1,N/4)$ *and* S_0 *stands for the Sobolev constant*:

$$S_0=\inf\big\{\|\nabla v\|_2^2\ \big|\ v\in C_0^\infty\big(\mathbf{R}^N\big),\ \|v\|_{p+1}=1\big\}.$$

We introduce the functionals

$$J_K(v)=\frac{1}{2}\|\nabla v\|^2_{2,K}-\frac{\lambda}{2}\|v\|^2_{2,K}-\frac{1}{p+1}\|v\|^{p+1}_{p+1,K}$$

and

$$I_K(v)=\|\nabla v\|^2_{2,K}-\lambda\|v\|^2_{2,K}-\|v\|^{p+1}_{p+1,K}$$

defined for $v\in H^1(K)$, where $\lambda=1/(p-1)$. Then, the potential depth of J_K in $H^1(K)$ is defined by

$$d_0=\inf\Big\{\sup_{\mu>0}J_K(\mu v)\ \big|\ v\in H^1(K)\setminus\{0\}\Big\},$$

and it holds that

$$d_0=\left(\frac{1}{2}-\frac{1}{p+1}\right)S_0^{(p+1)/(p-1)}=\frac{1}{N}S_0^{N/2}>0. \tag{11}$$

Here and henceforth, $\|\cdot\|_q$ indicates the standard L^q norm on $\mathbf{R}^N$. The stable and unstable sets to (4) are defined by

$$W_K=\big\{v\in H^1(K)\ \big|\ J_K(v)<d_0,\ I_K(v)>0\big\}\cup\{0\}$$

and

$$V_K=\big\{v\in H^1(K)\ \big|\ J_K(v)<d_0,\ I_K(v)<0\big\},$$

respectively.

Theorem 1 is proven by the study of the above defined stable and unstable sets. First, if the orbit enters in the stable set, then $v(\cdot,s)$ converges to 0 in infinite time.

Proposition 1.3. *If* $v=v(\cdot,s)$ *is the solution to* (4)–(5) *satisfying* $v(s_0)\in W_K$ *for some* $s_0\in[0,T)$, *then it holds that* $T=+\infty$ *and*

$$\big\|\nabla v(s)\big\|^2_{2,K}=O\big(e^{-\alpha s}\big) \tag{12}$$

as $s\uparrow+\infty$, *where* $\alpha\in(0,1)$.

If the orbit enters in the unstable set, on the contrary, then $v(\cdot,s)$ blows up in finite time; the following proposition implies Theorem 1.10(i) of [14].

Proposition 1.4. *If the solution $v = v(\cdot, s)$ to* (4)–(5) *satisfies $v(s_0) \in V_K$ for some $s_0 \in [0, T)$, then it holds that $T < +\infty$.*

The orbit $\{v(s)\}$ is, thus, *floating* in the other case than the ones treated in Propositions 1.3–1.4, i.e., $v(s) \notin (W_K \cup V_K)$ for $s \in [0, T)$. In spite of the possibility of the floating orbit blowing up in finite time, the following proposition is sufficient to classify the orbit to (1)–(2) as in Theorem 1.

Proposition 1.5. *If the orbit $\{v(s)\}$ is floating globally in time, then it holds that*

$$\lim_{s\uparrow+\infty} \|v(s)\|_{\infty,K} = +\infty. \tag{13}$$

This paper is composed of six sections. Section 2 takes preliminaries. We confirm inequality (11) and Proposition 1.1. In Section 3, we prove Propositions 1.3 and 1.4 by the study of stable and unstable sets. Section 4 describes the $L^q(K)$-theory of L. Using this, we study the floating orbit and show Proposition 1.5 in Section 5. The proof of Theorem 1 is completed in Section 6.

2. Preliminaries

This section is devoted to the preliminaries. First, we show (11), noting that

$$\sup_{\mu>0} J_K(\mu v) = J_K(\mu v)|_{\mu=\mu^*}$$

holds for

$$\mu^* = \left\{ \frac{\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2}{\|v\|_{p+1,K}^{p+1}} \right\}^{1/(p-1)} \quad \text{and} \quad v \in H^1(K) \setminus \{0\}.$$

Then, it follows that

$$\sup_{\mu>0} J_K(\mu v) = \frac{p-1}{2(p+1)} \left\{ \frac{\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2}{\|v\|_{p+1,K}^2} \right\}^{(p+1)/(p-1)}$$

and hence

$$d_0 = \left(\frac{1}{2} - \frac{1}{p+1}\right) S_\lambda^{(p+1)/(p-1)},$$

where

$$S_\lambda = \inf\left\{ \frac{\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2}{\|v\|_{p+1,K}^2} \;\middle|\; v \in H^1(K) \right\}.$$

We have, however,

$$\lambda = \frac{1}{p-1} = \frac{N-2}{4},$$

and therefore, $S_\lambda = S_0$ by Theorem 4.10 and Lemma 4.11 of [5]. This means (11).

Next, we confirm the operator theoretical feature of L in $L^2(K)$ described in the previous section. It is actually involved by the Schrödinger operator with harmonic oscillator;

$$K^{1/2} L K^{-1/2} = H \equiv -\Delta + V(y),$$

where

$$V(y) = \frac{N}{4} + \frac{|y|^2}{16}.$$

This H is associated with the bilinear form

$$\mathcal{A}(u,v)=\int_{\mathbf{R}^N}\big(\nabla u(y)\cdot\nabla v(y)+V(y)u(y)v(y)\big)\,dy$$

defined for $u, v \in H_1^1(\mathbf{R}^N)$ through the relation

$$\mathcal{A}(u,v)=(Hu,v)$$

for $u \in D(H) \subset H_1^1(\mathbf{R}^N)$ and $v \in H_1^1(\mathbf{R}^N)$, where

$$H_1^1\big(\mathbf{R}^N\big)=\big\{v\in H^1\big(\mathbf{R}^N\big)\ \big|\ |y|v\in L^2\big(\mathbf{R}^N\big)\big\}$$

and $(\ ,\ )$ denotes the L^2 inner product. It is realized as a positive self-adjoint operator in $L^2(\mathbf{R}^N)$ with the compact inverse, because the inclusion $H_1^1(\mathbf{R}^N) \subset L^2(\mathbf{R}^N)$ is compact. Now we show the following lemma.

Lemma 2.1. *The domain of H in $L^2(\mathbf{R}^N)$ is given by*

$$D(H)=H_2^2\big(\mathbf{R}^N\big)\equiv\big\{v\in H^2\big(\mathbf{R}^N\big)\ \big|\ |y|^2v\in L^2\big(\mathbf{R}^N\big)\big\}.$$

Proof. The inclusion $H_2^2(\mathbf{R}^N) \subset D(H)$ is obvious. We show that $v \in H_1^1(\mathbf{R}^N)$ with

$$-\Delta v+Vv=g\in L^2\big(\mathbf{R}^N\big) \tag{14}$$

implies $v \in H^2(\mathbf{R}^N)$. In fact, from the proof of Lemma 2.1 of [14] we have

$$\int_{\mathbf{R}^N}\big((\Delta v)^2+V|\nabla v|^2\big)\,dy\leqslant C_N\|g\|_2^2 \tag{15}$$

with a constant $C_N > 0$ determined by N. This implies $v \in H^2(\mathbf{R}^N)$, and then $Vv \in L^2(\mathbf{R}^N)$ by (14). We obtain $D(H) \subset H_2^2(\mathbf{R}^N)$ and the proof is complete. □

For later arguments, we describe the proof of (15) in short. First, (14) implies, formally, that

$$\int_{\mathbf{R}^N}\big((\Delta v)^2+Vv\cdot(-\Delta v)\big)\,dy=\int_{\mathbf{R}^N}g(-\Delta v)\,dy$$

and

$$\int_{\mathbf{R}^N}Vv\cdot(-\Delta v)\,dy=\int_{\mathbf{R}^N}\left(V|\nabla v|^2+\frac{1}{2}\nabla V\cdot\nabla v^2\right)dy=\int_{\mathbf{R}^N}\left(V|\nabla v|^2-\frac{1}{2}v^2\Delta V\right)dy.$$

Then, (15) holds by

$$\int_{\mathbf{R}^N}\big((\Delta v)^2+V|\nabla v|^2\big)\,dy\leqslant\frac{N}{16}\|v\|_2+\|g\|_2\cdot\|\Delta v\|_2.$$

To justify these calculations, we note that $v \in H^2_{loc}(\mathbf{R}^N)$ follows from (14). Next, taking $\varphi \in C_0^\infty(\mathbf{R}^N)$ such that

$$0\leqslant\varphi\leqslant1,\quad \operatorname{supp}\varphi\subset\big\{|y|<2\big\},\quad \varphi=1\quad\text{for } |y|\leqslant1,$$

we multiply $-\Delta v \cdot \varphi_n$ by (15), where $\varphi_n(y) = \varphi(y/n)$. Then, (14) is obtained by making $n \to +\infty$. Henceforth, such argument of justification will not be described explicitly.

