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Asymmetric potentials and motor effect: a homogenization approach

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Abstract

We provide a mathematical analysis for the appearance of motor effects, i.e., the concentration (as Dirac masses) at one side of the domain, for the solution of a Fokker–Planck system with two components, one with an asymmetric potential and diffusion and one with pure diffusion. The system has been proposed as a model for motor proteins moving along molecular filaments. Its components describe the densities of different conformations of proteins.

Contrary to the case with two asymmetric potentials, the case at hand requires a large number of periods in order for the motor effect to occur. It is therefore posed as a homogenization problem where the diffusion length is at the same scale as the period of the potential.

Our approach is based on the analysis of a Hamilton–Jacobi equation arising, at the zero diffusion limit, after an exponential transformation of the phase functions. The homogenization procedure yields an effective Hamiltonian whose properties are closely related to the concentration phenomena.

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1. Introduction

Living cells are able to generate motion as, for example, in muscle contraction. Even more elementary processes, known as "motor proteins," allow for intra-cellular material transport along various filaments that are part of the cytoskeleton. For example, myosins move along actin filaments and kinesins and dyneins move along micro-tubules. Experiments in the early 90's lead to an improved biological understanding of the biomotor process. The experimental observations made it possible to arrive at mathematical models for molecular motors [4,8,15,16,20,22,23,28]. The underlying principles are elementary. The filament provides an asymmetric potential (energy landscape) while the protein can reside in several different conformational states.

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In this paper we consider the following model: A bath of molecules that can reach two configurations with densities $n^{(1)}$ and $n^{(2)}$ is moving in an asymmetric potential seen only by the first configuration. Fuel consumption triggers a configuration change between the two states with rates $\nu^{(1)}$ and $\nu^{(2)}$. Diffusion, denoted by ε, is taken into account. The potential and the transition rates are assumed to be periodic and may oscillate at scale ε.

These considerations lead to the following simple Fokker–Planck system of elliptic equations for the densities $n_\varepsilon^{(1)}$ and $n_\varepsilon^{(2)}$:

$$\begin{cases} -\varepsilon n^{(1)}_{\varepsilon,xx} - \left(\psi_y\left(\dfrac{x}{\varepsilon}\right)n_\varepsilon^{(1)}\right)_x + \dfrac{1}{\varepsilon}\nu^{(1)}\left(\dfrac{x}{\varepsilon}\right)n_\varepsilon^{(1)} = \dfrac{1}{\varepsilon}\nu^{(2)}\left(\dfrac{x}{\varepsilon}\right)n_\varepsilon^{(2)}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } (0,1), \\ -\varepsilon n^{(2)}_{\varepsilon,xx} + \dfrac{1}{\varepsilon}\nu^{(2)}\left(\dfrac{x}{\varepsilon}\right)n_\varepsilon^{(2)} = \dfrac{1}{\varepsilon}\nu^{(1)}\left(\dfrac{x}{\varepsilon}\right)n_\varepsilon^{(1)} \\ \varepsilon n^{(1)}_{\varepsilon,x} + \psi_y\left(\dfrac{x}{\varepsilon}\right)n_\varepsilon^{(1)} = n^{(2)}_{\varepsilon,x} = 0 \quad \text{for } x = 0 \text{ or } 1. \end{cases} \tag{1.1}$$

We assume that

$$\text{the potential } \psi \text{ and the strictly positive rates } \nu^{(1)},\ \nu^{(2)} \text{ are smooth and 1-periodic} \tag{1.2}$$

and we choose $\varepsilon = 1/N$ for some integer N, so that

$$\psi\left(\frac{1}{\varepsilon}\right) = \psi(0) \quad \text{and} \quad \psi_y\left(\frac{1}{\varepsilon}\right) = \psi_y(0).$$

System (1.1) is an eigenvalue problem and we choose the solution normalized as

$$n_\varepsilon^{(1)} > 0 \quad \text{and} \quad n_\varepsilon^{(2)} > 0 \quad \text{in } [0,1] \quad \text{and} \quad \max_{[0,1]}\left(n_\varepsilon^{(1)} + n_\varepsilon^{(2)}\right) = 1. \tag{1.3}$$

The zero flux boundary conditions mean that the total number of molecules, in each state, is preserved by the transport but not the configuration exchange. Adding the equations in (1.1) and using the boundary conditions leads, after integration, to the total zero flux property

$$-\varepsilon\left(n_\varepsilon^{(1)} + n_\varepsilon^{(2)}\right)_x - \psi_y\left(\frac{x}{\varepsilon}\right)n_\varepsilon^{(1)} = 0 \quad \text{in } [0,1]. \tag{1.4}$$

Several biomotor models, including the one considered here, were analyzed, for fixed ε, in [10,11,21,24,25] using arguments from optimal transportation. The existence of steady state solutions of (1.1) satisfying (1.3) is proved in, among other places, [10]. The simplest way to find $(n_\varepsilon^{(1)}, n_\varepsilon^{(2)})$ is to consider the adjoint system which admits (trivial) constant solutions. As a result 0 is the first eigenvalue of the adjoint of (1.1) and, hence, (1.1). The Krein–Rutman theorem yields $(n_\varepsilon^{(1)}, n_\varepsilon^{(2)})$ as the eigenvector of (1.1) corresponding to the 0 eigenvalue.

The typical results obtained about biomotors [10,21,27] without oscillating potentials are that, for small diffusion ε and under some precise asymmetry assumptions on the potential and rates, the solutions tend to concentrate, as $\varepsilon \to 0$, as Dirac masses at either $x = 0$ or $x = 1$. This behavior is referred to as "motor effect."

The question we are asking here is for which potential ψ and rates $\nu^{(1)}$ and $\nu^{(2)}$ does (1.1) exhibit motor effect as $\varepsilon \to 0$. In particular, we are investigating for which ψ, $\nu^{(1)}$ and $\nu^{(2)}$ satisfying (1.2) we have, for $i = 1, 2$,

$$n_\varepsilon^{(i)} = \exp\left[-\frac{1}{\varepsilon}\left(R + o^{(i)}(1)\right)\right], \tag{1.5}$$

with an "effective rate" R having a strict minimum at one end of the domain—this gives the orientation of the molecular transport.

Restrictions on ψ, $\nu^{(1)}$ and $\nu^{(2)}$ are definitely necessary. Indeed for any periodic potential ψ choose the rates $\nu^{(1)} = \nu e^{\psi}$, $\nu^{(2)} = \nu$ for some constant $\nu > 0$. It is then immediate that

$$n_\varepsilon^{(1)}(x) = \exp\left(-\psi\left(\frac{x}{\varepsilon}\right)\right) \quad \text{and} \quad n_\varepsilon^{(2)}(x) = 1$$

and, clearly, there is no motor effect.

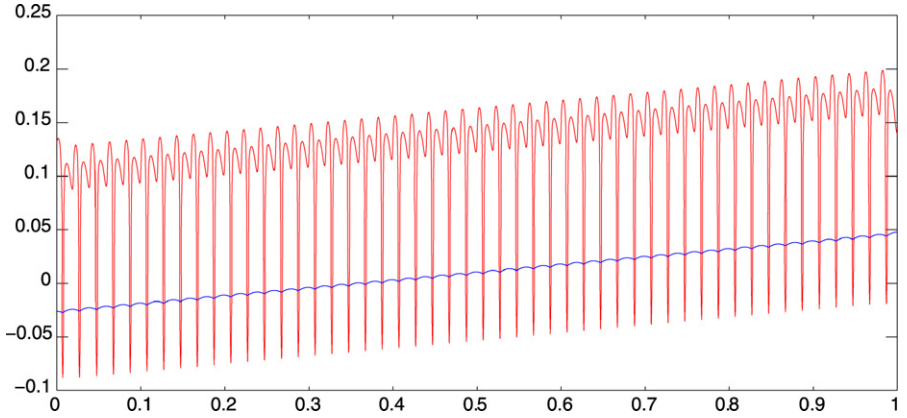

Fig. 1. Motor effect exhibited by the system (1.1) with 50 periods. The phase functions $R^{(1)}$ (highly oscillating) and $R^{(2)}$ (with nearly linear growth).

The motor effect can be induced by either asymmetric potentials or asymmetric transitions—we give an example for either case later in the paper. It is, however, far from easy to give a full description of the class of coefficients (potential and rates) that produce concentration effects. In this paper we provide a general rigorous characterization of what it is needed to have a motor effect. We then present two examples where it is possible to check this characterization. We do not know, however, any specific assumption on the coefficients that imply the general condition. This is different from the case of two potentials with a fixed number of periods (see [27]).

Our approach is based on the analysis of the homogenization limit, as $\varepsilon \to 0$, of the system satisfied by the rate functions $R_\varepsilon^{(1)}$ and $R_\varepsilon^{(2)}$ defined by the classical transformation

$$n_\varepsilon^{(1)} = \exp\left(-\frac{1}{\varepsilon} R_\varepsilon^{(1)}\right) \quad \text{and} \quad n_\varepsilon^{(2)} = \exp\left(-\frac{1}{\varepsilon} R_\varepsilon^{(2)}\right). \tag{1.6}$$

The common limit R of the $R_\varepsilon^{(1)}$ and $R_\varepsilon^{(2)}$ satisfies, in the Crandall–Lions viscosity sense, an averaged Hamilton–Jacobi equation. We show that the critical condition for motor effect is that the homogenized Hamiltonian has a nonzero root. Fig. 1 depicts these functions on an example.

