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Morse theory of causal geodesics in a stationary spacetime via Morse theory of geodesics of a Finsler metric

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Abstract

We show that the index of a lightlike geodesic in a conformally standard stationary spacetime $(\mathcal{M}_0 \times \mathbb{R}, g)$ is equal to the index of its spatial projection as a geodesic of a Finsler metric F on $\mathcal{M}_0$ associated to $(\mathcal{M}_0 \times \mathbb{R}, g)$. Moreover we obtain the Morse relations of lightlike geodesics connecting a point p to a curve $\gamma(s) = (q_0, s)$ by using Morse theory on the Finsler manifold $(\mathcal{M}_0, F)$. To this end, we prove a splitting lemma for the energy functional of a Finsler metric. Finally, we show that the reduction to Morse theory of a Finsler manifold can be done also for timelike geodesics.
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Résumé

On démontre que l'indice d'un rayon de lumière dans un espace-temps stationnaire $(\mathcal{M}_0 \times \mathbb{R}, g)$ conformément standard est égal à l'indice de sa projection spatiale vue comme une géodésique d'une métrique de Finsler F sur $\mathcal{M}_0$ associée à $(\mathcal{M}_0 \times \mathbb{R}, g)$. De plus, on obtient les relations de Morse de géodésiques isotropes reliant un point p à une courbe $\gamma(s) = (q_0, s)$ en utilisant la théorie de Morse sur la variété de Finsler $(\mathcal{M}_0, F)$. À cette fin, on démontre un lemme de séparation de la fonctionnelle de l'énergie d'une métrique de Finsler. Enfin, on montre que la réduction à la théorie de Morse d'une variété de Finsler peut être faite aussi pour les géodésiques temporelles.
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1. Introduction

Since the seminal paper [43], Morse theory has been applied successfully to spacetime geometry (Lorentzian manifolds) and global problems of general relativity. For instance, a consequence of the Morse relations of lightlike geodesics proved in [43] is that on a contractible globally hyperbolic spacetime, whose metric satisfies a suitable growth condition, the number of images produced by a gravitational lens is odd (or infinity) [29]. Gravitational lensing is the phenomenon where the gravitational field of a galaxy, located between an observer and a star, bends the light rays emitted by the star and focuses them at the same instant of observation, causing the observer to see multiple images of the same star (see e.g. [40]).

After the papers [43,29], Morse theory has been applied to compute the number of lightlike geodesics between an event and a timelike curve, on different classes of spacetimes and for different types of lenses and sources (see e.g. [14,15,17,21,36,37]).

We recall that a Lorentzian manifold $(\mathcal{M}, g)$ is a smooth connected manifold $\mathcal{M}$ endowed with a symmetric non-degenerate tensor field g of type $(0,2)$ having index 1. The geodesics of $(\mathcal{M}, g)$ are the critical points of the energy functional of the metric g

$$z \mapsto \frac{1}{2}\int_0^1 g(z)[\dot{z}, \dot{z}]\,\mathrm{d}s. \tag{1}$$

So they are the smooth curves $z:[a,b] \to \mathcal{M}$ satisfying the equation $\nabla_{\dot{z}}\dot{z} = 0$, where ∇ is the Levi-Civita connection of the metric g. If z is a geodesic, the function $s \mapsto g(z(s))[\dot{z}(s), \dot{z}(s)] := E_z$ is constant. According to the sign of E_z, a geodesic is said *timelike* if $E_\gamma < 0$, *lightlike* if $E_\gamma = 0$ and *spacelike* if $E_\gamma > 0$ or $\dot{z} = 0$. Timelike and lightlike geodesics are also called *causal*. Such a terminology is also used for any vector in any tangent space and for any piecewise smooth curve iff its tangent vector field has the same character at any point where it is defined.

A striking difference with the Riemannian case is that the energy functional of a Lorentzian metric is unbounded both from below and above and the Morse index of its critical points is $+\infty$. The common strategy used to develop Morse theory of a Lorentzian manifold is to consider only a particular type of geodesics (timelike or lightlike), to restrict the index form to the vector fields that are orthogonal to the geodesic, in the timelike case, and orthogonal modulo the vector fields pointwise collinear to the velocity vector field of the geodesic, in the lightlike case, and to use the length functional on a finite-dimensional approximation of the path space (see [7, Chapter 10] and the references therein). Another approach is to substitute the energy functional with a functional which has nice variational properties. This works for lightlike geodesics, which are the critical points of the *arrival time functional* (see [43]), or for particular kinds of Lorentzian manifolds as the standard stationary ones (see [6,25]).[3]

The aim of this paper is twofold: to show that for a standard stationary Lorentzian manifold, Morse theory of causal geodesics can be reduced to Morse theory of geodesics of a Finsler manifold of Randers type associated to the spacetime; to show that Morse theory for geodesics connecting two points on a Finsler manifold can be casted in a purely infinite-dimensional setting without using finite-dimensional approximations.

In regard to the first aim, we show that the number of conjugate instants (counted with their multiplicity) along a lightlike [resp. timelike] geodesic is equal to that of the corresponding Finslerian geodesic (Theorem 13) [resp. Theorem 18]. Moreover the Morse relations of lightlike [resp. timelike parametrized with respect to the proper time on a given interval] geodesics joining a point with a timelike curve on the spacetime can be obtained from the Morse relations of the geodesics joining two points on the Finsler manifold (Theorem 15) [resp. Theorem 18]. Although this reduction is very natural and convenient, stationary spacetimes seem to be the only type of spacetimes where it works fine, without leaving the realm of strongly convex Finsler metrics (cf. also [18]).

We recall that a *Finsler metric* F on a manifold M is a continuous function $F: TM \to [0, +\infty)$ such that

- F is smooth on $TM \setminus 0$;
- F is fiberwise positively homogeneous of degree one, that is $F(x, \lambda y) = \lambda F(x, y)$, for all $x \in M$, $y \in T_x M$ and $\lambda > 0$;

[3] For recent results about the Morse index theorem in the spacelike case and the Morse relations for all type of geodesics see respectively [38] and [2].

- F has fiberwise strongly convex square, that is

$$\mathbf{g}_{ij}(x,y) = \left[\frac{1}{2}\frac{\partial^2(F^2)}{\partial y^i \partial y^j}(x,y)\right]$$

is positively defined for any $(x,y) \in TM \setminus 0$.

By Euler's theorem we have that $F^2(x,y) = \mathbf{g}(x,y)[y,y]$. A Finsler metric is said of Randers type if

$$F(x,y) = \sqrt{\alpha(x)[y,y]} + \omega(x)[y],$$

where α is a Riemannian metric on M and ω is a 1-form on M having norm with respect to α strictly less than 1 (see [5, p. 17]).

The length of a piecewise smooth curve $\gamma : [a,b] \subset \mathbb{R} \to M$ with respect to the Finsler metric F is defined by $L(\gamma) = \int_a^b F(\gamma(s), \dot\gamma(s))\,\mathrm{d}s$. Thus the distance between two arbitrary points $p,\ q \in M$ is given by

$$\mathrm{dist}(p,q) = \inf_{\gamma \in C(p,q)} L(\gamma), \tag{2}$$

where $C(p,q)$ is the set of all piecewise smooth curves $\gamma : [a,b] \to M$ with $\gamma(a) = p$ and $\gamma(b) = q$. The distance function (2) is nonnegative and satisfies the triangle inequality, but it is not symmetric as F is non-reversible. Thus one has to distinguish the order of a pair of points in M when speaking about distance. As a consequence, one is naturally led to the notions of forward and backward Cauchy sequences and completeness (see [5, Section 6.2]): a sequence $\{x_n\} \subset M$ is called *forward* [resp. *backward*] *Cauchy sequence* if for all $\varepsilon > 0$ there exists $\nu \in \mathbb{N}$ such that, for all $\nu \leqslant i \leqslant j$, $\mathrm{dist}(x_i, x_j) \leqslant \varepsilon$ [resp. $\mathrm{dist}(x_j, x_i) \leqslant \varepsilon$]; (M,F) is *forward complete* [resp. *backward complete*] if all forward [resp. backward] Cauchy sequences converge.

The geodesics $x : [0,1] \to M$ of a Finsler manifold (M,F) parametrized with constant speed $F(x,\dot x)$ are the curves x satisfying the equation

$$D_{\dot x}\dot x = 0,$$

where $D_{\dot x}\dot x$ is the *Chern covariant derivative* of $\dot x$ along x with reference vector $\dot x$ (see [5, Chapter 5 and Exercise 5.2.5]). As it is shown for example in [8, Proposition 2.3], the geodesics parametrized with constant speed joining two given points $p_0, q_0 \in M$ coincide with the critical points of the energy functional

$$E(x) = \frac{1}{2}\int_0^1 F^2(x,\dot x)\,\mathrm{d}s$$

defined on the manifold $\Omega_{p_0,q_0}(M)$, which is the collection of the curves $x : [0,1] \to M$ such that $x(0) = p_0$, $x(1) = q_0$ and having H^1-regularity, that is x is absolutely continuous and the integral $\int_0^1 h(x)[\dot x, \dot x]\,\mathrm{d}s$ is finite. Here h is any complete Riemannian metric on M. It is well known that $\Omega_{p_0,q_0}(M)$ is a Hilbert manifold modeled on any of the equivalent Hilbert spaces of H^1-sections, with vanishing endpoints, of the pulled back bundle x^*TM, x any regular curve in M connecting p_0 to q_0 [23, Proposition 2.4.1]. The Riemannian metric on $\Omega_{p_0,q_0}(M)$ is given by

$$\langle X, Y\rangle = \int_0^1 h(x)\big[\nabla_{\dot x}^h X, \nabla_{\dot x}^h Y\big]\,\mathrm{d}s,$$

for every H_0^1-section, X and Y of x^*TM, ∇^h being the Levi-Civita connection of the metric h.

As in Riemannian geometry, Jacobi vector fields are the vector fields along the geodesic x which give rise to variations of x by means of geodesics (parametrized with constant Finslerian speed), see [5, Section 5.4] or [41, Section 11.2]. A conjugate instant $\bar s$ along the geodesic $x : [0,1] \to M$ is a value of the parameter s such that there exists a Jacobi vector field J, with $J(0) = J(\bar s) = 0$. The *multiplicity* of a conjugate instant is the dimension of the vector space of the Jacobi vector fields vanishing at 0 and $\bar s$. Two points p_0 and q_0 on M are said to be non-conjugate in (M,F) if $\bar s = 1$ is not a conjugate instant along any geodesic $x : [0,1] \to M$ such that $x(0) = p_0$ and $x(1) = q_0$. We observe that on a Randers manifold we can consider other type of Jacobi vector fields, given by variations of geodesics parametrized to have constant speed with respect to the Riemannian metric, but as commented in Remark 14, they generate the same conjugate points as the classical ones.

The function $G = F^2$ is smooth outside the zero section but it is only C^1 on the whole tangent bundle. It is C^2 on *TM* if and only if it is the square of the norm of a Riemannian metric (see [44]). Hence the lack of regularity on the zero section is a characteristic property of Finsler metrics. This has consequences on the level of regularity of the energy functional of a Finsler metric. It is easy to see that E is a $C^{1,1}$-functional on $\Omega_{p_0,q_0}(M)$, i.e. it is differentiable with locally Lipschitz differential (see [30, Theorem 4.1]) but it is well known that E is not a C^2-functional on $\Omega_{p_0,q_0}(M)$. This fact makes difficult the application of infinite-dimensional methods in Morse theory for Finsler manifolds and indeed approximations of $\Omega_{p_0,q_0}(M)$ (or of the free loop space $\Omega(M)$, in the closed geodesics problem) by finite-dimensional manifolds are commonly used to apply Morse theory to the energy functional of a Finsler metric, see for instance [41, Chapter 17] (or [27] and [4] in the periodic case).

