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Quasistatic evolution in the theory of perfect elasto-plastic plates. Part II: Regularity of bending moments

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Abstract

We study differentiability of solutions of quasistatic problems for perfect elasto-plastic plates. We prove that in the isotropic case bending moments has locally square-integrable first derivatives: $M \in L^{\infty}([0,T]; W^{1,2}_{loc}(\Omega; \mathbb{M}^{2\times 2}_{sym}))$. The result is based on discretization of time and uniform estimates of solutions of the incremental problems, which generalize the estimates in the static case of perfect elasto-plastic plates.

Keywords: Quasistatic evolution; Rate independent processes; Elasto-plastic plates; Regularity of solutions

1. Introduction

In this paper we study the regularity of the bending moments of the quasistatic evolution of clamped perfectly elasto-plastic plates under the action of a time-dependent transversal body force. Before introducing the regularity result, we describe the mechanical model. The reference configuration is a bounded open set $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary and the elastic domain $\mathbb{K}$ is a bounded closed convex subset of $\mathbb{M}^{2\times 2}_{sym}$ (the space of symmetric 2×2 matrices) with nonempty interior, whose boundary $\partial\mathbb{K}$ plays the role of the yield surface.

Given a scalar valued function $f(t,x)$ defined for $t \in [0,T]$ and $x \in \Omega$, which represents the transversal body force, the strong formulation of the evolution problem consists in finding a scalar valued function $u(t,x)$ (the vertical displacement) and three matrix-valued functions $e(t,x)$, $p(t,x)$ and $M(t,x)$ (the elastic and plastic curvatures and the bending moments) such that for every $t \in [0,T]$, for every $x \in \Omega$ the following conditions hold:

(1) kinematic admissibility: $D^2u(t,x) = e(t,x) + p(t,x)$ in Ω, $u(t,x) = 0$, $\frac{\partial u}{\partial \nu}(t,x) = 0$ on $\partial\Omega$,
(2) constitutive equation: $M(t,x) = \mathbb{C}e(t,x)$,
(3) equilibrium: $\operatorname{div}\operatorname{div} M(t,x) = f(t,x)$ in Ω,
(4) moment constraint: $M(t,x) \in \mathbb{K}$,
(5) associative flow rule: $\dot p(t,x) \in N_{\mathbb{K}}(M(t,x))$,

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where $\nu(x)$ is the outer unit normal to $\partial\Omega$ and $\mathbb{C}$ is the rigidity tensor. The symbol $N_{\mathbb{K}}(\xi)$ denotes the normal cone to the set $\mathbb{K}$ at the point ξ in the sense of convex analysis. The problem is supplemented by initial conditions at time $t=0$.

The boundary conditions $u=0$ and $\frac{\partial u}{\partial \nu}=0$ on $\partial\Omega$ reflect the mechanical assumption the plate is clamped.

For the regularity *we restrict ourselves* to the *isotropic case* where $\mathbb{K}$ *is a ball* and $\mathbb{C}$ *is a multiple of identity tensor* $\mathbf{1}$, which can be reduced to considering

$$\mathbb{K}=B_1(0), \quad \mathbb{C}=\mathbf{1}.$$

Existence of weak solutions to problems in perfect plasticity has been extensively studied during last decades (see, for example, [1,3–5,8,21]). Since the variational formulation of the problem used in the definition of weak solutions involves an integral with linear growth in D^2u, the natural functional spaces for the problem are $BH(\Omega)$ of functions with bounded Hessian for the vertical displacements u, and $L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})$ for the bending moments M.

We observe that the question of regularity of weak solutions to the problems in perfect plasticity was first addressed in [19], where the higher differentiability results have appeared for the first time.

However, in a similar problem for Prandtl–Reuss perfect plasticity it was shown in [2,6] that the stress (which is the counterpart of the bending moments) belongs to $W^{1,2}_{loc}(\Omega;\mathbb{M}^{2\times 2}_{sym})$ (see also [8,14–19] for similar results for some static models).

In the present paper we study the spatial regularity of the bending moments $M(t,\cdot)$ for the quasistatic problem for prefect elasto-plastic plates. The main result obtained (see Theorem 2.2 below) for the model under consideration is

$$M\in L^\infty\big([0,T];W^{1,2}_{loc}\big(\Omega;\mathbb{M}^{2\times 2}_{sym}\big)\big). \tag{1.1}$$

As in [6], our strategy for an evolutionary quasistatic problem relies on a regularity result for an analogous static problem, obtained in [16], where it was shown that in a static situation the bending moments enjoy the following differentiability condition:

$$M\in W^{1,2}_{loc}\big(\Omega;\mathbb{M}^{2\times 2}_{sym}\big).$$

We discretize time by points $(t^N_r)^N_{r=1}$

$$0=t^N_0<t^N_1<\cdots<T^N_N=T$$

and we approximate the original quasistatic problem by a sequence of incremental "static" problems, finding for each $r=1,\dots,N$ the updated values of $(u^N_r,e^N_r,p^N_r,M^N_r)$, provided that $(u^N_{r-1},\ e^N_{r-1},\ p^N_{r-1},\ M^N_{r-1})$ is already found. Shortly, the main idea is to generalize the estimates of [16] in order to take into account the influence of the previous steps.

To be more precise, following [7], we apply the standard method of constructing piecewise constant approximations

$$\big(u_N(t),e_N(t),p_N(t),M_N(t)\big)=\big(u^N_r,e^N_r,p^N_r,M^N_r\big) \quad \text{for } t^N_r\leqslant t<t^N_{r+1},$$

with $0\leqslant r<N$, of the continuous-time energy formulation of rate-independent processes (see [11] for the survey of this approach). Our aim is to get a uniform estimate of the form

$$\sup_{N\in\mathbb{N}}\sup_{t\in[0,T]}\big\|M_N(t)\big\|_{W^{1,2}_{loc}(\Omega;\mathbb{M}^{2\times 2}_{sym})}\leqslant C, \tag{1.2}$$

which clearly implies (1.1).

To get (1.2) we consider the updated values of $(u^N_r,e^N_r,p^N_r,M^N_r)$ as saddle points of some minimax problem, similar to the one considered in [8,16] for static cases in perfect plasticity. The main difference from the purely static problem is the presence of a term which takes into account the outcome of the preceding step. Approximating each incremental problem with a sequence of regularized problems, depending on a real parameter $\alpha\in(0,1)$, we obtain that their solutions M^α_r converge to M^N_r, a solution to the corresponding incremental problem, weakly in $L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})$, as $\alpha\to 0$. Then one can show, that for every incremental problem the bound

$$\sup_{\alpha>0}\big\|M^\alpha_r\big\|_{W^{1,2}(\Omega';\mathbb{M}^{2\times 2}_{sym})}\leqslant C_r$$

holds for any domain $\Omega'\Subset\Omega$, where the constant C_r depends on the discretization step and on Ω'. Thus, M^N_r is itself in $W^{1,2}_{loc}(\Omega;\mathbb{M}^{2\times 2}_{sym})$, and the compactness of Sobolev embedding improves the convergence of M^α_r to M^N_r. Then we do some analytical work to make the last estimate uniform in r and N, and we obtain (1.2).

Notice that all the arguments used below are purely local, and cannot be used for studying the behavior of bending moments up to the boundary $\partial\Omega$ (see [13] for the discussion of the global regularity issues in an analogous case of Hencky perfect plasticity). As far as we know, the only global regularity result in perfect plasticity was obtained in [9] for Hencky perfect plasticity.

The paper is organized as follows: in Section 2 we introduce the notation and state the main result. We present a weak formulation of the problem and prove a time-continuity result in Section 3. A minimax formulation of incremental problems in spirit of [8,16] is presented in Section 4. In Section 5 we introduce some regularized problems, depending on a real parameter $\alpha \in (0,1)$, whose solutions are smooth and "approximate", as $\alpha \to 0$, a solution $(u_r^N, e_r^N, p_r^N, M_r^N)$ of the corresponding incremental problem. We obtain $W^{1,2}_{loc}$ estimates of the solutions of regularized problems in Section 6, and conclude that

$$\sup_{t\in[0,T]} \left\| M_N(t) \right\|_{W^{1,2}(\Omega';\mathbb{M}^{2\times 2}_{sym})} \leqslant C(N,\Omega')$$

for each $\Omega' \Subset \Omega$ and $N \in \mathbb{N}$. Section 7 contains some analytical estimates, that will be used for making $W^{1,2}_{loc}$ estimates uniform with respect to N. Finally, in Section 8 we apply the results of Section 7 to obtain the uniform estimates of Sobolev norms and to conclude the proof of Theorem 2.2.

2. Preliminaries

2.1. Notation and definitions

We adopt the following notation:

$\mathbb{R}^n$ denotes the n-dimensional Euclidean space,

$\mathbb{M}^{2\times 2}_{sym}$ denotes the space of all 2×2 symmetric matrices, equipped with the Hilbert–Schmidt scalar product $\sigma : \xi = \sigma_{ij}\xi_{ij}$,

$a \odot b$ stands for the symmetrized tensor product of two vectors $a, b \in \mathbb{R}^n$, given by the formula $(a \odot b)_{ij} = \frac{1}{2}(a_i b_j + a_j b_i)$,

$L^p(\Omega;\mathbb{R}^m)$ is the Lebesgue space of functions from Ω into $\mathbb{R}^m$, having finite norm $(\int_\Omega |f|^p\,dx)^{1/p}$,

$W^{l,p}(\Omega;\mathbb{R}^m)$ is the Sobolev space of all functions from Ω into $\mathbb{R}^m$ with the norm

$$\|f\|_{l,p;\Omega} := \left(\int\limits_\Omega \sum_{\alpha=0}^{l} \left|\nabla^\alpha f\right|^p \right)^{1/p},$$

$\mathcal{L}^2$ stands for the Lebesgue measure on $\mathbb{R}^2$,

$\mathcal{H}^1$ is the one-dimensional Hausdorff measure,

$M_b(\overline{\Omega};\mathbb{R}^m)$ is the space of all bounded Radon measures on $\overline{\Omega}$ with values in $\mathbb{R}^m$.

For $\mu \in M_b(\overline{\Omega};\mathbb{R}^m)$, we denote its total variation by $|\mu|$, which is an element of $M_b(\overline{\Omega})$, with $\|\mu\|_{1;\overline{\Omega}} = |\mu|(\overline{\Omega})$, while by μ^a and μ^s we denote its absolutely continuous and singular part with respect to $\mathcal{L}^2$,

$BH(\Omega)$ is the space of all functions in $L^1(\Omega)$ such that $Du \in BV(\Omega;\mathbb{R}^2)$, with norm $\|u\|_{2,1;\Omega} := \|u\|_{1,1;\Omega} + \|D^2 u\|_{1;\Omega}$,

$\langle\cdot|\cdot\rangle$ denotes a duality product depending upon the context.