Lemma 2.1 establishes the operator theoretical profiles of L as is desired; it is realized as a positive self-adjoint operator in $L^2(K)$ with the compact inverse and the domain $D(L) = H^2(K)$. Here, we note the following lemma.

Lemma 2.2. *The multiplication $K^{1/2}$ induces the isomorphisms*

$$L^2(K)\cong L^2\big(\mathbf{R}^N\big),\qquad H^1(K)\cong H_1^1\big(\mathbf{R}^N\big),\qquad H^2(K)\cong H_2^2\big(\mathbf{R}^N\big).$$

Proof. It is obvious that $K^{1/2}: L^2(K) \to L^2(\mathbf{R}^N)$ is an isomorphism. To confirm that $K^{1/2}: H^1(K) \to H^1_1(\mathbf{R}^N)$ is an isomorphism, we take $v = K^{1/2}u \in L^2(\mathbf{R}^N)$, given $u \in H^1(K)$. Since

$$\nabla v = K^{1/2}\nabla u + \frac{y}{4}v, \tag{16}$$

it holds that

$$K|\nabla u|^2 = \left|\nabla v - \frac{y}{4}v\right|^2 = |\nabla v|^2 - \frac{y}{2}v\cdot\nabla v + \frac{|y|^2}{16}v^2 = |\nabla v|^2 - \frac{1}{4}y\cdot\nabla v^2 + \frac{|y|^2}{16}v^2.$$

Using

$$-\frac{1}{4}\int_{\mathbf{R}^N} y\cdot\nabla v^2\,dy = \frac{1}{4}\int_{\mathbf{R}^N}(\nabla\cdot y)v^2\,dy = \frac{N}{4}\int_{\mathbf{R}^N} v^2\,dy,$$

we obtain

$$\int_{\mathbf{R}^N}|\nabla u|^2K\,dy = \int_{\mathbf{R}^N}\left\{|\nabla v|^2 + \left(\frac{N}{4} + \frac{1}{16}|y|^2\right)v^2\right\}dy,$$

and therefore, $yv \in L^2(\mathbf{R}^N)^N$ and $\nabla v \in L^2(\mathbf{R}^N)^N$. This means $v = K^{1/2}u \in H^1_1(\mathbf{R}^N)$. Given $v \in H^1_1(\mathbf{R}^N)$, conversely, we take $u = K^{-1/2}v \in L^2(K)$. Then, $\nabla u \in L^2(K)^N$ follows from (16).

To show that $K^{1/2}: H^2(K) \to H^2_2(\mathbf{R}^N)$ is an isomorphism, first, we take $u \in H^2(K)$ and put $v = K^{1/2}u \in L^2(\mathbf{R}^N)$. Since $u \in H^1(K)$, it holds that $v \in H^1_1(\mathbf{R}^N)$ and hence $yv \in L^2(\mathbf{R}^N)^N$. This means

$$y_iu \in L^2(K) \quad (i = 1, 2, \ldots, N). \tag{17}$$

It also holds that $\nabla u \in H^1(K)^N$ and hence

$$y_i\frac{\partial u}{\partial y_j} \in L^2(K) \quad (i, j = 1, 2, \ldots, N) \tag{18}$$

follows similarly. The right-hand side of

$$\frac{\partial}{\partial y_j}(y_iu) = \delta_{ij}u + y_i\frac{\partial u}{\partial y_j},$$

therefore, belongs to $L^2(K)$, and hence $yu \in H^1(K)^N$ follows from (17). This implies

$$|y|^2u \in L^2(K) \tag{19}$$

similarly to (17) again, which means $|y|^2v \in L^2(\mathbf{R}^N)$. Finally, the right-hand side of

$$\frac{\partial^2 v}{\partial y_i\partial y_j} = \sqrt{K}\frac{\partial^2 u}{\partial y_i\partial y_j} + \frac{\sqrt{K}}{4}y_i\frac{\partial u}{\partial y_j} + \frac{\sqrt{K}}{16}y_iy_ju + \frac{\sqrt{K}}{4}y_j\frac{\partial u}{\partial y_i}$$

belongs to $L^2(\mathbf{R}^N)$ by (18) and (19). This means $v \in H^2(\mathbf{R}^N)$ and hence $v = K^{1/2}u \in H^2_2(\mathbf{R}^N)$.

Given $v \in H^2_2(\mathbf{R}^N)$, conversely, we take $u = K^{-1/2}v \in L^2(K)$. Since $v \in H^1_1(\mathbf{R}^N)$, it holds that $u \in H^1(K)$. The right-hand side of

$$\frac{\partial^2 u}{\partial y_i\partial y_j} = K^{-1/2}\frac{\partial^2 v}{\partial y_i\partial y_j} - \frac{K^{-1/2}}{4}y_i\frac{\partial v}{\partial y_j} + \frac{K^{-1/2}}{16}y_iy_jv - \frac{K^{-1/2}}{4}y_j\frac{\partial v}{\partial y_i},$$

on the other hand, belongs to $L^2(K)$ because of $v \in H^2_2(\mathbf{R}^N)$ and the following lemma. Then, we obtain $u = K^{-1/2}v \in H^2(K)$ and the proof is complete. □

Lemma 2.3. *If $v \in H^2_2(\mathbf{R}^N)$, then it holds that*

$$y_i\frac{\partial v}{\partial y_j} \in L^2(\mathbf{R}^N) \quad (i, j = 1, \ldots, N).$$

Proof. Henceforth, C denotes a generic large positive constant. It suffices to show

$$\|y\cdot\nabla v\|_2 \leqslant C\|v\|_{H_2^2(\mathbf{R}^N)} \tag{20}$$

for $v\in C_0^\infty(\mathbf{R}^N)$, where

$$\|v\|_{H_2^2(\mathbf{R}^N)} = \left\{\sum_{|\alpha|\leqslant 2}\left(\left\|D^\alpha v\right\|_2^2 + \left\|y^\alpha v\right\|_2^2\right)\right\}^{1/2}.$$

This inequality is obtained by

$$\begin{aligned}\int\limits_{\mathbf{R}^N} y_i^2\left(\frac{\partial v}{\partial y_i}\right)^2 dy &= -\int\limits_{\mathbf{R}^N}\frac{\partial}{\partial y_j}\left(y_i^2\frac{\partial v}{\partial y_j}\right)v\,dy = -\int\limits_{\mathbf{R}^N}\left(2\delta_{ij}y_i\frac{\partial v}{\partial y_j}v + y_i^2\frac{\partial^2 v}{\partial y_j^2}v\right)dy\\ &\leqslant 2\|y_i v\|_2\cdot\left\|\frac{\partial v}{\partial y_j}\right\|_2 + \left\|y_i^2 v\right\|_2\cdot\left\|\frac{\partial^2 v}{\partial y_j^2}\right\|_2,\end{aligned}$$

and the proof is complete. □

Lemma 2.4. *The operator $A = L - \frac{1}{p-1}$ in $L^2(K)$, positive, self-adjoint, and with the compact inverse in $L^2(K)$, is provided with the following properties*:

1. $D(A)\hookrightarrow L^q(K)$ *for*
$$q\begin{cases} = 2^{**} & (N\geqslant 5),\\ \in[2,\infty) & (N=4),\\ \in[2,\infty] & (N=3),\end{cases}$$
where $2^{**} = 2N/(N-4)$.
2. $\|v\|_{2p,K}\leqslant C\|A^{N/(N+2)}v\|_{2,K}$ *for* $v\in D(A^{N/(N+2)})$, *where* $C>0$ *is a constant.*
3. $\lim_{s\downarrow 0}s^\gamma\|A^\gamma e^{-sA}v\|_{2,K}=0$, *where* $\gamma>0$ *and* $v\in L^2(K)$.