Nevertheless, in several papers about geodesics on Finsler manifolds it is claimed that E is twice Frechet differentiable in the H^1-topology at any critical point (a geodesic). In particular, in [31] the authors prove an extension of the Morse Lemma to the case of a $C^{1,1}$-functional defined on a Hilbert manifold and twice Frechet differentiable at any non-degenerate critical point and then apply their result to cover Morse theory for closed geodesics of a bumpy Finsler metric on a compact manifold. Moreover in [13] the results of [31] are extended proving the splitting lemma at a degenerate and isolated critical point.

Unfortunately, as recently shown by A. Abbondandolo and M. Schwarz [3], the energy functional of a Finsler metric is twice differentiable at a critical point if and only if $F^2(x, y)$ is Riemannian along the critical point. Although [3, Proposition 2.3] deals with smooth time-dependent Lagrangians having at most quadratic growth in the velocities, the argument developed there works also for the square of a Finsler metric since it does not require continuity of the derivatives $\frac{\partial^2(F^2)}{\partial y^i \partial y^j}$ on the zero section. Moreover the proof in [3] concerns any curve in the manifold $H^1([0, 1], M)$ but it works, with minor modifications, also for curves satisfying periodic or fixed endpoint boundary conditions. Without the Morse Lemma, the computation of the critical groups, which are the local homotopic invariants describing the “nature” of an isolated critical point (see [39]), cannot be carried out in an infinite-dimensional setting.

In Section 2 we show (Theorem 7) that, in spite of the lack of twice differentiability, the splitting lemma holds for the energy functional E on the infinite-dimensional manifold $\Omega_{p_0,q_0}(M)$.

To this end, we use some ideas of K.-C. Chang who proved a splitting lemma (i.e. an extension of the Morse Lemma to the case of a degenerate critical point) for a C^2-functional J defined on a Banach space X immersed continuously as a dense subspace of a Hilbert space H and whose gradient is of the type $\nabla J(x) = x - K \cdot L(x)$, where L is a $C^{1,1}$-map from H to another Banach space E and K is a continuous linear operator from E to X (see [11] and also [10, Remark 5.1.15]). Such a result was extended in [22] for a $C^{1,1}$-functional on H which is C^2 on an open subset U of X. Similar ideas have also appeared in M. Struwe’s work about Plateau’s problem (cf. [42]) and in [32]. In particular the extension of the splitting lemma to Banach spaces proved in this last paper is suited also for the energy functional of a Finsler metric.

In fact, the energy functional E is C^2 on the manifold of the smooth regular curves, having fixed endpoints, endowed with the C^1-topology. After a *localization procedure* which allows us to work on the Hilbert space $H_0^1([0, 1], U)$, U being an open subset of $\mathbb{R}^n$, $n = \dim M$, the extension to $H_0^1([0, 1], \mathbb{R}^n)$ of the second Frechet derivative of E at a critical point $\bar{x}$ is given, with respect to a scalar product $(\cdot,\cdot)$ equivalent to the standard one, by $A(\bar{x}) = I + K(\bar{x})$, where I is the identity operator of $H_0^1([0, 1], \mathbb{R}^n)$ and $K(\bar{x})$ is a bounded linear operator from $H_0^1([0, 1], \mathbb{R}^n)$ to $C_0^1([0, 1], \mathbb{R}^n)$. More important, the gradient of E evaluated at the curves in $D \subset C_0^1([0, 1], U)$ is a field in $C_0^1([0, 1], \mathbb{R}^n)$. Here D is the open subset of $C_0^1([0, 1], U)$ which corresponds to the curves where the localized Lagrangian is regular (see the beginning of Section 2). Using the scalar product $(\cdot,\cdot)$ and the operator $A(\bar{x})$ to represent E, we obtain a splitting lemma for E restricted to $C_0^1([0, 1], U)$ (Theorem 7) and the Morse relations for the geodesics connecting two non-conjugate points in (M, F).

2. The splitting lemma for the energy functional of a Finsler metric and the Morse relations

By using a localization argument (see [1, Appendix A]), we can assume that the energy functional E is given in a coordinate system of the manifold $\Omega_{p_0,q_0}(M)$ by

$$\tilde{E}(x) = \frac{1}{2}\int_0^1 \tilde{G}\big(s, x(s), \dot{x}(s)\big)\,\mathrm{d}s, \tag{3}$$

where $\tilde{G}$ is a "time-dependent" (non-homogeneous) Lagrangian defined on an open subset U of $\mathbb{R}^n$, $n = \dim M$. The localization argument works as follows. Assume that $\bar{x} : [0,1] \to M$ is a differentiable curve of the Finsler metric F connecting the points p_0 and q_0. Let exp be the exponential map of the auxiliary Riemannian metric h, $\mu(p)$ be the injectivity radius of the point p in (M, h) and $\rho = \inf\{\mu(p)\colon\ p \in \bar{x}([0,1])\}$. Let $[0,1] \ni s \to \mathbf{E}(s) = (E_1(s), \ldots, E_n(s))$ be a parallel orthonormal frame along $\bar{x}$, $P_s : \mathbb{R}^n \to T_{\bar{x}(s)}M$ defined as $P_s(q_1, \ldots, q_r) = q_1 E_1(s) + \cdots + q_n E_n(s)$ and consider the Euclidean open ball U of radius $\rho/2$ and the map $\varphi(s,q) = \exp_{\bar{x}(s)} P_s(q)$. The map $\varphi_s : U \to M$, defined as $\varphi_s(q) = \varphi(s,q)$, is injective with invertible differential $\mathrm{d}\varphi_s(q)$, for every $s \in [0,1]$ and $q \in U$.

The Lagrangian $\tilde{G} : [0,1] \times U \times \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\tilde{G}(s,q,y) = F^2\big(\varphi(s,q), \mathrm{d}\varphi(s,q)\big[(1,y)\big]\big). \tag{4}$$

It is continuous on $[0,1] \times U \times \mathbb{R}^n$. The lack of regularity of F^2 on the zero section of TM is inherited by $\tilde{G}$ on the set $Z \subset [0,1] \times U \times \mathbb{R}^n$ given by all the points (s,q,y) such that $\mathrm{d}\varphi(s,q)[(1,y)] = 0$. Observe that for each $(s,q) \in [0,1] \times U$ there is only one $y \in \mathbb{R}^n$ such that $\mathrm{d}\varphi(s,q)[(1,y)] = 0$. In fact, $\mathrm{d}\varphi(s,q)[(1,y)] = \partial_s\varphi(s,q) + \partial_q\varphi(s,q)[y]$, where $\partial_s\varphi(s,q)$ and $\partial_q\varphi(s,q)$ are the partial differentials of φ with respect to the s and q variables; as $\partial_q\varphi(s,q)$ is injective, $y \in \mathbb{R}^n$ is the only vector such that $\partial_q\varphi(s,q)[y] = -\partial_s\varphi(s,q)$.

Since F^2 is fiberwise strictly convex, we have that $\tilde{G}_{yy}(s,q,y)$ is positive definite for all $(s,q,y) \in [0,1] \times U \times \mathbb{R}^n \setminus Z$.

Define the map

$$\varphi_* : H_0^1\big([0,1], U\big) \to \Omega_{p_0,q_0}(M), \quad \varphi_*(\xi)(s) = \varphi\big(s, \xi(s)\big). \tag{5}$$

Hence

$$\tilde{E} = E \circ \varphi_*. \tag{6}$$

Observe that the constant function $0 \in H_0^1([0,1], U)$ is mapped by φ_* to the geodesic $\bar{x}$.

Let D be the open subset of $C_0^1([0,1], U)$ made up by all the curves x such that $(s, x(s), \dot{x}(s)) \notin Z$, for all $s \in [0,1]$.

By a standard argument, it can be proved that $\tilde{E}$ is twice Frechet differentiable at x in D endowed with the C^1-topology.

Lemma 1. *$\tilde{E}$ admits second Frechet derivative $D^2\tilde{E}(x)$ at a curve $x \in D$, with respect to the C^1-topology and it is given by*

$$\begin{aligned} D^2\tilde{E}(x)[\xi_1,\xi_2] &= \frac{1}{2}\int_0^1 \big(\tilde{G}_{qq}(s,x,\dot{x})[\xi_1,\xi_2] + \tilde{G}_{yq}(s,x,\dot{x})[\xi_1,\dot{\xi}_2]\big)\,\mathrm{d}s \\ &\quad + \frac{1}{2}\int_0^1 \big(\tilde{G}_{qy}(s,x,\dot{x})[\dot{\xi}_1,\xi_2] + \tilde{G}_{yy}(s,x,\dot{x})[\dot{\xi}_1,\dot{\xi}_2]\big)\,\mathrm{d}s. \end{aligned} \tag{7}$$

Observe that the right-hand side of (7) can be extended to a bounded symmetric bilinear form B on $H_0^1([0,1], \mathbb{R}^n)$.

Observe also that the vector fields ν in the kernel N of B in $H_0^1([0,1], \mathbb{R}^n)$ correspond to the Jacobi fields along the geodesic x, vanishing at the endpoints. Therefore they are smooth and N is finite-dimensional.

Since $\tilde{G}$ is fiberwise strictly convex, the bilinear form

$$(\xi_1,\xi_2) \mapsto \frac{1}{2}\int_0^1 \tilde{G}_{yy}(s,0,0)[\dot{\xi}_1,\dot{\xi}_2]\,\mathrm{d}s \tag{8}$$

defines a scalar product $(\cdot,\cdot)$ on $H_0^1([0,1], \mathbb{R}^n)$ which is equivalent to the standard one.

Lemma 2. *Let B be the extension of $D^2\tilde{E}(0)$ to $H_0^1([0,1],\mathbb{R}^n)$. There exists a bounded linear operator $A : H_0^1([0,1],\mathbb{R}^n) \to H_0^1([0,1],\mathbb{R}^n)$ of the type $A = I + K$ where I is the identity operator and $K : H_0^1([0,1],\mathbb{R}^n) \to H_0^1([0,1],\mathbb{R}^n)$ is a bounded linear operator, such that B is represented with respect to the scalar product* (8) *by A. Moreover the range of K is contained in $C_0^1([0,1],\mathbb{R}^n)$ and, as an operator $H_0^1([0,1],\mathbb{R}^n) \to C_0^1([0,1],\mathbb{R}^n)$, K is bounded as well.*

Proof. From (7), K is the sum of the bounded linear operators $K_i : H_0^1([0,1],\mathbb{R}^n) \to C_0^1([0,1],\mathbb{R}^n)$, $i = 1,2,3$, defined as follows. For each $s \in [0,1]$, let $\tilde{G}^{yy}(s,0,0)$ be the inverse matrix of $\tilde{G}_{yy}(s,0,0)$. For any $\xi \in H_0^1([0,1],\mathbb{R}^n)$ let $K_1\xi$ be the C^1-vector field W_1 which solves the equation

$$\frac{\mathrm{d}}{\mathrm{d}s}\big(\tilde{G}_{yy}(s,0,0)\dot{W}_1\big) = -\tilde{G}_{qq}(s,0,0)\xi,$$

and vanishes at $s = 0, 1$, so that

$$\frac{1}{2}\int_0^1 \tilde{G}_{qq}(s,0,0)[\xi_1,\xi_2]\,\mathrm{d}s = \frac{1}{2}\int_0^1 \tilde{G}_{yy}(s,0,0)\left[\frac{\mathrm{d}}{\mathrm{d}s}\big(K_1(\xi_1)\big), \dot{\xi}_2\right]\mathrm{d}s.$$

Hence

$$\dot{W}_1 = -\tilde{G}^{yy}(s,0,0)\int_0^s \tilde{G}_{qq}(\tau,0,0)\xi\,\mathrm{d}\tau + \tilde{G}^{yy}(s,0,0)C_1(\xi), \tag{9}$$

where $C_1(\xi)$ is the constant vector equal to

$$C_1(\xi) = \left(\int_0^1 \tilde{G}^{yy}(s,0,0)\,\mathrm{d}s\right)^{-1}\left(\int_0^1 \tilde{G}^{yy}(s,0,0)\left(\int_0^s \tilde{G}_{qq}(\tau,0,0)\xi\,\mathrm{d}\tau\right)\mathrm{d}s\right) \tag{10}$$

(notice that since $\tilde{G}^{yy}(s,0,0)$ is positive definite for all $s \in [0,1]$, the matrix $\int_0^1 \tilde{G}^{yy}(s,0,0)\,\mathrm{d}s$ is positive definite and invertible).