Remark 2.1. We refer to [21, Chapter III] for the main properties of $BH(\Omega)$ and for the definition of weak* convergence in $BH(\Omega)$. Remark, that for $u \in BH(\Omega)$ we have the following embedding:

$$u \in C(\overline{\Omega}), \quad \nabla u \in L^2\big(\Omega;\mathbb{R}^2\big).$$

Let us introduce the notation

$$S(\Omega) = \big\{ M \in L^2\big(\Omega;\mathbb{M}^{2\times 2}_{sym}\big) \colon \operatorname{div}\operatorname{div} M \in L^2(\Omega) \big\},$$

$$\mathcal{K}(\Omega) = \big\{ m \in L^2\big(\Omega;\mathbb{M}^{2\times 2}_{sym}\big) \colon m(x) \in \mathbb{K} \text{ for a.e. } x \in \Omega \big\}.$$

2.2. The main result

We impose the following assumption on the data of the problem:

$$f \in AC\big([0,T]; L^2(\Omega)\big) \cap L^\infty\big([0,T]; W^{1,2}_{loc}(\Omega)\big). \tag{2.1}$$

We also assume the so-called uniform safe-load condition:

$$\begin{aligned} &\text{there exists a function } m^1 \in AC\big([0,T]; L^2\big(\Omega; \mathbb{M}^{2\times 2}_{sym}\big)\big), \text{ such that} \\ &\operatorname{div}\operatorname{div} m(t) = f(t) \quad \text{in } \Omega \text{ for every } t \in [0,T], \text{ and} \\ &\big|m^1(t,x)\big| \leqslant (1-\lambda) \quad \text{for some } 0<\lambda<1, \text{ a.e. } x \in \Omega, \text{ for every } t \in [0,T]. \end{aligned} \tag{2.2}$$

Here and in the rest of the paper div always denotes the divergence with respect to space variables.

The main result of the paper is the following regularity theorem.

Theorem 2.2. *Suppose that the set $\mathbb{K}$ is a ball and $\mathbb{C}$ is a multiple of the identity, and that assumptions* (2.1), (2.2) *are satisfied. Then for the solution (u,e,p) of the quasistatic problem, see Definition* 3.4 *below, we have*

$$M \in L^\infty\big([0,T]; W^{1,2}_{loc}\big(\Omega; \mathbb{M}^{2\times 2}_{sym}\big)\big),$$

with $M(t,x) = \mathbb{C}e(t,x)$.

Remark 2.3. As already mentioned, we consider the case $\mathbb{K} = B_1(0)$ and $\mathbb{C} = \mathbf{1}$. It means, that $M \equiv e$, and we will be using both notations M and e for the same object.

3. Weak formulation of the quasistatic problem

Below we give the possible definition of weak solution to the quasistatic problem. The formulation we use (see [7]) is expressed in terms of energy balance and energy dissipation.

3.1. Weak formulation: quasistatic evolution

Now we give the definition of a kinematically admissible triple. The first condition describes the additive decomposition, the second one gives the boundary conditions for u, while the third one reflects the boundary conditions for Du in a relaxed form, which is typical in the variational theory of functionals with linear growth.

Definition 3.1. A triple $(u,e,p) \in BH(\Omega) \times L^2(\Omega; \mathbb{M}^{2\times 2}_{sym}) \times M_b(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym})$ is called kinematically admissible, if the following conditions hold

$$\begin{aligned} &D^2 u = e + p \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \partial\Omega, \\ &p = -\nabla u \odot \nu \mathcal{H}^1 \quad \text{on } \partial\Omega. \end{aligned}$$

Definition 3.2. For a kinematically admissible triple (u,e,p) and $M \in S(\Omega)$ we define a measure $[M:p] \in M_b(\overline{\Omega})$ by putting

$$\begin{aligned} &[M:p] = M:p^a + \big[M:D^2u\big]^s = \big[M:D^2u\big] - M:e \quad \text{in } \Omega, \\ &[M:p] = -\frac{\partial u}{\partial \nu} M_{ij}\nu_i\nu_j \, d\mathcal{H}^1 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

where the measure $[M:D^2u]$ is defined in [5].

Thus, a duality pairing between $S(\Omega)$ and $\Pi(\Omega)$ is defined by

$$\langle M | p\rangle := [M:p](\overline{\Omega}). \tag{3.2}$$

One can prove the following integration by parts formula (see [7, Proposition 2.8]).

Proposition 3.3. *For a kinematically admissible triple (u, e, p) and $M \in S(\Omega)$ with* $\operatorname{div}\operatorname{div} M = f \in L^2(\Omega)$ *we have*

$$[M : p](\overline{\Omega}) = \int_\Omega uf\,dx - \int_\Omega (M : e)\,dx. \tag{3.3}$$

Let us define the functionals which appear in the energy formulation of the problem. The quadratic form $\mathcal{Q}: L^2(\Omega; \mathbb{M}^{2\times 2}_{sym}) \to \mathbb{R}$, corresponding to the stored elastic energy, is defined by

$$\mathcal{Q}(e) = \frac{1}{2}\int_\Omega e : e\,dx.$$

Since in our case the function H considered in [7, Section 2.1] reduces to the norm in $\mathbb{M}^{2\times 2}_{sym}$, the dissipation in any time interval $[s, t] \subset [0, T]$ is defined by

$$\mathcal{D}(p; s, t) = \sup\left\{\sum_{j=1}^{M} \left\| p(t_j) - p(t_{j-1})\right\|_{1;\overline{\Omega}} : s = t_0 \leqslant \cdots \leqslant t_M = t,\ M \in \mathbb{N}\right\}.$$

Now we are in a position to give a variational formulation of the quasistatic problem. In the following definition $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2(\Omega)$.

Definition 3.4. A quasistatic evolution is a function $t \mapsto (u(t), e(t), p(t))$ from $[0, T]$ into $BH(\Omega) \times L^2(\Omega; \mathbb{M}^{2\times 2}_{sym}) \times M_b(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym})$ which satisfies the following conditions:

(qs1) for every $t \in [0, T]$ the triple $(u(t), e(t), p(t))$ is kinematically admissible and

$$\mathcal{Q}\big(e(t)\big) - \big\langle f(t), u(t)\big\rangle \leqslant \mathcal{Q}(\eta) + \left\| q - p(t)\right\|_{1;\overline{\Omega}} - \big\langle f(t), v\big\rangle \tag{3.4}$$

for every kinematically admissible triple (v, η, q);

(qs2) the function $t \mapsto p(t)$ from $[0, T]$ into $M_b(\Omega; \mathbb{M}^{2\times 2}_{sym})$ has bounded variation and for every $t \in [0, T]$

$$\mathcal{Q}\big(e(t)\big) + \mathcal{D}(p; 0, t) - \big\langle f(t), u(t)\big\rangle = \mathcal{Q}\big(e(0)\big) - \big\langle f(0), u(0)\big\rangle - \int_0^t \big\langle \dot f(s), u(s)\big\rangle\,ds. \tag{3.5}$$

3.2. Existence result and time-discretization

The following theorem establishes the existence of a solution to the quasistatic problem in perfect plasticity.

Theorem 3.5. *Let a kinematically admissible triple (u_0, e_0, p_0) satisfy the stability condition*

$$Q(e_0) - \big\langle f(0), u_0\big\rangle \leqslant Q(\eta) + \|q - p_0\|_{1;\overline{\Omega}} - \big\langle f(0), v\big\rangle,$$

for every kinematically admissible triple (v, η, q). Then there exists a quasistatic evolution

$$\big(u(t), e(t), p(t)\big),$$

such that

$$u(0) = u_0, \qquad e(0) = e_0, \qquad p(0) = p_0.$$

Moreover, the elastic part $t \mapsto e(t)$ of $D^2u(t)$ is unique and a quasistatic evolution (u, e, p) as a function from $[0, T]$ to $BH(\Omega) \times L^2(\Omega; \mathbb{M}^{2\times 2}_{sym}) \times M_b(\overline{\Omega}; \mathbb{M}^{2\times 2}_{sym})$ is absolutely continuous in time.

In [7] this theorem is proved by a discretization of time. We divide the interval $[0, T]$ into N equal parts of length T/N by points $(t_r^N)_{r=0,\dots,N}$. For $r = 0, \dots, N$ we set

$$f_N^r = f\big(t_r^N\big) \quad \text{and} \quad \big(m^1\big)_r^N = m^1\big(t_r^N\big). \tag{3.6}$$

For every N we define u_r^N, e_r^N and p_r^N by induction. We set

$$\big(u_0^N, e_0^N, p_0^N\big) = (u_0, e_0, p_0),$$

while for every $r = 1, \ldots, N$ we define (u_r^N, e_r^N, p_r^N) as a solution to the incremental problem

$$\min_{(u,e,p)} \left\{ \mathcal{Q}(e) + \|p - p_{r-1}^N\|_{1;\overline{\Omega}} - \int_\Omega f_r^N u \, dx \right\}, \tag{3.7}$$

where the minimization is carried out over all kinematically admissible triples (see Definition 3.1).

Remark 3.6. We note, that (u, e, p) is a solution to (3.7) if and only if one of the following conditions holds:

(1) for every kinematically admissible triple (v, η, q) one has

$$-\|q\|_{1;\overline{\Omega}} \leqslant \langle e, \eta \rangle - \langle f_r^N, v \rangle \leqslant \|q\|_{1;\overline{\Omega}};$$

(2) $e \in S(\Omega) \cap \mathcal{K}$ with $\operatorname{div}\operatorname{div} e = -f_r^N$.

For $r = 0, \ldots, N$ we set $M_r^N = e_r^N$ and for every $t \in [0, T]$ we define piecewise constant interpolations

$$u_N(t) = u_r^N, \qquad e_N(t) = e_r^N, \qquad p_N(t) = p_r^N, \qquad M_N(t) = M_r^N, \qquad f_N(t) = f_r^N, \qquad m_N^1(t) = (m^1)_r^N,$$

where r is the largest integer such that $t_r^N \leqslant t$. By definition $(u_N(t), e_N(t), p_N(t))$ is kinematically admissible for every $t \in [0, T]$.

In the proof of the existence, it was shown that for approximate solutions one has the estimate

$$\sup_{t \in [0,T]} \big\|e_N(t)\big\|_{2;\Omega} + \operatorname{Var}(p_N; 0, T) + \sup_{t \in [0,T]} \|u_N\|_{2,1;\Omega} \leqslant C, \tag{3.8}$$

which is uniform with respect to N, and it was established that these functions converge pointwise (with respect to t) to a solution of the quasistatic evolution problem.

3.3. Continuity estimates of solutions of the incremental problems

In [7] it was established that the quasistatic evolution is absolutely continuous in time. However, as we will deal precisely with the solutions of the time-discretized problems, we would need the continuity estimates of solutions at the level of incremental problems.

The following notation will be often used below: given a function $h : [0, T] \to X$,

$$\delta h_r^N := h\big(t_r^N\big) - h\big(t_{r-1}^N\big). \tag{3.9}$$

We also consider the increment of the data of the problem, defined by

$$D_r^N := \big\|\delta\big(m^1\big)_r^N\big\|_{2;\Omega} + \big\|\delta f_r^N\big\|_{2;\Omega}. \tag{3.10}$$

By (2.1) and (2.2), after time reparametrization, we may assume that

$$f \in \operatorname{Lip}\big([0, T]; L^2(\Omega)\big) \cap L^\infty\big([0, T]; W_{loc}^{1,2}(\Omega)\big),$$

and

$$m^1 \in \operatorname{Lip}\big([0, T]; L^2\big(\Omega; \mathbb{M}_{sym}^{2\times 2}\big)\big).$$

Indeed, every absolutely continuous function can be made Lipschitz just by time reparametrization, and this leads to a corresponding reparametrization of the solutions, the problem being rate-independent. In other words, we may suppose, that

$$D_r^N \leqslant \frac{C}{N}. \tag{3.11}$$

Theorem 3.7. *For solutions of the incremental problems* (u_r^N, e_r^N, p_r^N) *the following inequality holds*:

$$\|\delta e_r^N\|_{2;\Omega} + \|\delta p_r^N\|_{1;\overline{\Omega}} + \|\delta u_r^N\|_{2,1;\Omega} \leqslant D_r^N, \tag{3.12}$$

where δh_r^N *in understood as in* (3.9) *and* D_r^N *denotes the increment of the data of the problem, defined by* (3.10).

