Analyticity and large time behavior for the Burgers equation and the quasi-geostrophic equation, the both with the critical dissipation

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Abstract

This paper is concerned with the Cauchy problem of the Burgers equation with the critical dissipation. The well-posedness and analyticity in both of the space and the time variables are studied based on the frequency decomposition method. The large time behavior is revealed for any large initial data. As a result, it is shown that any smooth and integrable solution is analytic in space and time as long as time is positive and behaves like the Poisson kernel as time tends to infinity. The corresponding results are also obtained for the quasi-geostrophic equation.
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1. Introduction

We consider the Cauchy problem for the Burgers equation with the fractional Laplacian:

$$\begin{cases} \partial_t u + \Lambda u + u \partial_x u = 0, & t > 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $\Lambda := \mathcal{F}^{-1}|\xi|\mathcal{F}$. We study the local solvability and the analyticity for initial data in the largest space of the Sobolev and Besov spaces which are related to the scaling invariant spaces and the large time behavior of solutions for any large initial data. It will be shown that the similar argument also works for the quasi-geostrophic equation.

It is known that the equation (1.1) is solvable locally in time (see [19,28,31]). In fact, for any initial data in $H^{\frac{1}{2}}(\mathbb{R})$ or $\dot{B}^{\frac{1}{p}}_{p,1}(\mathbb{R})$ with $p < \infty$, there exists a unique local solution. Also, the analyticity in space is proved in the papers [19,28], while that in time variable has been left open up to now. The first purpose of this paper is to construct

solutions for initial data in the Besov space $B^0_{\infty,\infty}(\mathbb{R})$, which is analytic in both of the space and the time variables for positive time.

Before recalling the results on the large time behavior, let us mention about the global regularity. Consider the problem with the fractional Laplacian of order γ.

$$\partial_t u + \Lambda^\gamma u + u \partial_x u = 0. \tag{1.2}$$

It was proved that the value $\gamma = 1$ is the threshold for the occurrence of singularity in finite time or the global regularity (see [3,19,21,28]). In fact, if $\gamma < 1$, it is shown that the gradient of the solution blows up in finite time for some continuous initial data. On the other hand, if $\gamma \geq 1$, such singularity does not appear, so that solution always exists globally in time. We refer to the results on the global regularizing effects in the subcritical case $\gamma > 1$ (see [21]), and on the non-uniqueness of weak solutions in the supercritical case $\gamma < 1$ (see [2]) and on the global regularizing effects for the n-dimensional Burgers equation in the critical case $\gamma = 1$ (see [11]). We should also mention about papers concerned with the quasi-geostrophic equation [10,27,29], where the method of [10] is inspired by Di-Giorgi iterative estimates, the approach of [27] involves control of Hölder norms using appropriate test functions, and the proof in [29] is based on a nonlocal maximum principle and to investigate a certain modulus of continuity of solutions. The global regularity is not the aim of this paper, but we apply their results to guarantee the global existence.

As for the large time behavior, Biler-Karch-Woyczynski [5] considered the equation with the semigroup generated by $\Lambda^\gamma - \Delta$ $(0 < \gamma < 2)$ to study the asymptotic expansion of solutions (see also [6–8]). For the equation (1.2), Karch-Miao-Xu [25] considered the subcritical case $1 < \gamma < 2$ to study that the large time asymptotics is described by the rarefaction waves with some condition in the distance. Alibaud-Imbert-Karch [1] investigated that the entropy solution converges to the self-similar solution for the critical case $\gamma = 1$, and the nonlinearity is negligible in the asymptotic expansion of solutions for the supercritical case $\gamma < 1$. In the previous paper [24], it was proved that the solution behaves like the Poisson kernel if initial data is integrable and small in the Besov space $\dot{B}^0_{\infty,1}(\mathbb{R})$. However, for large initial data, the large time behavior of smooth solutions has been left open up to now. In our theorem below, we show that any smooth and integrable solution behaves like the Poisson kernel without smallness condition for initial data.

Before stating our results, let us recall the definition of Besov spaces.

Definition *(Besov spaces).* Let $\{\psi\} \cup \{\phi_j\}_{j \in \mathbb{Z}}$ be such that

$$\operatorname{supp} \widehat{\psi} \subset \{|\xi| \leq 2\}, \quad \operatorname{supp} \widehat{\phi_j} \subset \{2^{j-1} \leq |\xi| \leq 2^{j+1}\} \text{ for any } j \in \mathbb{Z},$$

$$\widehat{\psi}(\xi) + \sum_{j=1}^{\infty} \widehat{\phi_j}(\xi) = 1 \quad \text{for any } \xi \in \mathbb{R}^d, \qquad \sum_{j \in \mathbb{Z}} \widehat{\phi_j}(\xi) = 1 \quad \text{for any } \xi \in \mathbb{R}^d \setminus \{0\}. \tag{1.3}$$

For $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, we define the Besov spaces as follows.

(i) $B^s_{p,q}(\mathbb{R}^d)$ is defined by

$$B^s_{p,q}(\mathbb{R}^d) := \{u \in \mathcal{S}'(\mathbb{R}^d) \mid \|u\|_{B^s_{p,q}} < \infty\},$$

where

$$\|u\|_{B^s_{p,q}} := \|\psi * u\|_{L^p} + \left\| \left\{ 2^{sj} \|\phi_j * u\|_{L^p} \right\}_{j \in \mathbb{N}} \right\|_{\ell^q(\mathbb{N})}.$$

(ii) $\dot{B}^s_{p,q}(\mathbb{R}^d)$ is defined by

$$\dot{B}^s_{p,q}(\mathbb{R}^d) := \{u \in \mathcal{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d) \mid \|u\|_{\dot{B}^s_{p,q}} < \infty\},$$

where $\mathcal{P}(\mathbb{R}^d)$ is the set of all polynomials and

$$\|u\|_{\dot{B}^s_{p,q}} := \left\| \left\{ 2^{sj} \|\phi_j * u\|_{L^p} \right\}_{j \in \mathbb{Z}} \right\|_{\ell^q(\mathbb{Z})}.$$

We also introduce the standard spaces $L^r(0, T; B^s_{p,q}(\mathbb{R}^d))$ and the Chemin-Lerner spaces $\widetilde{L}^r(0, T; B^s_{p,q}(\mathbb{R}^d))$ which is defined by the set of all distributions u such that

$$\|u\|_{\widetilde{L}^r(0,T;B^s_{p,q})} := \|\psi * u\|_{L^r(0,T;L^p)} + \left\|\left\{2^{sj}\|\phi_j * u\|_{L^r(0,T;L^p)}\right\}_{j\in\mathbb{N}}\right\|_{\ell^q(\mathbb{N})} < \infty.$$

We also define the Poisson kernel $P_t(x)$.

$$P_t(x) := \mathcal{F}^{-1}\big[e^{-t|\xi|}\big](x) = \frac{\Gamma(\frac{d+1}{2})\,t^{-d}}{\pi^{\frac{d+1}{2}}(1+|\frac{x}{t}|^2)^{\frac{d+1}{2}}} \quad \text{for } t > 0, x \in \mathbb{R}^d,$$

where $\Gamma(\cdot)$ is the Gamma function. The following is our main theorem.

Theorem 1.1. *Let u_0 be such that*

$$u_0 \in B^0_{\infty,\infty}(\mathbb{R}) \quad \text{and} \quad \lim_{j\to\infty}\|\phi_j * u_0\|_{L^\infty} = 0. \tag{1.4}$$

(i) *There exists $T > 0$ and a unique solution u of* (1.1) *such that*

$$u \in C([0,T], B^0_{\infty,\infty}(\mathbb{R})) \cap \widetilde{L}^\infty(0,T; B^0_{\infty,\infty}(\mathbb{R})) \cap \widetilde{L}^1(0,T; B^1_{\infty,\infty}(\mathbb{R})),$$
$$\lim_{j\to\infty}\|\phi_j * u(t)\|_{L^\infty} = 0 \quad \text{for any } t > 0.$$

Furthermore $u(t,x)$ is real analytic in space and time if $t > 0$.

(ii) *If in addition $u_0 \in L^1(\mathbb{R})$, the solution u from* (i) *is global in time, satisfies that $u(t) \in L^1(\mathbb{R})$ for any $t \geq 0$, and that for any $1 \leq p \leq \infty$*

$$\lim_{t\to\infty} t^{1-\frac{1}{p}}\left\|u(t) - P_t\int_{\mathbb{R}}u_0(y)dy\right\|_{L^p} = 0. \tag{1.5}$$

Furthermore, for any $\alpha > 0$ there exists $C > 0$ such that

$$\||\nabla|^\alpha u(t)\|_{L^p} \leq Ct^{-(1-\frac{1}{p})-\alpha} \quad \text{for any } t \geq 1. \tag{1.6}$$

Remark. The higher order asymptotic expansion in Theorem 1.2 (ii) in [24] are able to be proved also for any large initial data, since we already have the decay estimates of all regularity norms as (1.6). Namely, the following assertion

$$\lim_{t\to\infty} t^{1+d(1-\frac{1}{p})}\bigg\|u(t) - P_t M + \partial_x P_t \int_{\mathbb{R}} y u_0(y)dy$$
$$+ \frac{1}{2}(\partial_x P_t)\int_0^t\int_{\mathbb{R}} u(\tau,y)^2 dy d\tau + \frac{1}{2}\int_0^t (\partial_x P_{t-\tau}) * (M P_{\tau+1})^2 d\tau$$
$$- \frac{1}{2}(\partial_x P_t)\int_0^t\int_{\mathbb{R}} (M P_{\tau+1}(y))^2 dy d\tau\bigg\|_{L^p} = 0$$

is true provided that $u_0 \in L^1(\mathbb{R})$, $\int_{\mathbb{R}}|y u_0(y)|dy < \infty$ and (1.4) is satisfied, where $M = \int_{\mathbb{R}}u_0(y)dy$.

Let us give remarks on the proof. Based on the frequency decomposition method (see [31,34]), we develop the local solvability to obtain solutions in more general spaces which include non-decaying functions, while they [31,34] considered function spaces where the Schwartz class is dense. In Proposition 2.1 below, another frequency localized maximum principle is established, which enables us to work with the iterative method. On the analyticity, the existing method is to consider the direct computation in the frequency space through the Plancherel theorem (see e.g. [19,28]), or to apply the Fourier multiplier theorem to multipliers $e^{t\Lambda}$, $e^{t(|\partial_{x_1}|+\cdots+|\partial_{x_d}|)}$ (see [4,19,30]), which requires the boundedness of the Riesz transformation or singular integral operators. On the other hand, we simply consider the derivatives with the weight of time variable, which does not require the Fourier multiplier theorem, and the analyticity of Gevery type with order one is verified for not only space but also time variable (see Propositions 3.2, 3.3 and the proof of analyticity thereunder).

As for the large time behavior (1.5) without smallness of initial data, the most important point is to get an integrability of solutions

$$\int_0^\infty \|u(t)\|_{\dot{B}^1_{\infty,1}} \, dt < \infty, \tag{1.7}$$

which can be seen in the previous paper [24] and is useful to work via the Duhamel formula. To handle large initial data, we develop the time decay estimate in $L^\infty(\mathbb{R})$ along the paper [15], which assures the smallness of $u(t_0)$ for some large t_0. Then we can apply the argument for small data for $t \geq t_0$, while the integrability for $[0, t_0]$ is guaranteed by $u \in L^1_{loc}([0, \infty), \dot{B}^1_{\infty,1}(\mathbb{R}))$ by the result [31]. In addition, we also establish decay estimates with arbitrary positive regularity (1.6) for any large initial data (see also Proposition 4.3).

We next consider the quasi-geostrophic equation.

$$\begin{cases} \partial_t \theta + \Lambda^\gamma \theta + (u \cdot \nabla)\theta = 0, & t > 0, x \in \mathbb{R}^2, \\ u = (-R_2\theta, R_1\theta), & t > 0, x \in \mathbb{R}^2, \\ \theta(0, x) = \theta_0(x), & x \in \mathbb{R}^2, \end{cases} \tag{1.8}$$

where $\theta : \mathbb{R}^2 \to \mathbb{R}$ is the unknown function, R_1, R_2 are the Riesz transforms in $\mathbb{R}^2$. The equation is an important model in geophysical fluid dynamics, which describes the evolution of a surface temperature field in a rotating and stratified fluid. We obtain the following result by modifying the method for the Burgers equation to handle the Riesz transform especially for the low frequency part. For the sake of simplicity, we consider initial data in $L^1(\mathbb{R}^2) \cap B^0_{\infty,\infty}(\mathbb{R}^2)$ to avoid the complexity to get the following result.