Analogously $K_2\xi$ and $K_3\xi$ are the curves W_2 and W_3 in $C_0^1([0,1],\mathbb{R}^n)$ which solve respectively the equations

$$\dot{W}_2 = \tilde{G}^{yy}(s,0,0)\cdot\tilde{G}_{yq}(s,0,0)\xi + \tilde{G}^{yy}(s,0,0)C_2(\xi), \tag{11}$$

$$\dot{W}_3 = -\tilde{G}^{yy}(s,0,0)\int_0^s \tilde{G}_{qy}(\tau,0,0)\dot{\xi}\,\mathrm{d}\tau + \tilde{G}^{yy}(s,0,0)C_3(\xi), \tag{12}$$

where $C_2(\xi)$ is the constant vector equal to

$$C_2(\xi) = -\left(\int_0^1 \tilde{G}^{yy}(s,0,0)\,\mathrm{d}s\right)^{-1}\left(\int_0^1 \tilde{G}^{yy}(s,0,0)\tilde{G}_{yq}(s,0,0)\xi\,\mathrm{d}s\right) \tag{13}$$

and

$$C_3(\xi) = \left(\int_0^1 \tilde{G}^{yy}(s,0,0)\,\mathrm{d}s\right)^{-1}\left(\int_0^1 \tilde{G}^{yy}(s,0,0)\left(\int_0^s \tilde{G}_{qy}(\tau,0,0)\dot{\xi}\,\mathrm{d}\tau\right)\mathrm{d}s\right). \qquad \square \tag{14}$$

Remark 3. By the compact embedding of $H_0^1([0,1],\mathbb{R}^n)$ in $C_0^0([0,1],\mathbb{R}^n)$, it follows by (9), (10), (11), (13) that K_1 and K_2 are compact operators in $H_0^1([0,1],\mathbb{R}^n)$, moreover from the Ascoli–Arzelà theorem and (12), (14), also K_3 is compact and then A is a Fredholm operator in $H_0^1([0,1],\mathbb{R}^n)$ with closed range equal to $N^\perp$.

Remark 4. Since N is contained in $C_0^1([0,1],\mathbb{R}^n)$, every $\xi \in C_0^1([0,1],\mathbb{R}^n)$, as a vector field in $H_0^1([0,1],\mathbb{R}^n)$, has projection $P\xi$ on $N^\perp$ which is also in $C_0^1([0,1],\mathbb{R}^n)$. Hence $C_0^1([0,1],\mathbb{R}^n)$ is the topological direct sum of the closed subspaces N and $N^\perp \cap C_0^1([0,1],\mathbb{R}^n)$.

Let us define by $\tilde{P} : N \oplus (N^\perp \cap C_0^1([0,1],\mathbb{R}^n)) \to N^\perp \cap C_0^1([0,1],\mathbb{R}^n)$ the projection operator. In the following, we denote by $\|\cdot\|$ and $\|\cdot\|_{C^1}$ respectively the norm of $H_0^1([0,1],\mathbb{R}^n)$ endowed with the scalar product (8) and the norm of the C^1-topology in $C_0^1([0,1],\mathbb{R}^n)$.

Lemma 5. *The restriction $\tilde{A}$ of A to the subspace $N^\perp \cap C_0^1([0,1],\mathbb{R}^n)$ is an invertible operator $\tilde{A} : N^\perp \cap C_0^1([0,1],\mathbb{R}^n) \to N^\perp \cap C_0^1([0,1],\mathbb{R}^n)$ with bounded inverse.*

Proof. Let $\eta \in N^\perp \cap C_0^1([0,1],\mathbb{R}^n)$. Observe that as a curve in $H_0^1([0,1],\mathbb{R}^n)$, $A\eta$ belongs to $N^\perp$. Since $A\eta = \eta + K\eta$ and $R(K) \subset C_0^1([0,1],\mathbb{R}^n)$, $A\eta \in C_0^1([0,1],\mathbb{R}^n)$. Moreover

$$\|A\eta\|_{C^1} \leqslant \|\eta\|_{C^1} + \|K\eta\|_{C^1} \leqslant \|\eta\|_{C^1} + \|K\|\|\eta\| \leqslant \|\eta\|_{C^1} + \|K\|\|\eta\|_{C^1}.$$

Therefore A is bounded from $N^\perp \cap C_0^1([0,1],\mathbb{R}^n)$ to $N^\perp \cap C_0^1([0,1],\mathbb{R}^n)$. Moreover for any $\bar{\eta} \in N^\perp \cap C_0^1([0,1],\mathbb{R}^n)$ let $\eta \in N^\perp$ such that $A\eta = \bar{\eta}$. Hence $\eta = \bar{\eta} - K\eta \in C_0^1([0,1],\mathbb{R}^n)$ that is $\tilde{A}$ is surjective and by the open mapping theorem it has bounded inverse. □

Lemma 6. *Let $x \in D$, then $\nabla\tilde{E}(x) \in C_0^1([0,1]),\mathbb{R}^n)$. Moreover the map $x \in D \mapsto \nabla\tilde{E}(x)$ is continuous in the C^1-topology.*

Proof. Let $x \in D$, the differential of $\tilde{E}$ at x in $H_0^1([0,1],U)$ is given by

$$\mathrm{d}\tilde{E}(x)[\xi] = \frac{1}{2}\int_0^1 \left(\tilde{G}_q(s,x,\dot{x})\xi + \tilde{G}_y(s,x,\dot{x})\dot{\xi}\right)\mathrm{d}s$$

for all $\xi \in H_0^1([0,1],\mathbb{R}^n)$. Recalling that we are using the scalar product (8) on $H_0^1([0,1],\mathbb{R}^n)$, $\nabla\tilde{E}(x)$ is the curve $W \in H_0^1([0,1],\mathbb{R}^n)$ such that $(W,\xi) = \mathrm{d}\tilde{E}(x)[\xi]$, that is

$$\frac{1}{2}\int_0^1 \tilde{G}_{yy}(s,0,0)[\dot{W},\dot{\xi}] = \frac{1}{2}\int_0^1 \left(\tilde{G}_q(s,x,\dot{x})\xi + \tilde{G}_y(s,x,\dot{x})\dot{\xi}\right)\mathrm{d}s.$$

Thus

$$\frac{1}{2}\int_0^1 \tilde{G}_{yy}(s,0,0)[\dot{W},\dot{\xi}] - \frac{1}{2}\int_0^1 \left(-\int_0^s \tilde{G}_q(\tau,x,\dot{x})\,\mathrm{d}\tau + \tilde{G}_y(s,x,\dot{x})\right)\dot{\xi}\,\mathrm{d}s = 0$$

and this equality is satisfied for all $\xi \in H_0^1([0,1],\mathbb{R}^n)$ if and only if there exists a constant (depending on x) $C = C(x)$ such that

$$\dot{W} = \tilde{G}^{yy}(s,0,0)\left(-\int_0^s \tilde{G}_q(\tau,x,\dot{x})\,\mathrm{d}\tau + \tilde{G}_y(s,x,\dot{x}) + C(x)\right). \tag{15}$$

As W must vanish at $s=0$ and $s=1$, $C(x)$ has to be equal to

$$C(x) = \left(\int_0^1 \tilde{G}^{yy}(s,0,0)\,\mathrm{d}s\right)^{-1}\int_0^1 \tilde{G}^{yy}(s,0,0)\left(\int_0^s \tilde{G}_q(\tau,x,\dot{x})\,\mathrm{d}\tau - \tilde{G}_y(s,x,\dot{x})\right)\mathrm{d}s. \tag{16}$$

From (15) and (16), we see that $\nabla\tilde{E}(x) \in C_0^1([0,1],\mathbb{R}^n)$ and using uniform continuity of the vector fields $\tilde{G}_q(s,q,y)$ and $\tilde{G}_y(s,q,y)$ we get that $\|\nabla E(x_n) - \nabla E(x)\|_{C^1} \to 0$ if $x_n \to x$ in the C^1-topology. □

We are now ready to prove the splitting lemma for $\tilde{E}$ at a critical point. From (6), since the map φ_* is smooth and injective, we obtain the splitting lemma (or in case the geodesic x_0 is non-degenerate, the Morse Lemma) for E. A proof of the splitting lemma for Finsler manifolds was established in [27, Lemma 4.2], see also [41, Section 17.4], using a finite-dimensional reduction on the manifold of piecewise minimizing geodesics. Here we present an infinite-dimensional proof in the spirit of the papers of Gromoll and Meyer [19,20], see also [10,22,32].

Theorem 7. *Let x_0 be a geodesic of the Finsler manifold (M,F) connecting two points p_0 and q_0 in M and consider the function φ_* defined in* (5) *associated to x_0. Then there exist a ball $B(0,r)$ in $C^1_0([0,1],U)$ centered at* 0*, a local homeomorphism $\phi: B(0,r)\to\phi(B(0,r))\subset D$, $\phi(0)=0$, a C^1 map $h: B(0,r)\cap N\to D\cap N^\perp$, where N is the kernel of A such that*

$$\tilde{E}\big(\phi(\xi)\big)=\frac{1}{2}(A\eta,\eta)+\tilde{E}\big(\nu+h(\nu)\big), \tag{17}$$

$\xi=\eta+\nu$ with $\nu\in N$ and $\eta\in N^\perp$, where $N^\perp$ is the orthogonal of N with respect to $(\cdot,\cdot)$.