Proof. As the triple

$$\left(u_{r-1}^N, e_{r-1}^N, p_{r-1}^N\right)$$

is kinematically admissible, the minimality condition (3.7) and the integration by parts formula (3.3) imply

$$\mathcal{Q}\left(e_r^N\right) - \int_\Omega \left(m^1\right)_r^N : e_r^N\,dx + \|p_r^N - p_{r-1}^N\|_{1;\overline{\Omega}} - \left\langle \left(m^1\right)_r^N, p_r^N - p_{r-1}^N\right\rangle \leqslant \mathcal{Q}\left(e_{r-1}^N\right) - \int_\Omega \left(m^1\right)_r^N : e_{r-1}^N\,dx.$$

Developing the quadratic form in the right-hand side we arrive at:

$$\begin{aligned} &\frac{1}{2}\int_\Omega M_r^N : e_r^N\,dx - \frac{1}{2}\int_\Omega M_{r-1}^N : e_{r-1}^N\,dx + \mathcal{H}\left(p_r^N - p_{r-1}^N\right) \\ &\quad \leqslant \left\langle \left(m^1\right)_r^N, p_m^N - p_{m-1}^N\right\rangle - \int_\Omega \left(m^1\right)_r^N : e_{r-1}^N\,dx + \int_\Omega \left(m^1\right)_r^N : e_r^N\,dx. \end{aligned} \tag{3.13}$$

Now consider the functions

$$\begin{aligned} &v = u_r^N - u_{r-1}^N, \qquad \eta = e_r^N - e_{r-1}^N, \\ &q = p_r^N - p_{r-1}^N. \end{aligned}$$

Since (v, η, q) is kinematically admissible and $(u_{r-1}^N, e_{r-1}^N, p_{r-1}^N)$ is a solution of the corresponding minimum problem at the previous step, we obtain, by means of Remark 3.6 and the integration by parts formula (3.3)

$$-\int_\Omega M_{r-1}^N : \left(e_r^N - e_{r-1}^N\right)dx + \int_\Omega \left(m^1\right)_{r-1}^N : \left(e_r^N - e_{r-1}^N\right)dx + \left\langle \left(m^1\right)_{r-1}^N, p_r^N - p_{r-1}^N\right\rangle \leqslant \mathcal{H}\left(p_r^N - p_{r-1}^N\right). \tag{3.14}$$

By combining (3.13) and (3.14) we get the following

$$\begin{aligned} \mathcal{Q}\left(e_r^N - e_{r-1}^N\right) &= \frac{1}{2}\int_\Omega M_r^N : e_r^N\,dx - \frac{1}{2}\int_\Omega M_{r-1}^N : e_{r-1}^N\,dx - \int_\Omega M_{r-1}^N : \left(e_r^N - e_{r-1}^N\right)dx \\ &\leqslant \left\langle \left(m^1\right)_r^N, p_r^N - p_{r-1}^N\right\rangle - \int_\Omega \left(m^1\right)_r^N : e_{r-1}^N\,dx + \int_\Omega \left(m^1\right)_r^N : e_r^N\,dx \\ &\quad - \int_\Omega \left(m^1\right)_{r-1}^N : \left(e_r^N - e_{r-1}^N\right)dx - \left\langle \left(m^1\right)_{r-1}^N, p_r^N - p_{r-1}^N\right\rangle, \end{aligned} \tag{3.15}$$

where $\langle\cdot,\cdot\rangle$ is the duality defined in (3.2).

Let us apply the integration by parts formula (3.3) to compute $\langle (m^1)_m^N, p_m^N - p_{m-1}^N\rangle$:

$$\left\langle \left(m^1\right)_r^N, p_r^N - p_{r-1}^N\right\rangle = -\int_\Omega \left(m^1\right)_r^N : \left(e_r^N - e_{r-1}^N\right)dx + \int_\Omega f_r^N\left(u_r^N - u_{r-1}^N\right)dx. \tag{3.16}$$

A similar formula holds for $\langle (m^1)_{r-1}^N, p_r^N - p_{r-1}^N\rangle$.

Putting (3.16) into (3.15) we end up with the estimate

$$\mathcal{Q}\left(e_r^N - e_{r-1}^N\right) \leqslant \int_\Omega \left(f_r^N - f_{r-1}^N\right)\cdot\left(u_r^N - u_{r-1}^N\right)dx + \|f_r^N - f_{r-1}^N\|_{2;\Omega}\|u_r^N - u_{r-1}^N\|_{2,1;\Omega}. \tag{3.17}$$

Now let us estimate $\|p_r^N - p_{r-1}^N\|_{1;\overline{\Omega}}$ in terms of the data of the problem. First of all, the safe load condition yields

$$\lambda\|p_r^N - p_{r-1}^N\|_{1;\overline{\Omega}} \leqslant \mathcal{H}(p_r^N - p_{r-1}^N) - \langle (m^1)_r^N, p_r^N - p_{r-1}^N\rangle.$$

Now, the relation (3.13) and the boundedness of $\|\varrho_r^N\|_{\infty;\Omega}$, $\|e_r^N\|_{2;\Omega}$ and $\|p_r^N\|_{1;\overline{\Omega}}$ imply

$$\|p_r^N - p_{r-1}^N\|_{1;\overline{\Omega}} \leqslant C(\|e_r^N - e_{r-1}^N\|_{2;\Omega} + D_r^N). \tag{3.18}$$

Taking into account the inequality

$$\|u_r^N - u_{r-1}^N\|_{2,1;\Omega} \leqslant C(\|e_r^N - e_{r-1}^N\|_{2;\Omega} + \|p_r^N - p_{r-1}^N\|_{1;\overline{\Omega}}),$$

proved in [7, estimate (3.9)], the estimate

$$\|p_r^N - p_{r-1}^N\|_{1;\overline{\Omega}} + \|e_r^N - e_{r-1}^N\|_{2;\Omega} \leqslant C D_r^N \tag{3.19}$$

follows now from (3.17), (3.18) and the application of the Cauchy inequality.

To prove

$$\|D^2 u_r^N - D^2 u_{r-1}^N\|_{1;\Omega} \leqslant C D_r^N, \tag{3.20}$$

we recall the additive decomposition $D^2 u = e + p$ and make use of (3.19).

Finally to show the validity of (3.12), it remains to estimate $\|u_r^N - u_{r-1}^N\|_{1;\Omega}$. By the Poincaré inequality for BH the result follows from (3.11), (3.19), (3.20) and the latter inequality. □

4. Minimax problem

In this section we briefly discuss the minimax formulation of the incremental problem. We follow the general scheme, described in [8], which was applied in [16] for studying the regularity of solutions of static problems in the theory of perfect elasto-plastic plates.

We refer to [8, Chapter 1] for the complete exposition of an abstract theory and to [6, Section 4] for its short presentation. The following calculations follow closely [6, Section 5], making use of constructions developed in [16].

Recall that the time-discretization procedure, that provides us a way of constructing approximate solutions to the quasistatic problem for perfect elasto-plastic plates, leads one to solving a sequence of the following incremental problems:

$$\min_{(u,M,p)} \{\mathcal{Q}(M) + \|p - p_{r-1}^N\|_{1;\overline{\Omega}} - \langle f_r^N, u\rangle\}, \tag{4.1}$$

where the minimum is taken over all kinematically admissible triples (see Definition 3.1), with p_{r-1}^N be a solution of the corresponding incremental problem, obtained at the previous step.

4.1. Functional setting of the problem

We set

$$V_0 = W_0^{2,1}(\Omega), \qquad U = W_0^{1,2}(\Omega),$$

U^* is the dual space of U. If $1 < p \leqslant +\infty$, the space $L^p(\Omega)$, is embedded in U^* by the usual identification

$$\langle f, u\rangle = \int_\Omega f\, u\, dx \quad \text{for any } u \in W_0^{1,2}(\Omega).$$

Put

$$P = L^1(\Omega; \mathbb{M}_{sym}^{2\times 2}) \qquad P^* = L^\infty(\Omega; \mathbb{M}_{sym}^{2\times 2}).$$

We have the following

$$\begin{aligned}&\text{the embedding of } V_0 \text{ into } U \text{ is continuous,}\\ &V_0 \text{ is dense in } U.\end{aligned} \tag{4.2}$$

Let us introduce the functionals $G: P \to \overline{\mathbb{R}}$ and $\mathcal{L}: U \to \overline{\mathbb{R}}$ by

$$G(m) = \int_\Omega g\big(m + e_{r-1}^N\big)\,dx, \quad m \in P,$$

$$\mathcal{L}(v) = -\int_\Omega f_r^N \cdot v\,dx, \quad v \in U. \tag{4.3}$$

Thus, G and $\mathcal{L}$ are continuous and it is easy to see that the Legendre transform of G is

$$G^*(M) = \int_\Omega \big(g^*(M) - M : e_{r-1}^N\big)\,dx, \quad \text{for } M \in P^*. \tag{4.4}$$

Here $g: \mathbb{M}^{2\times 2}_{sym} \to \mathbb{R}$ has the form

$$g(m) \equiv g_0\big(|m|\big) := \begin{cases} \frac{1}{2}|m|^2, & \text{if } |m| \leqslant 1; \\ |m| - \frac{1}{2}, & \text{if } |m| > 1, \end{cases} \tag{4.5}$$

while its Legendre transform $g^*: \mathbb{M}^{2\times 2}_{sym} \to \overline{\mathbb{R}}$ is given by the formula

$$g^*(M) = \begin{cases} \frac{1}{2}|M|^2, & \text{if } |M| \leqslant 1; \\ +\infty, & \text{otherwise.} \end{cases}$$

4.2. Saddle-point problem in its strong formulation

Introduce the continuous linear operator $A: V_0 \to P$ as

$$Av := D^2 v, \quad v \in V_0.$$

We define the Lagrangian $\ell: V \times \mathcal{K}(\Omega) \to \overline{\mathbb{R}}$ by

$$\ell(v, m) = \int_\Omega \big(D^2 v : m + m : e_N^{r-1}\big)\,dx - \int_\Omega g^*(m)\,dx + \mathcal{L}(v),$$

and consider the following minimax problem

$$\begin{cases} \text{find a pair } (u, M) \in V_0 \times \mathcal{K}(\Omega) \text{ such that} \\ \ell(u, m) \leqslant \ell(u, M) \leqslant \ell(v, M), \quad \text{for all } v \in V_0,\ m \in \mathcal{K}(\Omega). \end{cases} \tag{4.6}$$

The minimax problem (4.6) generates a pair of dual problems, the primal one

$$\begin{cases} \text{find } \delta u_r^N \in V_0 \text{ such that} \\ I(\delta u_r^N) = \inf\{I(v):\ v \in V_0\}, \end{cases} \tag{4.7}$$

where the functional I is given by

$$I(v) = G(Av) + \mathcal{L}(v) = \int_\Omega g\big(D^2 v + e_N^{r-1}\big)\,dx - \int_\Omega f_r^N \cdot v\,dx,$$

and the dual one

$$\begin{cases} \text{find } M_r^N \in Q_{f_r^N} \cap \mathcal{K}(\Omega) \text{ such that} \\ R(M_r^N) = \sup\{R(m):\ m \in Q_{f_r^N} \cap \mathcal{K}(\Omega)\}, \end{cases} \tag{4.8}$$

where

$$R(m) = \begin{cases} \ell(0, m) = \int_\Omega (m : e_{r-1}^N - g^*(m))\,dx, & m \in Q_{f_r^N} \cap \mathcal{K}(\Omega); \\ -\infty, & m \notin Q_{f_r^N} \cap \mathcal{K}(\Omega), \end{cases} \quad \text{for } m \in \mathcal{K}(\Omega),$$

with $Q_{f_r^N}$ being defined as

$$Q_{f_r^N} = \big\{m \in S(\Omega):\ \operatorname{div}\operatorname{div} m = f_r^N\big\}.$$

The following theorem (see [8, Chapter 1]) shows that under very mild assumptions the dual problem (4.8) has a solution and one can exchange inf and sup signs.