Theorem 1.2. *Let* $\gamma = 1$, u_0 *satisfy* $u_0 \in L^1(\mathbb{R}^2) \cap B^0_{\infty,\infty}(\mathbb{R}^2)$, $\|\phi_j * u_0\|_{L^\infty} \to 0$ *as* $j \to \infty$. *Then, there exists a unique global solution* u *of* (1.8) *such that*

$$u \in C([0, \infty), B^0_{\infty,\infty}(\mathbb{R}^2)) \cap \widetilde{L}^\infty_{loc}(0, \infty; B^0_{\infty,\infty}(\mathbb{R}^2)) \cap \widetilde{L}^1_{loc}([0, \infty); B^1_{\infty,\infty}(\mathbb{R}^2)),$$

$$\lim_{j \to \infty} \|\phi_j * u(t)\|_{L^\infty} = 0 \quad \textit{for any } t > 0,$$

and $u(t, x)$ *is real analytic in space and time if* $t > 0$. *Furthermore, for any* $1 \leq p \leq \infty$

$$\lim_{t \to \infty} t^{2(1-\frac{1}{p})} \left\| u(t) - P_t \int_{\mathbb{R}^2} u_0(y) dy \right\|_{L^p} = 0,$$

and for any $\alpha > 0$ *there exists* $C > 0$ *such that*

$$\||\nabla|^\alpha u(t)\|_{L^p} \leq C t^{-2(1-\frac{1}{p})-\alpha} \quad \textit{for any } t \geq 1.$$

Let us compare with known results. There are a lot of known results on the global solvability and the asymptotic behavior. The solvability in the case when $\gamma = 1$ was studied for small initial data in $L^\infty(\mathbb{R}^2)$ in [12] and that for arbitrarily large data has been settled in the papers [10,27,29]. The well-posedness is also studied in Besov spaces in [34], where spaces are defined by the completion of the Schwartz class. Here we refer the recent papers [9,16,23,26] on regularity of super-critical case $\gamma < 1$, where the global regularity is open. As for the large time behavior, the subcritical case $\gamma > 1$ was resolved by [14,20,33]. For the critical case $\gamma = 1$, the time decay estimates in $L^p(\mathbb{R}^2)$ for some p of weak solutions are known in [12,18,32]. We also refer to the recent papers [22] for modified equations, including not only critical case but also super-critical case, [13] on the existence of a compact global attracter with a time-independent force, and see also references therein. The contribution of Theorem 1.2 is the analyticity in the space and the time variables and to reveal the large time behavior of smooth solutions for any large initial data.

This paper is organized as follows. In section 2, we prepare lemmas on the frequency localized maximum principle. On the proof of theorems, we prove Theorem 1.1 only, since Theorem 1.2 follows analogously. Section 3 is devoted to showing the local solvability and the analyticity for Burgers equation. In section 4, we verify the large time behavior of solutions in Theorem 1.1.

2. Preliminary

In this section, we prepare the frequency localized maximum principle for non-decaying smooth functions, which is motivated by the paper [34]. The Fourier multiplier theorem for the Poisson kernel and the continuity property of linear solutions are also investigated.

Proposition 2.1 *(Frequency localized maximum principle). Let u, v, f be smooth functions on $(0, \infty) \times \mathbb{R}^d$ such that $u, \partial_t u, \partial_t^2 u, v \in L^\infty((0, \infty) \times \mathbb{R}^d)$.*
(i) Let $j \in \mathbb{Z}$. If u satisfies $\partial_t(\phi_j * u) + (v \cdot \nabla)(\phi_j * u) + \Lambda(\phi_j * u) = f$, then there exists a positive constant c independent of u, v, f, j such that for almost every $t \geq 0$

$$\partial_t \|\phi_j * u\|_{L^\infty} + c2^j \|\phi_j * u\|_{L^\infty} \leq \|f\|_{L^\infty}. \tag{2.1}$$

(ii) If u satisfies $\partial_t(\psi * u) + (v \cdot \nabla)(\psi * u) + \Lambda(\psi * u) = f$, then for almost every $t \geq 0$

$$\partial_t \|\psi * u\|_{L^\infty} \leq \|f\|_{L^\infty}. \tag{2.2}$$

In order to prove the above proposition, we prepare two lemmas.

Lemma 2.2. *Let $u(t, x)$ be a smooth function with $u, \partial_t u, \partial_t^2 u \in L^\infty((0, \infty) \times \mathbb{R}^d)$. Then $\partial_t \|u(t)\|_{L^\infty}$ exists for almost every $t \geq 0$. Furthermore, for almost every $t \geq 0$, there exists a sequence $\{x_{t,n}\}_{n=1}^\infty \subset \mathbb{R}^d$ such that*

$$\|u(t)\|_{L^\infty} = \lim_{n \to \infty} u(t, x_{t,n}) \operatorname{sgn}(u(t, x_{t,n})), \tag{2.3}$$

$$\partial_t \|u(t)\|_{L^\infty} = \lim_{n \to \infty} (\partial_t u)(t, x_{t,n}) \operatorname{sgn}(u(t, x_{t,n})), \tag{2.4}$$

where $\operatorname{sgn} u$ is a sign function of u.

Proof. Our proof is based on the proof of Lemma 3.2 in the paper [34], but there needs some modification to handle non-decaying functions. In what follows, let $t \geq 0$ and $|h| < 1$ such that $t + h \geq 0$.

The existence of $\partial_t \|u(t)\|_{L^\infty}$ for almost every t is proved by

$$\left| \|u(t+h)\|_{L^\infty} - \|u(t)\|_{L^\infty} \right| \leq \|u(t+h) - u(t)\|_{L^\infty} \leq \sup_{|\tau - t| \leq 1} \|\partial_\tau u(\tau)\|_{L^\infty} |h|,$$

since this inequality implies the Lipschitz continuity of $\|u(t)\|_{L^\infty}$.

We turn to prove the latter assertion. Let us consider non-negative u for the sake of simplicity. For each $t + h$, there exists a sequence $\{x_{t+h,n}\}_{n=1}^\infty$ such that

$$u(t+h, x_{t+h,n}) \to \|u(t+h)\|_{L^\infty} \quad \text{as } n \to \infty. \tag{2.5}$$

By considering the limit as $h \to 0$, we can take sequences $\{h_m\}_{m=1}^\infty$ and $\{x_{t+h_m,n_m}\}_{m=1}^\infty$ such that

$$h_m \to 0, \quad u(t+h_m, x_{t+h_m,n_m}) \to \|u(t)\|_{L^\infty} \quad \text{as } m \to \infty,$$

$$\|u(t+h_m)\|_{L^\infty} - u(t+h_m, x_{t+h_m,n_m}) \leq h_m^2, \tag{2.6}$$

$$\|u(t)\|_{L^\infty} - u(t, x_{t+h_m,n_m}) \leq h_m^2, \tag{2.7}$$

$$\lim_{m \to \infty} (\partial_t u)(t, x_{t+h_m,n_m}) \text{ exists.}$$

Put $x_m' := x_{t+h_m,n_m}$. It follows from the inequality (2.6) and smoothness of u in the time variable that

$$
\begin{aligned}
\frac{\|u(t+h_m)\|_{L^\infty} - \|u(t)\|_{L^\infty}}{h_m} &\leq \frac{u(t+h_m, x_m') + h_m^2 - u(t, x_m')}{h_m} \\
&\leq \frac{u(t+h_m, x_m') - u(t, x_m')}{h_m} + h_m \\
&\to \lim_{m \to \infty} (\partial_t u)(t, x_m'),
\end{aligned}
$$

as $m \to \infty$, which implies

$$\partial_t \|u(t)\|_{L^\infty} \le \lim_{m \to \infty} (\partial_t u)(t, x'_m). \tag{2.8}$$

On the other hand, we have from (2.7) that

$$\frac{\|u(t + h_m)\|_{L^\infty} - \|u(t)\|_{L^\infty}}{h_m} \ge \frac{u(t + h_m, x'_m) - u(t, x'_m) - h_m^2}{h_m}$$

$$\ge \frac{u(t + h_m, x'_m) - u(t, x'_m)}{h_m} - h_m$$

$$\to \lim_{m \to \infty} (\partial_t u)(t, x'_m),$$

as $m \to \infty$, which proves the inequality in the opposite direction of (2.8). Hence, the assertion (2.4) of (a) together with (2.3) is proved for non-negative u. For general functions u, the analogous argument also works well by replacing u with $-u$ at which u takes the negative value. Therefore we complete the proof. $\square$

Lemma 2.3. *Let*

$$\mathcal{A} := \left\{ g \in L^\infty(\mathbb{R}^d) \mid \|g\|_{L^\infty} = 1, \ \operatorname{supp} \widehat{g} \subset \{\xi \in \mathbb{R}^d \mid 2^{-1} \le |\xi| \le 2\} \right\}.$$

There exists $c > 0$ such that for all $g \in \mathcal{A}$ and $\{x_n\}_{n=1}^\infty$ such that

$$\lim_{n \to \infty} g(x_n) \operatorname{sgn}(g(x_n)) = \|g\|_{L^\infty},$$

we have

$$\lim_{n \to \infty} \left(\Lambda g(x_n) \right) \operatorname{sgn}(g(x_n)) \ge c. \tag{2.9}$$

Proof. If the statement of the above lemma is not true, then there exist $\{g_n\}_{n=1}^\infty$ and $\{x_n\}_{n=1}^\infty$ such that

$$g_n(x_n) \ge 1 - \frac{1}{n}, \quad \limsup_{n \to \infty} (\Lambda g_n)(x_n) \le 0, \tag{2.10}$$

noting that it suffices to consider $-g_n$ instead of g_n if g_n can be negative. Let

$$\tilde{g}_n(x) := g_n(x + x_n).$$

Noting that $\tilde{g}_n(0) \to 1$ as $n \to \infty$ and $\|\nabla g_n\|_{L^\infty} \le C \|g_n\|_{L^\infty} = C$, by taking a subsequence we see that there exists $g \in L^\infty(\mathbb{R}^d)$ with $g(0) = 1$ such that $\tilde{g}_n(x) \to g(x)$ as $n \to \infty$ uniformly with respect to x in each bounded set of $\mathbb{R}^d$ by the Ascoli-Arzelá theorem and $g \in C^\infty(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, since $\widehat{g}_n$ is supported in the bounded set $\{\xi \in \mathbb{R}^d \mid 2^{-1} \le |\xi| \le 2\}$ and so is g. This g also satisfies $g \not\equiv 1$, since $1 = g(0) = \left((\phi_{-1} + \phi_0 + \phi_1) * g \right)(0)$ but the constant function 1 satisfies $(\phi_{-1} + \phi_0 + \phi_1) * 1 = 0$. Here we recall the following formula:

$$\Lambda g(x) = C_d \int_{\mathbb{R}^d} \frac{2g(x) - g(x + y) - g(x - y)}{|x - y|^{d+1}} \, dy, \tag{2.11}$$

where C_d is a positive constant depending on the dimensions (see e.g. Lemma 3.2 in [17]). The above formula yields that

$$\operatorname{sgn}\left(g(0) \right) \Lambda g(0) = \operatorname{sgn}\left(g(0) \right) C_d \int_{\mathbb{R}^d} \frac{2g(0) - g(y) - g(-y)}{|y|^{d+1}} \, dy > 0,$$

since g is not a constant function. Here we can show that

$$\lim_{n \to \infty} \int_{\mathbb{R}^d} \frac{2\tilde{g}_n(0) - \tilde{g}_n(y) - \tilde{g}_n(-y)}{|y|^{d+1}} \, dy = \int_{\mathbb{R}^d} \frac{2\tilde{g}(0) - \tilde{g}(y) - \tilde{g}(-y)}{|y|^{d+1}} \, dy.$$

In fact, for any $\delta > 0$, it follows from the dominated convergence theorem that

$$\lim_{n\to\infty} \int\limits_{|y|>\delta} \frac{2\tilde{g}_n(0) - \tilde{g}_n(y) - \tilde{g}_n(-y)}{|y|^{d+1}} \, dy = \int\limits_{|y|>\delta} \frac{2\tilde{g}(0) - \tilde{g}(y) - \tilde{g}(-y)}{|y|^{d+1}} \, dy,$$

while the uniform boundedness of $\|\nabla^2 g_n\|_{L^\infty}$ with respect to n gives that

$$\left| \int\limits_{|y|<\delta} \frac{2\tilde{g}_n(0) - \tilde{g}_n(y) - \tilde{g}_n(-y)}{|y|^{d+1}} \, dy \right| \leq \|\nabla^2 g_n\|_{L^\infty} \int\limits_{|y|<\delta} \frac{|y|^2}{|y|^{d+1}} dy$$

$$\leq C\|g_n\|_{L^\infty}\delta = C\delta.$$

Therefore, by the assumption (2.10), we find that

$$0 < \operatorname{sgn}\big(g(0)\big) \Lambda g(0) \leq \operatorname{sgn}\big(g(0)\big) \lim_{n\to\infty} C_d \int\limits_{\mathbb{R}^d} \frac{2\tilde{g}_n(0) - \tilde{g}_n(y) - \tilde{g}_n(-y)}{|y|^{d+1}} \, dy$$

$$= \operatorname{sgn}\big(g(0)\big) \liminf_{n\to\infty} \Lambda g_n(x_n)$$

$$\leq 0,$$

which is a contradiction. □

We are ready to prove Proposition 2.1.