Proof. Consider the equation

$$\tilde{P}\cdot\nabla\tilde{E}(\nu+\eta)=0, \tag{18}$$

where $(\nu,\eta)\in(D\cap N)\times(D\cap N^\perp)$ and $\tilde{P}$ was defined in Remark 4. The function $(\nu,\eta)\in(D\cap N)\times(D\cap N^\perp)\mapsto\tilde{F}(\nu+\eta)=\tilde{P}\nabla\tilde{E}(\nu+\eta)\in N^\perp\cap C^1_0([0,1],\mathbb{R}^n)$ is continuous by Lemma 6, moreover using (15) and (16), one can prove by a standard argument that it is differentiable with respect to the C^1-topology with continuous differential. In particular $\tilde{F}$ is differentiable with respect to η and its partial derivative $\mathrm{d}_\eta\tilde{F}(0,0): N^\perp\cap C^1_0([0,1],\mathbb{R}^n)\to N^\perp\cap C^1_0([0,1],\mathbb{R}^n)$ is the bounded invertible operator $\tilde{A}$. Namely, since $\tilde{P}$ is a bounded linear operator and $\tilde{A}$ assumes values in $N^\perp\cap C^1_0([0,1],R^n)$, it is enough to prove that

$$\frac{\|\nabla\tilde{E}(\eta)-\tilde{A}\eta\|_{C^1}}{\|\eta\|_{C^1}}\to 0,\quad \text{as } \|\eta\|_{C^1}\to 0.$$

From (9), (11), (12) and (15), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\big(\nabla\tilde{E}(\eta)-\tilde{A}\eta\big)=\tilde{G}^{yy}(s,0,0)&\left(-\int_0^s\tilde{G}_q(\tau,\eta,\dot\eta)\,\mathrm{d}\tau+\tilde{G}_y(s,\eta,\dot\eta)+C(\eta)\right)\\ &-\tilde{G}^{yy}(s,0,0)\tilde{G}_{yy}(s,0,0)\dot\eta+\tilde{G}^{yy}(s,0,0)\int_0^s\tilde{G}_{qq}(\tau,0,0)\eta\,\mathrm{d}\tau-\tilde{G}^{yy}(s,0,0)C_1(\eta)\\ &-\tilde{G}^{yy}(s,0,0)\cdot\tilde{G}_{yq}(s,0,0)\eta-\tilde{G}^{yy}(s,0,0)C_2(\eta)\\ &+\tilde{G}^{yy}(s,0,0)\int_0^s\tilde{G}_{qy}(\tau,0,0)\dot\eta\,\mathrm{d}\tau-\tilde{G}^{yy}(s,0,0)C_3(\eta).\end{aligned} \tag{19}$$

Since the constant curve of constant value 0 is a critical point of $\tilde{E}$, i.e. $0=\nabla\tilde{E}(0)$, also the derivative of the curve $\nabla\tilde{E}(0)$ is constant and equal to zero and then from (15) (with $x=0$) we get

$$0=\tilde{G}^{yy}(s,0,0)\left(-\int_0^s\tilde{G}_q(\tau,0,0)\,\mathrm{d}\tau+\tilde{G}_y(s,0,0)+C(0)\right), \tag{20}$$

and we can add the function on the right-hand side of (20) in the equality (19). By using the mean value theorem, for each $s\in[0,1]$ and to each component of the function

$$t\in[0,1]\mapsto-\int_0^s\tilde{G}_q\big(\tau,t\eta(\tau),t\dot\eta(\tau)\big)\,\mathrm{d}\tau+\tilde{G}_y\big(s,t\eta(s),t\dot\eta(s)\big)$$

and the uniform continuity in $[0,1]\times U\times\mathbb{R}^n\setminus Z$ of the second derivatives of the function $\tilde G$, we get that for all $\varepsilon>0$ there exists $\delta>0$ such that for all $\eta\in N^\perp\cap C_0^1([0,1],\mathbb{R}^n)$ with $\|\eta\|_{C^1}<\delta$

$$\Bigg\|\tilde G^{yy}(s,0,0)\Bigg(-\int_0^s\tilde G_q(\tau,\eta,\dot\eta)\,\mathrm{d}\tau+\int_0^s\tilde G_q(\tau,0,0)\,\mathrm{d}\tau+\tilde G_y(s,\eta,\dot\eta)$$
$$-\tilde G_y(s,0,0)-\tilde G_{yy}(s,0,0)\dot\eta+\int_0^s\tilde G_{qq}(s,0,0)\eta\,\mathrm{d}\tau$$
$$-\tilde G_{yq}(s,0,0)\eta+\int_0^s\tilde G_{qy}(\tau,0,0)\dot\eta\,\mathrm{d}\tau\Bigg)\Bigg\|_{C^0}<\varepsilon\|\eta\|_{C^1},$$

where $\|\cdot\|_{C^0}$ is the norm in the C^0-topology. Analogously, since

$$\left(\int_0^1\tilde G^{yy}(s,0,0)\,\mathrm{d}s\right)^{-1}\int_0^1\tilde G^{yy}(s,0,0)\tilde G_{yy}(s,0,0)\dot\eta\,\mathrm{d}s=0,$$

recalling (10), (13), (14) and (16), we have

$$\Bigg\|\tilde G^{yy}(s,0,0)\big(C(\eta)-C(0)-C_1(\eta)-C_2(\eta)-C_3(\eta)\big)$$
$$+G^{yy}(s,0,0)\left(\int_0^1\tilde G^{yy}(\tau,0,0)\,\mathrm{d}\tau\right)^{-1}\int_0^1\tilde G^{yy}(\tau,0,0)\tilde G_{yy}(\tau,0,0)\dot\eta\,\mathrm{d}\tau\Bigg)\Bigg\|_{C^0}<\varepsilon\|\eta\|_{C^1}.$$

Therefore, by the implicit function theorem there exists a C^1 map $h:B(0,r_1)\cap N\to B(0,\delta_1)\cap N^\perp$ such that

$$\tilde P\cdot\nabla\tilde E\big(\nu+h(\nu)\big)=0, \tag{21}$$

where $B(0,r_1)$ and $B(0,\delta_1)$ are two balls in D centered at 0.

Now we consider the Cauchy problem in $H_0^1([0,1],\mathbb{R}^n)$

$$\begin{cases}\dot\psi(s)=-\frac{A\psi(s)}{\|A\psi(s)\|},\\ \psi(0)=u\end{cases} \tag{22}$$

where $u\in B(0,r_1)$. Observe that, as $\|\psi(s)-u\|\leqslant|s|$, we have $\|\psi(s,u)\|\geqslant\|u\|-|s|$, thus the flow ψ is well defined for $|s|<\|u\|$ and $\psi(s,u)\in N^\perp$ if $u\in N^\perp$. By Lemma 5, we can solve the above ODE in the Banach space $N^\perp\cap C_0^1([0,1],\mathbb{R}^n)$ (observe that, since the function $u\in N^\perp\cap C_0^1([0,1],\mathbb{R}^n)\mapsto(Au,Au)$ is continuous with respect to the C^1-topology, the right-hand side of the equation in (22) is locally Lipschitz in $N^\perp\cap C_0^1([0,1],\mathbb{R}^n)\setminus\{0\}$ with the C^1-topology).

Let us call ζ the flow of (22) in $N^\perp\cap C_0^1([0,1],\mathbb{R}^n)$. By the uniqueness of the solutions of the Cauchy problem (22), we have $\zeta(s,u)=\psi(s,u)$ for all $s\in[0,\|u\|)$ and $u\in B(0,r_1)$. Moreover the map $(s,u)\mapsto\psi(s,u)$ is continuous, on the subset of $\mathbb{R}\times B(0,r_1)$ where it is defined, with respect to the product topology of $\mathbb{R}$ and $B(0,r_1)$ with the C^1-topology. Thus we can adapt the proof of the splitting lemma in [10, Theorem 5.1.13] to get the thesis. Namely, consider the functions

$$\mathcal{F}(u,\nu)=\tilde E(\nu+\eta)-\tilde E\big(\nu+h(\nu)\big),\qquad \mathcal{F}_2(u)=\frac12(Au,u),$$

where $u=\eta-h(\nu)\in N^\perp\cap C_0^1([0,1],\mathbb{R}^n)$. Observe that $\mathcal{F}(0,\nu)=0$ and $\mathrm{d}_u\mathcal{F}(0,\nu)=\mathrm{d}_\eta\tilde E(\nu+h(\nu))$. Since in the C^1-topology,

$$\mathrm{d}\tilde E(x)[\xi]=\frac12\int_0^1\big(\tilde G_q(s,x,\dot x)\xi+\tilde G_y(s,x,\dot x)\dot\xi\big)\,\mathrm{d}s=\big(\nabla\tilde E(x),\xi\big),$$

from (21) we get that $\mathrm{d}_\eta \tilde{E}(\nu + h(\nu))[\xi] = 0$ for all $\xi \in N^\perp \cap C_0^1([0,1], \mathbb{R}^n)$. Observe also that the second Frechet derivative of $\mathcal{F}$ at 0 in D with respect to the variable u is equal to

$$\mathrm{d}_u^2 \mathcal{F}(0,0) = \mathrm{d}_\eta^2 \tilde{E}(0,0).$$

As before $\mathrm{d}^2 \tilde{E}(0)[\xi_1, \xi_2] = (A\xi_1, \xi_2)$ and therefore $\mathrm{d}_\eta^2 \tilde{E}(0,0)[\xi_1, \xi_2] = (\tilde{A}\xi_1, \xi_2) = (A\xi_1, \xi_2)$ for all $\xi_1, \xi_2 \in N^\perp \cap C_0^1([0,1], \mathbb{R}^n)$.

Since $\mathcal{F}$ is C^2 on D with respect to the C^1-topology and $\tilde{G}$ is C^2 on $[0,1] \times U \times \mathbb{R}^n \setminus Z$, taking (7) into account and using the uniform continuity of the second partial derivatives of $\tilde{G}$, we can state that for all $\varepsilon > 0$ there exists a ball $B(0, r_2) \subset D$, with $r_2 < r_1$ such that

$$\begin{aligned} \left|\mathcal{F}(u,\nu) - \mathcal{F}_2(u)\right| &= \left|\mathcal{F}(u,\nu) - \mathcal{F}(0,\nu) - \mathrm{d}_u\mathcal{F}(0,\nu)[u] - \mathcal{F}_2(u)\right| \\ &= \left| \int_0^1 (1-s)\big(\mathrm{d}_u^2 \mathcal{F}(su,\nu) - \mathrm{d}_u^2\mathcal{F}(0,0)\big)[u,u]\,\mathrm{d}s \right| \\ &< \varepsilon \|u\|^2, \end{aligned} \tag{23}$$

for all $(u,\nu) \in (B(0,r_2) \cap N^\perp) \times (B(0,r_2) \cap N)$. Moreover

$$\begin{aligned} \left|\mathcal{F}_2\big(\psi(t,u)\big) - \mathcal{F}_2(u)\right| &= \left| \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} \mathcal{F}_2\big(\psi(s,u)\big)\,\mathrm{d}s \right| = \left| \int_0^t \big(\nabla \mathcal{F}_2(\psi), \dot{\psi}\big)\,\mathrm{d}s \right| \\ &= \int_0^{|t|} \left\| A\psi(s) \right\| \mathrm{d}s \geqslant C \int_0^{|t|} \left\|\psi(s)\right\| \mathrm{d}s \geqslant C\left(\|u\||t| - \frac{t^2}{2} \right), \end{aligned} \tag{24}$$

where C is a positive constant depending only on the spectral decomposition of A in $N^\perp$. As $\mathcal{F}_2(\psi(t,u))$ is strictly decreasing in t, from (23) and (24) we get that, if $\varepsilon < \frac{C}{4}$,

$$\mathcal{F}_2\big(\psi(-t,u)\big) > \mathcal{F}(u,\nu) > \mathcal{F}_2\big(\psi(t,u)\big)$$

holds, for all t such that

$$\|u\| \left(1 - \sqrt{1 - \frac{2\varepsilon}{C}} \right) \leqslant |t| < \|u\|$$

and for all $u \in B(0, r_2)$. Therefore, by continuity, there exists a unique $\bar{t} = \bar{t}(u,\nu)$, with

$$\left|\bar{t}(u,\nu)\right| \leqslant \|u\| \left(1 - \sqrt{1 - \frac{2\varepsilon}{C}} \right),$$

such that

$$\mathcal{F}_2\big(\psi\big(\bar{t}(u,\nu), u\big)\big) = \mathcal{F}(u,\nu). \tag{25}$$

By the implicit function theorem, the function $\bar{t} = \bar{t}(u,\nu)$ has to be continuous in the C^1-topology. Therefore the map ϕ is given by the inverse of the map $\theta = (u,\nu) \in V \mapsto (\vartheta(u,\nu), \nu)$, where ϑ is defined as

$$\vartheta(u,\nu) = \begin{cases} 0 & \text{if } u = 0, \\ \psi(\bar{t}(u,\nu), u) & \text{if } u \neq 0 \end{cases}$$

and $V = \theta^{-1}(B(0,r))$ where $B(0,r) \subset \theta(B(0,r_2))$. Eq. (17) then follows from (25). □

Remark 8. By the localization argument, the energy functional of a Finsler metric is treated as the action functional of a Lagrangian which is smooth outside the closed set Z and it is strictly convex in the velocities. Therefore the splitting lemma above also holds for the action functional of any smooth Lagrangian of this type or any such a Lagrangian which is non-smooth only on a closed subset of *TM* which does not intersect the support of the critical point x and its velocity vector field.