Theorem 4.1. *Suppose that the following two conditions hold*

$$C := \inf\{I(v):\ v \in V_0\} \in \mathbb{R}, \tag{4.9}$$

$$\begin{aligned} &\text{there exists } u_1 \in V \text{ such that } G(Au_1) < +\infty,\ \mathcal{L}(u_1) < +\infty \\ &\text{and the function } p \mapsto G(Au_1 + p) \text{ is continuous at zero.} \end{aligned} \tag{4.10}$$

Then problem (4.8) *has at least one solution and the identity*

$$C = \sup\{R(m):\ m \in P^*\}$$

is valid.

Condition (4.10) is obviously satisfied. It is easy to see, that the safe load condition (2.2) yields condition (4.9) and the coercivity of the functional I with respect to the norm of V_0. However, as the space V_0 is not reflexive, one needs to construct a suitable relaxation of the variational problem (4.6), (4.7).

4.3. The relaxed problem

We construct a variational extension of the problem. To this aim we construct a relaxation of problem (4.6). We will make use of an auxiliary space D, defined in the following way: a function m belongs to D if and only if there exists $u^* \in U^*$ such that

$$\int_\Omega u^* v\,dx = \int_\Omega m : D^2 v\,dx \quad \text{for all } v \in V_0.$$

Thus,

$$D = \{M \in P^*:\ \operatorname{div}\operatorname{div} M \in U^*\}.$$

According to the general procedure (see [8, Chapter 1]) we define an extension V_+ of the space V as

$$V_+ = \left\{v \in U:\ \sup_{\|M\|_{\infty;\Omega} \leqslant 1,\ M \in D} \int_\Omega v \operatorname{div}\operatorname{div} M\,dx < +\infty\right\}.$$

In particular, taking the test fields $M \in C_0^\infty(\Omega; \mathbb{M}^{2\times 2}_{sym})$ we conclude that $v \in BH(\Omega)$.

Introduce the relaxed Lagrangian

$$L(v, m) = \int_\Omega \left(\operatorname{div}\operatorname{div} m - f_N^r\right) v\,dx + \int_\Omega m : e_N^{r-1}\,dx - \int_\Omega g^*(m)\,dx$$

for $v \in V_+$ and $m \in \mathcal{K}(\Omega) \cap D$. Consider the minimax problem for this relaxed Lagrangian L:

$$\begin{cases} \text{find a pair } (u, M) \in V_+ \times \left(\mathcal{K}(\Omega) \cap D\right) \text{ such that} \\ L(u, m) \leqslant L(u, M) \leqslant L(v, M), \quad \text{for all } v \in V_+,\ m \in \mathcal{K}(\Omega) \cap D. \end{cases} \tag{4.11}$$

Arguing as in [8, Chapter 1] and [6, Section 5] we conclude that the following result holds (see Theorem 4.2 below): there exists a saddle point $u \in V_+$ and $M \in \mathcal{K}(\Omega) \cap D$ of the Lagrangian L on the set $V_+ \times (\mathcal{K}(\Omega) \cap D)$. In this case the tensor M is the unique solution of the problem (4.8) and $w \in V_+$ is a solution of the problem

$$\begin{cases} \text{find } w \in V_+ \text{ such that} \\ \Phi(w) = \inf\{\Phi(v):\ v \in V_+\}, \end{cases} \tag{4.12}$$

where

$$\Phi(v) := \sup\{L(v, m):\ m \in \mathcal{K}(\Omega) \cap D\}.$$

The precise result is expressed in the following theorem, which is a consequence of [8, Theorem 1.2.2] (see also [16, assertions (2.7)–(2.9)] for a similar construction in the static problem).

Theorem 4.2. *Suppose that $f_r^N \in L^2(\Omega)$ and condition (2.2) holds. Then there exists at least one solution $(\delta u_r^N, M_r^N)$ to the minimax problem (4.11) in $V_+ \times (\mathcal{K} \cap D)$. Moreover, M_r^N is the unique solution to the dual variational problem (4.8) and δu_r^N is a solution to (4.12). The identity*

$$\Phi(\delta u_r^N) = R(M_r^N)$$

holds. Finally, every minimizing sequence of the problem (4.7) contains a subsequence which converges to some solution of (4.12) weakly in U and strongly in $W_0^{1,p}(\Omega;\mathbb{R}^2)$ for $1 \leqslant p < 2$.

We note, that the following approximation result holds for the functions from $S(\Omega)$. Remark, that the proof presented in [21], Chapter III, contains an error. A correct proof was proposed by G. Seregin [20], and we present it here for completeness.

Lemma 4.3. *Let Ω be a bounded Lipshitz domain in $\mathbb{R}^2$ and let $M \in S(\Omega) \cap \mathcal{K}(\Omega)$. Then there exists a sequence $M_k \in C^\infty(\overline{\Omega};\mathbb{M}^{2\times 2}_{sym}) \cap \mathcal{K}(\Omega)$ satisfying*

$$\begin{aligned} &M_k \to M \quad \text{in } L^p(\Omega;\mathbb{M}^{2\times 2}_{sym}), \text{ for any } p < \infty, \\ &\operatorname{div}\operatorname{div} M_k \to \operatorname{div}\operatorname{div} M \quad \text{in } L^2(\Omega), \\ &\|M_k\|_{L^\infty} \leqslant C\,\|M\|_{L^\infty}. \end{aligned} \tag{4.13}$$

Proof. Denote the space $L^2(\Omega; M^{2\times 2}_{sym} \times \mathbb{R})$ of vector-valued functions by L. Let D be a subspace of $L^2(\Omega)$ such that $(m, \operatorname{div}\operatorname{div} m) \in L$. Let D_0 be the closure of $C^\infty(\overline{\Omega};\mathbb{M}^{2\times 2}_{sym})$ in the norm of the space L.

Assume that there exists an element $(m_*, \operatorname{div}\operatorname{div} m_*) \in D \setminus D_0$. As D_0 is closed and convex in L, by Hahn–Banach theorem there exists a pair $(u_1, u) \in L^2(\Omega; M^{2\times 2}_{sym}) \times L^2(\Omega, \mathbb{R})$ (i.e., simply from L) such that

$$\int_\Omega (m_* : u_1 + u \operatorname{div}\operatorname{div} m_*)\, dx = 1$$

and

$$\int_\Omega (m : u_1 + u \operatorname{div}\operatorname{div} m)\, dx = 0$$

for any $m \in D_0$. The last identity shows that $u_1 = u$. So, $u \in W_2^2(\Omega)$ and u has usual traces on $\partial\Omega$ (u and $\nu \cdot \nabla u$), where ν is the normal to $\partial\Omega$. Those traces of u are zero that follows from the second identity. So, if the domain Ω is not bad, for example Lipshitz, u belongs to the closure of $C_0^\infty(\Omega)$ in $W_2^2(\Omega)$ (see [10]). This means that there exist function $v \in C_0^\infty(\Omega)$ such that

$$\int_\Omega \left(m_* : \nabla^2 v + v \operatorname{div}\operatorname{div} m_*\right) dx > 1/2.$$

The left-hand side vanishes by definition of div div, which leads to a contradiction. □

4.4. Saddle points generate solutions of the incremental problems

Let us show, that if we interpret a saddle point $(\delta u_r^N, M_r^N)$ of (4.11) as the increment of u and the updated value of M, then we get a solution to the incremental problem (4.1).

Theorem 4.4. *Let $(\delta u_r^N, M_r^N) \in V_+ \times (D \cap \mathcal{K}(\Omega))$ be a saddle point of the relaxed Lagrangian L. Then the triple (u_r^N, e_r^N, p_r^N), constructed as*

$$\begin{aligned} &u_r^N = u_{r-1}^N + \delta u_r^N, \\ &e_r^N = M_r^N, \\ &p_r^N = D^2 u_r^N - e_r^N \quad \text{in } \Omega, \\ &p_r^N = -\nabla u_r^N \odot \nu \mathcal{H}^1 \quad \text{on } \partial\Omega \end{aligned}$$

is kinematically admissible and is a solution to the incremental problem (4.1).

Proof. First of all, kinematic admissibility of the triple (u_r^N, e_r^N, p_r^N) is obvious by its construction. Let us prove that it solves (4.1).

As $(\delta u_r^N, M_r^N) \in V_+ \times (D \cap \mathcal{K}(\Omega))$ is a saddle point of L, we have

$$L\big(\delta u_r^N, m\big) \leqslant L\big(\delta u_r^N, M_r^N\big) \leqslant L\big(v, M_r^N\big), \quad \text{for all } v \in V_+ \text{ and } m \in \big(D \cap \mathcal{K}(\Omega)\big). \tag{4.14}$$

Since $M_r^N \in D \cap \mathcal{K}(\Omega)$, we already know that $M_r^N \in \mathcal{K}(\Omega)$, while the second part of (4.14) implies

$$\operatorname{div}\operatorname{div} M_r^N = f_r^N \in L^2(\Omega). \tag{4.15}$$

The first part of inequality (4.14) yields

$$\int_\Omega \big[\operatorname{div}\operatorname{div} M_r^N \cdot \delta u_r^N - g^*\big(M_r^N\big) + M_r^N : M_{r-1}^N\big]\, dx \geqslant \int_\Omega \big[\operatorname{div}\operatorname{div} m \cdot \delta u_r^N - g^*(m) + m : M_{r-1}^N\big]\, dx \tag{4.16}$$

for every $m \in \mathcal{K}(\Omega) \cap D$.