Proof of Proposition 2.1. The inequality (2.1) is an immediate consequence of Lemmas 2.2, 2.3 with the scaling by 2^j and the fact that for any $\{x_{t,n}\}_{n=1}^\infty$ satisfying

$$\|u(t)\|_{L^\infty} = \lim_{n\to\infty} u(t, x_{t,n}) \operatorname{sgn}\big(u(t, x_{t,n})\big),$$

its gradient must satisfy

$$\lim_{n\to\infty} \nabla u(t, x_{t,n}) = 0.$$

It is also readily to show (2.2), since the same argument works well for $\partial_t(\psi * u)$ by the non-negativity $0 \leq \liminf_{n\to\infty} \Lambda \psi * u(t, x_{t,n}) \operatorname{sgn}(\psi * u(t, x_{t,n}))$ for $\{x_{t,n}\}_{n=1}^\infty$ satisfying $|\psi * u(t, x_{t,n})| \to \|\psi * u(t)\|_{L^\infty}$ as $n \to \infty$, which is seen from (2.11). □

The following is the Fourier multiplier theorem for the propagator defined with the Poisson kernel.

Lemma 2.4. (see e.g. [24,31]) *There exist $C > 0$, $0 < c < 1$ independent of u_0 such that*

$$ce^{-Ct2^j}\|\phi_j * u_0\|_{L^\infty} \leq \|\phi_j * (e^{-t\Lambda}u_0)\|_{L^\infty} \leq Ce^{-ct2^j}\|\phi_j * u_0\|_{L^\infty} \tag{2.12}$$

for all $j \in \mathbb{Z}$ and $u_0 \in B^0_{\infty,\infty}(\mathbb{R}^d)$.

Next lemma is concerned with the continuity of the linear solution.

Lemma 2.5. (i) *Let $u_0 \in B^0_{\infty,\infty}(\mathbb{R}^d)$. Then*

$$\lim_{t\to 0} e^{-t\Lambda}u_0 = u_0 \text{ in } B^0_{\infty,\infty}(\mathbb{R}^d) \quad \text{if and only if} \quad \lim_{j\to\infty} \|\phi_j * u_0\|_{L^\infty} = 0.$$

(ii) *Let $u_0 \in B^0_{\infty,\infty}(\mathbb{R}^d)$ be such that $\|\phi_j * u_0\|_{L^\infty} \to 0$ as $j \to \infty$. Then*

$$\lim_{T\to 0} \|e^{-t\Lambda}u_0\|_{\widetilde{L}^1(0,T;B^1_{\infty,\infty})} = 0. \tag{2.13}$$

Proof. Before proving, let us recall the proof of Proposition 2.4 (i) from [24], which would be helpful for the readers to understand details of the proof below.

We prove (i) first. Since for any $\xi \in \mathbb{R}^d$ with $2^{j-1} \leq |\xi| \leq 2^{j+1}$

$$
|e^{-t|\xi|} - 1| \simeq \begin{cases} 1 & \text{if } t2^j \geq 1, \\ t2^j & \text{if } t2^j \leq 1, \end{cases}
$$

we have from the Fourier multiplier theorem that

$$
\begin{aligned}
C^{-1} \min\{1, t2^j\} \|\phi_j * u_0\|_{L^\infty} &\leq \|\phi_j * \big((e^{-t\Lambda} - 1)u_0\big)\|_{L^\infty} \\
&\leq C \min\{1, t2^j\} \|\phi_j * u_0\|_{L^\infty}.
\end{aligned}
$$

The above first inequality implies the high frequency part of u_0 vanishing by the time continuity of $e^{-t\Lambda} u_0$, and the second one yields the time continuity under the condition that high frequency of u_0 vanishes.

Let us turn to prove (ii). For $j_0 \in \mathbb{N}$, it follows from (2.12) that

$$
\begin{aligned}
\|e^{-t\Lambda} u_0\|_{\widetilde{L}^1(0,T;B^1_{\infty,\infty})} &\leq T\Big(\|\psi * u_0\|_{L^\infty} + \sup_{1 \leq j \leq j_0} 2^j \|\phi_j * u_0\|_{L^\infty}\Big) \\
&\quad + C \sup_{j > j_0} \|e^{-ct2^j}\|_{L^1(0,T)} 2^j \|\phi_j * u_0\|_{L^\infty} \\
&\leq T 2^{j_0} \|u_0\|_{B^0_{\infty,\infty}} + C \sup_{j > j_0} \|\phi_j * u_0\|_{L^\infty}.
\end{aligned}
$$

By taking j_0 large and choosing T sufficiently small $T \ll 2^{-j_0} \|u_0\|_{B^0_{\infty,\infty}}$, we get (2.13). □

3. Local solvability and analyticity for Burgers equation

In this section, we prove the local in time solvability and analyticity in Theorem 1.1. Only in this section, let $\{\phi_j\}_{j=0}^\infty$ be the Littlewood Paley dyadic decomposition for inhomogeneous spaces, namely, ϕ_0 is taken as $\phi_0 = \psi$, where ψ satisfy (1.3). Put $S_j := \sum_{k=0}^j \phi_k *$ for $j = 0, 1, 2, \cdots$ and $S_j = 0$ for $j = -1, -2, \cdots$ for the sake of simplicity. Consider a sequence $\{u_n\}_{n=1}^\infty$ such that

$$
\begin{cases}
u_1 = e^{-t\Lambda} S_1 u_0, \\
\partial_t u_{n+1} + \Lambda u_{n+1} + \sum_{l \geq 0} (S_{l-3}u_n) \partial_x \phi_l * u_{n+1} \\
\quad = -\sum_{l \geq 0} \Big(\sum_{k \geq l+3} \phi_k * u_n\Big) \partial_x \phi_l * u_n - \frac{1}{2}\partial_x \sum_{|l-k| \leq 2} (\phi_k * u_n)(\phi_l * u_n), \\
u_n(0) = S_n u_0.
\end{cases}
\tag{3.1}
$$

Existence of u_{n+1} for given u_n is assured by smoothness of the initial data $S_n u_0$, and we need to obtain a priori estimate. It follows from the boundedness of $e^{-t\Lambda}$ and the maximal regularity estimate in $B^0_{\infty,\infty}(\mathbb{R})$ that

$$
\|u_1\|_{\widetilde{L}^\infty([0,T];B^0_{\infty,\infty}) \cap \widetilde{L}^1(0,T;B^1_{\infty,\infty})} + \|\partial_t u_1\|_{\widetilde{L}^1(0,T;B^0_{\infty,\infty})} \leq C\|u_0\|_{B^0_{\infty,\infty}}.
$$

We need to estimate u_n $(n = 2, 3, \cdots)$ and the difference $u_{m+1} - u_{n+1}$. For this purpose, we prepare the following lemma.

Lemma 3.1. *Let $T > 0$, $1 \leq q, r \leq \infty$, $s > -1$ and $0 < \delta < 1/2$. Then there exist positive constant C, c such that the following three inequalities hold.*

$$
\begin{aligned}
\|u_{n+1}\|_{\widetilde{L}^\infty(0,T;B^s_{\infty,q})} &\leq \|u_0\|_{B^s_{\infty,q}} + C\|u_n\|_{\widetilde{L}^2(0,T;B^{\frac{1}{2}}_{\infty,\infty})} \|u_{n+1}\|_{\widetilde{L}^2(0,T;B^{\frac{1}{2}+s}_{\infty,q})} \\
&\quad + C\|u_n\|_{\widetilde{L}^2(0,T;B^{\frac{1}{2}}_{\infty,\infty})} \|u_n\|_{\widetilde{L}^2(0,T;B^{\frac{1}{2}+s}_{\infty,q})},
\end{aligned}
\tag{3.2}
$$

$$
\begin{aligned}
\|u_{n+1}\|_{\widetilde{L}^r(0,T;B^{\frac{1}{r}+s}_{\infty,q})} &\leq \|e^{-tc\Lambda} u_0\|_{\widetilde{L}^r(0,T;B^{\frac{1}{r}+s}_{\infty,q})} + C\|u_n\|_{\widetilde{L}^2(0,T;B^{\frac{1}{2}}_{\infty,\infty})} \|u_{n+1}\|_{\widetilde{L}^2(0,T;B^{\frac{1}{2}+s}_{\infty,q})} \\
&\quad + C\|u_n\|_{\widetilde{L}^2(0,T;B^{\frac{1}{2}}_{\infty,\infty})} \|u_n\|_{\widetilde{L}^2(0,T;B^{\frac{1}{2}+s}_{\infty,q})},
\end{aligned}
\tag{3.3}
$$

$$
\begin{aligned}
\|u_{n+1} - u_{m+1}\|_{\widetilde{L}^2(0,T;B_{\infty,q}^{-\delta})} \leq & C \|e^{-tc\Lambda}(S_{n+1}u_0 - S_{m+1}u_0)\|_{\widetilde{L}^2(0,T;B_{\infty,q}^{-\delta})} \\
& + C \|u_n\|_{\widetilde{L}^2(0,T;B_{\infty,\infty}^{\frac{1}{2}})} \|u_{n+1} - u_{m+1}\|_{\widetilde{L}^2(0,T;B_{\infty,q}^{-\delta})} \\
& + C \big(\|u_n\|_{\widetilde{L}^2(0,T;B_{\infty,\infty}^{\frac{1}{2}})} + \|u_m\|_{\widetilde{L}^2(0,T;B_{\infty,\infty}^{\frac{1}{2}})} \\
& + \|u_{m+1}\|_{\widetilde{L}^2(0,T;B_{\infty,\infty}^{\frac{1}{2}})} \big) \|u_n - u_m\|_{\widetilde{L}^2(0,T;B_{\infty,q}^{-\delta})}.
\end{aligned}
\tag{3.4}
$$

Remark. The above lemma is concerned with the inhomogeneous Besov spaces, so that the constant C depends on the time $T > 0$. On the other hand, we can also consider the homogeneous Besov spaces. In that case, the constant C is independent of T.

Proof. First we show the inequality (3.2). For $j = 5, 6, 7, \cdots$, it follows from the recurrence relation (3.1) that

$$
\begin{aligned}
& \partial_t \phi_j * u_{n+1} + \Lambda \phi_j * u_{n+1} + (S_{j-3}u_n)\partial_x \phi_j * u_{n+1} \\
= & (S_{j-3}u_n)\partial_x \phi_j * u_{n+1} - \phi_j * \sum_{l \geq 0} (S_{l-3}u_n)\partial_x \phi_l * u_{n+1} \\
& - \phi_j * \sum_{l \geq 0} \Big(\sum_{k \geq l+3} \phi_k * u_n \Big) \partial_x \phi_l * u_n - \frac{1}{2}\phi_j * \partial_x \sum_{|k-l| \leq 2} (\phi_k * u_n)(\phi_l * u_n).
\end{aligned}
\tag{3.5}
$$

By Proposition 2.1, we get that

$$
\begin{aligned}
& \partial_t \|\phi_j * u_{n+1}\|_{L^\infty} + c2^j \|\phi_j * u_{n+1}\|_{L^\infty} \\
\leq & \Big\| (S_{j-3}u_n)\partial_x \phi_j * u_{n+1} - \phi_j * \sum_{l \geq 0} (S_{l-3}u_n)\partial_x \phi_l * u_{n+1} \Big\|_{L^\infty} \\
& + \Big\| \phi_j * \sum_{l \in \mathbb{Z}} \Big(\sum_{k \geq l+3} \phi_k * u_n \Big) \partial_x \phi_l * u_n \Big\|_{L^\infty} + \frac{1}{2} \Big\| \phi_j * \partial_x \sum_{|k-l| \leq 2} (\phi_k * u_n)(\phi_l * u_n) \Big\|_{L^\infty}.
\end{aligned}
$$

Noting that the left member is $e^{-tc2^j}\partial_t(e^{tc2^j}\|\phi_j * u_{n+1}\|_{L^\infty})$, multiplying by e^{tc2^j} and integrating in both sides, we have that

$$
\|\phi_j * u_{n+1}(t)\|_{L^\infty} \leq e^{-tc2^j} \|\phi_j * u_0\|_{L^\infty} + \mathrm{I}(t) + \mathrm{II}(t) + \mathrm{III}(t),
\tag{3.6}
$$

where

$$
\mathrm{I}(t) := \int_0^t e^{-(t-\tau)c2^j} \Big\| (S_{j-3}u_n)\partial_x \phi_j * u_{n+1} - \phi_j * \sum_{l \geq 0} (S_{l-3}u_n)\partial_x \phi_l * u_{n+1} \Big\|_{L^\infty} d\tau,
$$

$$
\mathrm{II}(t) := \int_0^t e^{-(t-\tau)c2^j} \Big\| \phi_j * \sum_{l \in \mathbb{Z}} \Big(\sum_{k \geq l+3} \phi_k * u_n \Big) \partial_x \phi_l * u_n \Big\|_{L^\infty} d\tau,
$$

$$
\mathrm{III}(t) := \int_0^t e^{-(t-\tau)c2^j} \frac{1}{2} \Big\| \phi_j * \partial_x \sum_{|k-l| \leq 2} (\phi_k * u_n)(\phi_l * u_n) \Big\|_{L^\infty} d\tau.
$$