Theorem 7 allows us to compute the critical groups of an isolated critical point as, for instance, in [10, Corollary 5.1.18]. In particular we can obtain the Morse relations of geodesics connecting two non-conjugate points in a Finsler manifold (Theorem 15).

Our reference about Morse theory for a $C^{1,1}$-functional defined on an infinite-dimensional manifold is [28]. Let Ω be a complete Hilbert manifold and $f:\Omega\to\mathbb{R}$ be a $C^{1,1}$-functional. Let us denote by $\mathcal{K}$ the set of the critical points of f. Let $u\in\mathcal{K}$ and U be a neighborhood of u such that $\mathcal{K}\cap U=\{u\}$. For each $n\in\mathbb{N}$, let us denote by $C_n(f,u)$ the n-th singular homology group of the pair $(f^c\cap U, f^c\cap U\setminus\{u\})$ over the field $\mathbb{K}$, where $c=f(u)$ and $f^c=f^{-1}((-\infty,c])$. Let $b,a\in\mathbb{R}$, $b>a$. We denote by $M(r,f^b,f^a)$ the formal series with coefficients in $\mathbb{N}\cup+\infty$ defined by $M(r,f^b,f^a)=\sum_{n=0}^{+\infty}M_n(f^b,f^a)r^n$, where $M_n(f^b,f^a)=\sum_{u\in\mathcal{K}\cap f^{-1}([a,b])}\dim C_n(f,u)$. We denote $M(r,f^b,\emptyset)$ and $M_n(f^b,\emptyset)$ by $M(r,f^b)$ and $M_n(f^b)$. Assume that:

(i) all the critical points of f are isolated,
(ii) f satisfies the Palais–Smale condition, i.e. any sequence $\{x_n\}\subset\Omega$ such that $f(x_n)$ is bounded and $\mathrm{d}f(x_n)\to 0$ as $n\to+\infty$ admits a convergent subsequence,
(iii) $M_n(f^b,f^a)$ is finite for every n and equal to zero for n large enough,

then there exists a polynomial $Q(r)$, with nonnegative integer coefficients, such that $M(r,f^b,f^a)=P(r,f^b,f^a)+(1+r)Q(r)$ where $P(r,f^b,f^a)$ is the Poincaré polynomial of the pair (f^b,f^a), i.e.

$$P\big(r,f^b,f^a\big)=\sum_{n=0}^{+\infty}B_n\big(f^b,f^a\big)r^n,$$

where $B_n(f^b,f^a)$ is the dimension of the n-th singular homology group of the pair (f^b,f^a) over the field $\mathbb{K}$.

Observe that under the assumptions (i) and (ii), f has only a finite number of critical points on the strip $f^{-1}([a,b])$. If in addition to (i)–(iii), we have also that

(iv) f is bounded from below,

then, choosing $a<\inf_\Omega f$, we get

$$M\big(r,f^b\big)=P\big(r,f^b\big)+(1+r)Q(r).$$

Theorem 9. *Let (M,F) be a Finsler manifold, and p_0,q_0 be two non-conjugate points in (M,F) and assume that F is forward or backward complete. Then there exists a formal series $Q(r)$ with coefficients in $\mathbb{N}\cup\{+\infty\}$ such that*

$$\sum_{x\in\Gamma}r^{\mu(x)}=P\big(r,\Omega_{p_0,q_0}(M)\big)+(1+r)Q(r), \tag{26}$$

where Γ is the set of all the geodesics connecting p_0 to q_0 and $\mu(x)$ is the number of conjugate instants, counted with their multiplicity, along the geodesic x.

Proof. Since the points p_0 and q_0 are non-conjugate in (M,F), any critical point x of E in $\Omega_{p_0,q_0}(M)$ is isolated and A has zero null space.

If (M,F) is forward or backward complete then E satisfies the Palais–Smale condition on $\Omega_{p_0,q_0}(M)$ (see [8, Theorem 3.1]) and it is bounded from below.

Using Theorem 7 we can compute the critical group $C_n(E,x)$. Let $\mathcal{O}_*$ be the image of the map φ_* in (5) associated to the critical point x and consider the functional $\tilde{E}$ in (3) associated to φ_*. Since the critical point x is non-degenerate, by Theorem 7, there exists a local homeomorphism $\phi: B=B(0,r)\to\phi(B)\subset D$ such that $\phi(0)=0$ and

$$\tilde{E}\big(\phi(\xi)\big)=\frac{1}{2}(A\xi,\xi)+\tilde{E}(0).$$

Let $O=\phi(B)$ and consider the deformation $\psi: O\times[0,1]\to O$ defined as

$$\psi(\xi,t)=\phi\big((1-t)\xi_+ +\xi_-\big),$$

where $\xi_+ + \xi_- = \phi^{-1}(\xi)$ and $\xi_+ \in H_+$ and $\xi_- \in H_-$, H_+ and H_- being the positive and the negative space of A according to its spectral decomposition in $H^1_0([0,1],\mathbb{R}^n)$ endowed with the scalar product (8). Since A is a compact deformation of the identity operator (see the proof of Lemma 2), we know that H_- is finite-dimensional.

Then ψ is a deformation retract of $\tilde{E}^c_{|X} \cap O$ to $\tilde{E}^c_{|X} \cap O_-$, where $O_- = \phi(B \cap H_-)$, $X = C^1_0([0,1],U)$ and $c = \tilde{E}(0)$. Therefore

$$H_n\big(\tilde{E}^c_{|X} \cap O, \tilde{E}^c_{|X} \cap O \setminus \{0\}\big) = H_n\big(\tilde{E}^c_{|X} \cap O_-, \tilde{E}^c_{|X} \cap O_- \setminus \{0\}\big) = \delta_{n,k}\mathbb{K}, \tag{27}$$

where k is the index of A as a bilinear form on $C^1_0([0,1],\mathbb{R}^n)$ or equivalently on $H^1_0([0,1],\mathbb{R}^n)$, that is $k = \dim(H_- \cap X) = \dim H_-$, and $\delta_{n,k}$ is Kronecker's delta. By [26, Theorems 41.1 and 43.2], we also have $k = \mu(x)$.

Since $C^1_0([0,1],\mathbb{R}^n)$ is immersed continuously in $H^1_0([0,1],U)$, by the excision property of the singular relative homology groups we have

$$H_n\big(\tilde{E}^c_{|X} \cap O, \tilde{E}^c_{|X} \cap O \setminus \{0\}\big) = H_n\big(\tilde{E}^c_{|X} \cap \tilde{O}^*, \tilde{E}^c_{|X} \cap \tilde{O}^* \setminus \{0\}\big) \tag{28}$$

where $\tilde{O}^*$ is any neighborhood of 0 in $H^1_0([0,1],U)$. On the other hand by [34, Theorems 16 and 17], and the fact that the map φ_* is a homeomorphism, we get (see also [12])

$$H_n\big(\tilde{E}^c_{|X} \cap \tilde{O}^*, \tilde{E}^c_{|X} \cap \tilde{O}^* \setminus \{0\}\big) = H_n\big(\tilde{E}^c \cap \tilde{O}^*, \tilde{E}^c \cap \tilde{O}^* \setminus \{0\}\big) = H_n\big(E^c \cap O^*, E^c \cap O^* \setminus \{x\}\big) \tag{29}$$

where $O^* = \varphi_*(\tilde{O}^*)$. Putting together Eqs. (27)–(29), we get

$$C_n(E,x) = \delta_{n,k}\mathbb{K}.$$

Therefore the assumptions (i)–(iv) are satisfied and the Morse relations

$$\sum_{z \in \Gamma \cap E^b} r^{\mu(z)} = P\big(r, E^b\big) + (1+r)Q(r) \tag{30}$$

hold. Finally, arguing as in the proof of [16, Theorem 1.7], we can pass to the limit on both sides of (30) obtaining (26), as $b \to +\infty$. □

Remark 10. Although the distance (2) associated to a Finsler metric is not a true distance due to the lack of symmetry, we can define a symmetric distance as

$$\mathrm{dist}_s(p,q) = \frac{1}{2}\big(\mathrm{dist}(p,q) + \mathrm{dist}(q,p)\big), \tag{31}$$

for every $p, q \in M$. Let us observe that if the closed balls for the symmetrized distance are compact, then the energy functional of the Finsler metric satisfies the Palais–Smale condition. This fact came out when studying the relation between causality and completeness of Fermat metrics (see [9, Theorems 4.3 and 5.2]). This condition is equivalent to have compact intersection $\bar{B}^+(p,r) \cap \bar{B}^-(p,r)$ for every $p \in M$ and $r > 0$, where $\bar{B}^+(p,r) = \{q \in M\colon \mathrm{dist}(p,q) \leqslant r\}$ and $\bar{B}^-(p,r) = \{q \in M\colon \mathrm{dist}(q,p) \leqslant r\}$ (see [9, Proposition 2.2]). If F is forward or backward complete, the Finslerian Hopf–Rinow theorem implies that F satisfies the above condition, but the reciprocal is not true (see [9, Example 4.6] for an example of a Finsler metric with compact symmetrized closed balls that is neither forward nor backward complete). In fact, the proof of the Palais–Smale condition for the Finslerian energy functional written out in [8, Proposition 3.1] works also under the above equivalent conditions (for example $\bar{B}^+(p,r) \cap \bar{B}^-(p,r)$ compact for every $p \in M$ and $r > 0$). For further details, see the comments before Theorem 5.2 in [9]. Then the Morse relations for geodesics connecting two non-conjugate points on a Finsler manifold (M,F) hold also under the more general assumption that the closed balls of the symmetrized distance associated to F are compact.

Remark 11. The restriction to C^1 curves, whose images are in a neighborhood of a given geodesic x, can be performed also for periodic boundary conditions. In the Finsler case, we have to take into account the equivariant action of $SO(2)$ on the free loop space $\Omega(M)$. Then a proof of the Morse relations for closed geodesics of a Finsler metric might be obtained along the same lines of [19, Lemma 4], considering the intersection of a tubular neighborhood of an isolated critical orbit $SO(2)x$ in $\Omega(M)$ with the Banach manifold $C^1(S,M)$.