For $\delta u_r^N \in BH(\Omega)$ with $\delta u_r^N = 0$ on $\partial\Omega$ and $m \in S(\Omega)$, the integration by parts formula [5, Proposition 2.3] takes the form

$$\int_\Omega \operatorname{div}\operatorname{div} m \cdot \delta u_r^N\, dx = \big[D^2\big(\delta u_r^N\big) : m\big](\Omega) - \int_{\partial\Omega} \frac{\partial(\delta u_r^N)}{\partial \nu} m_{ij} \nu_i \nu_j\, d\mathcal{H}^1.$$

Thus, from (4.16) we deduce

$$\big\langle D^2\big(\delta u_r^N\big), m - M_r^N\big\rangle - \frac{1}{2}\int_\Omega \big(|m|^2 - \big|M_r^N\big|^2\big)\, dx + \int_\Omega \big(m - M_r^N\big) : M_{r-1}^N\, dx$$
$$- \int_{\partial\Omega} \frac{\partial(\delta u_r^N)}{\partial \nu}(m_{ij} - M_{ij})\nu_i \nu_j\, d\mathcal{H}^1 \leqslant 0.$$

By taking $\widetilde{m} = M_r^N + \alpha(m - M_r^N) \in \mathcal{K} \cap D$ and letting $\alpha \to 0$ one obtains

$$\big\langle D^2\big(\delta u_r^N\big), m - M_r^N\big\rangle - \int_\Omega \big(m - M_r^N\big) : \delta e_r^N\, dx - \int_{\partial\Omega} \frac{\partial(\delta u_r^N)}{\partial \nu}(m_{ij} - M_{ij})\nu_i \nu_j\, d\mathcal{H}^1 \leqslant 0,$$

that is

$$\big\langle \delta p_r^N, m - M_r^N\big\rangle \leqslant 0$$

for all $m \in D \cap \mathcal{K}(\Omega)$. Hence, by [7, Proposition 2.3]

$$\big\|\delta p_r^N\big\|_{1;\overline{\Omega}} = \big\langle \delta p_r^N, M_r^N\big\rangle,$$

and we have

$$\big\|q + \delta p_r^N\big\|_{1;\overline{\Omega}} - \big\|\delta p_r^N\big\|_{1;\overline{\Omega}} - \big\langle q, M_r^N\big\rangle \geqslant 0$$

for every kinematically admissible triple (v, η, q). The latter inequality and (4.15) imply that (u_r^N, e_r^N, p_r^N) is a solution to problem (4.1). □

5. Approximations

In this section we show that some solutions of the relaxed minimax problem (4.11) can be approximated by more regular functions in a way that allows us to get higher regularity of bending moments.

We also prove some technical lemmas to be used in the rest of the paper.

5.1. Regularized problems

As in [16, Section 3] and [6, Section 6] we consider the family of variational problems, depending on a positive parameter $\alpha \in (0,1]$:

$$\begin{cases} \text{find } u_r^\alpha \in W_0^{2,2}(\Omega) \text{ such that} \\ I_\alpha(u_r^\alpha) = \inf\{I_\alpha(v)\colon\ v \in W_0^{2,2}(\Omega)\}, \end{cases} \tag{5.1}$$

where

$$\begin{aligned} I_\alpha(v) &= \frac{\alpha}{2}\int_\Omega \left|D^2 v + M_{r-1}^N\right|^2 dx + I(v) \\ &= \frac{\alpha}{2}\int_\Omega \left|D^2 v + M_{r-1}^N\right|^2 dx + \int_\Omega g\left(D^2 v + M_{r-1}^N\right) dx - \int_\Omega f_r^N v\, dx. \end{aligned} \tag{5.2}$$

It is easy to see that problem (5.1) has a unique solution $u_r^\alpha \in W_0^{2,2}(\Omega)$, which satisfies a nonlinear system of PDEs:

$$\int_\Omega M_r^\alpha : D^2 v\, dx = \int_\Omega f_r^N v\, dx \quad \text{for all } v \in C_0^\infty(\Omega), \tag{5.3}$$

that is

$$\operatorname{div}\operatorname{div} M_r^\alpha = f_r^N, \tag{5.4}$$

where

$$M_r^\alpha = \alpha\left(D^2 u_r^\alpha + M_{r-1}^N\right) + \frac{\partial g}{\partial \tau}\left(D^2 u_r^\alpha + M_{r-1}^N\right). \tag{5.5}$$

Lemma 5.1. *Under conditions* (2.1), (2.2) *and* (4.5) *the following estimates hold*

$$\sqrt{\alpha}\,\|u_r^\alpha\|_{2,2;\Omega} + \|u_r^\alpha\|_{2,1;\Omega} + \|u_r^\alpha\|_{1,2;\Omega} + \|u_r^\alpha\|_{\infty;\Omega} \leqslant C,$$

where the constant $C = C(\|f_r^N\|_{2;\Omega}, \|M_{r-1}^N\|_{2;\Omega;\mathbb{M}_{sym}^{2\times 2}})$ *does not depend on the parameter* α.

Proof. The safe-load condition (2.2) implies

$$\int_\Omega f_N^r u_r^\alpha\, dx = \int_\Omega m^1 : D^2 u_r^\alpha\, dx,$$

and using definition (5.2) of I_α we deduce the estimate

$$I_1(0) \geqslant I_\alpha\left(u_r^\alpha\right) \geqslant \int_\Omega \left\{\frac{\alpha}{2}\left|D^2 u_r^\alpha + M_{r-1}^N\right|^2 + g\left(D^2 u_r^\alpha + M_{r-1}^N\right) - c\left|D^2 u_r^\alpha\right|\right\} dx.$$

The claim now follows from the embedding theorems. $\square$

Lemma 5.2. *Under the conditions of Lemma* 5.1 *we can find subsequences, denoted by* u_r^α *and* M_r^α*, such that as* $\alpha \to 0$ *we have*

$$M_r^\alpha \rightharpoonup M_r^N \quad \textit{weakly in } L^2\left(\Omega; \mathbb{M}_{sym}^{2\times 2}\right), \tag{5.6}$$

$$u_r^\alpha \to \delta u_r^N \quad \textit{strongly in } W_0^{1,p}(\Omega), \textit{ for } 1 \leqslant p < 2, \tag{5.7}$$

$$u_r^\alpha \rightharpoonup \delta u_r^N \quad \textit{weakly in } W_0^{1,2}(\Omega), \tag{5.8}$$

$$\nabla u_r^\alpha \overset{*}{\rightharpoonup} \nabla \delta u_r^N \quad \textit{weakly* in } BV\left(\Omega; \mathbb{R}^2\right), \tag{5.9}$$

$$\alpha \int_\Omega \left|D^2 u_r^\alpha + M_{r-1}^N\right|^2 dx \to 0, \tag{5.10}$$

$$M_0^\alpha := \frac{\partial g}{\partial \tau}\left(D^2 u_r^\alpha + M_{r-1}^N\right) \overset{*}{\rightharpoonup} M_r^N \quad \textit{weakly* in } L^\infty\left(\Omega; \mathbb{M}^{2\times 2}_{sym}\right), \tag{5.11}$$

where the pair $(\delta u_r^N, M_r^N)$ *is a solution to problem* (4.11).

Proof. Assertions (5.6)–(5.9) and (5.11) follow from (5.5), Lemma 5.1 and embedding theorems.

Thus, it remains to prove (5.10) and that the pair $(\delta u_r^N, M_r^N)$ is a solution to problem (4.11).

As $M_0^\alpha \in \mathcal{K}(\Omega)$, the convergence (5.11) yields that $M_r^N \in \mathcal{K}(\Omega)$. Hence, from the Euler equation (5.3) and the convergence (5.6) we conclude that $M_r^N \in Q_{f_r^N} \cap \mathcal{K}(\Omega)$.

The duality relations imply that

$$M_0^\alpha : \left(D^2 u_r^\alpha + M_{r-1}^N\right) = g\left(D^2 u_r^\alpha + M_{r-1}^N\right) + g^*\left(M_0^\alpha\right) \quad \text{a.e. in } \Omega.$$

Therefore, by using the Euler equation (5.3) we can rewrite the functional I_α as

$$\begin{aligned} I_\alpha\left(u_r^\alpha\right) &= \int_\Omega \left[M_r^\alpha - \alpha\left(D^2 u^\alpha + M_{r-1}^N\right)\right] : \left(D^2 u_r^\alpha + M_{r-1}^N\right) dx \\ &\quad - \int_\Omega g^*\left(M_0^\alpha\right) dx - \int_\Omega f\, u_r^\alpha\, dx + \frac{\alpha}{2}\int_\Omega \left|D^2 u_r^\alpha + M_{r-1}^N\right|^2 dx \\ &= -\frac{\alpha}{2}\int_\Omega \left|D^2 u_r^\alpha + M_{r-1}^N\right|^2 dx - \int_\Omega g^*\left(M_0^\alpha\right) + \int_\Omega M_r^\alpha : M_{r-1}^N\, dx. \end{aligned}$$

By Theorem 4.1 applied to problems (4.7) and (4.8), we get

$$\begin{aligned} \sup\left\{R(m)\colon m \in Q_{f_r^N} \cap \mathcal{K}\right\} &= \inf\left\{I(v)\colon v \in V_0\right\} \leqslant I\left(u_r^\alpha\right) \leqslant I_\alpha\left(u_r^\alpha\right) \\ &= -\frac{\alpha}{2}\int_\Omega \left|D^2 u_r^\alpha + M_{r-1}^N\right|^2 dx - \int_\Omega g^*\left(M_0^\alpha\right) + \int_\Omega M_r^\alpha : M_{r-1}^N\, dx. \end{aligned} \tag{5.12}$$

As

$$-\int_\Omega g^*\left(M_r^N\right) + \int_\Omega M_r^N : M_{r-1}^N = R\left(M_r^N\right),$$

by making use of convergence (5.6) and (5.11) it follows, that

$$\lim_{\alpha\to 0} I_\alpha\left(u_r^\alpha\right) \leqslant R\left(M_r^N\right) - \limsup_{\alpha\to 0} \frac{\alpha}{2}\int_\Omega \left|D^2 u_r^\alpha + M_{r-1}^N\right|^2 dx.$$

According to (5.12) we have

$$\begin{aligned} \sup\left\{R(m)\colon m \in Q_{f_r^N} \cap \mathcal{K}\right\} &= \inf\left\{I(v)\colon v \in V_0\right\} \leqslant \liminf_{\alpha\to 0} I\left(u_r^\alpha\right) \\ &\leqslant \lim_{\alpha\to 0} I_\alpha\left(u_r^\alpha\right) \leqslant R\left(M_r^N\right) - \limsup_{\alpha\to 0} \frac{\alpha}{2}\int_\Omega \left|D^2 u_r^\alpha + M_{r-1}^N\right|^2 dx \leqslant R\left(M_r^N\right), \end{aligned}$$

which implies the relation (5.10) and ensures that M_r^N is a solution to problem (4.8).

Moreover, the identity

$$\lim_{\alpha\to 0} I\left(u_r^\alpha\right) = \inf\left\{I(v)\colon v \in V_0\right\} \tag{5.13}$$

yields that u_r^α is a minimizing sequence for problem (4.7), and therefore it converges to a solution of problem (4.12) as in Theorem 4.2. □

5.2. Convergence of variations

Now we show, that the approximating sequence enjoys better convergence properties, than those stated in Lemma 5.2.