We estimate the above three by the use of the Fourier multiplier theorem and the Hölder inequality. The first term I is estimated with a kind of commutator estimates (see e.g. (3.4) and (3.5) in [24]) as

$$
\begin{aligned}
\mathrm{I}(t) \leq &\, C \int_0^t \|\partial_x S_{j-3} u_n\|_{L^\infty} \sum_{\mu=-3}^{3} \|\phi_{j+\mu} * u_{n+1}\|_{L^\infty} d\tau \\
\leq &\, C \int_0^t \sum_{k \leq j-3} 2^k \|\phi_k * u_n\|_{L^\infty} \sum_{\mu=-3}^{3} \|\phi_{j+\mu} * u_{n+1}\|_{L^\infty} d\tau \\
\leq &\, C \sum_{k \leq j-3} 2^k \|\phi_k * u_n\|_{L^2(0,T;L^\infty)} \sum_{\mu=-3}^{3} \|\phi_{j+\mu} * u_{n+1}\|_{L^2(0,T;L^\infty)} \\
\leq &\, C \|u_n\|_{\widetilde{L}^2(0,T;B^{\frac{1}{2}}_{\infty,\infty})} \sum_{k \leq j-3} 2^{\frac{1}{2}k} \sum_{\mu=-3}^{3} \|\phi_{j+\mu} * u_{n+1}\|_{L^2(0,T;L^\infty)}.
\end{aligned}
$$

Above, we notice that there is a difference between S_{j-3} and S_{j-8} from [24], but the finite dyadic numbers of frequency component does not disturb the estimates, and we can estimate similarly. We multiply by 2^{sj} and take the sequence norm $\ell^q(\mathbb{Z})$ after the supremum for t to get

$$
\left\{ \sum_{j \geq 5} \left(2^{sj} \sup_{t \in (0,T)} \mathrm{I}(t) \right)^q \right\}^{\frac{1}{q}} \leq C \|u_n\|_{\widetilde{L}^2(0,T;B^{\frac{1}{2}}_{\infty,\infty})} \|u_{n+1}\|_{\widetilde{L}^2(0,T;B^{\frac{1}{2}+s}_{\infty,q})}.
$$

Since $\phi_j * \sum_{l \geq 0} \left(\sum_{k \geq l+3} \phi_k * u_n \right) \partial_x \phi_l * u_n = \phi_j * \left(\sum_{k=j-3}^{j+3} (\phi_k * u_n) \sum_{l \leq k-3} \partial_x \phi_l * u_n \right)$, the second term II is also handled in the similar way to the first one $\mathrm{I}(t)$:

$$
\left\{ \sum_{j \geq 5} \left(2^{sj} \sup_{t \in (0,T)} \mathrm{II}(t) \right)^q \right\}^{\frac{1}{q}} \leq C \|u_n\|_{\widetilde{L}^2(0,T;B^{\frac{1}{2}}_{\infty,\infty})} \|u_n\|_{\widetilde{L}^2(0,T;B^{\frac{1}{2}+s}_{\infty,q})}.
$$

As to the third term III, we also apply the Fourier multiplier theorem and the Hölder inequality to get that

$$
\begin{aligned}
\mathrm{III}(t) \leq &\, C \sum_{k \geq j-5} \sum_{l=k-2}^{k+2} \int_0^t 2^j \|\phi_k * u_n\|_{L^\infty} \|\phi_l * u_n\|_{L^\infty} d\tau \\
\leq &\, C \sum_{k \geq j-5} 2^{-(k-j)} 2^k \sum_{l=k-2}^{k+2} \|\phi_k * u_n\|_{L^2(0,T;L^\infty)} \|\phi_l * u_n\|_{L^2(0,T;L^\infty)} \\
\leq &\, C \sum_{m \geq -5} 2^{-m} \cdot 2^{j+m} \sum_{\mu=-2}^{2} \|\phi_{j+m} * u_n\|_{L^2(0,T;L^\infty)} \|\phi_{j+m+\mu} * u_n\|_{L^2(0,T;L^\infty)}.
\end{aligned}
$$

By multiplying by 2^{sj} and taking the sequence norm of $\ell^q(\mathbb{Z})$ after the supremum for t, we obtain that

$$
\begin{aligned}
&\left\{ \sum_{j \geq 5} \left(2^{sj} \sup_{t \in (0,T)} \mathrm{III}(t) \right)^q \right\}^{\frac{1}{q}} \\
\leq &\, C \sum_{m \geq -5} 2^{-(1+s)m} \left\{ \sum_{j \geq 5} \left(2^{(1+s)(j+m)} \sum_{\mu=-2}^{2} \|\phi_{j+m} * u_n\|_{L^2(0,T;L^\infty)} \|\phi_{j+m+\mu} * u_n\|_{L^2(0,T;L^\infty)} \right)^q \right\}^{\frac{1}{q}} \\
\leq &\, C \|u_n\|_{\widetilde{L}^2(0,T;B^{\frac{1}{2}}_{\infty,\infty})} \|u_n\|_{\widetilde{L}^2(0,T;B^{\frac{1}{2}+s}_{\infty,q})}.
\end{aligned}
$$

By the above estimates for I, II, III and the inequality (3.6), we obtain the inequality (3.2) for the frequency away from the origin. As to the frequency around the origin, we apply (2.1), (2.2) to (3.5) and integrate to get that for $j = 0, 1, 2, 3, 4$

$$\|\phi_j * u_{n+1}\|_{L^\infty} \leq \|\phi_j * u_0\|_{L^\infty} + \int_0^t \Big(\Big\| \phi_j * \sum_{l \geq 0} (S_{l-3}u_n) \partial_x \phi_l * u_{n+1} \Big\|_{L^\infty}$$

$$+ \Big\| \phi_j * \sum_{l \geq 0} \Big(\sum_{k \geq l+3} \phi_k * u_n \Big) \partial_x \phi_l * u_n \Big\|_{L^\infty}$$

$$+ \frac{1}{2} \Big\| \phi_j * \partial_x \sum_{|k-l| \leq 2} (\phi_k * u_n)(\phi_l * u_n) \Big\|_{L^\infty} \Big) d\tau$$

$$\leq \|\phi_j * u_0\|_{L^\infty} + C \int_0^t \Big(\sum_{l=0}^5 \sum_{k=0}^2 \|\phi_k * u_n\|_{L^\infty} \|\phi_l * u_{n+1}\|_{L^\infty}$$

$$+ \sum_{k=0}^5 \sum_{l=0}^2 \|\phi_k * u_n\|_{L^\infty} \|\phi_l * u_n\|_{L^\infty}$$

$$+ \sum_{|k-l| \leq 2} 2^{-(1+s)k} \cdot 2^{\frac{1}{2}k} \|\phi_k * u_n\|_{L^\infty} 2^{(\frac{1}{2}+s)l} \|\phi_l * u_n)\|_{L^\infty} \Big) d\tau. \tag{3.7}$$

Hence, we obtain that

$$\Big\{ \sum_{j=0}^4 \Big(2^{sj} \|\phi_j * u_{n+1}\|_{L^\infty(0,T;L^\infty)} \Big)^q \Big\}^{\frac{1}{q}} \leq \|u_0\|_{B_{\infty,q}^s} + C \|u_n\|_{\widetilde{L}^2(0,T;B_{\infty,\infty}^{\frac{1}{2}})} \|u_n\|_{\widetilde{L}^2(0,T;B_{\infty,q}^{\frac{1}{2}+s})}$$

$$+ C \|u_n\|_{\widetilde{L}^2(0,T;B_{\infty,\infty}^{\frac{1}{2}})} \|u_n\|_{\widetilde{L}^2(0,T;B_{\infty,q}^{\frac{1}{2}+s})},$$

which prove the estimate of (3.2) for low frequency part. We complete the proof of (3.2).

We next prove the estimate (3.3). By taking $L^r(0,T)$ norm for the estimate (3.6), applying the inequality (2.12) to the first term in the right member of (3.6) and order exchanging of integration for integrals of I, II and III, we have that for $j = 5, 6, \cdots$

$$\|\phi_j * u_{n+1}(t)\|_{L^r(0,T;L^\infty)} \leq C \Big(\|\phi_j * e^{-tc\Lambda} u_0\|_{\widetilde{L}^r(0,T;L^\infty)} + 2^{-\frac{1}{r}j} \big(\widetilde{I}(T) + \widetilde{II}(T) + \widetilde{III}(T) \big) \Big), \tag{3.8}$$

where $\widetilde{I}, \widetilde{II}, \widetilde{III}$ are similar to I, II, III such that $e^{-(t-\tau)c2^j}$ is replaced with 1. By multiplying this inequality by $2^{(\frac{1}{r}+s)j}$, and taking the sequence norm of $\ell^q(\mathbb{Z})$, the same argument as before enables us to get the estimate of high frequency part of (3.3). As to the low frequency, we can also apply the argument (3.7) to obtain the required estimate.

Let us prove the last estimate (3.4). By the recurrence relation (3.1), we write

$$\partial_t(u_{n+1} - u_{m+1}) + \Lambda(u_{n+1} - u_{m+1}) + \sum_{l \in \mathbb{Z}} (S_{l-3}u_n) \partial_x \phi_l * (u_{n+1} - u_{m+1})$$

$$= - \sum_{l \in \mathbb{Z}} \big(S_{l-3}(u_n - u_m) \big) \partial_x \phi_l * u_{m+1}$$

$$- \sum_{l \in \mathbb{Z}} \Big(\sum_{k \geq l+3} \phi_k * (u_n - u_m) \Big) \partial_x \phi_l * u_n - \sum_{l \in \mathbb{Z}} \Big(\sum_{k \geq l+3} \phi_k * u_m \Big) \partial_x \phi_l * (u_n - u_m)$$

$$- \frac{1}{2} \partial_x \sum_{|l-k| \leq 2} (\phi_k * (u_n - u_m))(\phi_j * u_n) - \frac{1}{2} \partial_x \sum_{|l-k| \leq 2} (\phi_k * u_m)(\phi_j * (u_n - u_m)).$$

The similar arguments to the proof of (3.2), (3.3) and Step 3 in the proof of Theorem 1.3 in [34] are applicable, each terms can be handled analogously to the previous estimates, and we obtain the estimate (3.4). □

Proof of unique solvability in Theorem 1.1. First we derive a uniform boundedness of $\{u_n\}_{n=1}^\infty$ to construct a solution. We consider the estimates that

$$\|u_n\|_{\widetilde{L}^\infty(0,T;B_{\infty,\infty}^0)} \leq C_0 \|u_0\|_{B_{\infty,\infty}^0}, \quad \|u_n\|_{\widetilde{L}^2(0,T;B_{\infty,\infty}^{\frac{1}{2}})} \leq 2\varepsilon, \tag{3.9}$$

where the constant $C_0 \geq 2$ is larger than absolute constants appearing in the propositions and lemmas and ε will be fixed as a small constant.

When $n = 1$, (3.9) is possible to be obtained, since the first one is just the boundedness of $e^{-t\Lambda}$ and the second one for small T is assured by (2.13) and the interpolation inequality,

$$\|f\|_{\widetilde{L}^2(0,T;B_{\infty,\infty}^{\frac{1}{2}})} \leq \|f\|_{\widetilde{L}^\infty(0,T;B_{\infty,\infty}^0)}^{\frac{1}{2}} \|f\|_{\widetilde{L}^1(0,T;B_{\infty,\infty}^1)}^{\frac{1}{2}},$$

which is on the controllability of the norm of $\widetilde{L}^2(0, T; B_{\infty,\infty}^{\frac{1}{2}}(\mathbb{R}))$ by $\widetilde{L}^\infty(0, T; B_{\infty,\infty}^0(\mathbb{R})) \cap \widetilde{L}^1(0, T; B_{\infty,\infty}^1(\mathbb{R}))$. We also take T smaller such that the inequality $\|e^{-tc\Lambda}u_0\|_{\widetilde{L}^2(0,T;B_{\infty,\infty}^{\frac{1}{2}})} \leq \varepsilon$ holds, where $c > 0$ is a small constant appearing in (3.3).

We fix $\varepsilon > 0$ small enough such that $\varepsilon \ll (C_0)^{-1}$, which gives the bounds

$$\|u_n\|_{\widetilde{L}^2(0,T;B_{\infty,\infty}^{\frac{1}{2}})} \leq \frac{1}{16C_0}.$$

Let us consider the estimates for u_{n+1} under the assumption (3.9) for u_n. It follows from the inequality (3.2) for $s = 0$ and the assumption for u_n that

$$\|u_{n+1}\|_{\widetilde{L}^\infty(0,T;B_{\infty,\infty}^0)} \leq \|u_0\|_{B_{\infty,\infty}^0} + C_0 \frac{1}{16C_0}\|u_{n+1}\|_{\widetilde{L}^2(0,T;B_{\infty,\infty}^{\frac{1}{2}})} + C_0\frac{1}{16C_0}\|u_n\|_{\widetilde{L}^2(0,T;B_{\infty,\infty}^{\frac{1}{2}})},$$

$$\|u_{n+1}\|_{\widetilde{L}^\infty(0,T;B_{\infty,\infty}^0)} \leq 2\|u_0\|_{B_{\infty,\infty}^0}.$$

We also apply (3.3) for $s = 0$ to get that

$$\|u_{n+1}\|_{\widetilde{L}^2(0,T;B_{\infty,\infty}^{\frac{1}{2}})} \leq \varepsilon + C_0 \cdot \frac{1}{16C_0}\|u_{n+1}\|_{\widetilde{L}^2(0,T;B_{\infty,\infty}^{\frac{1}{2}})} + C_0 \cdot \frac{1}{16C_0}\|u_n\|_{\widetilde{L}^2(0,T;B_{\infty,\infty}^{\frac{1}{2}-s})},$$

$$\|u_{n+1}\|_{\widetilde{L}^2(0,T;B_{\infty,\infty}^{\frac{1}{2}})} \leq 2\varepsilon.$$

Hence, the uniform estimates (3.9) is proved.