Proof. The last fact is a consequence of the general theory, because A is a positive self-adjoint operator in $L^2(K)$, see [9]. To show the first and the second facts, we use the relation derived from the interpolation theory, see [7], that is,

$$\begin{aligned}&D\left(A^\theta\right) = \left[L^2(K), H^2(K)\right]_\theta \equiv H^{2\theta}(K),\\ &K^{1/2}: H^{2\theta}(K)\to\left[L^2\left(\mathbf{R}^N\right), H_2^2\left(\mathbf{R}^N\right)\right]_\theta\quad\text{is an isomorphism,}\end{aligned}$$

where $\theta\in(0,1)$.

In fact, we have

$$\begin{aligned}&H^{2N/(N+2)}\left(\mathbf{R}^N\right) = \left[L^2\left(\mathbf{R}^N\right), H^2\left(\mathbf{R}^N\right)\right]_{N/(N+2)}\hookrightarrow L^{2p}\left(\mathbf{R}^N\right),\\ &\left[L^p(K), L^q(K)\right]_\theta = \left[K^{1/p}L^p\left(\mathbf{R}^N\right), K^{1/q}L^q\left(\mathbf{R}^N\right)\right]_\theta = K^{1/r}L^r\left(\mathbf{R}^N\right) = L^r(K),\\ &K^{1/(2p)}: H^{2p}(K)\to L^{2p}\left(\mathbf{R}^N\right)\quad\text{is an isomorphism,}\end{aligned}$$

and therefore,

$$\begin{aligned}D\left(A^{N/(N+2)}\right) &= H^{2N/(N+2)}(K)\cong K^{-1/2}\left[L^2\left(\mathbf{R}^N\right), H_2^2\left(\mathbf{R}^N\right)\right]_{N/(N+2)}\\ &\hookrightarrow K^{-1/2}\left[L^2\left(\mathbf{R}^N\right), H^2\left(\mathbf{R}^N\right)\right]_{N/(N+2)} = K^{-1/2}H^{2N/(N+2)}\left(\mathbf{R}^N\right)\\ &\hookrightarrow K^{-1/2}L^{2p}\left(\mathbf{R}^N\right)\cong K^{-1/2+1/(2p)}L^{2p}(K)\hookrightarrow L^{2p}(K).\end{aligned}$$

This means the second case.

The first case is proven as follows. If $N = 3$, we have $H^2(\mathbf{R}^N) \hookrightarrow L^r(\mathbf{R}^N)$ for $2 \leqslant r \leqslant \infty$. Then, it holds that

$$\begin{aligned} D(A) &= H^2(K) \cong K^{-1/2} H_2^2(\mathbf{R}^N) \hookrightarrow K^{-1/2} H^2(\mathbf{R}^N) \\ &\hookrightarrow K^{-1/2} L^r(\mathbf{R}^N) \hookrightarrow L^q(K) \end{aligned}$$

for $2 < q \leqslant r$, $r > 2$ and $r = q = 2$, and therefore,

$$H^2(K) \hookrightarrow L^q(\mathbf{R}^N)$$

for any $2 \leqslant q \leqslant \infty$.

If $N = 4$, we have $H^2(\mathbf{R}^N) \subset L^r(\mathbf{R}^N)$ for any $r \in [2, \infty)$. Then, it holds that

$$\begin{aligned} D(A) &= H^2(K) \cong K^{-1/2} H_2^2(\mathbf{R}^N) \hookrightarrow K^{-1/2} H^2(\mathbf{R}^N) \\ &\hookrightarrow K^{-1/2} L^r(\mathbf{R}^N) \hookrightarrow L^q(K) \end{aligned}$$

for $2 < q \leqslant r$, $r > 2$ and $r = q = 2$, and therefore,

$$H^2(K) \hookrightarrow L^q(K)$$

for any $2 \leqslant q < \infty$. Finally, if $N \geqslant 5$, it holds that

$$H^2(K) = K^{-1/2} H_2^2(\mathbf{R}^N) \hookrightarrow K^{-1/2} L^{2^{**}}(\mathbf{R}^N) = K^{-(1/2)+(1/2^{**})} L^{2^{**}}(K) \hookrightarrow L^{2^{**}}(K).$$

The proof is complete. □

Once Lemma 2.4 is proven, Proposition 1.1 is shown similarly to [9]. It suffices to use

$$\| f(u) - f(v) \|_{2,K} \leqslant C\big(\|u\|_{2p,K} + \|v\|_{2p,K}\big)^{p-1} \|u - v\|_{2p,K}$$

for $f(u) = |u|^{p-1}u$. The following proposition is also proven similarly.

Proposition 2.1. *If $u_0 \in H^1(K)$ and $v = v(\cdot, s)$ denotes the solution to* (4)–(5)*, then we have the following properties*:

1. $\int_s^t \|v_r(r)\|_{2,K}^2\, dr + J_K(v(t)) = J_K(v(s))$ *for* $t, s \in [0, S)$.
2. $s \in [0, S) \mapsto J_K(v(s))$ *is monotone decreasing.*
3. $v(s_0) \in W_K$ *(resp.* V_K*)* $\Rightarrow v(s) \in W_K$ *(resp.* V_K*) for* $s \in [s_0, S)$.
4. $\frac{1}{2}\frac{d}{ds}\|v(s)\|_{2,K}^2 + I_K(v(s)) = 0$ *for* $s \in [0, S)$.

The $H^1(K)$-solution $v = v(\cdot, s)$ to (4)–(5) is regarded as the $H^1(K)$-solution $u = u(\cdot, t)$ to (1)–(2) through the transformation (3). It is also the L^1-mild solution of [16], and is provided with the following properties.

Proposition 2.2. *If $u_0 \in H^1(K)$, we have*

$$\frac{d}{dt} J\big(u(t)\big) = -\big\|u_t(t)\big\|_2^2 \quad \text{and} \quad \frac{d}{dt}\big\|u(t)\big\|_2^2 = I\big(u(t)\big)$$

for the solution $u = u(\cdot, t)$ to (1)–(2)*, where*

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1} \quad \text{and} \quad I(u) = \|\nabla u\|_2^2 - \|u\|_{p+1}^{p+1}.$$

3. Stable and unstable sets

In this section we prove Propositions 1.3 and 1.4.

Lemma 3.1. *If $v = v(\cdot, s)$ is the solution to* (4)–(5) *satisfying $v(s_0) \in W_K$ for some $s_0 \in [0, T)$, then it holds that*

$$\big\|v(s)\big\|_{p+1,K}^{p+1} \leqslant (1-\gamma)\big(\big\|\nabla v(s)\big\|_{2,K}^2 - \lambda\big\|v(s)\big\|_{2,K}^2\big) \tag{21}$$

for $s \in [s_0, T)$, where $\gamma \in (0, 1)$ and $\lambda = \frac{1}{p-1}$.

Proof. First, Proposition 1.2 implies

$$S_0^{(p+1)/2}\|v\|_{p+1,K}^{p+1} \leqslant \left(\|\nabla v\|_{2,K}^2 - \lambda_*\|v\|_{2,K}^2\right)^{(p+1)/2}$$
$$\leqslant \left(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2\right)\left(\|\nabla v\|_{2,K}^2 - \lambda_*\|v\|_{2,K}^2\right)^{(p-1)/2}$$

for $v \in H^1(K)$. Next, if $v \in W_K$ it holds that

$$J_K(v) = \frac{p-1}{2(p+1)}\left(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2\right) + \frac{1}{p+1} I_K(v)$$
$$\geqslant \frac{p-1}{2(p+1)}\left(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2\right)$$
$$\geqslant \frac{p-1}{2(p+1)}\left(\|\nabla v\|_{2,K}^2 - \lambda_*\|v\|_{2,K}^2\right) \geqslant 0.$$

Let $v = v(\cdot, s)$ be the solution to (4)–(5) satisfying $v(s_0) \in W_K$. Then $v(s) \in W_K$ holds for $s \in [s_0, T)$ by Proposition 2.1. Using the above inequalities, we obtain

$$S_0^{2^*/2}\left\|v(s)\right\|_{p+1,K}^{p+1} \leqslant \left\{\frac{2(p+1)}{p-1} J_K\big(v(s)\big)\right\}^{(p-1)/2}\left(\left\|\nabla v(s)\right\|_{2,K}^2 - \lambda\left\|v(s)\right\|_{2,K}^2\right)$$
$$\leqslant \left\{\frac{2(p+1)}{p-1} J_K\big(v(s_0)\big)\right\}^{(p-1)/2}\left(\left\|\nabla v(s)\right\|_{2,K}^2 - \lambda\left\|v(s)\right\|_{2,K}^2\right)$$

for $2^* = p+1$. Since

$$d_0 = \frac{p-1}{2(p+1)} S_0^{(p+1)/(p-1)} > J_K\big(v(s_0)\big),$$

it holds that

$$\gamma \equiv 1 - S_0^{-2^*/2}\left\{\frac{2(p+1)}{p-1} J_K\big(v(s_0)\big)\right\}^{(p-1)/2} \in (0, 1).$$