Lemma 5.3. *We have*

$$\left|D^2u_r^\alpha + M_{r-1}^N\right| \overset{*}{\rightharpoonup} \left|D^2\left(\delta u_r^N\right) + M_{r-1}^N\right| \quad \text{in } M_b(\Omega). \tag{5.14}$$

Proof. By Lemma 5.2, Theorem 4.2 and (5.13) we have

$$\lim_{\alpha\to 0} \Phi\left(u_r^\alpha\right) = \lim_{\alpha\to 0} I\left(u_r^\alpha\right) = \inf_{v\in V_0} I(v) = \inf_{V_+} \Phi(v) = \Phi\left(\delta u_r^N\right),$$

so that

$$\int_\Omega g_0\left(\left|D^2u_r^\alpha + M_{r-1}^N\right|\right) dx \to \int_\Omega g_0\left(\left|D^2\left(\delta u_r^N\right) + M_{r-1}^N\right|\right). \tag{5.15}$$

The sequence $|D^2u_r^\alpha + M_{r-1}^N|$ is bounded in $M_b(\Omega)$, therefore there exists a nonnegative measure $\lambda \in M_b(\Omega)$, such that

$$\left|D^2u_r^\alpha + M_{r-1}^N\right| \overset{*}{\rightharpoonup} \lambda \quad \text{weakly* in } M_b(\Omega), \text{ as } \alpha \to 0. \tag{5.16}$$

Thus, $\lambda \geqslant |D^2(\delta u_r^N) + M_{r-1}^N|$ in $M_b(\Omega)$, and the inequality holds true also for $\mathcal{L}^2$-absolutely continuous and singular parts:

$$\begin{aligned} &\lambda^a \geqslant \left|D^2\left(\delta u_r^N\right) + M_{r-1}^N\right|^a, \\ &\lambda^s \geqslant \left|D^2\left(\delta u_r^N\right) + M_{r-1}^N\right|^s. \end{aligned} \tag{5.17}$$

By the weak* lower-semicontinuity of convex functionals of measures, and using the explicit form of the recession function of g_0, which is $g_0^\infty(t) = t$, we obtain

$$\lim_{\alpha\to 0} \int_\Omega g_0\left(\left|D^2u_r^\alpha + M_{r-1}^N\right|\right) dx \geqslant \int_\Omega g_0(\lambda) = \int_\Omega g_0\left(\lambda^a\right) dx + \lambda^s(\Omega). \tag{5.18}$$

On the other hand we have

$$\lim_{\alpha\to 0} \int_\Omega g_0\left(\left|D^2u_r^\alpha + M_{r-1}^N\right|\right) dx = \int_\Omega g_0\left(\left|D^2\left(\delta u_r^N\right) + M_{r-1}^N\right|^a\right) dx + \left|D^2\left(\delta u_r^N\right) + M_{r-1}^N\right|^s(\Omega). \tag{5.19}$$

As the function g_0 is strictly monotone increasing, from (5.15)–(5.19) we conclude that

$$\lambda = \left|D^2\left(\delta u_r^N\right) + M_{r-1}^N\right|.$$

Now the result follows from (5.16). □

5.3. Technical estimates

By the definition (5.5) of M_r^α we have

$$M_r^\alpha = \alpha\left(D^2u_r^\alpha + M_{r-1}^N\right) + \begin{cases} D^2u_r^\alpha + M_{r-1}^N, & \text{if } |D^2u_r^\alpha + M_{r-1}^N| \leqslant 1; \\ \frac{D^2u_r^\alpha + M_{r-1}^N}{|D^2u_r^\alpha + M_{r-1}^N|}, & \text{if } |D^2u_r^\alpha + M_{r-1}^N| > 1. \end{cases} \tag{5.20}$$

According to the chain rule of [12] the following expression for the derivatives of M_r^α is valid

$$M_{r,k}^\alpha = \alpha\left(D^2u_{r,k}^\alpha + M_{r-1,k}^N\right) + \frac{\partial^2 g}{\partial\kappa^2}\left(D^2u_r^\alpha + M_{r-1}^N\right)\left(D^2u_{r,k}^\alpha + M_{r-1,k}^N\right). \tag{5.21}$$

Here and henceforth the subscript $_{,k}$ denotes the partial derivative with respect to x_k.

In what follows we adopt the notation

$$\tau_r^\alpha := D^2 u_r^\alpha + M_{r-1}^N. \tag{5.22}$$

Let us introduce two bilinear forms, that depend on α and implicitly on the point $x \in \Omega$:

$$E_1^\alpha(\varepsilon, \varkappa) = \left(\frac{\partial^2 g}{\partial \tau^2}(\tau_r^\alpha)\varepsilon\right) : \varkappa == \frac{g_0'(|\tau_r^\alpha|)}{|\tau_r^\alpha|}\varepsilon : \varkappa + \left(g_0''\big(\big|\tau_r^\alpha\big|\big) - \frac{g_0'(|\tau_r^\alpha|)}{|\tau_r^\alpha|}\right)\frac{\tau_r^\alpha : \varepsilon}{|\tau_r^\alpha|}\frac{\tau_r^\alpha : \varkappa}{|\tau_r^\alpha|} \tag{5.23}$$

and

$$E_2^\alpha(\varepsilon, \varkappa) = \alpha\varepsilon : \varkappa + E_1^\alpha(\varepsilon, \varkappa). \tag{5.24}$$

Below we establish some technical inequalities to be used in the remaining sections.

Lemma 5.4. *The following relations hold true*:

$$M_{r,k}^\alpha : \varkappa = E_2^\alpha\big(\tau_{r,k}^\alpha, \varkappa\big), \tag{5.25}$$

$$E_2^\alpha(\varkappa, \varkappa) \leqslant \alpha|\varkappa|^2 + \begin{cases} |\varkappa|^2, & \text{if } |\tau_r^\alpha| \leqslant 1, \\ \frac{|\varkappa|^2}{|\tau_r^\alpha|}, & \text{if } |\tau_r^\alpha| > 1 \end{cases} \tag{5.26}$$

for any $\varkappa \in \mathbb{M}_{sym}^{2\times 2}$.

Proof. Identity (5.25) and inequality (5.26) follow from (5.20)–(5.24) and the expression of g_0 as in (4.5). □

Corollary 5.5. *The following estimates are valid*

$$E_2^\alpha\big(M_{r,k}^\alpha, M_{r,k}^\alpha\big) \leqslant \alpha M_{r,k}^\alpha : M_{r,k}^\alpha + \begin{cases} M_{r,k}^\alpha : M_{r,k}^\alpha, & \text{if } |\tau_r^\alpha| \leqslant 1, \\ \frac{1}{|\tau_r^\alpha|} M_{r,k}^\alpha : M_{r,k}^\alpha, & \text{if } |\tau_r^\alpha| > 1. \end{cases} \tag{5.27}$$

In particular, we have

$$E_2^\alpha\big(M_{r,k}^\alpha, M_{r,k}^\alpha\big) \leqslant (1+\alpha) M_{r,k}^\alpha : M_{r,k}^\alpha. \tag{5.28}$$

Lemma 5.6.

$$-E_2^\alpha\big(\tau_{r,k}^\alpha, \tau_{r,k}^\alpha\big) = -M_{r,k}^\alpha : \tau_{r,k}^\alpha \leqslant \begin{cases} -M_{r,k}^\alpha : M_{r,k}^\alpha + \alpha E_2^\alpha(\tau_{r,k}^\alpha, \tau_{r,k}^\alpha), & \text{if } |\tau_r^\alpha| \leqslant 1; \\ -|\tau_r^\alpha| M_{r,k}^\alpha : M_{r,k}^\alpha + \alpha|\tau_r^\alpha| E_2^\alpha(\tau_{r,k}^\alpha, \tau_{r,k}^\alpha), & \text{if } |\tau_r^\alpha| > 1. \end{cases} \tag{5.29}$$

Proof. Suppose, $|\tau_r^\alpha| \leqslant 1$. Then $M_r^\alpha = \alpha\,\tau_r^\alpha + \tau_r^\alpha$, and thus

$$-M_{r,k}^\alpha : \tau_{r,k}^\alpha = -M_{r,k}^\alpha : \big(\alpha\tau_{r,k}^\alpha + \tau_{r,k}^\alpha\big) + \alpha M_{r,k}^\alpha : \tau_{r,k}^\alpha = -M_{r,k}^\alpha : M_{r,k}^\alpha + \alpha E_2^\alpha\big(\tau_{r,k}^\alpha, \tau_{r,k}^\alpha\big). \tag{5.30}$$

Now let $|\tau_r^\alpha| > 1$. Then $M_r^\alpha = \alpha\,\tau_r^\alpha + \frac{\tau_r^\alpha}{|\tau_r^\alpha|}$, and hence

$$M_{r,k}^\alpha = \alpha\tau_{r,k}^\alpha + \left[\frac{\tau_r^\alpha}{|\tau_r^\alpha|}\right]_{,k}. \tag{5.31}$$

Expressing $\tau_{r,k}^\alpha$ from the latter relation we get

$$\tau_{r,k}^\alpha = \big|\tau_{r,k}^\alpha\big|\big(M_{r,k}^\alpha - \alpha\,\tau_{r,k}^\alpha\big) + \tau_r^\alpha\,\frac{\tau_r^\alpha : \tau_{r,k}^\alpha}{|\tau_r^\alpha|},$$

which yields

$$\begin{aligned} -M_{r,k}^\alpha : \tau_{r,k}^\alpha &= -\big|\tau_r^\alpha\big| M_{r,k}^\alpha : M_{r,k}^\alpha + \alpha\big|\tau_r^\alpha\big| M_{r,k}^\alpha : \tau_{r,k}^\alpha - \frac{\tau_r^\alpha : \tau_{r,k}^\alpha}{|\tau_r^\alpha|^2} M_{r,k}^\alpha : \tau_r^\alpha \\ &\leqslant -\big|\tau_r^\alpha\big| M_{r,k}^\alpha : M_{r,k}^\alpha + \alpha\big|\tau_r^\alpha\big| M_{r,k}^\alpha : \tau_{r,k}^\alpha, \end{aligned} \tag{5.32}$$

where (5.31) and the orthogonality of $[\frac{\tau_r^\alpha}{|\tau_r^\alpha|}]_{,k}$ and τ_r^α was used:

$$-\frac{\tau_r^\alpha : \tau_{r,k}^\alpha}{|\tau_r^\alpha|^2} M_{r,k}^\alpha : \tau_r^\alpha = -\alpha \left(\frac{\tau_r^\alpha : \tau_{r,k}^\alpha}{|\tau_r^\alpha|} \right)^2 \leqslant 0.$$

The claim now follows from (5.30) and (5.32). $\square$

6. $W_{loc}^{1,2}$ estimates of bending moments in the incremental problems

In this section we deduce some iterative estimates for the L^2 norms of the gradients of the functions M_r^α, defined by means of (5.5), and we show that for every given r and N we have $M_r^N \in W_{loc}^{1,2}(\Omega; \mathbb{M}_{sym}^{2\times 2})$. We note that for the moment we are concerned only with the problem of regularity of each M_r^N, that is, we do not care about the uniformity of estimates with respect to r and N. Having obtained the L^2 bounds, we conclude that the approximate solutions M_r^α, which were known to converge to M_r^N weakly in $L^2(\Omega; \mathbb{M}_{sym}^{2\times 2})$, actually converge strongly.

Remark that in what follows C_r will denote a constant independent of α, which may change from line to line. This constant may depend on r, N, and, in case of local estimates, on a domain $\Omega' \Subset \Omega$. We will use the notation C only when this constant does not depend on r and N.

For the moment, our objective is the following estimate:

$$\int_{\Omega'} M_{r,k}^\alpha : M_{r,k}^\alpha \, dx \leqslant C(r, N, \Omega'), \tag{6.1}$$

valid for any $\Omega' \Subset \Omega$.