We next consider the convergence of u_n in $\widetilde{L}^2(0, T; B_{\infty,\infty}^{-\delta}(\mathbb{R}))$. For fixed $0 < \delta < 1/2$, we have from the inequality (3.4) that

$$\|u_{n+1} - u_{m+1}\|_{\widetilde{L}^2(0,T;B_{\infty,\infty}^{-\delta})} \leq C_0\|e^{-tc\Lambda}(S_{n+1}u_0 - S_{m+1}u_0)\|_{\widetilde{L}^2(0,T;B_{\infty,\infty}^{-\delta})}$$

$$+ C_0 \cdot \frac{1}{16C_0}\|u_{n+1} - u_{m+1}\|_{\widetilde{L}^2(0,T;B_{\infty,\infty}^{-\delta})}$$

$$+ C_0 \cdot \frac{3}{16C_0}\|u_n - u_m\|_{\widetilde{L}^2(0,T;B_{\infty,\infty}^{-\delta})},$$

$$\frac{15}{16}\|u_{n+1} - u_{m+1}\|_{\widetilde{L}^2(0,T;B_{\infty,\infty}^{-\delta})} \leq C\|(S_{n+1} - S_{m+1})u_0\|_{B_{\infty,\infty}^{-\frac{1}{2}-\delta}}$$

$$+ \frac{3}{16}\|u_n - u_m\|_{\widetilde{L}^2(0,T;B_{\infty,\infty}^{-\delta})},$$

$$\|u_{n+1} - u_{m+1}\|_{\widetilde{L}^2(0,T;B_{\infty,\infty}^{-\delta})} \leq C2^{-\min\{n,m\}}\|u_0\|_{B_{\infty,\infty}^0} + \frac{1}{5}\|u_n - u_m\|_{\widetilde{L}^2(0,T;B_{\infty,\infty}^{-\delta})}.$$

The above inequality yields that

$$\|u_{n+1} - u_n\|_{\widetilde{L}^2(0,T;B_{\infty,\infty}^{-\delta})} \leq C2^{-n}\|u_0\|_{B_{\infty,\infty}^0},$$

which prove the existence of the following limit

$$u := \lim_{n\to\infty} u_n = \lim_{n\to\infty} u_1 + \sum_{k=1}^{n-1}(u_{k+1} - u_k) \quad \text{in } \widetilde{L}^2(0, T; B_{\infty,\infty}^{-\delta}(\mathbb{R})).$$

It can be checked that u also satisfies the same inequality as (3.9), and the standard limit argument ensures that u satisfies (1.1) in the sense of distribution. The uniqueness is proved by applying the inequality like (3.4) for the same initial data. $\square$

We next study the analyticity of solutions. Let us prepare a lemma and a proposition.

Proposition 3.2. *Let* $\alpha, \beta \in \mathbb{N} \cup \{0\}$ *and* $1 \le q \le \infty$. *Then there exists a positive constant* C_0 *independent of* α, β *such that for any* $u_0 \in B^0_{\infty,q}(\mathbb{R})$

$$\|t^{\alpha+\beta} \partial_t^\alpha \partial_x^\beta e^{-t\Lambda} u_0\|_{B^0_{\infty,q}} + \|t^{\alpha+\beta} \partial_t^\alpha \partial_x^\beta e^{-t\Lambda} u_0\|_{\widetilde{L}^1(0,T;B^1_{\infty,q})} \le C_0^{\alpha+\beta} (\alpha+\beta)! \, \|u_0\|_{B^0_{\infty,q}}. \tag{3.10}$$

Proof. It follows from $\partial_t e^{-t\Lambda} = -\Lambda e^{-t\Lambda}$ that

$$\|\partial_t^\alpha \partial_x^\beta e^{-t\Lambda} u_0\|_{B^0_{\infty,q}} = \|\Lambda^\alpha \partial_x^\beta e^{-t\Lambda} u_0\|_{B^0_{\infty,q}} = \|(\Lambda e^{-\frac{t}{\alpha+\beta}\Lambda})^\alpha (\partial_x e^{-\frac{t}{\alpha+\beta}\Lambda})^\beta u_0\|_{B^0_{\infty,q}}.$$

The smoothing effect of $e^{-t\Lambda}$ implies that

$$\|\partial_t^\alpha \partial_x^\beta e^{-t\Lambda} u_0\|_{B^0_{\infty,q}} \le C^{\alpha+\beta} \Big(\frac{\alpha+\beta}{t}\Big)^{\alpha+\beta} \|u_0\|_{B^0_{\infty,q}}.$$

The above inequality and Stirling's approximation yield the desired estimate of the first term in the left member of (3.10). As for the second term, we also have from the above estimate and the maximal regularity estimate in the Chemin-Lerner spaces that

$$
\begin{aligned}
\|t^{\alpha+\beta} \partial_t^\alpha \partial_x^\beta e^{-t\Lambda} u_0\|_{\widetilde{L}^1(0,T;B^1_{\infty,q})} &= \|(\Lambda e^{-\frac{t}{2(\alpha+\beta)}\Lambda})^\alpha (\partial_x e^{-\frac{t}{2(\alpha+\beta)}\Lambda})^\beta e^{-\frac{t}{2}\Lambda} u_0\|_{\widetilde{L}^1(0,T;B^1_{\infty,q})} \\
&\le C^{\alpha+\beta} (\alpha+\beta)^{\alpha+\beta} \|e^{-\frac{t}{2}\Lambda} u_0\|_{\widetilde{L}^1(0,T;B^1_{\infty,q})} \\
&\le C C^{\alpha+\beta} (\alpha+\beta)^{\alpha+\beta} \|u_0\|_{B^0_{\infty,q}}.
\end{aligned}
$$

Therefore we obtain the inequality (3.10). $\square$

Based on the proof of Lemma 3.1, we show the following nonlinear estimates to obtain the analyticity.

Proposition 3.3. *Let* $\partial_{t,x}^\alpha = \partial_t^{\alpha_0} \partial_x^{\alpha_1}$, $\alpha! := \alpha_0! \alpha_1!$ *and* $|\alpha| = \alpha_0 + \alpha_1$ *for* $\alpha = (\alpha_0, \alpha_1) \in (\mathbb{N} \cup \{0\})^2$. *Assume that* $|\alpha| \ge 1$. *Then there exists a positive constant* C *independent of* α *such that for any* $1 \le q \le \infty$

$$
\begin{aligned}
&\|t^{|\alpha|} \partial_{t,x}^\alpha u_{n+1}\|_{\widetilde{L}^\infty(0,T;B^0_{\infty,q}) \cap \widetilde{L}^1(0,T;B^1_{\infty,q})} \\
&\le C|\alpha| \|t^{|\alpha|-1} \partial_{t,x}^\alpha u_{n+1}\|_{\widetilde{L}^1(0,T;B^0_{\infty,q})} \\
&\quad + C\|u_n\|_{\widetilde{L}^1(0,T;B^1_{\infty,1})} \|t^{|\alpha|} \partial_{t,x}^\alpha u_{n+1}\|_{\widetilde{L}^\infty(0,T;B^0_{\infty,q})} \\
&\quad + C \sum_{\beta \ne 0, \, \beta+\gamma=\alpha} \frac{\alpha!}{\beta! \gamma!} \|t^{|\beta|} \partial_{t,x}^\beta u_n\|_{\widetilde{L}^\infty(0,T;B^0_{\infty,1})} \|t^{|\gamma|} \partial_{t,x}^\gamma u_{n+1}\|_{\widetilde{L}^1(0,T;B^1_{\infty,q})} \\
&\quad + C \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta! \gamma!} \|t^{|\beta|} \partial_{t,x}^\beta u_n\|_{\widetilde{L}^2(0,T;B^{\frac{1}{2}}_{\infty,\infty})} \|t^{|\gamma|} \partial_{t,x}^\gamma u_n\|_{\widetilde{L}^2(0,T;B^{\frac{1}{2}}_{\infty,q})},
\end{aligned}
\tag{3.11}
$$

Proof. By applying $\partial_{t,x}^\alpha$ to the recurrence relation (3.1) and the similar argument as in (3.5), we write

$$\partial_t \phi_j * (\partial_{t,x}^\alpha u_{n+1}) + \Lambda \phi_j * (\partial_{t,x}^\alpha u_{n+1}) + (S_{j-3}u_n)\partial_x \phi_j * (\partial_{t,x}^\alpha u_{n+1})$$

$$= (S_{j-3}u_n)\partial_x \phi_j * (\partial_{t,x}^\alpha u_{n+1}) - \phi_j * \sum_{l \in \mathbb{Z}} (S_{l-3}u_n)\partial_x \phi_l * (\partial_{t,x}^\alpha u_{n+1})$$

$$+ \phi_j * \sum_{l \in \mathbb{Z}} (S_{j-3}u_n)\partial_x \phi_l * (\partial_{t,x}^\alpha u_{n+1}) - \partial_{t,x}^\alpha \phi_j * \sum_{l \in \mathbb{Z}} (S_{j-3}u_n)\partial_x \phi_l * u_{n+1}$$

$$- \partial_{t,x}^\alpha \phi_j * \sum_{l \in \mathbb{Z}} \Big(\sum_{k \geq l+3} \phi_k * u_n \Big) \partial_x \phi_l * u_n - \partial_{t,x}^\alpha \frac{1}{2} \phi_j * \partial_x \sum_{|k-l| \leq 2} (\phi_k * u_n)(\phi_l * u_n).$$

It follows from Proposition 2.1 and the Leibniz rule that for $j = 5, 6, \cdots$

$$\partial_t \|\phi_j * (\partial_{t,x}^\alpha u_{n+1})\|_{L^\infty} + c2^j \|\phi_j * (\partial_{t,x}^\alpha u_{n+1})\|_{L^\infty}$$

$$\leq \Big\| (S_{j-3}u_n)\partial_x \phi_j * (\partial_{t,x}^\alpha u_{n+1}) - \phi_j * \sum_{l \in \mathbb{Z}} (S_{l-3}u_n)\partial_x \phi_l * (\partial_{t,x}^\alpha u_{n+1}) \Big\|_{L^\infty}$$

$$+ \sum_{\beta \neq 0,\ \beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \Big\| \phi_j * \sum_{l \in \mathbb{Z}} (S_{l-3}\partial_{t,x}^\beta u_n)\partial_x \phi_l * (\partial_{t,x}^\gamma u_{n+1}) \Big\|_{L^\infty}$$

$$+ \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \Big\| \phi_j * \sum_{l \in \mathbb{Z}} \Big(\sum_{k \geq l+3} \phi_k * \partial_{t,x}^\beta u_n \Big) \partial_x \phi_l * (\partial_{t,x}^\gamma u_n) \Big\|_{L^\infty}$$

$$+ \frac{1}{2} \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \Big\| \phi_j * \partial_x \sum_{|k-l| \leq 2} (\phi_k * \partial_{t,x}^\beta u_n)(\phi_l * \partial_{t,x}^\gamma u_n) \Big\|_{L^\infty}.$$

Multiplying the above inequality by $t^{|\alpha|}$ and noting that $t^{|\alpha|}\partial_t f = \partial_t(t^{|\alpha|}f) - |\alpha|t^{|\alpha|-1}f$ with $f = \|\phi_j * (\partial_{t,x}^\alpha u_{n+1})\|_{L^\infty}$ and $t^{|\alpha|} = t^{|\beta|}t^{|\gamma|}$, we have that

$$\partial_t \|\phi_j * (t^{|\alpha|}\partial_{t,x}^\alpha u_{n+1})\|_{L^\infty} + c2^j \|\phi_j * (t^{|\alpha|}\partial_{t,x}^\alpha u_{n+1})\|_{L^\infty}$$

$$\leq |\alpha| \|\phi_j * (t^{|\alpha|-1}\partial_{t,x}^\alpha u_{n+1})\|_{L^\infty}$$

$$+ \Big\| (S_{j-3}u_n)\partial_x \phi_j * (t^{|\alpha|}\partial_{t,x}^\alpha u_{n+1}) - \phi_j * \sum_{l \in \mathbb{Z}} (S_{l-3}u_n)\partial_x \phi_l * (t^{|\alpha|}\partial_{t,x}^\alpha u_{n+1}) \Big\|_{L^\infty}$$

$$+ \sum_{\beta \neq 0,\ \beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \Big\| \phi_j * \sum_{l \in \mathbb{Z}} (S_{l-3}t^{|\beta|}\partial_{t,x}^\beta u_n)\partial_x \phi_l * (t^{|\gamma|}\partial_{t,x}^\gamma u_{n+1}) \Big\|_{L^\infty}$$

$$+ \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \Big\| \phi_j * \sum_{l \in \mathbb{Z}} \Big(\sum_{k \geq l+3} \phi_k * t^{|\beta|}\partial_{t,x}^\beta u_n \Big) \partial_x \phi_l * (t^{|\gamma|}\partial_{t,x}^\gamma u_n) \Big\|_{L^\infty}$$

$$+ \frac{1}{2} \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \Big\| \phi_j * \partial_x \sum_{|k-l| \leq 2} (\phi_k * t^{|\beta|}\partial_{t,x}^\beta u_n)(\phi_l * t^{|\gamma|}\partial_{t,x}^\gamma u_n) \Big\|_{L^\infty}.$$

By $t^{|\alpha|}\partial_{t,x}^\alpha u_{n+1} = 0$ for $t = 0$ and the analogous argument to the estimates of I, II, III and $\widetilde{\text{I}}, \widetilde{\text{II}}, \widetilde{\text{III}}$ appearing in (3.6) and (3.8), we obtain the desired inequality (3.11) for the frequency away from the origin. As for the frequency around the origin, we apply the similar inequality to (3.7) to get the required inequality. The proof is finished. $\quad\square$

Let us turn to prove the analyticity.