Viscosity solutions are the correct class of weak solutions for first- and second-order fully nonlinear degenerate elliptic PDE. We refer to [13] and the references therein for a good introduction to the theory. Our arguments combine ideas from the methods used to study front propagation and large deviations [5,19] and homogenization [18].

The appearance of Dirac concentrations in different areas of biology, for example, trait selection in evolution theory, relies also on introducing a phase function and the study of the viscosity solutions of an appropriate Hamilton–Jacobi equation [6,14,26].

Other homogenization problems, as in fluid dynamics through porous media [3], are also known to yield an oriented drift and concentration effects. A single state is enough in this case, a fact which is different from the process we study here as well as the concentration effect we are interested in.

Another example [1,17,9] is the homogenization of growth-diffusion eigenvalue systems like

$$\begin{cases} -\varepsilon^2 \operatorname{div}\left(A_i\left(\dfrac{x}{\varepsilon}\right) D n_\varepsilon^{(i)}\right) + \Sigma\left(\dfrac{x}{\varepsilon}\right) n_\varepsilon = \mu^\varepsilon \sigma\left(\dfrac{x}{\varepsilon}\right) n_\varepsilon \quad \text{in } \Omega \ (i = 1, \dots, k), \\ u_\varepsilon = 0 \quad \text{on } \partial\Omega. \end{cases}$$

The settings and approaches of [1,17,9] are different than ours. In particular $\sigma \not\equiv 0$ and the boundary conditions are different. Moreover, the matrix Σ satisfies conditions that guarantee that the first eigenvalue μ^ε is nonzero. Note that in the problem we study here there is no σ and the eigenvalue problem (1.1) has 0 as an eigenvalue. The effective Hamiltonian we derive here also appears in [1,9] but plays a different role. The critical growth in [1,9] corresponds to the maximum of the effective Hamiltonian not the nontrivial root. Notice that contrary to the system we are considering here, a single equation is enough to yield, in certain cases, concentration phenomena provided the diffusion depends on the small scale. To illustrate this issue, we give an explicit example in Section 6 where this also happens in our case.

Substantially different analysis is needed for a seemingly closely related problem which is set in $\mathbb{R}$. Then the issue is to compute the asymptotic propagation speed for long times. This has been done in [7,12] for the case of flashing rachets.

The paper is organized as follows: In Section 2 we use some formal arguments to introduce the relevant (effective) Hamilton–Jacobi equation and we state the main result. Section 3 is devoted to the rigorous study of the phase functions and their limit and the connection with the effective Hamiltonian. In Section 4 we discuss two particular examples of coefficients. In Section 5 we prove that the general condition introduced in Section 2 yields a motor effect. We discuss possible extensions in Section 6.

2. The main result

We begin with a formal discussion which both motivates and introduces the necessary terminology for the statement of the main result.

The first observation is that, after multiplication by ε, at the limit $\varepsilon \to 0$, we must have, in the distributional sense,

$$\nu^{(1)} n_\varepsilon^{(1)} - \nu^{(2)} n_\varepsilon^{(2)} \to 0,$$

a fact that implies that Dirac mass type concentration must occur on both densities if at all.

A simple computation yields that the phase functions $R_\varepsilon^{(1)}$ and $R_\varepsilon^{(2)}$, defined by (1.6), solve the nonlinear system

$$\begin{cases} -\varepsilon R_{\varepsilon,xx}^{(1)} + \left|R_{\varepsilon,x}^{(1)}\right|^2 - \psi_y\left(\dfrac{x}{\varepsilon}\right) R_{\varepsilon,x}^{(1)} + \psi_{yy}\left(\dfrac{x}{\varepsilon}\right) + \nu^{(2)}\left(\dfrac{x}{\varepsilon}\right) \exp\left[\varepsilon^{-1}\left(R_\varepsilon^{(1)} - R_\varepsilon^{(2)}\right)\right] = \nu^{(1)}\left(\dfrac{x}{\varepsilon}\right), \\ -\varepsilon R_{\varepsilon,xx}^{(2)} + \left|R_{\varepsilon,x}^{(2)}\right|^2 - \nu^{(1)}\left(\dfrac{x}{\varepsilon}\right) \exp\left[\varepsilon^{-1}\left(R_\varepsilon^{(2)} - R_\varepsilon^{(1)}\right)\right] = \nu^{(2)}\left(\dfrac{x}{\varepsilon}\right) \\ R_{\varepsilon,x}^{(1)} = \psi_y\left(\dfrac{x}{\varepsilon}\right) \quad \text{and} \quad R_{\varepsilon,x}^{(2)} = 0 \quad \text{at } x = 0 \text{ or } 1. \end{cases} \quad \text{in } (0,1), \tag{2.1}$$

The observation at the beginning of the section, the nonlinear terms $\exp(\pm\varepsilon^{-1}(R_\varepsilon^{(1)} - R_\varepsilon^{(2)}))$ and the positivity of the transition rates suggest that $R_\varepsilon^{(1)}$ and $R_\varepsilon^{(2)}$ must converge, as $\varepsilon \to 0$, along subsequences, if at all, to the same limit R, i.e., as $\varepsilon \to 0$, we must have

$$R_\varepsilon^{(1)}, R_\varepsilon^{(2)} \to R \quad \text{in } (0,1).$$

The oscillations in (2.1) suggest then the formal expansion, for $i = 1, 2$,

$$R_\varepsilon^{(i)}(x) = R(x) + \varepsilon \phi^{(i)}\left(\frac{x}{\varepsilon}\right) + O^{(i)}\left(\varepsilon^2\right),$$

with $\phi^{(1)}$ and $\phi^{(2)}$ 1-periodic (see Fig. 1 for an illustration).

Substituting in (2.1) and looking at the terms multiplying ε^0 we find that the 1-periodic vector $(\phi^{(1)}, \phi^{(2)})$ must solve the system

$$\begin{cases} -\phi_{yy}^{(1)} + \left|\phi_y^{(1)} + R_x\right|^2 - \psi_y(y)\left(\phi_y^{(1)} + R_x\right) + \psi_{yy}(y) + \nu^{(2)}(y) \exp\left(\phi^{(1)} - \phi^{(2)}\right) = \nu^{(1)}(y), \\ -\phi_{yy}^{(2)} + \left|\phi_y^{(2)} + R_x\right|^2 + \nu^{(1)}(y) \exp\left(\phi^{(2)} - \phi^{(1)}\right) = \nu^{(2)}(y) \end{cases} \quad \text{in } (0,1), \tag{2.2}$$

where we write y for the fast variable x/ε.

As it is usually done in the homogenization theory, we seek a "compatibility"-type condition on $p = R_x$ so that (2.2) has a periodic solution. This leads to the homogenized equation. Finding the compatibility condition and, hence, $\phi^{(1)}$ and $\phi^{(2)}$ is known as solving the cell problem. In the case at hand the cell problem is described as follows:

For each $p \in \mathbb{R}$ there exists a unique constant $\bar{H}(p)$ such that the system

$$\begin{cases} -\phi_{yy}^{(1)} + \left|\phi_y^{(1)} + p\right|^2 - \psi_y\left(\phi_y^{(1)} + p\right) + \psi_{yy} + \nu^{(2)} \exp\left(\phi^{(1)} - \phi^{(2)}\right) = \nu^{(1)} + \bar{H}(p), \\ -\phi_{yy}^{2} + \left|\phi_y^{(2)} + p\right|^2 + \nu^{(1)} \exp\left(\phi^{(2)} - \phi^{(1)}\right) = \nu^{(2)} + \bar{H}(p) \end{cases} \quad \text{in } \mathbb{R}, \tag{2.3}$$

has an 1-periodic solution $(\phi^{(1)}, \phi^{(2)})$.

Observe that (2.3) is really a standard Hamilton–Jacobi-type nonlinear eigenvalue problem (see [5]). The connection with a "real" eigenvalue follows from the observation that, if, for $i = 1, 2$,

$$\chi^{(i)}(y) = e^{-py}\phi^{(i)}(y),$$

then

$$\begin{cases} -\chi_{yy}^{(1)} - \left(\psi_y \chi^{(1)}\right)_y + \nu^{(1)}\chi^{(1)} - \nu^{(2)}\chi^{(2)} = -\bar{H}(p)\chi^{(1)}, \\ -\chi_{yy}^{(2)} + \nu^{(2)}\chi^{(2)} - \nu^{(1)}\chi^{(1)} = -\bar{H}(p)\chi^{(2)} \end{cases} \quad \text{in } \mathbb{R}, \tag{2.4}$$

complemented by the (boundary) condition, for $i = 1, 2$,

$$y \mapsto e^{py}\chi^{(i)}(y) \text{ is 1-periodic and } \chi^{(i)} > 0. \tag{2.5}$$

For future reference we also introduce the normalization

$$\int_0^1 \left(\chi^{(1)} + \chi^{(2)}\right)(y)\, dy = 1. \tag{2.6}$$

The previous formal analysis then yields that R must solve the equation

$$\bar{H}(R_x) = 0 \quad \text{in } (0, 1). \tag{2.7}$$

It turns out—we show this later in the paper—that $\bar{H}$ is strictly convex and $\bar{H}(0) = 0$. The goal then is to show that, for a class of potentials ψ and rates $\nu^{(1)}, \nu^{(2)}$, it is possible for $\bar{H}(p) = 0$ to also have a nonzero solution $\bar{p}$ and to prove that in this case $R_x = \bar{p}$.