Then, inequality (21) follows. □

Proof of Proposition 1.3. First, we show $T = +\infty$, assuming $v(s_0) \in W_K$ for some $s_0 \in [0, T)$. In fact, if $v \in W_K$ is the case, then

$$\|v\|_{p+1}^{p+1} < \|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2 < \frac{2}{p+1}\|v\|_{p+1,K}^{p+1} + 2d_0$$

and hence

$$\|v\|_{p+1}^{p+1} < S_0^{n/2}$$

by $p = \frac{n+2}{n-2}$ and $d_0 = \frac{1}{n} S^{n/2}$. This implies

$$\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2 < \frac{2}{p+1}\|v\|_{p+1,K}^{p+1} + 2d_0 < S_0^{n/2}.$$

Since $v(s_0) \in W_K$, we obtain

$$\limsup_{s \uparrow T}\left\{\left\|\nabla v(s)\right\|_{2,K}^2 - \lambda\left\|v(s)\right\|_{2,K}^2\right\} < S_0^{n/2},$$
$$\limsup_{s \uparrow T}\left\|v(s)\right\|_{p+1}^{p+1} < S_0^{n/2} \tag{22}$$

and hence

$$\limsup_{t \uparrow T}\left\|\nabla v(s)\right\|_{2,K} < +\infty \tag{23}$$

by Proposition 1.2.

Inequality (23) then implies

$$S_0\|u(t)\|_{p+1}^2 \leqslant \|\nabla u(t)\|_2^2 = \|\nabla v(s)\|_2^2 \leqslant \|\nabla v(s)\|_{2,K}^2 \leqslant C$$

with a constant $C > 0$ independent of $t \in [0, T_0)$, where $T_0 = e^T - 1$, and therefore,

$$\lim_{t\uparrow T_0} J\big(u(t)\big) > -\infty$$

for $J(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1}$. Then, the blow-up analysis guarantees

$$\limsup_{t\uparrow T_0}\|u(t)\|_{p+1}^{p+1} \geqslant S_0^{N/2}$$

similarly to the bounded domain case, see [11], and therefore,

$$\limsup_{s\uparrow T}\|v(s)\|_{p+1,K}^{p+1} \geqslant \limsup_{s\uparrow T}\|v(s)\|_{p+1}^{p+1} = \limsup_{t\uparrow T_0}\|u(t)\|_{p+1}^{p+1} \geqslant S_0^{N/2},$$

a contradiction to (22). Thus, $T = +\infty$ follows.

The proof of (12) is similar to that of [9]. First, $v \in W_K$ implies

$$J_K(v) \geqslant \frac{p-1}{2(p+1)}(\lambda_* - \lambda)\|v\|_{2,K}^2 \tag{24}$$

with $0 < \lambda = \frac{1}{p-1} < \lambda_*$ by (22) and Lemma 2.1. If $v = v(\cdot, s)$ is the solution satisfying $v(s_0) \in W_K$ for some $s_0 \in [0, +\infty)$, then it holds that

$$\begin{aligned}\int_s^{s_1} I_K\big(v(r)\big)\,dr &= \frac{1}{2}\big(\|v(s)\|_{2,K}^2 - \|v(s_1)\|_{2,K}^2\big)\\ &\leqslant \frac{1}{2}\|v(s)\|_{2,K}^2 \leqslant \frac{p+1}{(p-1)(\lambda_*-\lambda)} J_K\big(v(s)\big)\end{aligned} \tag{25}$$

by Proposition 2.1 and (24), where $s_0 \leqslant s \leqslant s_1$. Since Lemma 3.1 implies

$$\gamma\big(\|\nabla v(s)\|_{2,K}^2 - \lambda\|v(s)\|_{2,K}^2\big) \leqslant I_K\big(v(s)\big)$$

by $v(s) \in W_K$, it follows that

$$\begin{aligned}J_K\big(v(s)\big) &= \frac{p-1}{2(p+1)}\big(\|\nabla v(s)\|_{2,K}^2 - \lambda\|v(s)\|_{2,K}^2\big) + \frac{1}{p+1} I_K\big(v(s)\big)\\ &\leqslant \left(\frac{p-1}{2\gamma(p+1)} + \frac{1}{p+1}\right) I_K\big(v(s)\big).\end{aligned} \tag{26}$$

Inequalities (25) and (26) imply

$$\int_s^{+\infty} J_K\big(v(r)\big)\,dr \leqslant M J_K\big(v(s)\big)$$

with a constant $M > 0$ independent of $s \in [s_0, +\infty)$. Then, inequality (12) is obtained by Komornik's method, see [17], and the proof is complete. □

Lemma 3.2. *If $v = v(\cdot, s)$ is an $H^1(K)$-solution to (4)–(5), satisfying $v(s_0) \in V_K$ for some $s_0 \in [0, T)$, then there is $\delta > 0$ such that*

$$\|v(s)\|_{p+1,K}^{p+1} \geqslant (1+\delta)\big\{\|\nabla v(s)\|_{2,K}^2 - \lambda\|v(s)\|_{2,K}^2\big\} \tag{27}$$

for $s \in [s_0, T)$.

Proof. If $v \in V_K$, we have $I_K(v) < 0$, and therefore,

$$\frac{2(p+1)}{p-1}d_0 \leqslant \frac{\{\|\nabla v\|_{2,K}^2 - \lambda_* \|v\|_{2,K}^2\}^{(p+1)/(p-1)}}{\{\|v\|_{2^*,K}^{p+1}\}^{2/(p-1)}}$$
$$\leqslant \frac{\{\|\nabla v\|_{2,K}^2 - \lambda \|v\|_{2,K}^2\}^{(p+1)/(p-1)}}{\{\|\nabla v\|_{2,K}^2 - \lambda \|v\|_{2,K}^2\}^{2/(p-1)}} = \|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2 \tag{28}$$

by Lemma 1.2, (11), and $\lambda_* > \lambda = 1/(p-1)$. Proposition 2.1, on the other hand, implies

$$-I_K(v(s)) = \frac{p-1}{2}\left(\|\nabla v(s)\|_{2,K}^2 - \lambda\|v(s)\|_{2,K}^2\right) - (p+1)J_K(v(s))$$
$$\geqslant \frac{p-1}{2}\left(\|\nabla v(s)\|_{2,K}^2 - \lambda\|v(s)\|_{2,K}^2\right) - (p+1)J_K(v(s_0))$$
$$= \frac{p-1}{2}\left(\|\nabla v(s)\|_{2,K}^2 - \lambda\|v(s)\|_{2,K}^2\right) - (p+1)(1-\epsilon_0)d_0 \tag{29}$$

for $s \geqslant s_0$, where

$$\epsilon_0 = 1 - \frac{J_K(v(s_0))}{d_0} > 0.$$

We shall show

$$\frac{p-1}{2}\left(\|\nabla v(s)\|_{2,K}^2 - \lambda\|v(s)\|_{2,K}^2\right) - (p+1)(1-\epsilon_0)d_0$$
$$\geqslant \delta\left(\|\nabla v(s)\|_{2,K}^2 - \lambda\|v(s)\|_{2,K}^2\right) \tag{30}$$

for $\delta = (p-1)\epsilon_0/2 > 0$. Then, (27) will follow from (29).