Proof of analyticity in Theorem 1.1. The solution u constructed in the previous proof of unique solvability satisfies $u(t) \in B_{\infty,1}^0(\mathbb{R})$ for almost every $t \in (0, T]$ since $\widetilde{L}^1(0, T; B_{\infty,\infty}^1(\mathbb{R}))$ is embedded to $L^1(0, T; B_{\infty,1}^0(\mathbb{R}))$. Hence, the problem is reduced to the one for initial data $u_0 \in B_{\infty,1}^0(\mathbb{R})$.

For initial data $u_0 \in B_{\infty,1}^0(\mathbb{R})$, we consider the sequence $\{u_n\}_{n=1}^\infty$ in (3.1) again to obtain the uniform boundedness in $\widetilde{L}^\infty(0, T; B_{\infty,1}^0(\mathbb{R})) \cap \widetilde{L}^1(0, T; B_{\infty,1}^1(\mathbb{R}))$ like (3.9):

$$\|u_n\|_{\widetilde{L}^{\infty}([0,T];B^0_{\infty,1})} \leq C_0 \|u_0\|_{B^0_{\infty,1}}, \quad \|u_n\|_{\widetilde{L}^1(0,T;B^1_{\infty,1})} \leq 2\varepsilon, \tag{3.12}$$

with small fixed constant $\varepsilon > 0$. By Lemma 3.1 and the similar argument to unique solvability, we obtain the uniform boundedness of $\{u_n\}_{n=1}^{\infty}$ in the space $\widetilde{L}^{\infty}(0,T;B^0_{\infty,q}(\mathbb{R})) \cap \widetilde{L}^1(0,T;B^1_{\infty,q}(\mathbb{R}))$ for $q = 1, \infty$. In addition, we also derive another boundedness for analyticity, namely, we will show that there exist $C_0, C_1 > 0$ such that

$$\|t^{|\alpha|} \partial^{\alpha}_{t,x} u_n\|_{\widetilde{L}^{\infty}(0,T;B^0_{\infty,1}) \cap \widetilde{L}^1(0,T;B^1_{\infty,1})} \leq \frac{C_0^{|\alpha|-1} C_1^{|\alpha|} \alpha!}{(1+|\alpha|)^4} \tag{3.13}$$

for any $n \in \mathbb{N}$, $\alpha = (\alpha_0, \alpha_1) \in (\mathbb{N} \cup \{0\})^2$, where $\partial^{\alpha}_{t,x} = \partial^{\alpha_0}_t \partial^{\alpha_1}_x$, C_0 and C_1 are fixed sufficiently large and depend on $\|u_0\|_{B^0_{\infty,1}}$.

We prove (3.13) by induction argument with respect to n and α. When $n = 1$, (3.13) is proved by Proposition 3.2. When $|\alpha| = 0$, the inequality (3.13) is true by (3.12). Here take $n' \in \mathbb{N}$ and $\alpha' \in (\mathbb{N} \cup \{0\})^2 \setminus \{(0,0)\}$ and let us assume that (3.13) is true for $u_{n'}$ with all $\alpha \in (\mathbb{N} \cup \{0\})^2$ and for $u_{n'+1}$ with all α such that $|\alpha| \leq |\alpha'| - 1$. We need to consider the estimate (3.13) for $u_{n'+1}$ with $\alpha = \alpha'$. It follows from the inequality (3.11) and the assumption of induction and (3.12) that

$$\begin{aligned}
&\|t^{|\alpha'|} \partial^{\alpha'}_{t,x} u_{n'+1}\|_{\widetilde{L}^{\infty}(0,T;B^0_{\infty,1}) \cap \widetilde{L}^1(0,T;B^1_{\infty,1})} \\
&\leq C|\alpha'| \|t^{|\alpha'|-1} \partial^{\alpha'}_{t,x} u_{n'+1}\|_{\widetilde{L}^1(0,T;B^0_{\infty,1})} + C \|u_{n'}\|_{\widetilde{L}^1(0,T;B^1_{\infty,1})} \|t^{|\alpha'|} \partial^{\alpha'}_{t,x} u_{n'+1}\|_{\widetilde{L}^{\infty}(0,T;B^0_{\infty,1})} \\
&\quad + C \sum_{\beta \neq 0, \beta+\gamma=\alpha'} \frac{\alpha'!}{\beta!\gamma!} \frac{C_0^{|\beta|-1} C_1^{|\beta|} \beta!}{(1+|\beta|)^4} \frac{C_0^{|\gamma|-1} C_1^{|\gamma|} \gamma!}{(1+|\gamma|)^4} \\
&\quad + C \sum_{\beta+\gamma=\alpha'} \frac{\alpha'!}{\beta!\gamma!} \frac{C_0^{|\beta|-1} C_1^{|\beta|} \beta!}{(1+|\beta|)^4} \frac{C_0^{|\gamma|-1} C_1^{|\gamma|} \gamma!}{(1+|\gamma|)^4}. \\
&\leq C|\alpha'| \|t^{|\alpha'|-1} \partial^{\alpha'}_{t,x} u_{n'+1}\|_{\widetilde{L}^1(0,T;B^0_{\infty,1})} + C \cdot 2\varepsilon \|t^{|\alpha'|} \partial^{\alpha'}_{t,x} u_{n'+1}\|_{\widetilde{L}^{\infty}(0,T;B^0_{\infty,1})} \\
&\quad + \frac{C C_0^{|\alpha'|-2} C_1^{|\alpha'|} \alpha'!}{(1+|\alpha'|)^4},
\end{aligned}$$

and

$$\begin{aligned}
&\|t^{|\alpha'|} \partial^{\alpha'}_{t,x} u_{n'+1}\|_{\widetilde{L}^{\infty}(0,T;B^0_{\infty,1}) \cap \widetilde{L}^1(0,T;B^1_{\infty,1})} \\
&\leq C|\alpha'| \|t^{|\alpha'|-1} \partial^{\alpha}_{t,x} u_{n'+1}\|_{\widetilde{L}^1(0,T;B^0_{\infty,1})} + \frac{C}{C_0} \cdot \frac{C_0^{|\alpha'|-1} C_1^{|\alpha'|} \alpha'!}{(1+|\alpha'|)^4},
\end{aligned}$$

since we can take ε sufficiently small. Hence, we need to estimate $|\alpha'| \|t^{|\alpha'|-1} \partial^{\alpha'}_{t,x} u_{n'+1}\|_{\widetilde{L}^1(0,T;B^0_{\infty,q})}$. For $\alpha' = (\alpha'_0, \alpha'_1)$, we have

$$\begin{aligned}
&|\alpha'| \|t^{|\alpha'|-1} \partial^{\alpha'}_{t,x} u_{n'+1}\|_{L^1(0,T;B^0_{\infty,1})} \\
&= \alpha'_0 \|t^{\alpha'_0-1+\alpha'_1} \partial_t (\partial^{\alpha'_0-1}_t \partial^{\alpha'_1}_x) u_{n'+1}\|_{L^1(0,T;B^0_{\infty,1})} + \alpha'_1 \|t^{\alpha'_0+\alpha'_1-1} \partial_x (\partial^{\alpha'_0}_t \partial^{\alpha'_1-1}_x) u_{n'+1}\|_{L^1(0,T;B^0_{\infty,1})}.
\end{aligned}$$

For the first term in the right member, by operating $t^{\alpha'_0-1+\alpha'_1} \partial^{\alpha'_0}_t \partial^{\alpha'_1}_x$ to (3.1), estimating directly with taking the norm of $L^1(0,T;B^0_{\infty,1}(\mathbb{R}))$, and applying the assumption of the induction for α with $|\alpha| \leq (\alpha'_0 - 1) + \alpha'_1 = |\alpha'| - 1$, we get that

$$\alpha_0' \|t^{\alpha_0'-1+\alpha_1'} \partial_t(\partial_t^{\alpha_0'-1}\partial_x^{\alpha_1'})u_{n'+1}\|_{L^1(0,T;B_{\infty,1}^0)}$$

$$\leq C\alpha_0' \|t^{\alpha_0'-1+\alpha_1'}\partial_t^{\alpha_0'-1}\partial_x^{\alpha_1'}u_{n'+1}\|_{L^1(0,T;B_{\infty,1}^1)}$$

$$+ C \sum_{\beta+\gamma\leq(\alpha_0'-1,\alpha_1')} \frac{(\alpha_0'-1)!\alpha_1'!}{\beta!\gamma!}\|t^{|\beta|}\partial_{t,x}^\beta u_{n'}\|_{\widetilde{L}^\infty(0,T;B_{\infty,1}^0)}\|t^{|\gamma|}\partial_{t,x}^\gamma u_{n'+1}\|_{\widetilde{L}^1(0,T;B_{\infty,1}^1)}$$

$$+ C \sum_{\beta+\gamma\leq(\alpha_0'-1,\alpha_1')} \frac{(\alpha_0'-1)!\alpha_1'!}{\beta!\gamma!}\|t^{|\beta|}\partial_{t,x}^\beta u_{n'}\|_{\widetilde{L}^\infty(0,T;B_{\infty,1}^0)}\|t^{|\gamma|}\partial_{t,x}^\gamma u_{n'}\|_{\widetilde{L}^1(0,T;B_{\infty,1}^1)}$$

$$\leq C\alpha_0' \cdot \frac{C_0^{|\alpha'|-2}C_1^{|\alpha'|-1}(\alpha_0'-1)!\alpha_1'!}{(1+|\alpha'|-1)^4} + \frac{C}{C_0}\cdot\frac{C_0^{|\alpha'|-2}C_1^{|\alpha'|-1}\alpha'!}{(1+|\alpha'|)^4}$$

$$= \frac{2C}{C_0^2C_1}\cdot\frac{C_0^{|\alpha'|-1}C_1^{|\alpha'|}\alpha'!}{(1+|\alpha'|)^4}.$$

As for the second, since ∂_x is a mapping from $B_{\infty,1}^1(\mathbb{R})$ to $B_{\infty,1}^0(\mathbb{R})$, the following holds:

$$\alpha_1' \|t^{\alpha_0'+\alpha_1'-1}\partial_x(\partial_t^{\alpha_0'}\partial_x^{\alpha_1'-1})u_{n'+1}\|_{L^1(0,T;B_{\infty,1}^0)} \leq C\alpha_1' \|t^{\alpha_0'+\alpha_1'-1}\partial_t^{\alpha_0'}\partial_x^{\alpha_1'-1}u_{n'+1}\|_{L^1(0,T;B_{\infty,1}^1)}$$

$$\leq C\alpha_1' \cdot \frac{C_0^{|\alpha'|-2}C_1^{|\alpha'|-1}\alpha_0'!(\alpha_1'-1)!}{(1+|\alpha'|-1)^4}$$

$$= \frac{C}{C_0C_1}\cdot\frac{C_0^{|\alpha'|-1}C_1^{|\alpha'|}\alpha'!}{(1+|\alpha'|)^4}.$$

Hence, the above four inequalities yield that

$$\|t^{|\alpha'|}\partial_{t,x}^{\alpha'}u_{n'+1}\|_{\widetilde{L}^\infty(0,T;B_{\infty,1}^0)\cap\widetilde{L}^1(0,T;B_{\infty,1}^1)} \leq \left(\frac{C}{C_0}+\frac{C}{C_0^2C_1}\right)\frac{C_0^{|\alpha'|-1}C_1^{|\alpha'|}\alpha'!}{(1+|\alpha'|)^4}$$

$$\leq \frac{C_0^{|\alpha'|-1}C_1^{|\alpha'|}\alpha'!}{(1+|\alpha'|)^4},$$

where C_0, C_1 are taken sufficiently large. Therefore, we obtain the estimate (3.13) for all $n \in \mathbb{N}$ and $\alpha \in (\mathbb{N}\cup\{0\})^2$.