Note that if $\bar{H} = 0$ has only the trivial solution $p = 0$, then we must have $R_x = 0$, in which case for $i = 1, 2$,

$$n_\varepsilon^{(i)}(x) = \exp\left(-\frac{c + o^{(i)}(1)}{\varepsilon} - \phi^{(i)}\left(\frac{x}{\varepsilon}\right)\right),$$

with $\phi^{(i)}$ 1-periodic and $c \in \mathbb{R}$. It is then possible that the $n_\varepsilon^{(i)}$'s either do not converge to a Dirac mass or they do, but with rate weaker than we are interested in here.

We continue now with a rigorous discussion about $\bar{H}$ and some of its properties. The Krein–Rutman theorem yields that, for each $p \in \mathbb{R}$, there exist unique $\bar{H}(p)$ and $(\chi^{(1)}, \chi^{(2)})$ satisfying (2.4), (2.5) and (2.6).

Adding the equations in (2.4) and integrating in y we find that there exits a constant $F(p)$, which we call the total flux associated with (2.2), (2.5) and (2.6), such that, for all $y \in \mathbb{R}$,

$$F(p) = -\chi_y^{(1)} - \chi_y^{(2)} - \psi_y\chi^{(1)} + \bar{H}(p)\int_0^y \left(\chi^{(1)} + \chi^{(2)}\right)(y)\, dy. \tag{2.8}$$

The following lemma, which we prove at the end of this section, summarizes the key properties of $\bar{H}$ and F that are needed for our analysis.

We have:

Lemma 2.1. *For $p \in \mathbb{R}$, let $\bar{H}(p)$ be the eigenvalue of* (2.4) *with eigenvector $(\chi^{(1)}, \chi^{(2)})$ satisfying* (2.5) *and* (2.6). *Then*: *$\bar{H} \in C^1(\mathbb{R})$, $\bar{H}(0) = 0$, $\bar{H}(p) = F(p)(e^{-p} - 1)$, $\bar{H}$ is strictly convex and $\bar{H}(p) \to \infty$ as $|p| \to \infty$.*

We continue introducing the notion of "asymmetric potential-transition rate." In Section 4 we give two examples.

Definition 2.2. The triplet $(\psi, \nu^{(1)}, \nu^{(2)})$ is said to be asymmetric if one of the following three equivalent conditions hold:

(i) There exists $\bar{p} \neq 0$ such that $\bar{H}(\bar{p}) = 0$,
(ii) $\bar{H}'(0) \neq 0$,
(iii) $F(0) \neq 0$.

The equivalence of (i), (ii) and (iii) above is indeed obvious from Lemma 2.1, since $\bar{H}'(0) = -F(0)$ and $\bar{H}$ is strictly convex.

Recalling the standard notation δ_a for the Dirac mass at $x = a$, we proceed with the main result which is:

Theorem 2.3. *Assume that coefficients ψ, $\nu^{(1)}$ and $\nu^{(2)}$ satisfy* (1.2) *and are asymmetric in the sense of Definition* 2.2. *Then a concentration effect takes place, i.e., for $i = 1, 2$, there exist $\rho^{(i)} > 0$ such that, as $\varepsilon \to 0$,*

$$\text{either} \quad n_\varepsilon^{(i)} \rightharpoonup \rho^{(i)}\delta_0 \quad \text{if } \bar{p} > 0, \quad \text{or} \quad n_\varepsilon^{(i)} \rightharpoonup \rho^{(i)}\delta_1 \quad \text{if } \bar{p} < 0.$$

Moreover,

$$n_\varepsilon^{(i)}(x) = \exp\left[-\frac{1}{\varepsilon}\bar{p}x - \phi^{(i)}\left(\frac{x}{\varepsilon}\right) + \frac{o^{(i)}(1)}{\varepsilon}\right].$$

We conclude this section with the

Proof of Lemma 2.1. The eigenvalue problem adjoint to (2.4) is

$$\begin{cases} -v_{yy}^{(1)} + \psi_y(y)v_y^{(1)} + \nu^{(1)}v^{(1)} = \nu^{(1)}v^{(2)} - \bar{H}(p)v^{(1)}, \\ -v_{yy}^{(2)} + \nu^{(2)}v^{(2)} = \nu^{(2)}v^{(1)} - \bar{H}(p)v^{(2)} \qquad \text{in } \mathbb{R}, \\ y \mapsto v^{(i)}(y)e^{-py} \text{ 1-periodic and } v^{(i)} > 0 \quad \text{for } i = 1, 2. \end{cases} \tag{2.9}$$

It has obviously $(1, 1)$ as a solution for $p = 0$ and this proves that $\bar{H}(0) = 0$ by uniqueness of the positive eigenfunction.

We evaluate next, using (2.6), the total flux equation (2.8) at $y = 0$ and $y = 1$ to find

$$-\big(\chi^{(1)} + \chi^{(2)}\big)_y(0) - \psi_y(0)\chi^{(1)}(0) = F(p) = -\big(\chi^{(1)} + \chi^{(2)}\big)_y(1) - \psi_y(1)\chi^{(1)} + \bar{H}(p).$$

Since, in view of (2.5),

$$\chi^{(1)}(1) = e^p\chi^{(1)}(0), \qquad \chi_y^{(1)}(0) = \chi_y^{(1)}(1)e^p,$$

we conclude that

$$F(p) = F(p)e^{-p} - \bar{H}(p).$$

The proof of the strict convexity of $\bar{H}$ is better seen at the level of the Hamilton–Jacobi system (2.3). Indeed if $\bar{H}$ were not strictly convex, there would exist $p_1, p_2 \in \mathbb{R}$ such that $p_1 \neq p_2$ and

$$\bar{H}\left(\frac{p_1 + p_2}{2}\right) \geqslant \frac{1}{2}\big(\bar{H}(p_1) + \bar{H}(p_2)\big).$$

Let $(\phi_1^{(1)}, \phi_1^{(2)})$ and $(\phi_2^{(1)}, \phi_2^{(2)})$ be periodic solutions of (2.1) for p_1 and p_2 respectively. Then

$$\begin{aligned} &-\frac{1}{2}\big(\phi_1^{(1)} + \phi_2^{(1)}\big)_{yy} + \left|\frac{(\phi_1^{(1)} + \phi_2^{(1)})_y}{2} + \frac{p_1 + p_2}{2}\right|^2 - \psi_y\left(\left(\frac{\phi_1^{(1)} + \phi_2^{(1)}}{2}\right)_y + \frac{p_1 + p_2}{2}\right) + \psi_{yy} \\ &\quad + \nu^{(2)} \exp\frac{1}{2}\big[\big(\phi_1^{(1)} + \phi_2^{(1)}\big) - \big(\phi_1^{(2)} + \phi_2^{(2)}\big)\big] \\ &< -\left(\frac{\phi_1^{(1)} + \phi_2^{(1)}}{2}\right)_{yy} + \frac{|\phi_{1,y}^{(1)} + p_1|^2 + |\phi_{2,y}^{(1)} + p_2|^2}{2} - \psi_y\left(\left(\frac{\phi_1^{(1)} + \phi_1^{(1)}}{2}\right)_y + \frac{p_1 + p_2}{2}\right) + \psi_{yy} \\ &\quad + \frac{\nu^{(2)}}{2}\big[\exp\big(\phi_1^{(1)} - \phi_1^{(2)}\big) + \exp\big(\phi_2^{(1)} - \phi_2^{(2)}\big)\big] \end{aligned}$$

$$= \nu^{(1)} + \frac{1}{2}\big(\bar{H}(p_1) + \bar{H}(p_2)\big) \leqslant \nu^{(1)} + \bar{H}\left(\frac{p_1+p_2}{2}\right),$$

and, similarly,

$$-\left(\frac{\phi_1^{(2)}+\phi_2^{(2)}}{2}\right)_{yy} + \left|\left(\frac{\phi_1^{(2)}+\phi_2^{(2)}}{2}\right)_y + \frac{p_1+p_2}{2}\right|^2 + \nu^{(1)} \exp\frac{1}{2}\big[\big(\phi_1^{(2)}+\phi_2^{(2)}\big) - \big(\phi_1^{(1)}+\phi_2^{(1)}\big)\big]$$
$$< \nu^{(2)} + \frac{1}{2}\big(\bar{H}(p_1) + \bar{H}(p_2)\big) \leqslant \nu^{(2)} + \bar{H}\left(\frac{p_1+p_2}{2}\right).$$

It follows that $(\frac{\phi_1^{(1)}+\phi_2^{(1)}}{2}, \frac{\phi_1^{(2)}+\phi_2^{(2)}}{2})$ is a periodic strict subsolution of (2.1) for $p = \frac{p_1+p_2}{2}$. This is, however, a contradiction since elementary maximum principle considerations imply that an eigenvalue problem cannot have strict subsolutions.