For this purpose, we use

$$\frac{p-1}{2}\left(\|\nabla v(s)\|_{2,K}^2 - \lambda\|v(s)\|_{2,K}^2\right) - (p+1)(1-\epsilon_0)d_0 - \delta\left(\|\nabla v(s)\|_{2,K}^2 - \lambda\|v(s)\|_{2,K}^2\right)$$
$$= \left(\frac{p-1}{2} - \delta\right)\left(\|\nabla v(s)\|_{2,K}^2 - \lambda\|v(s)\|_{2,K}^2\right) - (p+1)(1-\epsilon_0)d_0$$
$$= \frac{p-1}{2}\cdot\frac{1}{d_0}\cdot J_K(v(s_0))\left(\|\nabla v(s)\|_{2,K}^2 - \lambda\|v(s)\|_{2,K}^2\right) - (p+1)J_K(v(s_0))$$
$$= \frac{p-1}{2}\cdot\frac{1}{d_0}\cdot J_K(v(s_0))\left(\|\nabla v(s)\|_{2,K}^2 - \lambda\|v(s)\|_{2,K}^2 - \frac{2(p+1)d_0}{p-1}\right).$$

The right-hand side of the above inequality is non-negative by (28), and therefore, (30) follows. □

Proof of Proposition 1.4. The argument of [9] for the case of the bounded domain is not valid here, because $L^{p+1}(K) \subset L^2(K)$ does not arise. To avoid this difficulty, we suppose the contrary, $T = +\infty$. It holds that $v(s) \in V_K$ for $s \in [s_0, +\infty)$ by Proposition 2.1, and hence

$$-I_K(v(s)) \geqslant \delta\left(\|\nabla v(s)\|_{2,K}^2 - \lambda\|v(s)\|_{2,K}^2\right) \geqslant \delta\left(\frac{N}{2} - \frac{1}{p-1}\right)\|v(s)\|_{2,K}^2$$

by (6) and Lemma 3.2. This means

$$\frac{1}{2}\cdot\frac{d}{ds}\|v(s)\|_{2,K}^2 \geqslant \delta_1\|v(s)\|_{2,K}^2$$

by Proposition 2.1, where

$$\delta_1 = \delta\left(\frac{N}{2} - \frac{1}{p-1}\right) > 0,$$

and therefore,

$$\left\|v(s)\right\|_{2,K}^{2} \geqslant \left\|v(s_0)\right\|_{2,K}^{2} e^{2\delta_1(s-s_0)}$$

is obtained for $s \geqslant s_0$.

We have, on the other hand,

$$\left\|v(s)\right\|_{2,K} \leqslant \left\|v(s_0)\right\|_{2,K} + (s-s_0)^{1/2}\left\{\int_{s_0}^{s} \left\|v_s(r)\right\|_{2,K}^{2} dr\right\}^{1/2},$$

which implies

$$\left\|v(s)\right\|_{2,K}^{2} \leqslant 2\left\|v(s_0)\right\|_{2,K}^{2} + 2(s-s_0)\int_{s_0}^{s} \left\|v_s(r)\right\|_{2,K}^{2} dr,$$

and hence

$$J_K\big(v(s_0)\big) - J_K\big(v(s)\big) = \int_{s_0}^{s} \left\|v_s(r)\right\|_{2,K}^{2} dr \geqslant \frac{\|v(s_0)\|_{2,K}^{2}(e^{2\delta_1(s-s_0)} - 2)}{2(s-s_0)}$$

for $s > s_0$. In particular, there exists $s_1 > s_0$ such that

$$J_K\big(v(s_1)\big) < 0,$$

which, however, induces $T < +\infty$ by Theorem 1.10 of [14], a contradiction. □

4. L^q-theory of the generator

To study the floating orbit in detail, the L^q-theory of L is useful. Henceforth, we define the operator L_q in $L^q(K)$ by

$$D(L_q) = \left\{u \in L^q(K) \,\middle|\, Lu \in L^q(K)\right\}$$

and $L_q u = Lu$ for $u \in D(L_q)$, where $Lu = -\frac{1}{K}\nabla\cdot(K\nabla u)$ and $1 < q < \infty$. This definition is consistent to $L = L_2$, the positive self-adjoint operator in $L^2(K)$ with the domain $D(L_2) = H^2(K)$. Here, we recall the following facts, see [14].

First, $-L_2$ generates a holomorphic semigroup in $L^2(K)$, denoted by $\{e^{-sL_2}\}_{s\geqslant 0}$. Second, $\lambda_1 = N/2$ is the first eigenvalue of L_2, and hence the semigroup $\{e^{-s(L_2-\lambda_1)}\}_{s\geqslant 0}$ is bounded in $L^2(K)$. Finally,

$$\left\|e^{-sL}u_0\right\|_{L^q(K)} \leqslant \|u_0\|_{L^q(K)} \tag{31}$$

is valid for $q \in (1,\infty)$ and $u_0 \in L^q(K) \cap L^2(K)$, and therefore, $-L_q$ generates a (C_0) contraction semigroup in $L^q(K)$, compatible to $\{e^{-sL_2}\}_{s\geqslant 0}$ in $L^2(K)$.

The following theorem has its own interest, where

$$W^{m,q}(K) = \left\{v \in L^q(K) \,\middle|\, D^\alpha v \in L^q(K),\ |\alpha| \leqslant m\right\}.$$

Theorem 2. *The operator $-L_q + \lambda_1$ generates a bounded holomorphic semigroup in $L^q(K)$ with the domain*

$$D(L_q) = W^{2,q}(K) \tag{32}$$

for each $1 < q < \infty$.

The fact that $-L_q + \lambda_1$ generates a bounded holomorphic semigroup in $L^q(K)$ compatible to $\{e^{-s(L_2-\lambda_1)}\}_{s\geqslant 0}$ in $L^2(K)$ is a consequence of the general theory of Markov semigroup, see Theorems 1.4.1 and 1.4.2 of [4]. This semigroup in $L^q(K)$ is henceforth denoted by $\{e^{-s(L_q-\lambda_1)}\}_{s\geqslant 0}$. Now we shall show (32).

Lemma 4.1. *It holds that*

$$\int_{\mathbf{R}^N} |y|^q |v|^q K(y)\,dy \leqslant C_q \int_{\mathbf{R}^N} |\nabla v|^q K(y)\,dy \tag{33}$$

for $v \in W^{1,q}(K)$, where $1 < q < \infty$ and $C_q > 0$ is a constant.

Proof. In terms of the polar coordinate $y = r\omega$ with $r = |y|$, the left-hand side of (33) is equal to

$$\int_{|\omega|=1} d\omega \cdot \int_0^\infty r^q K(r) r^{N-1} \big|v(r\omega)\big|^q\,dr = -\int_{|\omega|=1} d\omega \cdot \int_0^\infty I(r)\big|v(r\omega)\big|^{q-2} v(r\omega) v_r(r\omega)\,dr,$$

where $K(r) = e^{r^2/4}$ and

$$I(r) = q\int_0^r s^{N+q-1} K(s)\,ds = q\int_0^r s^{N+q-1} e^{s^2/4}\,ds.$$

We have

$$\int_{\mathbf{R}^N} |y|^q |v|^q K(y)\,dy \leqslant \left\{ \int_{|\omega|=1} d\omega \cdot \int_0^\infty r^{\alpha q'} K(r)^{\beta q'} \big|v(r\omega)\big|^q\,dr \right\}^{1/q'} \cdot \left\{ \int_{|\omega|=1} d\omega \cdot \int_0^\infty \frac{I(r)^q}{r^{\alpha q} K(r)^{\beta q}} \big|v_r(r\omega)\big|^q\,dr \right\}^{1/q}$$

for $(1/q) + (1/q') = 1$ and $\alpha, \beta \in \mathbf{R}$. Putting $\beta = 1/q'$ and $\alpha = (N+q-1)/q'$, we obtain

$$\int_{\mathbf{R}^N} |y|^q |v|^q K(y)\,dy \leqslant \left\{ \int_{|\omega|=1} \int_0^\infty \frac{I(r)^q |v_r|^q}{r^{(q-1)(N+q-1)} K^{q-1}}\,dr\,d\omega \right\}^{1/q} \cdot \left\{ \int_{|\omega|=1} \int_0^\infty r^{N+q-1} K(r) |v|^q\,dr\,d\omega \right\}^{1/q'},$$

and therefore, inequality (33) will follow from

$$\frac{I(r)^q}{r^{(q-1)(N+q-1)} K(r)^{q-1}} \leqslant C_q r^{N-1} K(r),$$

or equivalently,

$$I(r) \leqslant C_q^{1/q} \cdot r^{N+q-2} K(r). \tag{34}$$

Inequality (34) is immediate, because

$$\frac{I(r)}{q} = \frac{2}{q}\left\{ r^{N+q-2} K(r) - (N+q-2)\int_0^r s^{N+q-3} K(s)\,ds \right\} \leqslant \frac{2}{q} r^{N+q-2} K(r).$$

The proof is complete. □

Lemma 4.2. *The multiplication $K^{1/q}$ induces the equivalences*

$$L^q(K) \cong L^q(\mathbf{R}^N), \tag{35}$$
$$W^{1,q}(K) \cong W_1^{1,q}(\mathbf{R}^N) = \{v \in W^{1,q}(\mathbf{R}^N) \mid y_i v \in L^q(\mathbf{R}^N),\ 1 \leqslant i \leqslant N\}, \tag{36}$$
$$W^{2,q}(K) \cong W_2^{2,q}(\mathbf{R}^N) = \{v \in W^{2,q}(\mathbf{R}^N) \mid |y|^2 v \in L^q(\mathbf{R}^N)\} \tag{37}$$

for $1 < q < \infty$.