The estimates (3.12) implies the solvability of solution in the space $C([0,T], B_{\infty,1}^0(\mathbb{R}))\cap\widetilde{L}^1(0,T;B_{\infty,1}^1(\mathbb{R}))$, and (3.13) for all $n \in \mathbb{N}$ enables us to get the same boundedness of the solution, which verifies the analyticity of solution in space-time. The proof of analyticity is completed. □

4. Large time behavior for Burgers equation

We start by considering the initial data $u_0 \in L^1(\mathbb{R})\cap B_{\infty,1}^0(\mathbb{R})$, since initial data in $L^1(\mathbb{R})$ with (1.4) can be approximated by functions in $L^1(\mathbb{R})\cap B_{\infty,1}^0(\mathbb{R})$. Let $u_0 \in L^1(\mathbb{R})\cap B_{\infty,1}^0(\mathbb{R})$ and we consider a solution u of (1.1) such that

$$u \in C([0,\infty), B_{\infty,1}^0(\mathbb{R}))\cap L_{loc}^1([0,\infty), B_{\infty,1}^1(\mathbb{R})), \tag{4.1}$$

noting that the argument of solvability in section 3 is applicable to that in $B_{\infty,1}^0(\mathbb{R})$ and the global regularity (see e.g. [28,31]) assures the global existence. We prepare the following lemma to guarantee the integrability of the solution for all the positive time.

Lemma 4.1. Let $u_0 \in L^1(\mathbb{R})\cap B_{\infty,1}^0(\mathbb{R})$. Then there exists a unique global solution $u \in C([0,\infty), L^1(\mathbb{R})\cap B_{\infty,1}^0(\mathbb{R}))\cap L_{loc}^1(0,\infty; B_{\infty,1}^1(\mathbb{R}))$ of the integral equation

$$u(t) = e^{-t\Lambda} u_0 - \frac{1}{2} \int_0^t e^{-(t-\tau)\Lambda} \partial_x u(\tau)^2 \, d\tau. \tag{4.2}$$

Proof. We start by the local solvability in $L^1(\mathbb{R}) \cap B^0_{\infty,1}(\mathbb{R})$. We consider a sequence of approximating solutions in the framework of $B^1_{\infty,1}(\mathbb{R})$ by the similar argument of the local solvability in Section 3 together with the boundedness in $L^1(\mathbb{R})$. Let $\{u_n\}_{n=1}^\infty$ be defined by (3.1). On the estimates in $L^1(\mathbb{R})$, we estimate the corresponding integral equation in (3.1) that

$$\|u_1(t)\|_{L^1} = \|e^{-t\Lambda} u_0\|_{L^1} \leq \|u_0\|_{L^1},$$

$$\|u_{n+1}(t)\|_{L^1} \leq \|u_0\|_{L^1} + C \int_0^t \|u_n\|_{L^1} (\|u_{n+1}\|_{\dot{B}^1_{\infty,1}} + \|u_n\|_{\dot{B}^1_{\infty,1}}) d\tau$$

$$\leq \|u_0\|_{L^1} + C(\|u_{n+1}\|_{L^1(0,t;\dot{B}^1_{\infty,1})} + \|u_n\|_{L^1(0,t;\dot{B}^1_{\infty,1})}) \|u\|_{L^\infty(0,t;L^1)}.$$

Following the proof of unique solvability in section 3, we have from Lemma 3.1 for $q = 1$ that

$$\|u_n\|_{\widetilde{L}^\infty(0,T;B^0_{\infty,1})} \leq C \|u_0\|_{B^0_{\infty,1}}, \quad \|u_n\|_{\widetilde{L}^1(0,T;B^1_{\infty,1})} \leq 2\varepsilon,$$

$$\|u_{n+1} - u_{m+1}\|_{\widetilde{L}^2(0,T;B^{-\delta}_{\infty,1})} \leq C 2^{-\min\{n,m\}} \|u_0\|_{B^0_{\infty,1}} + \frac{1}{5} \|u_n - u_m\|_{\widetilde{L}^2(0,T;B^{-\delta}_{\infty,1})},$$

where $\varepsilon > 0$ is a fixed sufficiently small constant, T is small and is depending on u_0. Hence we deduce from the above four estimates that for some fixed small constant $\varepsilon > 0$ and small T, u_n satisfies that

$$\|u_n\|_{L^\infty(0,T;L^1)} \leq C \|u_0\|_{L^1}, \quad \|u_n\|_{\widetilde{L}^\infty(0,T;B^0_{\infty,1})} \leq C \|u_0\|_{B^0_{\infty,1}}, \quad \|u_n\|_{\widetilde{L}^1(0,T;B^1_{\infty,1})} \leq 2\varepsilon,$$

$$\|u_{n+1} - u_n\|_{\widetilde{L}^2(0,T;B^{-\delta}_{\infty,1})} \leq C 2^{-n}.$$

So we get a solution $u \in L^\infty(0, T; B^0_{\infty,1}(\mathbb{R})) \cap L^1(0, T; B^1_{\infty,1}(\mathbb{R}))$. Noting that u_n is a bounded sequence in $L^\infty(\mathbb{R})$ the dual of $L^1(\mathbb{R})$, we see that u_n may converge to u in the topology of dual weak sense in $L^\infty(\mathbb{R})$ by taking a subsequence if necessary, so $u_n(t, x)$ converges to $u(t, x)$ for almost everywhere $x \in \mathbb{R}$. The Fatou lemma yields that

$$\|u(t)\|_{L^1} \leq \liminf_{n \to \infty} \|u_n(t)\|_{L^1} \leq C \|u_0\|_{L^1}.$$

This inequality proves $u(t) \in L^1(\mathbb{R})$, and we have that u satisfies the integral equation (4.2) thanks to the following estimate of nonlinear term

$$\int_0^t \|e^{-(t-\tau)\Lambda} \partial_x u(\tau)^2\|_{L^1 \cap B^0_{\infty,1}} \, d\tau \leq C \int_0^t \|u\|_{L^1 \cap B^0_{\infty,1}} \|u\|_{B^1_{\infty,1}} \, d\tau$$

$$\leq C \|u\|_{L^\infty(0,T;L^1 \cap B^0_{\infty,1})} \|u\|_{L^1(0,T;B^1_{\infty,1})},$$

although we omit the detail. This estimates also implies the time continuity of u in $L^1(\mathbb{R}) \cap B^0_{\infty,1}(\mathbb{R})$, since the linear part $e^{-t\Lambda} u_0$ is continuous in $L^1(\mathbb{R}) \cap B^1_{\infty,1}(\mathbb{R})$. To verify larger existence time, we apply the global regularity; The boundedness of the derivative (see Corollary 3.5 from [28], page 1690 from [28,31]):

$$L := \sup_{t > T_0} \|\partial_x u(t)\|_{L^\infty} < \infty, \quad \text{for each } T_0 > 0.$$

Let us fix T_0. The above boundedness and the L^∞-maximum principle allow us to obtain local solution with the data $u(t_0)$ $(t_0 \geq T_0)$ in a time interval $[t_0, t_0 + \widetilde{T}]$ with a constant length $\widetilde{T}$ independent of t_0. In fact, we can control the linear part as follows.

$$\|e^{-t\Lambda} u(t_0)\|_{L^1(0,\widetilde{T};B^1_{\infty,1})} \leq C \|t^{-\frac{1}{2}} u(t_0)\|_{B^{\frac{1}{2}}_{\infty,\infty}} \|_{L^1(0,\widetilde{T})} \leq C \widetilde{T}^{\frac{1}{2}} \sup_{t > T_0} \|u(t)\|_{W^{1,\infty}}$$

by which we can perform the fixed point argument any number of times. We then have $u \in L^1_{loc}([0, \infty), \dot{B}^1_{\infty,1}(\mathbb{R}))$, which proves Lemma 4.1. $\quad\square$

The following proposition is essential to handle the large time behavior of solutions.

Proposition 4.2. *Let u be a solution of the integral equation* (4.2) *obtained in Lemma 4.1. Then $u \in L^1(0, \infty;$ $\dot{B}^1_{\infty,1}(\mathbb{R}))$.*

Sketch of proof. We start by proving the decay estimate of the solution in $L^\infty(\mathbb{R})$ along the paper [15]. By taking x_t such that $|u(t, x_t)| = \|u(t)\|_{L^\infty}$, we have from the equation that

$$\partial_t \|u(t)\|_{L^\infty} \leq -\Lambda u(t, x_t) \, \mathrm{sgn}(u(t, x_t)). \tag{4.3}$$

Without loss of generality, we can assume that $u(t, x_t) \geq 0$, since it suffices to consider $-u(t, x_t)$ otherwise, so let $u(t, x_t) \geq 0$. We also put

$$B_\delta(x_t) := \{y \in \mathbb{R} \,|\, |y - x_t| \leq \delta\}, \quad \Omega_1 := \{y \in B_\delta(x_t) \,|\, u(t, x_t) - u(t, y) \geq u(t, x_t)/2\},$$

where δ will be taken later. Then it follows from $u(t, x_t) \geq u(t, y)$ for all $y \in \mathbb{R}$ that

$$\Lambda u(t, x_t) \geq P.V. \int_{\Omega_1} \frac{u(t, x_t) - u(t, y)}{|x_t - y|^2} \, dy \geq \frac{u(t, x_t)}{2\delta^2} \int_{\Omega_1} dy.$$

On the other hand, we have from the maximum principle in $L^1(\mathbb{R})$ that

$$\|u_0\|_{L^1} \geq \|u(t)\|_{L^1} \geq \int_{B_\delta(x_t) \backslash \Omega_1} |u(t, y)| dy \geq \frac{u(t, x_t)}{2} \Big(\int_{B_\delta(x_t)} dy - \int_{\Omega_1} dy \Big).$$

The above inequalities, (4.3) and $\int_{B_\delta(x_t)} dy = 2\delta$ imply that

$$\partial_t \|u(t)\|_{L^\infty} \leq -\frac{u(t, x_t)}{2\delta^2} \int_{\Omega_1} dy \leq \frac{u(t, x_t)}{2\delta^2} \Big(\frac{2\|u_0\|_{L^1}}{u(t, x_t)} - 2\delta \Big).$$

By taking $\delta = 2\|u_0\|_{L^1}/u(t, x_t)$, we get that

$$\partial_t \|u(t)\|_{L^\infty} \leq -\frac{u(t, x_t)}{2\delta^2} \frac{2\|u_0\|_{L^1}}{u(t, x_t)} = -\frac{\|u\|^2_{L^\infty}}{4\|u_0\|_{L^1}},$$

$$\|u(t)\|_{L^\infty} \leq \frac{\|u_0\|_{L^\infty}}{1 + \frac{\|u_0\|_{L^\infty}}{4\|u_0\|_{L^1}} t}, \tag{4.4}$$

which completes the proof of time decay estimate of u in $L^\infty(\mathbb{R})$.

We turn to prove $u \in L^1(0, \infty; \dot{B}^1_{\infty,1}(\mathbb{R}))$, dividing into two cases; $\|u_0\|_{L^\infty} \leq \delta_0$, and $\|u_0\|_{L^\infty} > \delta_0$, where δ_0 is a small constant which will be taken later.

If $\|u_0\|_{L^\infty} \leq \delta_0$, the solution also satisfies $\|u(t)\|_{L^\infty} \leq \delta$ thanks to the maximum principle in $L^\infty(\mathbb{R})$. Then we can estimate directly the integral equation by the maximal regularity estimate $\dot{B}^0_{\infty,1}(\mathbb{R})$ and the bilinear estimate $\|u^2\|_{\dot{B}^1_{\infty,1}} \leq \|u\|_{L^\infty} \|u\|_{\dot{B}^1_{\infty,1}}$ that

$$\|u\|_{L^\infty(0,\infty;\dot{B}^0_{\infty,1}) \cap L^1(0,\infty;\dot{B}^1_{\infty,1})} \leq C\|u_0\|_{\dot{B}^0_{\infty,1}} + C \int_0^\infty \|\partial_x u^2\|_{\dot{B}^0_{\infty,1}} \, d\tau$$

$$\leq C\|u_0\|_{\dot{B}^0_{\infty,1}} + C\|u\|_{L^\infty(0,\infty;L^\infty)} \|u\|_{L^1(0,\infty;\dot{B}^0_{\infty,1})}$$

$$\leq C\|u_0\|_{\dot{B}^0_{\infty,1}} + C\delta_0 \|u\|_{L^\infty(0,\infty;\dot{B}^0_{\infty,1}) \cap L^1(0,\infty;\dot{B}^1_{\infty,1})}.$$

By taking δ_0 such that $C\delta_0 \leq 1/2$, we have that

$$\|u\|_{L^\infty(0,\infty;\dot{B}^0_{\infty,1})\cap L^1(0,\infty;\dot{B}^1_{\infty,1})} \leq 2C\|u_0\|_{\dot{B}^0_{\infty,1}},$$

which proves $u \in L^1(0,\infty;\dot{B}^1_{\infty,1}(\mathbb{R}))$.