Suppose, by induction, that we have already proved that $M_{r-1}^N \in W_{loc}^{1,2}(\Omega; \mathbb{M}_{sym}^{2\times 2})$. To simplify the notation, in this section we sometimes omit writing the index N for the solutions of the incremental problem (4.1). Let us examine the regularized problem (5.1). Since u_r^α is a solution of the nonlinear elliptic system (5.4) with $f_r^N \in L^2(\Omega)$ and $e_{r-1}^N \in W_{loc}^{1,2}(\Omega; \mathbb{M}_{sym}^{2\times 2})$, one can show, by working with difference quotients, that

$$\begin{aligned} &u_m^\alpha \in W_{loc}^{3,2}\big(\Omega; \mathbb{M}_{sym}^{2\times 2}\big), \\ &M_m^\alpha, D^2 u_m^\alpha \in W_{loc}^{1,2}\big(\Omega; \mathbb{M}_{sym}^{2\times 2}\big). \end{aligned} \tag{6.2}$$

By using formula (5.25) , estimate (5.28) and the definition (5.22) of τ_r^α we obtain

$$\begin{aligned} M_{r,k}^\alpha : M_{r,k}^\alpha &= E_2^\alpha\big(\tau_{r,k}^\alpha, M_{r,k}^\alpha\big) \leqslant \big[E_2^\alpha\big(\tau_{r,k}^\alpha, \tau_{r,k}^\alpha\big)\big]^{1/2}\big[E_2^\alpha\big(M_{r,k}^\alpha, M_{r,k}^\alpha\big)\big]^{1/2} \\ &\leqslant \frac{1}{2} E_2^\alpha\big(\tau_{r,k}^\alpha, \tau_{r,k}^\alpha\big) + \frac{1}{2} E_2^\alpha\big(M_{r,k}^\alpha, M_{r,k}^\alpha\big) \\ &\leqslant \frac{1}{2} M_{r,k}^\alpha : \tau_{r,k}^\alpha + \left(\frac{1}{2} + \frac{\alpha}{2}\right) M_{r,k}^\alpha : M_{r,k}^\alpha \\ &\leqslant \frac{1}{2} M_{r,k}^\alpha : D^2 u_{r,k}^\alpha + \frac{1}{2} M_{r,k}^\alpha : M_{r-1,k}^N + \left(\frac{1}{2} + \frac{\alpha}{2}\right) M_{r,k}^\alpha : M_{r,k}^\alpha. \end{aligned} \tag{6.3}$$

By applying the Cauchy inequality to $M_{r,k}^\alpha : M_{r-1,k}^N$ we get

$$(1-\alpha) M_{r,k}^\alpha : M_{r,k}^\alpha \leqslant M_{r-1,k}^N : M_{r-1,k}^N + 2 M_{r,k}^\alpha : D^2 u_{r,k}^\alpha. \tag{6.4}$$

Thus, it remains to prove the boundedness in $L_{loc}^1(\Omega)$ of the second summand of

Let us introduce the notation

$$M^\alpha := M_r^\alpha, \qquad f := f_r^N, \qquad u^\alpha := u_r^\alpha,$$

omitting index m for further convenience. Let $\varphi \in C_0^3(\Omega)$ be an arbitrary cut-off function, such that $\varphi \equiv 1$ on Ω', and $\operatorname{supp} \varphi \subset \Omega'' \Subset \Omega$. By (6.2) we can put the function

$$v = \varphi^4 u_{,k}^\alpha$$

into the Euler equation (5.3).

We start by

$$\int_\Omega M^\alpha_{,k} : D^2\big(\varphi^4 u^\alpha_{,k}\big)\,dx = \int_\Omega \varphi^4 \nabla f\cdot\nabla u^\alpha\,dx.$$

This equality can be expressed in the following way

$$J^\alpha_r := \int_\Omega \varphi^4 M^\alpha_{,k} : D^2 u^\alpha_{,k}\,dx = \int_\Omega \varphi^4 f_{,k}\,u^\alpha_{,k}\,dx - 2\int_\Omega M^\alpha_{ij,k}\varphi^4_{,j}\,u^\alpha_{,ki}\,dx - \int_\Omega M^\alpha_{ij,k}\,\varphi^4_{,ij}\,u^\alpha_{,k}\,dx. \tag{6.5}$$

Thus, we have

$$J^\alpha_r \leqslant I^\alpha_1 + I^\alpha_2 + I^\alpha_3, \tag{6.6}$$

with

$$I^\alpha_1 := \int_\Omega \varphi^4 f_{,k} u^\alpha_{,k}\,dx, \qquad I^\alpha_2 := -2\int_\Omega M^\alpha_{ij,k}\,\varphi^4_{,j}\,u^\alpha_{,ki}\,dx,$$
$$I^\alpha_3 := -\int_\Omega M^\alpha_{ij,k}\varphi^4_{,ij}\,u^\alpha_{,k}\,dx. \tag{6.7}$$

Estimate of I^α_1.

$$\big|I^\alpha_1\big| \leqslant \|f\|_{1,2;\Omega''}\big\|u^\alpha\big\|_{1,2;\Omega} \leqslant C_r. \tag{6.8}$$

Estimate of I^α_2. Let us introduce the matrices $S^{(k)} = (S^{(k)}_{ij})$ defined by

$$S^{(k)}_{ij} := -\varphi_{,j}\,u^\alpha_{,ki}. \tag{6.9}$$

Then by using (5.25), (5.27), (5.22) and the fact that $\|M^N_{r-1}\|_{\infty;\Omega} \leqslant 1$ we obtain

$$\begin{aligned}
I^\alpha_2 &= -2\int_\Omega M^\alpha_{ij,k}\,\varphi^4_{,j}\,u^\alpha_{,ki}\,dx = 8\int_\Omega \varphi^3 E^\alpha_2\big(\tau^\alpha_{r,k}, S^{(i)}\big)\,dx \\
&\leqslant \frac{1}{100}\int_\Omega \varphi^4 E^\alpha_2\big(\tau^\alpha_{r,k}, \tau^\alpha_{r,k}\big)\,dx + C_r\int_\Omega \varphi^2 E^\alpha_2\big(S^{(k)}, S^{(k)}\big)\,dx \\
&\leqslant \frac{1}{100}\int_\Omega \varphi^4 M^\alpha_{r,k} : \tau^\alpha_{r,k} + \alpha C_r\int_\Omega \varphi^2\big|S^{(k)}\big|^2\,dx + C_r\int_{|\tau^\alpha_r|\leqslant 1} \varphi^2\big|S^{(k)}\big|^2\,dx + C_r\int_{|\tau^\alpha_r|>1} \frac{\varphi^2|S^{(k)}|^2}{|\tau^\alpha_r|}\,dx \\
&\leqslant \frac{1}{100}\bigg(J^\alpha_r + \int_\Omega \varphi^4\,M^\alpha_{r,k} : M^\alpha_{r,k}\,dx + \int_\Omega \varphi^4\,M^N_{r-1,k} : M^N_{r-1,k}\,dx\bigg) \\
&\quad + C_r\int_{|\tau^\alpha_r|\leqslant 1} \varphi^2\big|D^2u^\alpha_r\big|^2\,dx + C_r\int_{|\tau^\alpha_r|>1} \frac{\varphi^2|D^2u^\alpha_r|^2}{|\tau^\alpha_r|}\,dx + \alpha C_r\big\|D^2u^\alpha_r\big\|^2_{2;\Omega} \\
&\leqslant \frac{1}{100}\bigg(J^\alpha_r + \int_\Omega \varphi^4 M^\alpha_{r,k} : M^\alpha_{r,k}\,dx + \int_\Omega \varphi^4\,M^N_{r-1,k} : M^N_{r-1,k}\,dx\bigg) + C_r\int_\Omega \big|D^2u^\alpha_r\big|\,dx + C_r.
\end{aligned} \tag{6.10}$$

Estimate of I^α_3. Using (5.25) and Lemma 5.1

$$
\begin{aligned}
I_3^\alpha &= -\int_\Omega M^\alpha_{ij,k}\varphi^4_{,ij}u^\alpha_{,k}\,dx = -4\int_\Omega \varphi^3 u^\alpha_{,k}E_2^\alpha\big(\tau^\alpha_{r,k},\nabla^2\varphi\big)\,dx - 12\int_\Omega \varphi^2 u^\alpha_{,k}E_2^\alpha\big(\tau^\alpha_{r,k},\nabla\varphi\otimes\nabla\varphi\big)\,dx \\
&\leqslant \frac{1}{100}\int_\Omega \varphi^4 E_2^\alpha\big(\tau^\alpha_{r,k},\tau^\alpha_{r,k}\big)\,dx + C_r\int_\Omega \big|\nabla u^\alpha\big|^2\big(\varphi^2E_2^\alpha\big(\nabla^2\varphi,\nabla^2\varphi\big)+E_2^\alpha(\nabla\varphi\otimes\nabla\varphi,\nabla\varphi\otimes\nabla\varphi)\big)\,dx \\
&\leqslant \frac{1}{100}\bigg(J_r^\alpha + \int_\Omega \varphi^4 M^\alpha_{r,k}:M^\alpha_{r,k}\,dx + \int_\Omega \varphi^4 M^N_{r-1,k}:M^N_{r-1,k}\,dx\bigg) + C_r.
\end{aligned}
\tag{6.11}
$$

So, (6.5), (6.6), (6.8)–(6.11), and the regularity of M^N_{r-1} proved at the previous step, imply that

$$
J_r^\alpha \leqslant C_r + \frac{2}{100}J_r^\alpha + \frac{2}{100}\int_\Omega \varphi^4\, M^\alpha_{r,k}:M^\alpha_{r,k}\,dx.
$$

Therefore, (6.4) allows us to conclude that (6.1) holds for every $k = 1, 2$, and thus

$$
\limsup_{\alpha\to 0}\big\|\nabla M_r^\alpha\big\|_{2;\Omega'} \leqslant C(r,N,\Omega'). \tag{6.12}
$$

Remark 6.1. Inequality (6.12) and the convergence $M_r^\alpha \rightharpoonup M_r^N$ in $L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})$, see (5.6), imply that

$$
\begin{aligned}
&M_r^N \in W^{1,2}_{loc}\big(\Omega;\mathbb{M}^{2\times2}_{sym}\big),\\
&M_r^\alpha \rightharpoonup M_r^N \quad \text{in } W^{1,2}_{loc}\big(\Omega;\mathbb{M}^{2\times2}_{sym}\big), \quad \text{and}\\
&M_r^\alpha \to M_r^N \quad \text{in } L^2_{loc}\big(\Omega;\mathbb{M}^{2\times2}_{sym}\big),
\end{aligned}
\tag{6.13}
$$

where the strong convergence in $L^2(\Omega;\mathbb{M}^{2\times2}_{sym})$ is guaranteed by Sobolev embedding.

7. Auxiliary estimates

In this section we prove a fine convergence estimate for the approximate solutions of regularized problems (Lemmas 7.1 and 7.3) and get analytic estimates, which are the core of the proof of the uniform boundedness of M_r^N in $W^{1,2}_{loc}(\Omega;\mathbb{M}^{2\times2}_{sym})$ (Lemmas 7.4, 7.5 and Corollary 7.6).

In these estimates it is crucial that the constants C does not depend on r and N, although they might depend on φ.

In the rest of the paper $\omega_r(\alpha)$ will denote a generic function, converging to 0 as $\alpha\to 0$, which may change from line to line and may depend on r and N.

7.1. Fine properties of approximating sequence

Lemma 7.1. *For any function $\psi \in C_0(\Omega)$ with $0 \leqslant \psi \leqslant 1$, we have*

$$
\int_{|\tau_r^\alpha|>1} \psi\big(\big|D^2u_r^\alpha + M^N_{r-1}\big| - \big|M^N_{r-1}\big|\big)\,dx \leqslant \frac{C}{N} + \omega_r(\alpha), \tag{7.1}
$$

where the constant C and the quantity $\omega_r(\alpha)$ may depend on the properties of ψ.