Proof. Relation (35) is obvious. Given $u \in W^{1,q}(K)$, we take $v = K^{1/q}u$ and obtain

$$\nabla v = K^{1/q}\nabla u + \frac{y}{2q}v \in L^q(\mathbf{R}^N)^N \tag{38}$$

by Lemma 4.1. This implies $v \in W_1^{1,q}(\mathbf{R}^N)$. If $v \in W_1^{1,q}(\mathbf{R}^N)$, conversely, then $\frac{\partial u}{\partial y_i} \in L^q(K)$ by (38), and hence (36) follows. Finally, relation (37) is obtained similarly to the case $q = 2$ of Lemma 2.2, because the following lemma is valid. □

Lemma 4.3. *If $v \in W_2^{2,q}(\mathbf{R}^N)$, it holds that*

$$y_i \frac{\partial v}{\partial y_j} \in L^q(\mathbf{R}^N)$$

for $1 \leqslant i, j \leqslant N$, where $1 < q < \infty$.

Proof. The proof is similar to that of Lemma 2.3. It suffices to use

$$\begin{aligned}
\int_{\mathbf{R}^N} \left| y_i \frac{\partial v}{\partial y_j}\right|^q dy &= -\int_{\mathbf{R}^N} \frac{\partial}{\partial y_j}\left[|y_i|^q \left|\frac{\partial v}{\partial y_j}\right|^{q-2} \frac{\partial v}{\partial y_j}\right] \cdot v\, dy \\
&= -\int_{\mathbf{R}^N} \left[q|y_i|^{q-2} y_i \delta_{ij} \left|\frac{\partial v}{\partial y_j}\right|^{q-2} \frac{\partial v}{\partial y_j} \cdot v + |y_j|^{q-2}(q-1)\left|\frac{\partial v}{\partial y_j}\right|^{q-2} \frac{\partial^2 v}{\partial y_j^2} \cdot y_j^2 v\right] dy \\
&\leqslant q\left\| y_i \frac{\partial v}{\partial y_j}\right\|_q^{q-1} \cdot \|v\|_q + (q-1)\left\| y_j \frac{\partial v}{\partial y_j}\right\|_q^{q-2} \cdot \left\|\frac{\partial^2 v}{\partial y_j^2}\right\|_q \cdot \|y_j^2 v\|_q
\end{aligned}$$

for $v \in C_0^\infty(\mathbf{R}^N)$. □

Proof of Theorem 2. We have $K^{1/q} L K^{-1/q} = H_q$ for

$$H_q v = -\Delta v - \frac{1}{2}\left(1 - \frac{2}{q}\right) y \cdot \nabla v + V_q(y) v,$$

where $V_q(y) = \frac{\alpha N}{2} + \frac{\alpha^2}{4}|y|^2$ with $\alpha = 1 - \frac{1}{q}$. If $v \in W_2^{2,q}(\mathbf{R}^N)$, it holds that $H_q v \in L^q(\mathbf{R}^N)$ by Lemma 4.2. Thus, it suffices to show that $v \in L^q(\mathbf{R}^N)$ and $H_q v = g \in L^q(\mathbf{R}^N)$ imply $v \in W_2^{2,q}(\mathbf{R}^N)$.

In fact, we obtain

$$\int_{\mathbf{R}^N} H_q v \cdot \langle y\rangle^{2(q-1)} |v|^{q-2} v\, dy$$

with $\langle y\rangle = \sqrt{1 + |y|^2}$. First, we have

$$
\begin{aligned}
\int_{\mathbf{R}^N} (-\Delta v)\langle y\rangle^{2(q-1)}|v|^{q-2}v\,dy &= \int_{\mathbf{R}^N} \nabla v\cdot\nabla\big(\langle y\rangle^{2(q-1)}|v|^{q-2}v\big)\,dy \\
&= (q-1)\int_{\mathbf{R}^N} \big(\langle y\rangle^2|v|\big)^{q-2}\nabla v\cdot\nabla\big(\langle y\rangle^2 v\big)\,dy \\
&= (q-1)\int_{\mathbf{R}^N} \big(\langle y\rangle^{2q}|v|^{q-2}|\nabla v|^2 + 2\big(\langle y\rangle^2|v|\big)^{q-2} v y\cdot\nabla v\big)\,dy \\
&\geqslant \frac{2(q-1)}{q}\int_{\mathbf{R}^N} \langle y\rangle^{2(q-2)} y\cdot\nabla|v|^q\,dy \\
&= -\frac{2(q-1)}{q}\int_{\mathbf{R}^N} |v|^q\nabla\cdot\big(\langle y\rangle^{2(q-2)}y\big)\,dy
\end{aligned}
$$

with

$$
\nabla\cdot\big(\langle y\rangle^{2(q-2)}y\big) = \langle y\rangle^{2(q-3)}\big\{2(q-2)|y|^2 + n\langle y\rangle^2\big\} \leqslant \big(2(q-2)_+ + N\big)\langle y\rangle^{2(q-2)}
$$

and

$$
\int_{\mathbf{R}^N} \langle y\rangle^{2(q-2)}|v|^q\,dy \leqslant \|v\|_q\cdot\big\|\langle y\rangle^2 v\big\|_q^{q-1}.
$$

Next, we have

$$
0 \leqslant \nabla\cdot\big(\langle y\rangle^{2(q-1)}y\big) = \langle y\rangle^{2(q-2)}\big\{2(q-1)|y|^2 + N\langle y\rangle^2\big\} \leqslant \big(2(q-1)+N\big)\langle y\rangle^{2(q-1)}
$$

and hence

$$
\begin{aligned}
0 &\leqslant -\int_{\mathbf{R}^N} (y\cdot\nabla v)\langle y\rangle^{2(q-1)}|v|^{q-2}v\,dy = -\frac{1}{q}\int_{\mathbf{R}^N} \langle y\rangle^{2(q-1)} y\cdot\nabla|v|^q\,dy \\
&= \frac{1}{q}\int_{\mathbf{R}^N} |v|^q\nabla\cdot\big(\langle y\rangle^{2(q-1)}y\big)\,dy \leqslant \frac{2(q-1)+N}{q}\int_{\mathbf{R}^N} |v|^q\langle y\rangle^{2(q-1)}\,dy \\
&\leqslant \frac{2(q-1)+N}{q}\|v\|_q\cdot\big\|\langle y\rangle^2 v\big\|_q^{q-1}.
\end{aligned}
$$

Since

$$
V_q(y) = \frac{\alpha^2}{4}|y|^2 + \frac{\alpha N}{2} \geqslant \min\left\{\frac{\alpha^2}{4}, \frac{\alpha N}{2}\right\}\langle y\rangle^2,
$$

it follows that

$$
\begin{aligned}
\min\left\{\frac{\alpha^2}{4}, \frac{\alpha N}{2}\right\}\big\|\langle y\rangle^2 v\big\|_q^q &\leqslant \frac{2(q-1)}{q}\big(2(q-2)_+ + N\big)\|v\|_q\cdot\big\|\langle y\rangle^2 v\big\|_q^{q-1} \\
&\quad + \frac{1}{2}\left(1-\frac{2}{q}\right)_+\frac{2(q-1)+N}{q}\|v\|_q\cdot\big\|\langle y\rangle^2 v\big\|_q^{q-1}.
\end{aligned}
$$

This implies $V_q v \in L^q(\mathbf{R}^N)$ and hence $v \in W_2^{2,q}(\mathbf{R}^N)$. $\square$

5. Floating orbit

This section is devoted to the proof of Proposition 1.5. First, Theorem 2 guarantees that each $m = 1, 2, 3, \ldots$ admits $\delta > 0$ and $C > 0$ such that

$$\left\| A^m e^{-sA} v \right\|_{q,K} \leqslant C s^{-m} e^{-s\delta} \|v\|_{q,K} \tag{39}$$

for $s > 0$. Next,

$$K^{1/q} : W^{m,q}(K) \to W_m^{m,q}\left(\mathbf{R}^N\right) \equiv \left\{ v \in W^{m,q}\left(\mathbf{R}^N\right) \;\middle|\; |y|^m v \in L^q\left(\mathbf{R}^N\right) \right\} \tag{40}$$

is an isomorphism for $m = 0, 1, 2, \ldots$. In fact, this relation with $m = 0, 1, 2$ is proven in Lemma 4.2, and the other case is obtained by an induction based on Lemmas 4.1 and 4.3.