If $\|u_0\|_{L^\infty} > \delta_0$, we apply the estimate (4.4) to get that

$$\|u(t_0)\|_{L^\infty} \leq \delta_0 \quad \text{for } t_0 = \frac{4\|u_0\|_{L^1}}{\|u_0\|_{L^\infty}}\left(\frac{\|u_0\|_{L^\infty}}{\delta_0} - 1\right).$$

Hence, the previous case of small initial data implies that $u \in L^1(t_0,\infty;\dot{B}^1_{\infty,1}(\mathbb{R}))$, which assures $u \in L^1(0,\infty; \dot{B}^1_{\infty,1}(\mathbb{R}))$. $\square$

We next prove the decay estimates of solutions.

Proposition 4.3. *Let u be a solution of the integral equation* (4.2) *obtained in Lemma* 4.1 *such that* $u \in C([0,\infty),\dot{B}^s_{\infty,1}(\mathbb{R}))$ *for any $s > 0$. Let $\alpha > 0$ and $1 \leq p \leq \infty$. Then*

$$\|u(t)\|_{L^p} \leq Ct^{-(1-\frac{1}{p})} \quad \text{for any } t \geq 1. \tag{4.5}$$

$$\int_0^\infty \|t^\alpha u(t)\|_{\dot{B}^{1+\alpha}_{\infty,1}}\,dt < \infty \quad \text{and} \quad \||\nabla|^\alpha u(t)\|_{L^p} \leq Ct^{-\alpha-(1-\frac{1}{p})} \quad \text{for any } t \geq 1. \tag{4.6}$$

Proof. The decay estimate (4.5) is a consequence of (4.4) or Proposition 5.2 in [24], since $u \in L^1(0,\infty;\dot{B}^1_{\infty,}(\mathbb{R}))$ by Proposition 4.2. We also notice that the decay estimate of $\||\nabla|^\alpha u(t)\|_{L^\infty}$ in (4.6) can be proved by Proposition 5.2 in [24] once the integrability in (4.6) is obtained. Hence, all we have to do is to prove the former part of (4.6).

We prove the integrability in (4.6). Since u is smooth, it is sufficient to consider the integrability for large t. By the decay estimate (4.5), we see that for any $\delta > 0$, there exists $t_\delta > 0$ such that $\|u(t)\|_{L^\infty} \leq \delta$ for any $t \geq t_\delta$, so we may consider the integrability on the time interval with smallness of L^∞ norm. Put

$$v(t) := u(t_\delta + t) \quad \text{for } t \geq 0.$$

For any $T > 0$, we estimate the integral equation for v similarly to the proof of Proposition 5.2 in [24]. For the linear part, it follows from the smoothing effect and the maximal regularity for $e^{-t\Lambda}$ that

$$\int_0^T \|t^\alpha e^{-t\Lambda} v(0)\|_{\dot{B}^{1+\alpha}_{\infty,1}}\,dt \leq C\int_0^T \|e^{-\frac{t}{2}\Lambda} v(0)\|_{\dot{B}^1_{\infty,1}}\,dt \leq C\|v(0)\|_{\dot{B}^0_{\infty,1}}.$$

As to the nonlinear part, we decompose $[0,t]$ into $[0,t/2]$ and $[t/2,t]$ to have that

$$\int_0^T \left\| t^\alpha \int_0^{\frac{t}{2}} e^{-(t-\tau)\Lambda} \partial_x v(\tau)^2\,d\tau \right\|_{\dot{B}^{1+\alpha}_{\infty,1}} dt$$

$$\leq C\int_0^T t^\alpha \int_0^{\frac{t}{2}} (t-\tau)^{-\alpha}\|e^{-\frac{t-\tau}{2}\Lambda} v(\tau)^2\|_{\dot{B}^2_{\infty,1}}\,d\tau\,dt \leq C\int_0^T \int_{2\tau}^T \|e^{-\frac{t-\tau}{2}\Lambda} v(\tau)^2\|_{\dot{B}^2_{\infty,1}}\,dt\,d\tau$$

$$\leq C\int_0^T \|v(\tau)^2\|_{\dot{B}^1_{\infty,1}}\,d\tau \leq C\|v\|_{L^\infty(0,T;L^\infty)}\|v\|_{L^1(0,T;\dot{B}^1_{\infty,1})},$$

where we have used the smoothing effect and the maximal regularity estimate for $e^{-\frac{t-\tau}{2}\Lambda}$ and the bilinear estimate $\|v^2\|_{\dot{B}^1_{\infty,1}} \leq \|v\|_{L^\infty}\|v\|_{\dot{B}^1_{\infty1}}$. On the integral on $[t/2,t]$, it follows from $t^\alpha \leq 2^\alpha \tau^\alpha$ for $t/2 \leq \tau \leq t$, the maximal regularity estimate in $\dot{B}^{\alpha+1}_{\infty,1}(\mathbb{R})$ and the bilinear estimate $\|v^2\|_{\dot{B}^{1+\alpha}_{\infty,1}} \leq C\|v\|_{L^\infty}\|v\|_{\dot{B}^{1+\alpha}_{\infty,1}}$ that

$$\int\limits_0^T \Big\| t^\alpha \int\limits_{\frac{t}{2}}^t e^{-(t-\tau)\Lambda} \partial_x v(\tau)^2 \, d\tau \Big\|_{\dot{B}^{1+\alpha}_{\infty,1}} dt \le C \int\limits_0^T \int\limits_{\frac{t}{2}}^t \| e^{-(t-\tau)\Lambda} \tau^\alpha v(\tau)^2 \|_{\dot{B}^{2+\alpha}_{\infty,1}} \, d\tau \, dt$$

$$\le C \int\limits_0^T \int\limits_\tau^T \| e^{-(t-\tau)\Lambda} \tau^\alpha v(\tau)^2 \|_{\dot{B}^{2+\alpha}_{\infty,1}} \, dt \, d\tau$$

$$\le C \int\limits_0^T \| \tau^\alpha v(\tau)^2 \|_{\dot{B}^{1+\alpha}_{\infty,1}} \, dt \, d\tau$$

$$\le C \|v\|_{L^\infty(0,T;L^\infty)} \int\limits_0^T \tau^\alpha \|v\|_{\dot{B}^{1+\alpha}_{\infty,1}} \, d\tau.$$

We also have on the norm of $L^1(0, T; \dot{B}^1_{\infty,1}(\mathbb{R}))$ from the maximal regularity estimate in $\dot{B}^0_{\infty,1}(\mathbb{R})$ and the bilinear estimate $\|v^2\|_{\dot{B}^1_{\infty,1}} \le C\|v\|_{L^\infty}\|v\|_{\dot{B}^1_{\infty,1}}$ that

$$\|v\|_{L^1(0,T;\dot{B}^1_{\infty,1})} \le C\|v(0)\|_{\dot{B}^0_{\infty,1}} + C\|v\|_{L^\infty(0,T;L^\infty)}\|v\|_{L^1(0,T;\dot{B}^1_{\infty,1})}.$$

The above four estimates and $\|v(t)\|_{L^\infty} \le \delta$ yield that

$$\|t^\alpha v(t)\|_{L^1_t(0,T;\dot{B}^{1+\alpha}_{\infty,1})} + \|v(t)\|_{L^1(0,T;\dot{B}^1_{\infty,1})}$$
$$\le C\|v(0)\|_{\dot{B}^0_{\infty,1}} + C\delta\big(\|t^\alpha v(t)\|_{L^1_t(0,T;\dot{B}^{1+\alpha}_{\infty,1})} + \|v(t)\|_{L^1(0,T;\dot{B}^1_{\infty,1})}\big). \tag{4.7}$$

Here taking δ such that $C\delta \le 1/2$, where C is a constant appearing in the above estimate, we obtain

$$\|t^\alpha v(t)\|_{L^1_t(0,T;\dot{B}^{1+\alpha}_{\infty,1})} + \|v(t)\|_{L^1(0,T;\dot{B}^1_{\infty,1})} \le 2C\|v(0)\|_{\dot{B}^0_{\infty,1}} \quad \text{for any } T > 0.$$

Hence the integrability in (4.6) is verified and we finish to prove Proposition 4.3. □

Based on the lemma and the propositions, we prove the large time behavior.

Proof of large time behavior (1.5). Let u be a global solution, which is obtained by (i) of Theorem 1.1 and the global regularity, with initial data u_0 satisfying (1.4) and $u_0 \in L^1(\mathbb{R})$. We see that $u(t) \in B^s_{\infty,1}(\mathbb{R})$ for any $s \ge 0$, $t > 0$, since we can have regularity as much as it is needed thanks to the analyticity and the global regularity. Once $u(t) \in L^1(\mathbb{R})$ for all t near 0 is proved, we are able to apply Lemma 4.1 and Propositions 4.3 by regarding the initial data as $u(t_0)$ for time $t_0 > 0$ near 0 to obtain the decay estimates (4.5) and (4.6), which proves the large time behavior (1.5) in the same argument of the paper [24] (see the proof of (1.4)). All we have to do is to prove that $u(t) \in L^1(\mathbb{R})$ for all t near 0.

Put $u_{0,N} := S_N u_0$, where $S_N = (\psi + \sum_{j=1}^N \phi_j)\ast$. We denote by u_N, u the solutions with the initial data $u_{0,N}$, u_0, respectively. It follows from $u_{0,N} \in L^1(\mathbb{R}) \cap B^0_{\infty,1}(\mathbb{R})$ and Lemma 4.1 that u_N satisfies the energy identity in $L^2(\mathbb{R})$

$$\|u_N(t)\|_{L^2}^2 + 2\int\limits_0^t \|\Lambda^{\frac{1}{2}} u_N\|_{L^2}^2 d\tau \le \|u_{0,N}\|_{L^2}^2,$$

and hence,

$$\|u_N(t)\|_{L^2}^2 \le \|u_{0,N}\|_{L^2}^2 \le \|u_0\|_{L^2}^2.$$

Since $L^2(\mathbb{R})$ is a Hilbert space, we can take a subsequence of $\{u_N(t)\}_{N=1}^\infty$, denoted by the same, $u_N(t)$ converges to an element $\widetilde{u}(t)$ of $L^2(\mathbb{R})$ in the weak topology of $L^2(\mathbb{R})$. On the other hand, we can prove that $u_N(t)$ tends to $u(t)$ in $\mathcal{S}'(\mathbb{R})$ for almost every t. In fact, it follows from the same proof of (3.4) that for $q = \infty$

$$\|u_N - u\|_{\widetilde{L}^2(0,T;B_{\infty,q}^{-\delta})}$$
$$\leq C\|e^{-tc\Lambda}(u_{0,N} - u_0)\|_{\widetilde{L}^2(0,T;B_{\infty,q}^{-\delta})} \tag{4.8}$$
$$+ C\big(\|u_N\|_{\widetilde{L}^2(0,T;B_{\infty,\infty}^{\frac{1}{2}})} + \|u\|_{\widetilde{L}^2(0,T;B_{\infty,\infty}^{\frac{1}{2}})}\big)\|u_N - u\|_{\widetilde{L}^2(0,T;B_{\infty,q}^{-\delta})}.$$

We note that

$$\lim_{T\to 0}\Big(\sup_N \|u_N\|_{\widetilde{L}^2(0,T;B_{\infty,\infty}^{\frac{1}{2}})} + \|u\|_{\widetilde{L}^2(0,T;B_{\infty,\infty}^{\frac{1}{2}})}\Big) = 0$$

which is assured by that $u_{0,N}$ is defined by restricting the frequency of u_0. Hence, by taking $T = T_0 > 0$ sufficiently small, the inequality (4.8) yields that

$$\|u_N - u\|_{\widetilde{L}^2(0,T_0;B_{\infty,\infty}^{-\delta})} \leq 2C\|e^{-tc\Lambda}(S_N u_0 - u_0)\|_{\widetilde{L}^2(0,T_0;B_{\infty,\infty}^{-\delta})} \leq 2CT_0^{\frac{1}{2}}\|u_{0,N} - u_0\|_{B_{\infty,\infty}^{-\delta}} \to 0$$

as $N \to \infty$. So $u_N(t,x)$ converges to $u(t,x)$ as $N \to \infty$ in $\mathcal{D}'((0,T_0) \times \mathbb{R})$. Therefore for almost every t, $u(t) = \widetilde{u}(t)$ in $\mathcal{S}'(\mathbb{R})$ and $u_N(t,x)$ tends to $u(t,x)$ for almost every $x \in \mathbb{R}$ by the weak convergence in $L^2(\mathbb{R})$ of $u_N(t)$ to $u(t)$. By applying the Fatou Lemma, we have that

$$\|u(t)\|_{L^1} \leq \liminf_{N\to\infty}\|u_N(t)\|_{L^1} \leq \liminf_{N\to\infty}\|u_{0,N}\|_{L^1} \leq C\|u_0\|_{L^1} < \infty,$$

where we have used the maximum principle in $L^1(\mathbb{R})$, namely, $\|u_N(t)\|_{L^1} \leq \|u_{0,N}\|_{L^1}$ obtained by multiplying the equation by $u/|u|$, integrating and the integration by parts. Therefore, $u(t) \in L^1(\mathbb{R})$ for almost every $t \leq T_0$ and the smoothness of u assures that $u(t) \in L^1(\mathbb{R})$ for all $t \leq T_0$. We complete the proof of the large time behavior in Theorem 1.1. □

Declaration of competing interest

There is no competing of interest.

Acknowledgements

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