3. Morse theory of lightlike geodesics

A conformally standard stationary spacetime is a Lorentzian manifold $(\mathcal{M}, g)$ such that $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ and

$$g(x,t)\big[(y,\tau),(y,\tau)\big] = a(x,t)\big(g_0(x)[y,y] + 2g_0(x)\big[\delta(x),y\big]\tau - \beta(x)\tau^2\big), \tag{32}$$

where $(x,t) \in \mathcal{M}_0 \times \mathbb{R}$, $(y,\tau) \in T_x\mathcal{M}_0 \times \mathbb{R}$, g_0 is a Riemannian metric on $\mathcal{M}_0$ and δ, β and a are, respectively, a smooth vector field on $\mathcal{M}_0$, a smooth positive function on $\mathcal{M}_0$ and a smooth positive function on $\mathcal{M}$. We denote by $\tilde{g}_0$ the conformal Riemannian metric g_0/β. Since lightlike geodesics and conjugate points along lightlike geodesics are preserved under conformal changes of the metric we can divide (32) by $a\beta$, and so we can assume that the metric g is given by

$$g(x,t)\big[(y,\tau),(y,\tau)\big] = \tilde{g}_0(x)[y,y] + 2\tilde{g}_0(x)\big[\delta(x),y\big]\tau - \tau^2. \tag{33}$$

By definition, a smooth lightlike curve $[a,b] \ni s \to \gamma(s) = (x(s), t(s)) \in \mathcal{M}$ has to satisfy the equation

$$\tilde{g}_0(x)[\dot{x},\dot{x}] + 2\tilde{g}_0(x)\big[\delta(x),\dot{x}\big]\dot{t} - \dot{t}^2 = 0,$$

and therefore the derivative of the t component is given by

$$\dot{t} = \sqrt{\tilde{g}_0(x)[\dot{x},\dot{x}] + \tilde{g}_0(x)\big[\delta(x),\dot{x}\big]^2} + \tilde{g}_0(x)\big[\delta(x),\dot{x}\big] \tag{34}$$

or

$$\dot{t} = -\sqrt{\tilde{g}_0(x)[\dot{x},\dot{x}] + \tilde{g}_0(x)\big[\delta(x),\dot{x}\big]^2} + \tilde{g}_0(x)\big[\delta(x),\dot{x}\big].$$

Notice that in the first case $\dot{t} > 0$ (the lightlike curve is future-pointing) while in the second one $\dot{t} < 0$ (the lightlike curve is past-pointing). The right-hand side of the first equation and minus the right-hand side of the second one define two Randers metrics on $\mathcal{M}_0$ that are denoted, respectively, by F and F_-:

$$\begin{aligned} F(x,y) &= \sqrt{\tilde{g}_0(x)[y,y] + \tilde{g}_0(x)\big[\delta(x),y\big]^2} + \tilde{g}_0(x)\big[\delta(x),y\big], \\ F_-(x,y) &= \sqrt{\tilde{g}_0(x)[y,y] + \tilde{g}_0(x)\big[\delta(x),y\big]^2} - \tilde{g}_0(x)\big[\delta(x),y\big]. \end{aligned} \tag{35}$$

Such Randers metrics play an important role in the study of lightlike and timelike geodesics on a conformally standard stationary spacetime as we see below and, moreover, they give a lot of information about the causal structure of such type of spacetimes (see [8]). As in [8,9] we call the Randers metric F the *Fermat metric* associated to $(\mathcal{M}, g)$.

Lightlike geodesics connecting an event p on a spacetime with a timelike curve $\gamma : (a,b) \to \mathcal{M}$ can be characterized by a variational principle (a *Fermat principle*) stating that, among all the future-pointing (or past-pointing) lightlike curves $z : [0,1] \to \mathcal{M}$ such that $z(0) = 0$ and $z(1) \in \gamma((a,b))$, the lightlike geodesics are all and only the curves making stationary the arrival time functional T that is the functional $z \mapsto T(z) = \gamma^{-1}(z(1))$. This is a fairly well known fact since the beginning of general relativity, but a precise formulation with the above generality and a rigorous proof was given only in the '90s by I. Kovner and V. Perlick (see [24,35]). In the case of a conformally standard stationary spacetime, if we consider an observer whose world line γ is the vertical line $\mathbb{R} \ni s \to (x_1, s) \in \mathcal{M}$, the arrival time T coincides with the value of the global time coordinate t at the endpoint of the curve $[0,1] \ni s \to z(s) = (x(s), t(s)) \in \mathcal{M}$. Therefore, for a future-pointing lightlike curve, from (34) we get

$$T(z) \equiv T(x) = t_0 + \int_0^1 F(x,\dot{x})\,\mathrm{d}s,$$

hence $T(z)$ is equal, up to an additive constant, to the length with respect to F of the projection of z on $\mathcal{M}_0$. Kovner's Fermat principle can be formulated as follows (see [8, Theorem 4.4]).

Proposition 12 *(Fermat principle). Let* $(\mathcal{M}, g)$ *be a standard stationary spacetime,* $p = (p_0, t_0) \in \mathcal{M}$*,* $[0,1] \ni s \to \gamma(s) = (q_0, s) \in \mathcal{M}$*,* $p_0, q_0 \in \mathcal{M}_0$*. A curve* $[0,1] \ni s \to z(s) = (x(s), t(s)) \in \mathcal{M}$ *is a future-pointing lightlike* *geodesic of* $(\mathcal{M}, g)$ *as in* (33) *if and only if* $[0,1] \ni s \to x(s) \in \mathcal{M}_0$ *is a geodesic of the Fermat metric* F*, parametrized with constant Riemannian speed* $\tilde{g}_0(x)[\delta(x), \dot{x}]^2 + \tilde{g}_0(x)[\dot{x}, \dot{x}] = const.$*, and* $t(s)$ *is given by*

$$t(s) = t_0 + \int_0^s F(x, \dot{x})\,\mathrm{d}v.$$

By the Fermat principle the search of lightlike geodesics in a stationary spacetime can be reduced to the search of geodesics in the Finsler manifold $(\mathcal{M}_0, F)$.

Let $z = (x,t) : [0,1] \to \mathcal{M}$ be a future-pointing lightlike geodesic. By Proposition 12, x is a geodesic in $(\mathcal{M}_0, F)$. We denote by $\mu(z)$ (resp. $\mu(x)$) the *geometric index* of z (resp. x), that is the number of conjugate points along z (resp. x) counted with their multiplicity.

We recall that on a Lorentzian manifold $(\mathcal{M}, g)$ the notions of Jacobi vector field, conjugate instant and non-conjugate points are given, as on a Riemannian manifold, using the Levi-Civita connection and the Riemannian curvature tensor (see for instance [33]).

We are going to show that the geometric index of z coincides with the geometric index of its spatial projection x as a geodesic of the Fermat metric. This fact allows us to bring the Morse theory for Finsler geodesics to the Morse theory of lightlike geodesics.

Theorem 13. *Let* $(\mathcal{M}, g)$ *be a conformally standard stationary spacetime,* $[0,1] \ni s \to z(s) = (x(s), t(s)) \in \mathcal{M}$ *be a future-pointing lightlike geodesic. Let* F *be the Fermat metric associated to* $(\mathcal{M}, g)$*. Then the points* $x(0)$ *and* $x(1)$ *are non-conjugate along the geodesic* x *in* $(\mathcal{M}_0, F)$ *if and only if the points* $z(0)$ *and* $z(1)$ *are non-conjugate along the lightlike geodesic* z *in* $(\mathcal{M}, g)$*. Moreover*

$$\mu(z) = \mu(x). \tag{36}$$

Proof. As conjugate points of lightlike geodesics are preserved by conformal changes with their multiplicity, we can consider the metric g as in (33), which can be expressed as

$$g(x)[v,v] = \alpha(x)[v,v] - \big(\tau - \alpha(x)[v,\eta]\big)^2, \tag{37}$$

where $\alpha(x)[v,v] = \tilde{g}_0(x)[v,v] + \tilde{g}_0(x)[v,\delta(x)]^2$ and $\alpha(x)[v,\eta(x)] = \tilde{g}_0(x)[v,\delta(x)]$ for every $v \in T_xM$. Let $\bar{\nabla}$ be the Levi-Civita connection of the metric α and consider the $(1,1)$-tensor field Ω on $\mathcal{M}_0$ defined as

$$\Omega[\dot{x}] = (\bar{\nabla}\eta)[\dot{x}] - (\bar{\nabla}\eta)^*[\dot{x}],$$

where $(\bar{\nabla}\eta)[\dot{x}] = \bar{\nabla}_{\dot{x}}\eta$ and $(\bar{\nabla}\eta)^*$ is the adjoint with respect to α of $\bar{\nabla}\eta$.

Since ∂_t is a Killing vector field for $(\mathcal{M}, g)$, we know that for any geodesic $z = (x,t)$ in $(\mathcal{M}, g)$ there exists a constant C_z such that

$$C_z = \dot{t} - \alpha(x)[\dot{x}, \eta]. \tag{38}$$

Then considering variation vector fields having vanishing t component, from (37), one can easily see that the x component of a geodesic z of $(\mathcal{M}, g)$, as a critical point of the functional (1), has to satisfy the equation

$$\bar{\nabla}_{\dot{x}}\dot{x} = -C_z\Omega[\dot{x}]. \tag{39}$$

The linearized equations of this system (38)–(39) are

$$\begin{aligned} J'' &= -R(J,\dot{x})\dot{x} - C_{J,W}\Omega[\dot{x}] - C_z(\bar{\nabla}_J\Omega)[\dot{x}] - C_z\Omega\big[J'\big],\\ W' &= C_{J,W} + \alpha(x)\big[J',\eta\big] + \alpha(x)[\dot{x}, \bar{\nabla}_J\eta]. \end{aligned} \tag{40}$$

On the other hand, the Fermat metric can be expressed as $F(x,v) = \sqrt{\alpha(x)[v,v]} + \alpha(x)[v,\eta]$ and its geodesics with constant α-Riemannian speed are determined by

$$\bar{\nabla}_{\dot{x}}\dot{x} = -C_x\Omega[\dot{x}], \tag{41}$$

where $C_x = \sqrt{\alpha(\dot{x},\dot{x})}$ (see [5, boxed formula at p. 297]). The linearized equation of (41) is

$$J'' = -R(J,\dot{x})\dot{x} - \frac{\alpha(x(0))[\dot{x}(0), J'(0)]}{C_x}\Omega[\dot{x}] - C_x(\bar{\nabla}_J\Omega)[\dot{x}] - C_x\Omega\big[J'\big]. \tag{42}$$

If (J, W) is a Jacobi field of z satisfying (40) with $(J(0), W(0)) = (J(s_0), W(s_0)) = (0, 0)$, then from (40), using integration by parts, Eq. (39) and the fact that $\alpha(x)[\dot{x}, J']$ is constant along x (as can be verified by a direct computation, taking into account that the operators Ω and $\bar{\nabla}_J\Omega$ are skew-symmetric), we obtain the following chain of identities:

$$\begin{aligned} W(s_0) &= s_0C_{J,W} + \int_0^{s_0}\big(\alpha(x)\big[J', \eta\big] + \alpha(x)[\dot{x}, \bar{\nabla}_J\eta]\big)\,\mathrm{d}s \\ &= s_0C_{J,W} + \int_0^{s_0}\alpha(x)\big[\Omega[\dot{x}], J\big]\,\mathrm{d}s \\ &= s_0C_{J,W} - \frac{1}{C_z}\int_0^{s_0}\alpha(x)[\bar{\nabla}_{\dot{x}}\dot{x}, J]\,\mathrm{d}s \\ &= s_0C_{J,W} + \frac{s_0}{C_z}\alpha\big(x(0)\big)\big[\dot{x}(0), J'(0)\big]. \end{aligned}$$

We observe that $C_z \neq 0$ because z is lightlike. As $W(s_0) = 0$, last formula implies that $C_{J,W} = -(\alpha(x(0))[\dot{x}(0), J'(0)])/C_z$ and therefore J satisfies (42) taking $C_x = C_z$. Analogously, we can show that if J satisfies (42) and has a conjugate instant s_0, then we can construct a Jacobi vector field (J, W) satisfying the system (40) with $C_z = C_x$, $C_{J,W} = -(\alpha(x(0))[\dot{x}(0), J'(0)])/C_z$ and having a conjugate instant in s_0. In conclusion, there is a bijection between the Jacobi vector fields of z vanishing in 0 and s_0 and the Jacobi vector fields of x (as a Fermat geodesic) that are zero in 0 and s_0. This concludes the proof. □