From

$$\begin{aligned} D\left(A_q^{1/2}\right) &= \left[L^q(K), W^{2,q}(K)\right]_{1/2} \cong \left[K^{-1/q} L^q\left(\mathbf{R}^N\right), K^{-1/q} W_2^{2,q}\left(\mathbf{R}^N\right)\right]_{1/2} \\ &\hookrightarrow \left[K^{-1/q} L^q\left(\mathbf{R}^N\right), K^{-1/q} W^{2,q}\left(\mathbf{R}^N\right)\right]_{1/2} = K^{-1/q}\left[L^q\left(\mathbf{R}^N\right), W^{2,q}\left(\mathbf{R}^N\right)\right]_{1/2} \\ &= K^{-1/q} W^{1,q}\left(\mathbf{R}^N\right) \hookrightarrow K^{-1/q} L^\infty\left(\mathbf{R}^N\right) \hookrightarrow L^\infty\left(\mathbf{R}^N\right) \end{aligned}$$

for $q > N$, e.g., it follows that

$$v(s) \in L^\infty\left(\mathbf{R}^N\right), \quad 0 < s < T, \tag{41}$$

where $v = v(\cdot, s)$ denotes the solution to (4)–(5). Thus, we may assume $u_0 \in H^1(K) \cap L^\infty(\mathbf{R}^N)$ without loss of generality.

Lemma 5.1. *Any $A > 0$ admits $\delta(A) > 0$ such that $\|u_0\|_{\infty,K} \leqslant A$ implies*

$$\|v(s)\|_{\infty,K} \leqslant 2A, \quad s \in \left[0, \delta(A)\right],$$

where $v = v(\cdot, s)$ is the solution to (4)–(5) *with $u_0 \in H^1(K)$.*

Proof. We apply the L^∞-energy method of [19]. Taking $m \geqslant 1$, we multiply $|v|^{m-1} vK$ to (4). It follows that

$$\begin{aligned} \frac{1}{m}\frac{d}{ds}\|v\|_{m,K}^m &= -m \int_{\mathbf{R}^N} |\nabla v|^2 |v|^{m-1} K\, dy - \frac{1}{p-1} \int_{\mathbf{R}^N} |v|^{m+1} v K\, dy + \int_{\mathbf{R}^N} |v|^{m+p} K\, dy \\ &\leqslant \|v\|_{m+p,K}^{m+p} \leqslant \|v\|_\infty^p \|v\|_{m,K}^m, \end{aligned}$$

and therefore,

$$\frac{d}{ds}\left\|v(s)\right\|_{m,K} \leqslant \left\|v(s)\right\|_\infty^p \left\|v(s)\right\|_{m,K}.$$

This implies

$$\left\|v(s)\right\|_{m,K} \leqslant \left\|v(0)\right\|_{m,K} \exp\left(\int_0^s \left\|v(r)\right\|_\infty^p dr\right).$$

Sending $m \uparrow +\infty$, we obtain

$$\left\|v(s)\right\|_{\infty,K} \leqslant \left\|v(0)\right\|_{\infty,K} \exp\left(\int_0^s \left\|v(r)\right\|_\infty^p dr\right)$$

by (9). The assertion thus follows from $\|v\|_{\infty,K} = \|v\|_\infty$. $\square$

To complete the proof of Proposition 5.1, we use the following proposition obtained by [5].

Proposition 5.1. *If $\lambda \leqslant N/4$ and $p = \frac{N+2}{N-2}$, $N \geqslant 3$, there is no non-trivial solution $v = v(y) \in H^1(\mathbf{R}^N)$ to*

$$-\Delta v - \frac{1}{2} y \cdot \nabla v = |v|^{p-1} v + \lambda v \quad \text{in } \mathbf{R}^N. \tag{42}$$

We apply also the fact that $T = +\infty$ in (4)–(5) implies

$$\limsup_{s \uparrow +\infty} \|v(s)\|_{2,K} < +\infty. \tag{43}$$

This is derived from Poincaré's inequality (10), similarly to the prescaled case on the bounded domain, see [18,2,23].

Proof of Proposition 1.5. From the assumption, we have $T = +\infty$ and

$$\beta_K \equiv \lim_{s \uparrow +\infty} J_K\big(v(s)\big) \geqslant d_0 > 0. \tag{44}$$

If (13) is not the case, we have $A > 0$ and $s_k \uparrow +\infty$ such that

$$\|v(s_k)\|_\infty \leqslant A, \quad k = 1, 2, \ldots,$$

and then we obtain

$$\|v(s + s_k)\|_\infty \leqslant 2A, \quad s \in \big[0, \delta(A)\big], \tag{45}$$

for $\delta(A)$ prescribed by Lemma 5.1. We may assume

$$s_{k+1} > s_k + \delta(A), \quad k = 1, 2, \ldots,$$

and in this case it follows that

$$\lim_{k \to \infty} \int_{s_k}^{s_k + \delta(A)} \|v_s(s)\|_{2,K}^2 \, ds = 0$$

from (44). Thus, there is $s_k' \in [s_k, s_k + \delta(A)]$ such that

$$\lim_{k \to \infty} \|v_s(s_k')\|_2 = 0.$$

By (43), on the other hand, we obtain

$$\big|I_K\big(v(s_k')\big)\big| = \big|\big(v(s_k'), v_s(s_k')\big)_K\big| \leqslant C \|v_s(s_k')\|_{2,K},$$

where

$$(f, g)_K = \int_{\mathbf{R}^N} f(y) g(y) K(y) \, dy,$$

and hence it holds that

$$\lim_{k \to \infty} I_K\big(v(s_k')\big) = 0. \tag{46}$$

Thus,

$$\beta_K = \lim_{s \uparrow +\infty} J_K\big(v(s)\big) = \frac{p-1}{2(p+1)} \lim_{k \to +\infty} \big\{ \|\nabla v(s_k')\|_{2,K}^2 - \lambda \|v(s_k')\|_{2,K}^2 \big\} \tag{47}$$

follows from

$$J_K(v) = \frac{p-1}{2(p+1)} \big\{ \|\nabla v\|_{2,K}^2 - \lambda \|v\|_{2,K}^2 \big\} + \frac{1}{p+1} I_K(v),$$

where $\lambda = \frac{1}{p-1}$.

Since Proposition 1.2 implies

$$\|\nabla v(s_k')\|_{2,K}^2 - \lambda\|v(s_k')\|_{2,K}^2 \geqslant \mu\|\nabla v(s_k')\|_{2,K}^2$$

with $\mu = 1 - \lambda/\lambda_1 > 0$, we now obtain

$$\|\nabla v(s_k')\|_{2,K}^2 = O(1).$$

Here, Rellich's type of compactness theorem holds in the inclusion $H^1(K) \hookrightarrow L^2(K)$ by Lemma 2.2, and therefore, we have

$$v(s_k') \rightharpoonup v \quad \text{weakly in } H^1(K),$$
$$v(s_k') \to v \quad \text{strongly in } L^2(K)$$

passing to a subsequence. This $v \in H^1(K)$ is a critical point of J_K, and therefore, $v = 0$ by Proposition 5.1. We have, furthermore, $\|v(s_k')\|_\infty \leqslant C$ by (45), and therefore,

$$\|v(s_k') - v(s_\ell')\|_{p+1,K}^{p+1} \leqslant \|v(s_k') - v(s_\ell')\|_{2,K}^2 \|v(s_k') - v(s_\ell')\|_\infty^{p-1} \to 0$$

as $k', \ell' \to \infty$. Thus, it holds that

$$v(s_k') \to 0 \quad \text{in } L^{p+1}(K) \text{ and } L^2(K).$$

We now obtain

$$\|\nabla v(s_k')\|_{2,K}^2 = \frac{1}{p-1}\|v(s_k')\|_{2,K}^2 + \|v(s_k')\|_{p+1,K}^{p+1} + \big(v(s_k'), v_s(s_k')\big)_k \to 0$$

by (46), and therefore,

$$\beta_K = \lim_{k\to\infty} J_K\big(v(s_k')\big) = 0,$$

a contradiction. □

6. Proof of Theorem 1

If the orbit enters in the unstable set V_K, then $T < +\infty$ occurs by Proposition 1.4. For the floating orbit global in time, next, Proposition 1.5 is applicable. It holds that