Notice that the strict inequalities above are due to the strict convexity of both $q \mapsto |q|^2$ and $z \mapsto e^z$ and the fact that, if $p_1 \neq p_2$, then $(\phi_1^{(1)}, \phi_1^{(2)}) \neq (\phi_2^{(1)}, \phi_2^{(2)})$.

To establish the coercivity of $\bar{H}$ we rewrite (2.3) as

$$\begin{cases} -\phi_{yy}^{(1)} + \big(\phi_y^{(1)}\big)^2 - (\psi_y - 2p)\phi_y^{(1)} - \psi_y p + \psi_{yy} + \nu^{(2)} \exp\big(\phi^{(1)} - \phi^{(2)}\big) = \nu^{(1)} + \bar{H}(p) - |p|^2, \\ -\phi_{yy}^{(2)} + \big|\phi_y^{(2)}\big|^2 + 2p\phi_y^{(2)} + \nu^{(1)} \exp\big(\phi^{(2)} - \phi^{(1)}\big) = \nu^{(2)} + \bar{H}(p) - |p|^2 \end{cases} \quad \text{in } \mathbb{R},$$

and evaluate it at a $y_0 \in [0,1]$ where $\max_{[0,1]}(\phi^{(1)}, \phi^{(2)})$ is attained.

If $\phi^{(1)}(y_0) \geqslant \phi^{(2)}(y_0)$, then $-\phi_{yy}^{(1)}(y_0) \geqslant 0$ and $\phi_y^{(1)}(y_0) = 0$ and the first equation yields

$$\bar{H}(p) \geqslant |p|^2 - \|\psi_y\||p| - \nu^{(1)} - \|\psi_{yy}\|,$$

while, if $\phi^{(1)}(y_0) < \phi^{(2)}(y_0)$, then $\phi_{yy}^{(2)}(y_0) \geqslant 0$, $\phi_y^{(1)}(y_0) = 0$, and the second equation gives

$$\bar{H}(p) \geqslant |p|^2 - \nu^{(2)}.$$

In either case we clearly have, for some $C > 0$, $\bar{H}(p) \geqslant |p|^2 - C$. □

3. The homogenized equation

We study here the properties of the phase functions $R_\varepsilon^{(1)}$ and $R_\varepsilon^{(2)}$ defined by (1.6) as well as their behavior as $\varepsilon \to 0$. For the analysis it is convenient to introduce a phase function S_ε for the total density $n_\varepsilon^{(1)} + n_\varepsilon^{(2)}$, i.e.,

$$n_\varepsilon^{(1)} + n_\varepsilon^{(2)} = \exp\left(-\frac{1}{\varepsilon} S_\varepsilon\right). \tag{3.1}$$

We have:

Theorem 3.1. *Let $R_\varepsilon^{(i)}$, for $i = 1, 2$, and S_ε be defined by* (1.6) *and* (3.1) *respectively and assume that ψ, $\nu^{(1)}$ and $\nu^{(2)}$ and $n_\varepsilon^{(1)}$ and $n_\varepsilon^{(2)}$ satisfy* (1.2) *and* (1.3)*. Then, for $i = 1, 2$,*

(i) *$R_\varepsilon^{(i)}$ and S_ε are bounded and Lipschitz continuous in* $[0,1]$ *uniformly on ε,*

(ii) *there exists $C > 0$, independent of ε, such that*

$$\max_{[0,1]} \big|R_\varepsilon^{(2)} - R_\varepsilon^{(1)}\big| \leqslant C\varepsilon, \tag{3.2}$$

(iii) *along subsequences $\varepsilon \to 0$, $R_\varepsilon^{(1)}$, $R_\varepsilon^{(2)}$ and S_ε converge uniformly to a bounded and Lipschitz continuous R satisfying, in the viscosity sense,*

$$\bar{H}(R_x) = 0 \quad \textit{in } (0,1) \quad \textit{and} \quad \min_{(0,1)}(-\psi_y) \leqslant R_x \leqslant \max_{(0,1)}(-\psi_y).$$

Proof. Inserting (3.1) in the total flux equation (1.4) we find

$$S_{\varepsilon,x} = -\psi_y \frac{n_\varepsilon^{(1)}}{n_\varepsilon^{(1)} + n_\varepsilon^{(2)}}, \tag{3.3}$$

and, hence,

$$\left|S_{\varepsilon,x}\right| \leqslant \|\psi_y\|.$$

We also observe that the normalization (1.3) implies that $\min_{[0,1]} S_\varepsilon = 0$. Therefore, we also get

$$|S_\varepsilon| \leqslant \|\psi_y\| \quad \text{in } [0,1].$$

To obtain a Lipschitz bound for $R_\varepsilon^{(1)}$ and $R_\varepsilon^{(2)}$, in view of the boundary conditions, it suffices to consider what happens at interior extrema points of $R_{\varepsilon,x}^{(1)}$ and $R_{\varepsilon,x}^{(2)}$. Since at such points we have $R_{\varepsilon,xx}^{(1)} = 0$ and $R_{\varepsilon,xx}^{(2)}$, the equations yield

$$\max_{[0,1]} \left|R_{\varepsilon,x}^{(1)}\right| \leqslant \left\|\nu^{(1)}\right\| + \|\psi_{yy}\| \quad \text{and} \quad \max_{[0,1]} \left|R_{\varepsilon,x}^{(2)}\right| \leqslant \left\|\nu^{(2)}\right\|.$$

The bounds on $R_\varepsilon^{(1)}$ and $R_\varepsilon^{(2)}$ come from the bounds on S_ε and the obvious inequalities

$$\max\left(n_\varepsilon^{(1)}, n_\varepsilon^{(2)}\right) \leqslant n_\varepsilon^{(1)} + n_\varepsilon^{(2)} \leqslant 2\max\left(n_\varepsilon^{(1)}, n_\varepsilon^{(2)}\right),$$

which imply

$$S_\varepsilon \leqslant \min\left(R_\varepsilon^{(1)}, R_\varepsilon^{(2)}\right) \leqslant S_\varepsilon + \varepsilon \ln 2.$$

It follows that $\min(R_\varepsilon^{(1)}, R_\varepsilon^{(2)})$ is bounded, uniformly in ε. Moreover, along subsequences, $\varepsilon \to 0$, S_ε and $\min(R_\varepsilon^{(1)}, R_\varepsilon^{(2)})$ converge, uniformly in $[0,1]$, to the same limit R.

Next, we add the equations of the system (2.1), integrate over $[0,1]$, use the boundary conditions and the Lipschitz bounds. This way, because the exponential with a positive argument dominates, we find an exponential decay that implies, for some $C > 0$, that

$$\int_0^1 \left|R_\varepsilon^{(1)} - R_\varepsilon^{(2)}\right|^2 \leqslant C\varepsilon^2.$$

Combining all the above it is now possible to show that both $R_\varepsilon^{(1)}$ and $R_\varepsilon^{(2)}$ converge, along subsequences, uniformly to R which, as a consequence of (3.3), is Lipschitz continuous. We refer to [27] for the details.

Next we prove that there exists $C > 0$, independent of ε, such that

$$\max_{[0,1]}\left(R_\varepsilon^{(2)} - R_\varepsilon^{(1)}\right) \leqslant C\varepsilon.$$

A similar argument gives an upper bound for $\max(R_\varepsilon^{(1)} - R_\varepsilon^{(2)})$, and, hence, (3.2).

Subtracting the first equation from the second in (2.1) we find

$$\begin{aligned} &-\varepsilon\left(R_\varepsilon^{(2)} - R_\varepsilon^{(1)}\right)_{xx} + \left(R_{\varepsilon,x}^{(2)} + R_{\varepsilon,x}^{(1)}\right)\left(R_\varepsilon^{(2)} - R_\varepsilon^{(1)}\right)_x - \psi_y\left(R_\varepsilon^{(1)}\right)_x + \psi_{yy} + \nu^{(1)}\left(\exp\left(\varepsilon^{-1}\left(R_\varepsilon^{(2)} - R_\varepsilon^{(1)}\right)\right) + 1\right) \\ &\quad \leqslant \nu^{(2)}\left(\exp\left(\varepsilon^{-1}\left(R_\varepsilon^{(1)} - R_\varepsilon^{(2)}\right)\right) + 1\right). \end{aligned} \tag{3.4}$$