Remark 14. The conjugate points of a Fermat geodesic x, when it is parametrized with constant α-Riemannian speed, coincide with the conjugate points when it is parametrized with constant Fermat speed. Indeed, let J be a Jacobi vector field and $\Gamma : [0,1] \times (-\varepsilon, \varepsilon) \to \mathcal{M}_0$ be a variation, by means of geodesics parametrized with constant Finsler speed, generating J. Then we can consider a variation of geodesics with constant α-Riemannian speed as $[0,1] \times (-\varepsilon, \varepsilon) \ni (s, w) \to \tilde{\Gamma}(s, w) = \Gamma(\psi_w(s), w) \in \mathcal{M}_0$, where $[0,1] \ni s \to \psi_w(s) \in [0,1]$ is the reparametrization giving geodesics with constant α-Riemannian speed. Consequently, the Jacobi vector field $\tilde{J}$ corresponding to the variation $\tilde{\Gamma}$ can be expressed as $\tilde{J}(s) = \sigma(s)\dot{x}(s) + J(\psi_0(s))$ for every $s \in [0,1]$ (here $x(s) = \tilde{\Gamma}(s, 0)$). We observe that if $[0,1] \ni s \to \sigma(s)\dot{x}(s)$ is a Jacobi vector field along x, then σ must be an affine function. This can be easily seen, using that $\alpha(x)[\dot{x}, J']$ is constant. It follows that if we take J such that $J(0) = J(t_0) = 0$ with $t_0 = \psi_0(s_0)$, then $\bar{J}(s) = \tilde{J}(s) - \frac{\sigma(s_0)}{s_0}s\,\dot{x}(s)$ satisfies $\bar{J}(0) = \bar{J}(s_0) = 0$ and it is the unique Jacobi vector field of the type $\bar{J}(s) = \tilde{J}(s) + \sigma(s)\dot{x}(s)$ satisfying this property. Conversely, if $\bar{J}$ is a Jacobi vector field generated by a variation of geodesics with constant α-Riemannian speed and such that $\bar{J}(0) = \bar{J}(s_0) = 0$, there exists a function $\sigma : [0,1] \to \mathbb{R}$ and a family of reparametrizations $[0,1] \ni t \to \phi_w(t) \in [0,1]$ such that $J(t) = \sigma(t)\dot{x}(t) + \bar{J}(\phi_w(t))$ is a Jacobi vector field corresponding to a variation of geodesics parametrized with constant Finslerian speed. Using again the fact that $\sigma(t)\dot{x}(t)$ is a Jacobi vector field if and only if $\sigma(t)$ is an affine function (which can be seen now directly by the Jacobi equation in Finsler geometry, see for instance [41, formula (6.1)]) we conclude as before that there exists a unique Jacobi field J corresponding to $\bar{J}$ such that $J(0) = J(t_0) = 0$, where $t_0 = \phi_0(s_0)$. Therefore there is a bijection between the conjugate points preserving the points in the geodesic and the order of conjugacy.

We pass now to study the Morse relations for lightlike geodesics connecting $p = (p_0, 0)$ to the curve $\mathbb{R} \ni s \to \gamma(s) = (q_0, s) \in \mathcal{M}_0 \times \mathbb{R}$, $p_0, q_0 \in \mathcal{M}_0$.

Theorem 15. *Let $(\mathcal{M}, g)$ be a globally hyperbolic conformally standard stationary spacetime, $p = (p_0, t_0) \in \mathcal{M}$ and $\mathbb{R} \ni s \to \gamma(s) = (q_0, s) \in \mathcal{M}$. Assume that for each $s \in \mathbb{R}$ the points p and (q_0, s) are non-conjugate along* *every future-pointing lightlike geodesic connecting them. Then there exists a formal series $Q(r)$ with coefficients in $\mathbb{N}\cup\{+\infty\}$ such that*

$$\sum_{z\in G_{p,\gamma}} r^{\mu(z)} = P\big(r, \Omega_{p_0,q_0}(\mathcal{M}_0)\big) + (1+r)Q(r), \tag{43}$$

where $G_{p,\gamma}$ is the set of all the future-pointing lightlike geodesics connecting p to γ.

Proof. Let F be the Fermat metric in (35). From Proposition 12, any geodesic x in $(\mathcal{M}_0, F)$ connecting p_0 to q_0 corresponds to a future-pointing lightlike geodesic $[0,1]\ni s\to z(s)=(x(s),t(s))\in\mathcal{M}$ connecting p to γ and vice versa. From Theorem 13, the points p_0 and q_0 are non-conjugate in $(\mathcal{M}_0, F)$ and $\mu(x)=\mu(z)$. Moreover, by [9, Theorem 4.3] as $(\mathcal{M}, g)$ is globally hyperbolic, the Fermat metric F has compact symmetrized closed balls. Then (43) comes directly from (26) and Remark 10. □

Remark 16. Observe that taking $r=1$ in (43) gives

$$\sum_{n=0}^{\infty} N_n = \sum_{n=0}^{\infty} B_n\big(\Omega_{p_0,q_0}(\mathcal{M}_0)\big) + 2Q(1),$$

where N_n is the number of future-pointing lightlike geodesics having index n. If $\mathcal{M}_0$ is contractible then $B_0(\Omega_{p_0,q_0}(\mathcal{M}_0))=1$ and $B_n(\Omega_{p_0,q_0}(\mathcal{M}_0))=0$ for all $n\geqslant 1$. Therefore the number of future-pointing lightlike geodesics joining p to γ is infinite or odd.

Remark 17. The Morse relations of lightlike geodesics connecting p to γ in a standard stationary spacetime were obtained in [14] by using the functional

$$\tilde{J}(x) = \int_0^1 g_0(x)\big[\delta(x),\dot{x}\big]\,\mathrm{d}s + \left(\int_0^1 \big(g_0(x)\big[\delta(x),\dot{x}\big]^2 + g_0(x)[\dot{x},\dot{x}]\big)\,\mathrm{d}s\right)^{\frac{1}{2}},$$

and the following Fermat principle: a curve $[0,1]\ni s\to z(s)=(x(s),t(s))\in\mathcal{M}$ is a future-pointing lightlike geodesic connecting $p=(p_0,t_0)$ and $\mathbb{R}\ni s\to\gamma(s)=(q_0,s)\in\mathcal{M}$ if and only if x is a critical point of $\tilde{J}$ and $t(s)=t_0+\int_0^s F(x,\dot{x})\,\mathrm{d}v$. In [14], it was also claimed that the Morse index of a critical point x of $\tilde{J}$ is equal to the geometrical index of the corresponding lightlike geodesic z, but there is a gap in the proof of that statement.

4. Morse theory of timelike geodesics

The reduction of Morse theory of lightlike geodesics connecting a point with a timelike line on a stationary spacetime $(\mathcal{M}_0\times\mathbb{R}, g)$ to Morse theory of geodesics of a Finsler metric on $\mathcal{M}_0$ can be also carried out for timelike geodesics. Namely timelike geodesics can be viewed as projections on $\mathcal{M}$ of lightlike geodesics in a one-dimensional higher stationary spacetime as follows.

Let $(\mathcal{M}, g)$ be a standard stationary spacetime (that is g is given by (32) and $a(x,t)=1$; since timelike geodesics are not invariant under conformal changes of the metric, this time we cannot divide g by β). We seek for timelike geodesics $z:[0,\bar{s}]\to\mathcal{M}$ connecting a point $(p_0,t_0)\in\mathcal{M}$ with a timelike curve $\mathbb{R}\ni s\to\gamma(s)=(q_0,s)\in\mathcal{M}$ and parametrized with respect to proper time i.e. $E_z=g(z)[\dot{z},\dot{z}]=-1$, for all $s\in[0,\bar{s}]$.

We extend the Riemannian manifold $\mathcal{M}_0$ to the manifold $\mathcal{N}_0=\mathcal{M}_0\times\mathbb{R}$ endowed with the metric $n_0=g_0+\mathrm{d}u^2$ where u is the natural coordinate on $\mathbb{R}$, and we associate to the manifold $\mathcal{N}_0$ the one-dimensional higher Lorentzian manifold $(\mathcal{N}, n)$, with the metric n defined as

$$n(x,u,t)\big[(y,v,\tau),(y,v,\tau)\big] = g_0(x)[y,y] + v^2 + 2g_0(x)\big[\delta(x),y\big]\tau - \beta(x)\tau^2. \tag{44}$$

Since ∂_u is a Killing vector field for the metric n, geodesics $\varsigma=(x,u,t)$ in $(\mathcal{N}, n)$ have to satisfy also the conservation law $n[\dot{\varsigma},\partial_u]=\text{const.}$, which implies that the u component of a geodesic is an affine function. Moreover the projection $[a,b]\ni s\to z(s)=(x(s),t(s))\in\mathcal{M}$ of ς is a geodesic for $(\mathcal{M}, g)$. In particular lightlike geodesics of the metric n satisfy the following equation

$$g_0[\dot{x},\dot{x}] + 2g_0[\delta,\dot{x}]\dot{t} - \beta\dot{t}^2 = -\dot{u}^2 = \text{const.}$$

Thus in order to find timelike geodesics $z = (x, t)$ in $(\mathcal{M}, g)$, parametrized with respect to proper time, it is enough to find lightlike geodesics in $(\mathcal{N}, n)$ whose u component has derivative equal to 1. The Fermat principle can be restated in $(\mathcal{N}, n)$, reducing future-pointing lightlike geodesics on $(\mathcal{N}, n)$ to geodesics for the Fermat metric $\tilde{F}$ on the manifold $\mathcal{N}_0$, where $\tilde{F}$ is given by

$$\tilde{F}\big((x,u),(y,v)\big) = \sqrt{\frac{1}{\beta(x)}\big(g_0[y,y]+v^2\big) + \frac{1}{\beta(x)^2} g_0\big[\delta(x),y\big]^2} + \frac{1}{\beta(x)} g_0\big[\delta(x),y\big],$$

for all $((x,u),(y,v)) \in T\mathcal{N}_0$. Summing up Theorems 13 and 15 we get:

Theorem 18. *Let $(\mathcal{M}, g)$ be a standard stationary spacetime, $[0,\bar{s}] \ni s \to z(s) = (x(s), t(s)) \in \mathcal{M}$ be a future-pointing timelike geodesic connecting the point $p = (p_0, t_0)$ to the curve $\mathbb{R} \ni s \to \gamma(s) = (q_0, s) \in \mathcal{M}$, $q_0 \in \mathcal{M}_0$. Let $\tilde{F}$ be the Fermat metric associated to $(\mathcal{N}, n)$. Then the points $(p_0, 0)$ and $(q_0, \bar{s})$ are non-conjugate along the geodesic $[0,\bar{s}] \ni s \to \tilde{x}(s) = (x(s), s)$ in $(\mathcal{N}_0, \tilde{F})$ if and only if the points p and $(q_0, t(\bar{s}))$ are non-conjugate along the timelike geodesic z in $(\mathcal{M}, g)$. Moreover*

$$\mu(z) = \mu(\tilde{x}).$$

If for each $s \in \mathbb{R}$, the points p and (q_0, s) are non-conjugate along every future-pointing timelike geodesic, parametrized with respect to proper time on the interval $[0,\bar{s}]$ and connecting them, and $(\mathcal{M}, g)$ is globally hyperbolic, then there exists a formal series $Q(r)$ with coefficients in $\mathbb{N} \cup \{+\infty\}$ such that

$$\sum_{z \in \mathcal{T}_{p,\gamma}} r^{\mu(z)} = P\big(r, \Omega_{p_0,q_0}(\mathcal{M}_0)\big) + (1+r)Q(r),$$

where $\mathcal{T}_{p,\gamma}$ is the set of all the future-pointing timelike geodesics $[0,\bar{s}] \ni s \to z(s) = (x(s), t(s)) \in \mathcal{M}$ parametrized with respect to proper time and such that $z(0) = p$ and $x(\bar{s}) = q_0$.