$$\lim_{t\uparrow+\infty} (1+t)^{1/(p-1)}\|u(t)\|_\infty = \lim_{s\uparrow+\infty}\|v(s)\|_\infty = \lim_{s\uparrow+\infty}\|v(s)\|_{\infty,K} = +\infty,$$

and therefore,

$$\lim_{t\uparrow+\infty} t^{(N-2)/4}\|u(t)\|_\infty = +\infty.$$

If the orbit $\{v(s)\}$ enters in the stable set W_K, finally, Proposition 1.3 is applicable. We have $T = +\infty$ and (12), and therefore,

$$\lim_{s\uparrow+\infty}\|v(s)\|_{H^1(K)} = \lim_{s\uparrow+\infty}\|v(s)\|_{p+1,K} = 0 \tag{48}$$

by (10). Then it follows that

$$\lim_{t\uparrow+\infty}\|u(t)\|_{p+1} = 0 \tag{49}$$

from

$$\|u(t)\|_{p+1} = \|v(s)\|_{p+1} \leqslant \|v(s)\|_{p+1,K}.$$

Here, we may assume

$$u_0 \in L^1\big(\mathbf{R}^N, \big(1+|x|\big)\,dx\big) \cap L^\infty\big(\mathbf{R}^N\big), \tag{50}$$

regarding (41) and

$$\int_{\mathbf{R}^n} \big(1+|x|\big)u_0(x)\,dx \leqslant \|u_0\|_{2,K}\left\{\int_{\mathbf{R}^N} \big(1+|x|\big)^2 K(x)^{-1}\,dx\right\}^{1/2} < +\infty. \tag{51}$$

Relations (49)–(50) imply

$$\lim_{t\uparrow+\infty} t^{N/2}\big\|u(t)\big\|_\infty < +\infty$$

by Theorem 4.1 of [16]. If

$$u_0 \geqslant 0 \quad \text{and} \quad u_0 \neq 0, \tag{52}$$

furthermore,

$$t^{N/2}\left\|u(t) - m_\infty(4\pi t)^{-N/2}\exp\left(-\frac{|x|^2}{4t}\right)\right\|_\infty \leqslant Ct^{-1/2} \tag{53}$$

holds by Theorem 4.1 of [15], where $C > 0$ is a constant and

$$0 < m_\infty = \sup_{t\geqslant 0}\big\|u(t)\big\|_1 < +\infty.$$

Then, (7) follows.

Remark 6.1. When (49)–(50) arises, we have

$$\sup_{t\geqslant 0}\big\|u(t)\big\|_1 < +\infty \tag{54}$$

by Theorem 4.1 of [16]. If (52) is the case, furthermore, it follows that

$$\lim_{t\uparrow+\infty} t^{(1-1/q)N/2}\left\|u(t) - m_\infty(4\pi t)^{-N/2}\exp\left(-\frac{|x|^2}{4t}\right)\right\|_q = 0$$

for $q \in [1,\infty]$, see Lemma 3 of [6]. This relation is refined as

$$t^{(1-1/q)N/2}\left\|u(t) - m_\infty(4\pi t)^{-N/2}\exp\left(-\frac{|x|^2}{4t}\right)\right\|_\infty \leqslant Ct^{-1/2} \tag{55}$$

by (54) and $|x|^2u_0 \in L^1(\mathbf{R}^N)$, see [10].

Remark 6.2. If

$$\int_{\mathbf{R}^N} \big(u_0^2 + |\nabla u_0|^2\big)K(x-a)\,dx < +\infty \tag{56}$$

is assumed for any $a \in \mathbf{R}^N$, we can argue similarly to the prescaled case [9] to derive

$$\lim_{s\uparrow+\infty}\big\|v(s)\big\|_{\infty,K} = 0 \tag{57}$$

from (48) directly.

In fact, we use the moment inequality, see [24],

$$\big\|A^\alpha v\big\|_{2,K} \leqslant C\|Av\|_{2,K}^{2\alpha-1}\big\|A^{1/2}v\big\|_{2,K}^{2-2\alpha},$$

where $\alpha \in (0,1)$. Applying Lemma 2.4 for $\alpha = N/(N+2)$, we obtain

$$\big\||v|^p\big\|_{2,K} \leqslant C\|Av\|_{2,K}\big\|A^{1/2}v\big\|_{2,K}^{\frac{4}{N-2}}. \tag{58}$$

Since

$$\big\|A^{1/2}v(s)\big\|_{2,K}^2 = \big\|\nabla v(s)\big\|_{2,K}^2 + \frac{1}{p-1}\big\|v(s)\big\|_{2,K}^2 \to 0$$

as $s \uparrow +\infty$ by (48) and also

$$\big\|Av(s)\big\|_{2,K} \leqslant \big\|v_s(s)\big\|_{2,K} + \big\|\big|v(s)\big|^p\big\|_{2,K}$$

by (4), inequality (58) implies

$$\big\|Av(s)\big\|_{2,K} \leqslant C\big\|v_s(s)\big\|_{2,K}, \quad s \gg 1. \tag{59}$$

We have, on the other hand,

$$\int_0^\infty \big\|v_s(s)\big\|_{2,K}^2\,ds < +\infty$$

by (48) and Proposition 2.2, and therefore,

$$\int_0^\infty \big\|Av(s)\big\|_{2,K}^2\,ds < +\infty. \tag{60}$$

Inequalities (59)–(60) imply

$$\int_0^\infty \big\|v(s)\big\|_{q,K}^2\,ds < +\infty$$

for some $q > 2^* = p+1$ by Lemma 2.4, and then

$$\limsup_{s\uparrow+\infty}\big\|v(s)\big\|_{q,K} < +\infty$$

follows similarly to the prescaled case, Lemma 6.5 of [9], from the L^q-theory developed in Section 4. This implies that relation (48) yields

$$\limsup_{s\uparrow+\infty}\big\|v(s)\big\|_{\infty,K} < +\infty \tag{61}$$

by the bootstrap argument using (39) with $D(A^m) = W^{2m,q}(K)$ and (40).

Relation (61) reads;

$$\sup_{t\geqslant 0}(1+t)^{1/(p-1)}\big\|u(t)\big\|_\infty < +\infty.$$

Given $s_k \uparrow +\infty$, we take $x_k \in \mathbf{R}^N$ such that

$$\big\|u(t_k)\big\|_\infty = \big|u(x_k,t_k)\big|$$

for $t_k = e^{s_k} - 1$. Then, $v_k = v_k(y,s)$ defined by

$$v_k(y,s) = (1+t)^{1/(p-1)}u(x,t), \quad t = e^s - 1, \quad x = x_k + (1+t)^{1/2}y$$

satisfies

$$v_{ks} + Lv_k = |v_k|^{p-1} v_k + \frac{1}{p-1} v_k \quad \text{in } \mathbf{R}^N \times (0, +\infty),$$
$$\sup_{s \geqslant 0} \| v_k(s) \|_\infty < +\infty,$$
$$\int_0^\infty \| v_{ks}(s) \|_{2,K}^2 \, ds \leqslant J_K\big(v_k(0)\big) < +\infty,$$

where relation (56) is used. There is $\{s'_k\} \subset \{s_k\}$, in particular, such that

$$\int_{s'_k - 1}^{s'_k + 1} \| v_{ks}(s) \|_{2,K}^2 \, ds < 1/k, \quad k = 1, 2, \ldots.$$

Putting $\tilde{v}_k(y, s) = v_k(y, s + s'_k)$, we apply the parabolic regularity. Thus, there is a subsequence, denoted by the same symbol, and $v_\infty = v_\infty(y, s)$ such that $\tilde{v}_k \to v_\infty$ locally uniformly in $\mathbf{R}^N \times (-1, 1)$, and furthermore,

$$\int_{-1}^{1} \| \tilde{v}_{ks}(s) \|_{2,K}^2 \, ds = \int_{s'_k - 1}^{s'_k + 1} \| v_{ks}(s) \|_{2,K}^2 \, ds \to 0.$$

Thus, we obtain $v_{\infty s} = 0$,

$$-\Delta v_\infty - \frac{1}{2} y \cdot \nabla v_\infty = |v_\infty|^{p-1} v_\infty + \frac{1}{p-1} v_\infty \quad \text{in } \mathbf{R}^N \times (-1, 1),$$

and hence $v_\infty = 0$ by Proposition 5.1. This implies

$$\| v(s'_k) \|_\infty = |\tilde{v}_k(0, 0)| \to 0$$

and therefore, (57) because $s_k \uparrow +\infty$ is arbitrary.

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