If $x_\varepsilon \in (0,1)$ is such that $(R_\varepsilon^{(2)} - R_\varepsilon^{(1)})(x_\varepsilon) = \max_{[0,1]}(R_\varepsilon^{(2)} - R_\varepsilon^{(1)})$, then from (3.4) and the Lipschitz bounds, we must have

$$\nu^{(1)}\left(\exp\left(\varepsilon^{-1}\left(R_\varepsilon^{(2)} - R_\varepsilon^{(1)}\right)\right)(x_\varepsilon) + 1\right) \leqslant \nu^{(2)}\left(\exp\left(\varepsilon^{-1}\left(R_\varepsilon^{(1)} - R_\varepsilon^{(2)}\right)(x_\varepsilon)\right) + 1\right) + C,$$

and, therefore, for some other $C > 0$,

$$\max_{[0,1]}\left(R_\varepsilon^{(2)} - R_\varepsilon^{(1)}\right) = \left(R_\varepsilon^{(2)} - R_\varepsilon^{(1)}\right)(x_\varepsilon) \leqslant \varepsilon C.$$

If $x_\varepsilon = 0$ or 1 or both, then the argument is more complicated. We begin with the latter case, i.e., we assume that

$$\big(R_\varepsilon^{(2)} - R_\varepsilon^{(1)}\big)(0) = \big(R_\varepsilon^{(2)} - R_\varepsilon^{(1)}\big)(1) = \max_{[0,1]}\big(R_\varepsilon^{(2)} - R_\varepsilon^{(1)}\big).$$

Then from the boundary conditions we must have

$$-\psi_y(0) = \big(R_\varepsilon^{(2)} - R_\varepsilon^{(1)}\big)_x(0) \leqslant 0 \leqslant \big(R_\varepsilon^{(2)} - R_\varepsilon^{(1)}\big)_x(1) = -\psi_y\left(\frac{1}{\varepsilon}\right) = -\psi_y(0),$$

and, hence,

$$\psi_y(0) = 0.$$

Consequently, $x = 0$ is a strict maximum of $R_\varepsilon^{(2)} - R_\varepsilon^{(1)} - \varepsilon^2 x^2$ over $[0, 1]$ and, for $\lambda > 0$, the function

$$W^\varepsilon(x) = \big(R_\varepsilon^{(2)} - R_\varepsilon^{(1)}\big)(x) - \varepsilon^2 x^2 + \lambda x$$

attains its global maximum at some $x_{\varepsilon,\lambda} \in (0, 1)$ such that $x_{\varepsilon,\lambda} \to 0$, as $\lambda \to 0$.

In addition

$$\begin{aligned}&-\varepsilon W^\varepsilon_{xx} + b^\varepsilon W^\varepsilon_x - \psi_y\big(R_\varepsilon^{(1)}\big)_x + \psi_{yy} + \nu^{(1)}\big(\exp\big(\varepsilon^{-1}\big(W^\varepsilon + \varepsilon^2 x^2 - \lambda x\big)\big) + 1\big)\\ &\quad\leqslant \nu^{(2)}\big(\exp\big(\varepsilon^{-1}\big(W^\varepsilon + \varepsilon^2 x^2 - \lambda x\big)\big) + 1\big) + (2\varepsilon + \lambda)\big\|b^\varepsilon\big\| + 2\varepsilon^3,\end{aligned}$$

where

$$b^\varepsilon = R^{(2)}_{\varepsilon,x} + R^{(1)}_{\varepsilon,x}.$$

At the maximum point $x_{\varepsilon,\lambda}$ we then have, for a uniform constant $C' > 0$,

$$W^\varepsilon\big(x_\varepsilon^\lambda\big) - \lambda x_{\varepsilon,\lambda} + \varepsilon^2 (x_{\varepsilon,\lambda})^2 \leqslant \varepsilon C'.$$

Therefore, for all $x \in [0, 1]$,

$$R_\varepsilon^{(2)}(x) - R_\varepsilon^{(1)}(x) + \lambda x - \varepsilon x^2 \leqslant \varepsilon C' + \lambda x_{\varepsilon,\lambda}.$$

The conclusion now follows, letting $\lambda \to 0$, for $C = C' + 1$.

A slight modification of the above argument yields the conclusion, if the maximum occurs only at either 0 or 1. We sketch some of the details if the maximum occurs at 0. In this case we must have

$$-\psi_y'(0) = \big(R_\varepsilon^{(2)} - R_\varepsilon^{(1)}\big)_x(0) \leqslant 0, \quad \text{i.e.,} \quad \psi_y(0) \geqslant 0.$$

Then, for any $\lambda > \psi_y(0)$, the function

$$W^\varepsilon(x) = \big(R_\varepsilon^{(2)} - R_\varepsilon^{(1)}\big)(x) + \lambda x,$$

which satisfies the inequality

$$\begin{aligned}&-\varepsilon W^\varepsilon_{xx} + b^\varepsilon W^\varepsilon_x - \psi_y\big(R_\varepsilon^{(1)}\big)_x + \psi_{yy} + \nu^{(1)}\big(\exp\big(\varepsilon^{-1}\big(W^\varepsilon(x) - \lambda x\big)\big) + 1\big)\\ &\quad\leqslant \nu^{(2)}\big(\exp\big(\varepsilon^{-1}\big(W^\varepsilon(x) - \lambda x\big)\big) + 1\big) + \lambda\big\|b^\varepsilon\big\|,\end{aligned}$$

attains its maximum over $[0, 1]$ at some $x_{\varepsilon,\lambda} \in (0, 1)$ such that $x_{\varepsilon,\lambda} \to 0$ as $\lambda \to \psi_y(0)$.

As before, at $x_{\varepsilon,\lambda}$, we must have, for some uniform $C' > 0$,

$$W^\varepsilon(x_{\varepsilon,\lambda}) - \lambda x_{\varepsilon,\lambda} \leqslant \varepsilon C'$$

and, hence,

$$R_\varepsilon^{(2)}(x) - R_\varepsilon^{(1)}(x) + \lambda x \leqslant \varepsilon C' + \lambda x_{\varepsilon,\lambda}.$$

The claim (ii) now follows letting $\lambda \to \psi_y(0)$ and using that $\psi_y(0) \geqslant 0$ and $x_{\varepsilon,\lambda} \to 0$.

Next, using the so-called perturbed test function method (see [18]) and following the strategy of [5], we prove that the common limit R satisfies the Hamilton–Jacobi equation

$$\bar{H}(R_x) = 0 \quad \text{in } (0,1).$$

Here we only show that R is a viscosity subsolution. The supersolution property follows in a similar way so we omit the details.

To this end, fix a smooth test function Φ and assume that $x_0 \in (0,1)$ is an interior maximum point of $R - \Phi$ in $(0,1)$, i.e.,

$$(R-\Phi)(x_0) = \max_{x\in(0,1)} (R-\Phi)(x).$$

Next we perturb the test function using the solution $(\phi^{(1)}, \phi^{(2)})$ of (2.2) corresponding to $p = \Phi'(x_0)$, and, for ε small enough, we consider a sequence of points $x_\varepsilon \in (0,1)$ (this is in fact true along a subsequence but, to keep the notation simple, we still write ε) such that, as $\varepsilon \to 0$, $x_\varepsilon \to x_0$ and

$$R_\varepsilon^{(1)}(x_\varepsilon) - \Phi(x_\varepsilon) - \varepsilon\phi^{(1)}\left(\frac{x_\varepsilon}{\varepsilon}\right) = \max_{(0,1)}\left[R_\varepsilon^{(1)} - \Phi - \varepsilon\phi^{(1)}\left(\frac{\cdot}{\varepsilon}\right)\right] \geqslant \max_{(0,1)}\left[R_\varepsilon^{(2)} - \Phi - \varepsilon\phi^{(2)}\left(\frac{\cdot}{\varepsilon}\right)\right],$$

and, thus,

$$R_\varepsilon^{(1)}(x_\varepsilon) - \Phi(x_\varepsilon) - \varepsilon\phi^{(1)}\left(\frac{x_\varepsilon}{\varepsilon}\right) = \max_{(0,1)}\left[R_\varepsilon^{(1)} - \Phi - \varepsilon\phi^{(1)}\left(\frac{\cdot}{\varepsilon}\right)\right] \geqslant R_\varepsilon^{(2)}(x_\varepsilon) - \Phi(x_\varepsilon) - \varepsilon\phi^{(2)}\left(\frac{x_\varepsilon}{\varepsilon}\right). \tag{3.5}$$

This is possible, inverting if necessary the indices 1 and 2, depending upon which of the two possible maximum brackets is the largest. Here we choose this order, but the argument is the same in the other case.