Proof. The first part of the theorem comes arguing as in Theorem 13, observing that a Jacobi vector field $\xi = (U, \Upsilon)$ along the lightlike geodesic $[0,\bar{s}] \ni s \to \varsigma(s) = (x(s), t(s), s)$ in $(\mathcal{N}, n)$, with vanishing endpoints, has Υ component equal to 0 and U component which is a Jacobi vector field along the timelike geodesic z.

The second part comes arguing as in Theorem 15, after having observed that if the Fermat metric F associated to $(\mathcal{M}, g)$ has compact symmetrized closed balls the same holds for $\tilde{F}$. Namely if $\{(x_n, u_n)\} \subset \mathcal{N}_0$ is contained in a symmetrized closed ball of center $(x, u) \in \mathcal{M}_0 \times \mathbb{R}$ and radius $r > 0$, then it is easy to see that x_n is contained in the symmetrized closed ball for F of center x and radius r, which is compact because $(\mathcal{M}, g)$ is globally hyperbolic (see [9, Theorem 4.3]). Therefore, there is a subsequence x_{n_k} of x_n that converges and β is bounded on this subsequence. Thus also u_{n_k} admits a convergent subsequence u_{n_l} in $\mathbb{R}$ and therefore $\{(x_{n_l}, u_{n_l})\}$ converges. Finally, observe that the manifold $\Omega_{(p_0,0),(q_0,\bar{s})}(\mathcal{N}_0)$ is homotopically equivalent to $\Omega_{p_0,q_0}(\mathcal{M}_0)$. □

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Appendix A

In a paper about Morse theory of causal geodesics in a globally hyperbolic Lorentzian manifold [43], K. Uhlenbeck introduced the following functional

$$J(z) = \int_0^1 \left(g(z)[\dot{z}, \dot{z}] + \left(\frac{\mathrm{d}Pz}{\mathrm{d}s}\right)^2 \right) \mathrm{d}s, \tag{45}$$

defined on the set of piecewise differentiable curves on $\mathcal{M}$ satisfying the constraint $g(z)[\dot{z}, \dot{z}] = 0$ and the boundary conditions $z(0) = p$, $z(1) = (q_0, Pz(1))$, where $P : \mathcal{M} \to \mathbb{R}$ is the natural projection on $\mathbb{R}$, and proved that critical points of J are all and only the lightlike geodesics connecting p to the line $s \mapsto (q_0, s)$.

In this appendix we study, whenever $\mathcal{M}$ is conformally standard stationary, the relation between J and the energy functional E of the Fermat metric. We show (Proposition 21) that the Morse index $m_J(z)$ of a critical point $z=(x,t)$ of J is equal to the Morse index $m_E(x)$ of x as a critical point of E.

This fact provides a variational and alternative proof of the equality (36) since by Theorem 19, $\mu(z)=m_J(z)$ and by [26, Theorems 41.1 and 43.2] we also have $\mu(x)=m_E(x)$.

As K. Uhlenbeck observed, the constraint equation $g(z)[\dot z,\dot z]=0$ does not define a smooth submanifold of the set of piecewise differentiable curves in $\mathcal{M}$, however J is differentiable if viewed as a functional on the set of piecewise differentiable regular (i.e. $\dot z(s)\neq 0$, where it is defined) curves on $\mathcal{M}_0$. This can be done after solving the constraint equation with respect to $\dot t$, for any fixed x. The Lorentzian metric considered in [43] is of the type $g(x,t)[(y,\tau),(y,\tau)]=g_0(x,t)[y,y]-\tau^2$ where, for any $t\in\mathbb{R}$, $g_0(\cdot,t)$ is a Riemannian metric on $\mathcal{M}_0$. The solutions t_x of the differential equation

$$\dot t=\sqrt{g_0(x,t)[\dot x,\dot x]},$$

arising from the constraint equation, are defined on the whole interval [0, 1] if a rather technical growth assumption on the metric g_0 is fulfilled. The critical points of J are exactly the lightlike geodesics $[0,1]\ni s\to z(s)=(x(s),t(s))\in\mathcal{M}$ connecting the point p to the curve $\mathbb{R}\ni s\to\gamma(s)=(q_0,s)\in\mathcal{M}$, parameterized with $\dot t$ constant. Moreover she proved the following Morse index theorem (see [43, Lemma 4.2]):

Theorem 19 *(Uhlenbeck). J is twice Gateaux differentiable at any critical point* (*a lightlike geodesic*). *Its second derivative at a critical point z is given by*

$$D^2J(z)[U,V]=\int_0^1\left(\frac{\mathrm{d}}{\mathrm{d}s}\Big(g(z)[\nabla P,U]\Big)\frac{\mathrm{d}}{\mathrm{d}s}\big(g(z)[\nabla P,V]\big)+c(s)\big(g(z)[\nabla_{\dot z}U,\nabla_{\dot z}V]-g(z)\big[R(\dot z,U)\dot z,V\big]\big)\right)\mathrm{d}s,\tag{46}$$

where U and V are piecewise smooth vector fields along z such that $U(0)=V(0)=0=V(1)=U(1)$, $g(z)[\dot z,U]=g(z)[\dot z,V]=0$, c is a function such that $\zeta=c(s)\dot z$, ζ the parallel transport of $\dot z(1)$ along z, ∇P is the gradient of P and R is the curvature tensor of $(\mathcal{M},g)$. A critical point is non degenerate if and only if its endpoints are non-conjugate. The index of a critical point is equal to its geometrical index as a lightlike geodesic, that is the number of conjugate points counted with their multiplicity.

Remark 20. The above theorem is based on the existence of a global time function P and on the fact that z is a lightlike geodesic. It does not depend on the form of the metric g, neither on assumptions on the metric coefficients, nor on topological assumptions as global hyperbolicity. This fact was exploited in the paper [16]. The key point in the proof of Theorem 19 is that, if z is a lightlike geodesic, the bilinear form at the right-hand side of (46) is a compact perturbation of a positive definite invertible operator on the H^1 completion of the space of piecewise smooth vector fields U along z satisfying the condition $g[\dot z,U]=0$.

Thus the Morse index of J at a critical point is finite and is equal to the sum of the dimensions of the kernels of the above bilinear forms along $z(s)$ which are isomorphic to the space of Jacobi vector fields along z which vanish at the initial point and in some other point $\bar s\in(0,1)$.

In the following proposition the equality between the Morse index of J and E is stated.

Proposition 21. *Under the assumptions of Theorem* 13, *we have that*

$$m_J(z)=m_E(x),\tag{47}$$

where $m_J(z)$ and $m_E(x)$ are respectively the Morse indexes of the functionals J and E at their critical points z and x.

Proof. Let us denote by I the functional given by (45) defined on the manifold $\Omega_{p,\gamma}(\mathcal{M})$ of H^1-curves in $\mathcal{M}$ connecting p to $\gamma(\mathbb{R})$. Observe that J is equal to the functional I restricted to the set $\Lambda_{p,\gamma}(\mathcal{M})\subset\Omega_{p,\gamma}(\mathcal{M})$ of future-pointing curves such that $g(z)[\dot z, \dot z] = 0$, a.e. on [0, 1]. Consider the map $\Psi : \Omega_{p_0,q_0}(\mathcal{M}_0) \to \Lambda_{p,\gamma}(\mathcal{M})$ defined by

$$\Psi(x)(s) = \left(x(s), t_0 + \int_0^s F(x, \dot x)\, \mathrm{d}\nu \right).$$

Observe that I is a smooth functional and Ψ is differentiable at any regular curve x. Clearly we have that $J(z) = (I \circ \Psi)(x) = 2E(x)$ and, for any $\xi, \eta \in T_x\Omega_{p_0,q_0}(\mathcal{M}_0)$, $\mathrm{d}\Psi(x)[\xi]$ is equal to

$$\mathrm{d}\Psi(x)[\xi](s) = \left(\xi(s), \int_0^s \big(F_x(x, \dot x)[\xi] + F_y(x, \dot x)[\dot \xi] \big)\, \mathrm{d}s \right), \tag{48}$$

hence $\mathrm{d}\Psi(x)$ is an injective map. Since x is a critical point of the length functional $x \mapsto \int_0^1 F(x, \dot x)\, \mathrm{d}s$, we have that for any $\xi \in T_x\Omega_{p_0,q_0}(\mathcal{M}_0)$, $\mathrm{d}\Psi(x)[\xi](0) = \mathrm{d}\Psi(x)[\xi](1) = 0$. Let now $U(s) = (U_0(s), \tau(s))$ be a vector field along z such that $U(0) = U(1) = 0$ and $g(z)[\dot z, U] = 0$. We are going to show that $\mathrm{d}\Psi(x)[U_0] = U$ and hence $\mathrm{d}\Psi(x)$ is an isomorphism between the space of piecewise smooth vector fields along x vanishing at the endpoints and the space of admissible variations for J (see Theorem 19). Observe that $g(z)[U, \dot z] = 0$ implies that

$$\tilde g_0(x)[U_0, \dot x] + \tilde g_0(x)\big[\delta(x), U_0\big]\dot t + \tilde g_0(x)\big[\delta(x), \dot x\big]\tau - \tau \dot t = 0.$$

By (34) we get

$$\tilde g_0(x)[U_0, \dot x] + \tilde g_0(x)\big[\delta(x), U_0\big]F(x, \dot x) - \tau\sqrt{\alpha(x)[\dot x, \dot x]} = 0,$$

where $\alpha(x)[\dot x, \dot x] = \tilde g_0(x)[\dot x, \dot x] + \tilde g_0(x)[\delta(x), \dot x]^2$. Hence

$$\tau = \frac{\tilde g_0(x)[U_0, \dot x] + \tilde g_0(x)[\delta(x), U_0]F(x, \dot x)}{\sqrt{\alpha(x)[\dot x, \dot x]}}.$$

From (48), since x is a geodesic for the metric F and $U_0(0) = 0$, the t component of the vector field $\mathrm{d}\Psi(x)[U_0]$ is equal to $F_y(x, \dot x)[U_0(s)]$, which is given by

$$\begin{aligned} F_y(x, \dot x)[U_0(s)] &= \tilde g_0(x)\big[\delta(x), U_0\big] + \frac{\tilde g_0(x)[\delta(x), U_0]\tilde g_0(x)[\delta(x), \dot x] + \tilde g_0(x)[U_0, \dot x]}{\sqrt{\alpha(x)[\dot x, \dot x]}} \\ &= \tau(s). \end{aligned}$$

Let $\varphi = \varphi(r, s) : (-\varepsilon, \varepsilon) \times [0, 1] \to \mathcal{M}$ be a variation defined by the admissible variational vector field $U = (U_0, \tau)$, and $\varphi_0 = \varphi_0(r, s) : (-\varepsilon, \varepsilon) \times [0, 1] \to \mathcal{M}_0$ be the one defined by U_0, we have that

$$\begin{aligned} D^2 J(z)[U, U] &= \frac{\mathrm{d}^2}{\mathrm{d}r^2} J\big(\varphi(r, \cdot)\big)_{|r=0} \\ &= \frac{\mathrm{d}^2}{\mathrm{d}r^2} I\big(\Psi\big(\varphi_0(r, \cdot)\big)\big)_{|r=0} = 2\frac{\mathrm{d}^2}{\mathrm{d}r^2} E\big(\varphi_0(r, \cdot)\big)_{|r=0} = 2D^2E(x)[U_0, U_0]. \end{aligned} \tag{49}$$

From (49), by polarization, we get the equality between (46) and the index form of the metric F and then the equality (47). □

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