Testing the first equation of (2.1) and setting $p_\varepsilon = \Phi_x(x_\varepsilon)$, we find

$$\begin{aligned} &-\varepsilon\Phi_{xx}(x_\varepsilon) - \phi_{yy}^{(1)}\left(\frac{x_\varepsilon}{\varepsilon}\right) + \left|p_\varepsilon + \phi_y^{(1)}\left(\frac{x_\varepsilon}{\varepsilon}\right)\right|^2 - \psi_y\left(\frac{x_\varepsilon}{\varepsilon}\right)\left(\phi_y^{(1)}\left(\frac{x_\varepsilon}{\varepsilon}\right) + p_\varepsilon\right) + \psi_{yy}\left(\frac{x_\varepsilon}{\varepsilon}\right) \\ &\quad + \nu_2\left(\frac{x_\varepsilon}{\varepsilon}\right)\exp\left(\varepsilon^{-1}\left(R_\varepsilon^{(1)} - R_\varepsilon^{(2)}\right)\right) \leqslant \nu_1\left(\frac{x_\varepsilon}{\varepsilon}\right). \end{aligned} \tag{3.6}$$

Since, in view of (3.5),

$$\varepsilon\left(\phi^{(1)}\left(\frac{x_\varepsilon}{\varepsilon}\right) - \phi^{(2)}\left(\frac{x_\varepsilon}{\varepsilon}\right)\right) \leqslant R_\varepsilon^{(1)}(x_\varepsilon) - R_\varepsilon^{(2)}(x_\varepsilon),$$

we finally obtain, for $y_\varepsilon = x_\varepsilon/\varepsilon$,

$$\begin{aligned} &-\varepsilon\Phi_{xx}(x_\varepsilon) - \phi_{yy}^{(1)}(y_\varepsilon) + \left|\Phi_x(x_\varepsilon) + \phi_y^{(1)}(y_\varepsilon)\right|^2 - \psi_y(y_\varepsilon)\left(\phi_y^{(1)}(y_\varepsilon) + \Phi_x(x_\varepsilon)\right) + \psi_{yy}(y_\varepsilon) \\ &\quad + \nu_2\exp\left(\phi^{(1)} - \phi^{(2)}\right)(y_\varepsilon) \leqslant \nu_1. \end{aligned} \tag{3.7}$$

Taking into account the equation that defines the effective Hamiltonian $\bar{H}$, we conclude that,

$$-\varepsilon\Phi_{xx}(x_\varepsilon) \leqslant -\bar{H}\left(\Phi_x(x_\varepsilon)\right),$$

and, in the limit $\varepsilon \to 0$,

$$\bar{H}(\Phi_x(x_0) \leqslant 0,$$

which proves the viscosity subsolution criterion. □

4. Examples of asymmetric triplets

We present here two examples of data $(\psi, \nu^{(1)}, \nu^{(2)})$ giving rise to effective Hamiltonians satisfying the asymmetry condition of Definition 2.2. Another example, based on periodic diffusion, is given in Section 6. In the first example, the potential ψ is "strongly" asymmetric and has a "sawtooth" shape while the rates are constant. In the second example, the potential is symmetric (even) but the rates $\nu^{(1)}$ and $\nu^{(2)}$ vary.

Before proceeding we point out that the "asymmetry condition" is indeed more than saying that ψ cannot be even. As the second example indicates, it involves some "correlation" between ψ and $(\nu^{(1)}, \nu^{(2)})$. Indeed, as already discussed in the Introduction, let

$$\nu^{(1)}(y) = \tilde{\nu}^{(1)} e^{\psi(y)}, \quad \text{with } \tilde{\nu}^{(1)}, \nu^{(2)}, \tilde{\chi}^{(1)}, \chi^{(2)} \text{ positive constants such that } \tilde{\nu}^{(1)}\tilde{\chi}^{(1)} = \nu^{(2)}\chi^{(2)}.$$

The solution of (2.4) satisfying (2.5) and (2.6) is

$$\chi^{(1)}(y) = \tilde{\chi}^{(1)} e^{-\psi(y)} \quad \text{and} \quad \chi^{(2)},$$

and the total flux $\bar{F}(0)$ is zero, i.e.,

$$F(0) = -\chi_y^{(1)} - \chi_y^{(2)} - \psi_y \chi^1 = 0.$$

As a consequence of the Definition 2.2, in this case the Hamiltonian $\bar{H}$ vanishes only at $p = 0$ and the limit R of the $R_\varepsilon^{(1)}$, $R_\varepsilon^{(2)}$ is constant.

4.1. Sawtooth potentials

We consider here (discontinuous) periodic sawtooth potentials ψ given by

$$\psi(y) = \alpha y \quad \text{in } (0, 1). \tag{4.1}$$

We have:

Lemma 4.1. *Assume that ψ is a periodic sawtooth potential given by* (4.1) *and $\nu^{(1)}$, $\nu^{(2)}$ are positive constants. Then, for sufficiently small $\alpha > 0$, the triplet $(\psi, \nu^{(1)}, \nu^{(2)})$ is asymmetric in the sense of Definition* 2.2.

Proof. In view of Lemma 2.1, it suffices to show that the unique solution $(\chi^{(1)}, \chi^{(2)})$ satisfying (2.4), (2.5) and (2.6) with $p = 0$ yields nonvanishing flux. We argue by contradiction and assume that there exist 1-periodic $\chi^{(1)}$ and $\chi^{(2)}$ such that

$$\begin{cases} \left(\chi^{(1)} + \chi^{(2)}\right)_y + \alpha\chi^{(1)} = 0 \quad \text{in } \mathbb{R}, \\ -\chi_{yy}^{(2)} + \nu^{(2)}\chi^{(2)} = \nu^{(1)}\chi^{(1)}. \end{cases} \tag{4.2}$$

The solution $\chi^{(1)}$ cannot be smooth (recall that ψ is not smooth) and the periodic boundary condition can be inferred from the divergence formulation

$$-\left(e^{-\psi}\left(e^{\psi}\chi^{(1)}\right)_y\right)_y = \nu^{(2)}\chi^{(2)} - \nu^{(1)}\chi^{(1)},$$

which, by elliptic regularity, yields that $(e^{\psi}\chi^{(1)})_y$ is smooth in $(0, 1)$ with discontinuities only at $y = 0$ and $y = 1$. Thus $e^{\psi}\chi^{(1)}$ is Lipschitz continuous.

Therefore we have

$$e^{\psi(1)}\chi^{(1)}(1) = e^{\psi(0)}\chi^{(1)}(0).$$

On the other hand, since $\chi_{yy}^{(2)}$ has the regularity of $\chi^{(1)}$, i.e., it is piecewise smooth, $\chi^{(2)} \in W^{2,\infty}(\mathbb{R})$ and, thus,

$$\chi^{(2)}(0) = \chi^{(2)}(1), \qquad \chi_y^{(2)}(0) = \chi_y^{(2)}(1).$$

Since (1.1) is equivalent to a third-order differential equation with constant coefficients on $(0, 1)$, the three-dimensional vector space of solutions is spanned by

$$\chi^{(1)}(y) = e^{\lambda y}, \qquad \chi^{(2)}(y) = ae^{\lambda y},$$

where a and λ satisfy the algebraic conditions

$$\lambda(1 + a) + \alpha = 0 \quad \text{and} \quad -a\lambda^2 + a\nu^{(2)} = \nu^{(1)},$$

which are equivalent to

$$\lambda^2(\lambda+\alpha)-\lambda\big(\nu^{(1)}+\nu^{(1)}\big)-\alpha\nu^{(2)}=0 \quad \text{with } a=-\lambda^{-1}(a+\lambda). \tag{4.3}$$

The polynomial in (4.3) takes positive value for $\lambda=-\alpha$. Hence it has three roots, two negative and one positive, that we denote by $\lambda_1, \lambda_2, \lambda_3$, with $\lambda_1<\lambda_2<0<\lambda_3$.

The generic solution $(\chi^{(1)}, \chi^{(2)})$ of (4.2) inside $(0,1)$ is given by the formula

$$\chi^{(1)}(y)=\sum_{i=1}^{3}\eta_i\lambda_i e^{\lambda_i y} \quad \text{and} \quad \chi^{(2)}(y)=\sum_{i=1}^{3}\eta_i a_i\lambda_i e^{\lambda_i y} \quad \text{with } a_i=-\lambda_i^{-1}a-1,$$

with the free parameters η_1, η_2, η_3 to be specified by the boundary conditions and periodicity. Notice that in the expressions for $\chi^{(1)}$ and $\chi^{(2)}$ we have multiplied by λ_i to simplify the expressions coming up below.

The periodicity conditions specified above give rise to the linear system

$$\sum_{i=1}^{3}\eta_i\lambda_i\big(e^{\lambda_i}-e^{-\alpha}\big)=0, \qquad \sum_{i=1}^{3}\eta_i a_i\lambda_i\big(e^{\lambda_i}-1\big)=0, \qquad \sum_{i=1}^{3}\eta_i a_i\lambda_i^2\big(e^{\lambda_i}-1\big)=0,$$

which, for $\alpha=0$, reads

$$\sum_{i=1}^{3}\eta_i\big(e^{\lambda_i}-1\big)=0, \qquad \sum_{i=1}^{3}\eta_i\lambda_i^2\big(e^{\lambda_i}-1\big)=0,$$

and has as solution $\eta_1=\eta_3=0$ and any η_2. It follows that $\chi^{(1)}, \chi^{(2)}$ are constants and $\lambda_2=0$.

We show next that, for sufficiently small $\alpha>0$, the system is invertible, by proving that the determinant $D(\alpha)$ of the matrix

$$\begin{pmatrix} \lambda_1(e^{\lambda_1}-e^{-\alpha}) & \lambda_2(e^{\lambda_2}-e^{-\alpha}) & \lambda_3(e^{\lambda_3}-e^{-\alpha}) \\ (\alpha+\lambda_1)(e^{\lambda_1}-1) & (\alpha+\lambda_2)(e^{\lambda_2}-1) & (\alpha+\lambda_3)(e^{\lambda_3}-1) \\ (\alpha+\lambda_1)\lambda_1(e^{\lambda_1}-1) & (\alpha+\lambda_2)\lambda_2(e^{\lambda_2}-1) & (\alpha+\lambda_3)\lambda_3(e^{\lambda_3}-1) \end{pmatrix}$$

is nonzero.

To this end, set

$$F(\alpha,\lambda)=\lambda(\alpha+\lambda)^{-1}\big(e^{\lambda}-1\big)^{-1}\big(e^{\lambda}-e^{-\alpha}\big) \quad \text{and} \quad A(\alpha,\lambda)=(\alpha+\lambda)\big(e^{\lambda}-1\big).$$

A straightforward computation yields

$$D(\alpha)=\bar{A}(\alpha)\bar{D}(\alpha),$$

with

$$\bar{D}(\alpha)=F(\alpha,\lambda_1)(\lambda_3-\lambda_2)+F(\alpha,\lambda_2)(\lambda_1-\lambda_3)+F(\alpha,\lambda_3)(\lambda_2-\lambda_1) \quad \text{and} \quad \bar{A}(\alpha)=\prod_{i=1}^{3}A(\alpha,\lambda_i).$$

For $\alpha=0$ we have

$$\lambda_1=-\sqrt{\nu^{(1)}+\nu^{(2)}}, \qquad \lambda_2=0, \qquad \lambda_3=\sqrt{\nu^{(1)}+\nu^{(2)}},$$

and

$$\frac{\partial\lambda_1}{\partial\alpha}=\frac{\partial\lambda_3}{\partial\alpha}=-\frac{\nu^{(1)}}{2(\nu^{(1)}+\nu^{(2)})}, \qquad \frac{\partial\lambda_2}{\partial\alpha}=-\frac{\nu^{(2)}}{\nu^{(1)}+\nu^{(2)}}.$$

Observe next that $\alpha=0$ is a zero of order three for $D(\alpha)$. It is easy, however, to see that $D(\alpha)$ does not vanish near $\alpha=0$.

Indeed, $F(0,\lambda_1)=F(0,\lambda_2)=F(0,\lambda_3)=1$, therefore $\bar{D}(0)=0$. Moreover,

$$\frac{\partial}{\partial\alpha}F(0,\lambda) = -\frac{1}{\alpha} + \frac{1}{e^{\lambda}-1}, \qquad \frac{\partial}{\partial\alpha}F(0,0) = -\frac{1}{2}, \qquad \frac{\partial}{\partial\lambda}F(0,\lambda) = 0, \quad \text{and}$$
$$\bar{D}'(0) = \sqrt{\nu^{(1)}+\nu^{(2)}}\left[1 + \frac{1}{e^{\sqrt{\nu^{(1)}+\nu^{(2)}}}-1} + \frac{1}{e^{-\sqrt{\nu^{(1)}+\nu^{(2)}}}-1}\right] \neq 0,$$

therefore

$$\bar{D}(\alpha) \neq 0 \quad \text{for small } \alpha > 0.$$

Finally if $\bar{A}(\alpha) = 0$, then, for some $i = 1, 2, 3$, we must have either $\lambda_i = -\alpha$ or $\lambda_i = 0$. It is, however, immediate that neither one is possible unless $\alpha = 0$. Consequently, the determinant and, hence, the flux do not vanish for $\alpha > 0$ small enough. □

4.2. Variable transition rates

We present an example of an even, but not constant, potential, and variable rates which form an asymmetric triplet. We construct such coefficients by means of a perturbation argument from a "symmetric" triplet.

To make the calculation explicit, consider the potential ψ and the transitions rates $\overline{\nu^{(1)}}$ and $\overline{\nu^{(2)}}$ given by

$$\psi(y) = \sin(2\pi y), \quad \overline{\nu^{(1)}}(y) = \overline{\nu^{(2)}}e^{\psi(y)} \quad \text{and} \quad \overline{\nu^{(2)}} \equiv 1.$$

As discussed earlier, with this choice we have $\bar{H}(0) = 0$, the cell problem (2.4) for $p = 0$ admits the obvious solution

$$\overline{\chi^{(1)}} = e^{-\psi(y)}, \qquad \overline{\chi^{(2)}} = 1,$$

and the associated total flux obviously vanishes, i.e.,

$$\bar{F}(0) = -\left(\overline{\chi^{(1)}} + \overline{\chi^{(2)}}\right)_y - \psi_y\overline{\chi^{(1)}} = 0.$$

For the same symmetric potential, we consider the family of transition rates

$$\nu^{(1)}(y) = \overline{\nu^{(1)}}(y) + \alpha q(y) \quad \text{and} \quad \nu^{(2)} = \overline{\nu^{(2)}},$$

where for $\alpha > 0$ small enough and the function q is chosen later.

Next we expand in α and set

$$\chi^{(1)} = \overline{\chi^{(1)}} + \alpha\eta^{(1)} + \mathrm{O}(\alpha^2) \quad \text{and} \quad \chi^{(2)} = \overline{\chi^{(2)}} + \alpha\eta^{(2)} + \mathrm{O}(\alpha^2).$$

The linearized cell problem is

$$\begin{cases} -\eta^{(1)}_{yy} + \left(\psi_y(y)\eta^{(1)}\right)_y + \overline{\nu^{(1)}}\eta^{(1)} + q\overline{\chi^{(1)}} = \overline{\nu^{(2)}}\eta^{(2)}, \\ -\eta^{(2)}_{yy} + \overline{\nu^{(2)}}\zeta^{(2)} = \overline{\nu^{(1)}}\eta^{(1)} + q\overline{\chi^{(1)}} \qquad \text{in } \mathbb{R}, \\ \eta^{(1)}, \eta^{(2)} \text{ is 1-periodic}, \end{cases} \tag{4.4}$$

and its total flux is given by

$$DF(0) = -\left(\eta^{(1)} + \eta^{(2)}\right)_y - \psi_y\eta^{(1)}.$$

Choose

$$\eta^{(1)} = \cos(2\pi y) \quad \text{and} \quad \left(\eta^{(1)} + \eta^{(2)}\right)_y = \sin^2(y) - \int_0^1 \sin^2(z)\,dz \tag{4.5}$$

so that

$$DF(0) = -1 + \int_0^1 \sin^2(y) \neq 0;$$

notice that (4.5) yields $\eta^{(2)}$ implicitly because $\eta^{(1)} + \eta^{(2)}$ is a periodic function.

The total flux associated with (4.4) satisfies, for α small enough,

$$F_\alpha(0) = 0 + \alpha DF(0) + \mathrm{O}(\alpha^2) \neq 0.$$

It remains to check that the above choices of $(\eta^{(1)}, \eta^{(2)})$ correspond to a possible solution to the cell problem. Since the flux equation is satisfied, it is enough, for instance, to check the equation on $\eta^{(2)}$ which gives the perturbation direction q. It is now possible to complete the construction.

5. The concentration phenomena

Before we present the proof of Theorem 2.3, we collect some preliminary material that are critical for the argument. For $\bar{p} \in \mathbb{R}$ such that $\bar{H}(\bar{p}) = 0$, the Krein–Rutman theorem yields that the adjoint system

$$\begin{cases} -\zeta^1_{yy} + \psi_y \zeta^{(1)}_y + \nu^{(1)} \zeta^{(1)} = \nu^{(1)} \zeta^{(2)}, \\ -\zeta^2_{yy} + \nu^{(2)} \zeta^{(2)} = \nu^{(2)} \zeta^{(1)} \end{cases} \quad \text{in } \mathbb{R}, \tag{5.1}$$

of (2.4) admits a unique, up to multiplication, solution $(\zeta^{(1)}, \zeta^{(2)})$ such that

$$y \mapsto \zeta^{(1)}(y) e^{-\bar{p}y} \quad \text{and} \quad y \mapsto \zeta^{(2)}(y) e^{-\bar{p}y} \quad \text{are 1-periodic, and } \zeta^{(1)} > 0 \text{ and } \zeta^{(2)} > 0. \tag{5.2}$$

Recall that for $\bar{p} = 0$, the solution of (5.1) satisfying (5.2) is simply $(1, 1)$. This is not, however, the case when $\bar{p} \neq 0$.

We have:

Lemma 5.1. *The solution* $(\zeta^{(1)}, \zeta^{(2)})$ *of* (5.1) *and* (5.2) *is increasing if* $\bar{p} > 0$ *and decreasing if* $\bar{p} < 0$.

Proof. We only consider here $\bar{p} > 0$, since the other case follows similarly.

We observe first that there does not exist $y_0 \in \mathbb{R}$ such that $\zeta^{(1)}_y(y_0) = \zeta^{(2)}_y(y_0) = 0$. Indeed if this were the case, then, after a translation, we could consider (5.1) with Neumann boundary conditions. It would then follow that constants are the only solutions, a fact which is incompatible with (5.2).

Let $y_1 \in \mathbb{R}$ be such that

$$\zeta^{(1)}(y_1) = \max_y \big(\zeta^{(1)}(y), \zeta^{(2)}(y)\big);$$

if the maximum is achieved at $\zeta^{(2)}(y_1)$ the argument is similar. It follows from the maximum principle that we must have $\zeta^{(2)}(y_1) \geqslant \zeta^{(1)}(y_1)$, and, hence, by the previous argument, again a contradiction.

Thus, if one of the $\zeta^{(1)}$ and $\zeta^{(2)}$ is not increasing—for definiteness we take it to be $\zeta^{(1)}$—and it has, for example, a first local maximum at y_1, then by the previous argument we must have $\zeta^{(2)}(y_1) > \zeta^{(1)}(y_1)$. Moreover, the local maximum has to be followed by a local minimum y_2, since the periodicity of $y \mapsto \zeta^{(i)} e^{-\bar{p}y}$ implies that $y \mapsto \zeta^{(i)}_y e^{-\bar{p}y}$ is also periodic.

This yields again a contradiction because it is impossible to have $\zeta^{(2)}(y_2) < \zeta^{(1)}(y_2)$. Indeed if this were the case, then $\zeta^{(2)}$ would have a local maximum larger than $\zeta^{(1)}$ between y_1 and y_2, a contradiction to the previous conclusion about the monotonicity of the maximum. At y_2 we also get a contradiction to the monotonicity of the minimum. □

We use the adjoint eigenvector to obtain a special identity. We have

Lemma 5.2. *The solution* $(n^{(1)}_\varepsilon, n^{(2)}_\varepsilon)$ *of* (1.1), (1.3) *satisfies the identity*

$$\zeta^{(1)}_y(0) e^{\bar{p}/\varepsilon} n^{(1)}_\varepsilon(1) + \zeta^{(2)}_y(0) e^{\bar{p}/\varepsilon} n^{(2)}_\varepsilon(1) = \zeta^{(1)}_y(0) n^{(1)}_\varepsilon(0) + \zeta^{(2)}_y(0) n^{(2)}_\varepsilon(0). \tag{5.3}$$

Proof. Notice that the pair $(\zeta^{(1)}(\frac{x}{\varepsilon}) e^{\bar{p}x/\varepsilon}, \zeta^{(2)}(\frac{x}{\varepsilon}) e^{\bar{p}x/\varepsilon})$ is a solution to (1.1), but with different, not no-flux, boundary conditions. Then a direct integration by parts yields (5.3). □

We now complete the

Proof of Theorem 2.3. Observe that (5.3) implies, after a renormalization, that there exists $\alpha > 0$ such that

$$\alpha e^{\bar{p}/\varepsilon} n_\varepsilon^{(1)} + (1-\alpha) e^{\bar{p}/\varepsilon} n_\varepsilon^{(2)}(1) = \alpha n_\varepsilon^{(1)}(0) + (1-\alpha) n_\varepsilon^{(2)}(0).$$

Rewriting this last identity using the phase functions $R_\varepsilon^{(1)}$ and $R_\varepsilon^{(2)}$ we find

$$\alpha e^{[-R_\varepsilon^{(1)}(1)+\bar{p}]/\varepsilon} + (1-\alpha) e^{[-R_\varepsilon^{(2)}(1)+\bar{p}]/\varepsilon} = \alpha e^{-R_\varepsilon^{(1)}(0)/\varepsilon} + (1-\alpha) e^{-R_\varepsilon^{(2)}(0)/\varepsilon}.$$

Since we already know that the $R_\varepsilon^{(i)}$'s converge uniformly to the same Lipschitz continuous limit R, we get that

$$-R(1) + \bar{p} = -R(0). \tag{5.4}$$

The strict convexity of the effective Hamiltonian $\bar{H}$ implies that R_x can only take the values 0 or $\bar{p}$. It then follows from (5.4) that $R_x \equiv \bar{p}$. □

6. Generalizations

Several possible extensions of our results are possible and we discuss some of them in this section. We indicate them keping in mind that for Dirichlet boundary conditions some localization effect can be expected, see [2].

6.1. Coefficients depending on x

The viscosity method we have used here is particularly well adapted to treat the case of coefficients depending on x in system (1.1), namely $v^{(i)}(x, \frac{x}{\varepsilon})$, $\psi(x, \frac{x}{\varepsilon})$. Then each cell problem (2.3) yields an effective Hamiltonian $\bar{H}(p, x)$. Moreover, the common uniform limit R satisfies in the viscosity sense

$$\bar{H}(R_x, x) = 0.$$

This Hamilton–Jacobi equation has always a trivial solution $R \equiv 0$ because $\bar{H}(0, x) = 0$. As before we may assume that, for all x the coefficients are asymmetric in the sense of Definition 2.2 and thus there is a nonzero root $\bar{p}(x)$, i.e., $\bar{H}(\bar{p}(x), x) = 0$.

But we still face the difficulty of proving that the limiting process gives the nonzero viscosity solution. With additional technical assumptions, one can certainly reach the particular case were the coefficients are such that, uniformly in x the second root $\bar{p}(x)$ of the effective Hamiltonian has a constant sign and stays away from 0 using the duality method in Section 5. Since the argument is global on (0, 1), it seems more difficult to handle the general case where $\bar{p}$ may change sign.

6.2. Periodic diffusion

We can make a further connection to the homogenization of eigenvalues in [1,9] by including a periodic diffusion in our model. The system then has the form

$$\begin{cases} -\varepsilon\left(a^{(1)}\left(\frac{x}{\varepsilon}\right) n_{\varepsilon,x}^{(1)}\right)_x - \left(\psi_y\left(\frac{x}{\varepsilon}\right) n_\varepsilon^{(1)}\right)_x + \frac{1}{\varepsilon} v^{(1)}\left(\frac{x}{\varepsilon}\right) n_\varepsilon^{(1)} = \frac{1}{\varepsilon} v^{(2)}\left(\frac{x}{\varepsilon}\right) n_\varepsilon^{(2)}, \\ -\varepsilon\left(a^{(2)}\left(\frac{x}{\varepsilon}\right) n_{\varepsilon,x}^{(2)}\right)_x + \frac{1}{\varepsilon} v^{(2)}\left(\frac{x}{\varepsilon}\right) n_\varepsilon^{(2)} = \frac{1}{\varepsilon} v^{(1)}\left(\frac{x}{\varepsilon}\right) n_\varepsilon^{(1)} \quad \text{in } (0,1), \\ \varepsilon a^{(1)}\left(\frac{x}{\varepsilon}\right) n_{\varepsilon,x}^{(1)} + \psi_y\left(\frac{x}{\varepsilon}\right) n_\varepsilon^{(1)} = a^{(2)}\left(\frac{x}{\varepsilon}\right) n_{\varepsilon,x}^{(2)} = 0 \quad \text{for } x = 0 \text{ or } 1, \end{cases} \tag{6.1}$$

with

$$a^{(1)}, a^{(2)} > 0 \text{ and 1-periodic.}$$

Again most of our analysis goes through in this case. However, instead of showing all the details, here we present an explicit computation that yields the motor effect.

It turns out that a single equation is enough in this case, therefore we consider the problem

$$-\varepsilon\left(a\left(\frac{x}{\varepsilon}\right)n_{\varepsilon,x}\right)_x - \left(\psi_y\left(\frac{x}{\varepsilon}\right)n_\varepsilon\right)_x = 0 \quad \text{in } (0,1) \tag{6.2}$$

with the zero flux condition

$$\varepsilon a\left(\frac{x}{\varepsilon}\right)n_{\varepsilon,x} + \psi_y\left(\frac{x}{\varepsilon}\right)n_\varepsilon = 0 \quad \text{for } x = 0 \text{ or } 1. \tag{6.3}$$

In this setting the asymmetry condition is reduced to the explicit assumption that

$$\bar{p} = \int_0^1 \frac{\psi'(y)}{a(y)}\,dy \neq 0. \tag{6.4}$$

To see this, we integrate (6.2) and find, using (6.3), that

$$-\varepsilon a\left(\frac{x}{\varepsilon}\right)n_{\varepsilon,x} - \psi_y\left(\frac{x}{\varepsilon}\right)n_\varepsilon = 0 \quad \text{and} \quad n_\varepsilon(x) = n_\varepsilon(0)\exp\left(-\int_0^{\frac{x}{\varepsilon}} \frac{\psi'(y)}{a(y)}\,dy\right).$$

This formula shows directly that

$$n_\varepsilon(x) = n_\varepsilon(0)e^{-\frac{R_\varepsilon(x)}{\varepsilon}} \quad \text{with } R_\varepsilon(x) = \bar{p}x + \varepsilon\phi\left(\frac{x}{\varepsilon}\right).$$

The corrector ϕ is thus explicitly known in this case and, up to an additive constant, we have

$$\phi(y) = \int_0^y \left(\frac{\psi'(z)}{a(z)} - \bar{p}\right)dz.$$

An immediate calculation shows that this ϕ is indeed the periodic solution to the corresponding cell problem

$$-a(y)\phi_{yy} + a(y)|p+\phi_y|^2 - \big(a'(y)+\psi'(y)\big)(p+\phi_y) + \psi_{yy} = \bar{H}(p) \quad \text{in } \mathbb{R}.$$

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