Proof. As $|M^N_{r-1}| \leqslant 1$, we have

$$
\begin{aligned}
&\int_{|\tau_r^\alpha|>1} \psi\big(\big|D^2u_r^\alpha + M^N_{r-1}\big| - \big|M^N_{r-1}\big|\big)\,dx \\
&\quad = \int_\Omega \psi\big(\big|D^2u_r^\alpha + M^N_{r-1}\big| - \big|M^N_{r-1}\big|\big)\,dx - \int_{|\tau_r^\alpha|\leqslant 1} \psi\big(\big|D^2u_r^\alpha + M^N_{r-1}\big| - \big|M^N_{r-1}\big|\big)\,dx.
\end{aligned}
\tag{7.2}
$$

Equality (5.20) implies that on the set $\{|\tau_r^\alpha| \leqslant 1\}$ one has $M_r^\alpha = \alpha\,\tau_r^\alpha + \tau_r^\alpha$. Thus, by (5.20), (5.22), Lemma 5.1, (6.13) and (3.12) we obtain

$$\begin{aligned}
&- \int\limits_{|\tau_r^\alpha|\leqslant 1} \psi\big(\big|D^2u_r^\alpha + M_{r-1}^N\big| - \big|M_{r-1}^N\big|\big)\,dx \\
&\quad\leqslant \int\limits_{|\tau_r^\alpha|\leqslant 1} \psi\big(\big|M_r^\alpha - M_{r-1}^N\big| + \alpha\big|\tau_r^\alpha\big|\big)\,dx \leqslant \int\limits_\Omega \psi\big|M_r^\alpha - M_{r-1}^N\big|\,dx + C\alpha \\
&\quad\leqslant \int\limits_\Omega \psi\big|M_r^N - M_{r-1}^N\big|\,dx + \int\limits_\Omega \psi\big|M_r^N - M_r^\alpha\big|\,dx + C\alpha \leqslant \frac{C}{N} + \omega_m(\alpha).
\end{aligned}$$

On the other hand, by (5.14) and (3.12)

$$\lim_{\alpha\to 0}\int\limits_\Omega \psi\big(\big|D^2u_r^\alpha + M_{r-1}^N\big| - \big|M_{r-1}^N\big|\big)\,dx = \big\langle\psi, \big|D^2\big(\delta u_r^N\big) + M_{r-1}^N\big| - \big|M_{r-1}^N\big|\cdot\mathcal{L}^2\big\rangle \leqslant \big|D^2\big(\delta u_r^N\big)\big|(\Omega) \leqslant \frac{C}{N}.$$

The estimate (7.1) follows from last two estimates and (7.2). □

As a corollary, we prove a local estimate for $|D^2u_r^\alpha|$.

Corollary 7.2. *We have*

$$\int\limits_\Omega \psi\big|D^2u_r^\alpha\big|\,dx \leqslant \frac{C}{N} + \omega_r(\alpha) \tag{7.3}$$

for every function $\psi \in C_0(\Omega)$ with $0 \leqslant \psi \leqslant 1$, where the constant C and the quantity $\omega_r(\alpha)$ may depend on ψ.

Proof. Introduce the notation

$$\widetilde{\Omega}_1 = \big\{\big|\tau_r^\alpha\big| \leqslant 1\big\}, \qquad \widetilde{\Omega}_2 = \big\{1 < \big|\tau_r^\alpha\big|\big\}.$$

Now we divide the integral over Ω into two integrals over $\widetilde{\Omega}_i$, $i = 1, 2$, and estimate each one of them, as in (7.3).

Estimate over $\widetilde{\boldsymbol{\Omega}}_{\mathbf{1}}$. According to (5.20) and (5.22) in the region $\widetilde{\Omega}_1$ we have

$$D^2u_r^\alpha = M_r^\alpha - M_{r-1}^N + \alpha\tau_r^\alpha,$$

and hence, by (6.13) and (3.12) we obtain

$$\int\limits_{\widetilde{\Omega}_1} \psi\big|D^2u_r^\alpha\big|\,dx \leqslant \int\limits_{\widetilde{\Omega}_1} \psi\big|M_r^\alpha - M_{r-1}^N\big|\,dx + \alpha\int\limits_{\widetilde{\Omega}_1} \psi\,dx \leqslant \frac{C}{N} + \omega_r(\alpha). \tag{7.4}$$

Estimate over $\widetilde{\boldsymbol{\Omega}}_{\mathbf{2}}$. By (5.20) and (5.22) in $\widetilde{\Omega}_2$ one has

$$D^2u_r^\alpha = M_r^\alpha\big(\big|\tau_r^\alpha\big| - 1\big) - \big(M_{r-1}^N - M_r^\alpha\big) - \alpha\tau_r^\alpha\big|\tau_r^\alpha\big|. \tag{7.5}$$

Again, by (5.20) and (5.22) we get

$$\big|M_r^\alpha\big|\big|\tau_r^\alpha\big| \leqslant \alpha\big|\tau_r^\alpha\big|^2 + \big|D^2u_r^\alpha + M_{r-1}^N\big|,$$

and the triangle inequality $|M_r^\alpha| \geqslant |M_{r-1}^N| - |M_{r-1}^N - M_r^\alpha|$ yields

$$-\big|M_r^\alpha\big| \leqslant -\big|M_{r-1}^N\big| + \big|M_{r-1}^N - M_r^\alpha\big|.$$

By the last two estimates, the relation (7.5) becomes

$$\int\limits_{\widetilde{\Omega}_2} \psi\big|D^2u_r^\alpha\big|\,dx \leqslant \int\limits_{\widetilde{\Omega}_2} \psi\big(\big|D^2u_r^\alpha + M_{r-1}^N\big| - \big|M_{r-1}^N\big|\big)\,dx + 2\int\limits_{\widetilde{\Omega}_2} \psi\big|M_{r-1}^N - M_r^\alpha\big|\,dx + 2\alpha\int\limits_{\widetilde{\Omega}_2} \big|\tau_r^\alpha\big|^2\,dx.$$

Using (7.1), (5.10), the convergence (6.13) and (3.12), by the last estimate we conclude, that

$$\int_{\tilde{\Omega}_2} \psi \left| D^2 u_r^\alpha \right| dx \leqslant \frac{C}{N} + \omega_r(\alpha). \tag{7.6}$$

Now the claim (7.3) follows from (7.4) and (7.6). $\square$

Lemma 7.3. *The following estimate holds*:

$$\int_{\Omega} \psi^2 \left| \nabla u_r^\alpha \right|^2 dx \leqslant \frac{C}{N^2} + \omega_r(\alpha), \tag{7.7}$$

for any function $\psi \in C_0^2(\Omega)$ *with* $0 \leqslant \psi \leqslant 1$. *Remark, that the constant* C *and the quantity* $\omega_r(\alpha)$ *depend upon* $\|\psi\|_{2,\infty;\Omega}$.

Proof. We begin by defining the functions $v_r^\alpha := u_r^\alpha \psi \in W_0^{2,2}(\Omega)$, which satisfy the following equalities:

$$\begin{aligned} &\nabla v_r^\alpha = \psi \nabla u_r^\alpha + u_r^\alpha \nabla \psi, \\ &D^2 v_r^\alpha = \psi D^2 u_r^\alpha + 2\nabla\psi \odot \nabla u_r^\alpha + u_r^\alpha \nabla^2 \psi. \end{aligned} \tag{7.8}$$

Then, by using (7.8), the Sobolev embeddings $W^{2,1}(\Omega) \hookrightarrow W^{1,2}(\Omega)$ and $W^{1,1}(\Omega) \hookrightarrow L^2(\Omega)$, and the Poincaré inequality for $W_0^{2,1}(\Omega)$ we can estimate the integral considered as follows

$$\begin{aligned} \int_{\Omega} \left| \psi \nabla u_r^\alpha \right|^2 dx &\leqslant 2 \int_{\Omega} \left| \psi \nabla u_r^\alpha + u_r^\alpha \nabla \psi \right|^2 dx + 2 \int_{\Omega} |u_r^\alpha \nabla \psi|^2 dx \\ &\leqslant C \int_{\Omega} \left| \nabla v_r^\alpha \right|^2 dx + C \int_{\Omega} \left| u_r^\alpha \right|^2 dx \leqslant C \left\| v_r^\alpha \right\|_{2,1;\Omega}^2 + C \left\| u_r^\alpha \right\|_{1,1;\Omega}^2 \\ &\leqslant C \left(\int_{\Omega} \left| D^2 v_r^\alpha \right| dx \right)^2 + C \left\| u_r^\alpha \right\|_{1,1;\Omega}^2 \\ &\leqslant C \left(\int_{\Omega} \psi \left| D^2 u_r^\alpha \right| dx + \int_{\Omega} \left| \nabla u_r^\alpha \right| dx + \int_{\Omega} \left| u_r^\alpha \right| dx \right)^2 + C \left\| u_r^\alpha \right\|_{1,1;\Omega}^2. \end{aligned} \tag{7.9}$$

Now we use (7.9), the estimate (7.3), the convergence $u_r^\alpha \to \delta u_r^N$ in $W^{1,1}(\Omega)$, as in (5.7), the embedding $BH(\Omega) \hookrightarrow W^{1,1}(\Omega)$, and (3.12) to obtain

$$\int_{\Omega} \psi^2 \left| \nabla u_r^\alpha \right|^2 dx \leqslant C \left(\int_{\Omega} \psi \left| D^2 u_r^\alpha \right| dx \right)^2 + C \left\| u_r^\alpha \right\|_{1,1;\Omega}^2 \leqslant \frac{C}{N^2} + C \left\| \delta u_r^N \right\|_{BH(\Omega)}^2 + \omega_r(\alpha) \leqslant \frac{C}{N^2} + \omega_r(\alpha).$$

The claim is proved. $\square$

7.2. Analytic estimates

Lemma 7.4. *The following inequality holds for* J_r^α *defined in* (6.5):

$$J_r^\alpha \leqslant -2 \int_{\Omega} M_{ij,k}^\alpha \varphi_{,j}^4 u_{,ki}^\alpha \, dx + \frac{1}{N} \int_{\Omega} \varphi^4 E_2^\alpha \left(\tau_{r,k}^\alpha, \tau_{r,k}^\alpha \right) + \frac{C}{N} + \omega_r(\alpha). \tag{7.10}$$

Proof. Recalling (6.6), we have $J_r^\alpha \leqslant I_1^\alpha + I_2^\alpha + I_3^\alpha$ with I_i^α, $i = 1, \ldots, 3$, defined in (6.7). We show, that I_1^α and I_3^α are of order $\frac{1}{N}$ when $\alpha \to 0$.

Estimate over I_1^α. Since $f_r^N \in W^{1,2}_{loc}(\Omega)$, one can employ the convergence (5.8) to pass to the limit in I_1^α, and use the estimates (3.12) of $\|\delta u_r^N\|_{BH(\Omega)}$ to obtain

$$\left|I_1^\alpha\right| \leqslant C\left(\|f\|_{L^\infty([0,T];W^{1,2}(\Omega''))}\right)\frac{1}{N} + \omega_r(\alpha). \tag{7.11}$$

Estimate over I_3^α. First of all, remark that the function

$$\varphi\left(E_2^\alpha\left(\nabla^2\varphi, \nabla^2\varphi\right) + E_2^\alpha(\nabla\varphi\otimes\nabla\varphi, \nabla\varphi\otimes\nabla\varphi)\right)$$

is bounded and has a compact support, which is a subset of $\operatorname{supp}\varphi$. Let us choose a function $\psi \in C_0^\infty(\Omega)$, such that

$$\psi \equiv 1 \quad \text{on } \operatorname{supp}\varphi \quad \text{and} \quad \operatorname{supp}\psi \subset \Omega'',$$

$$\begin{aligned} I_3^\alpha &= -\int_\Omega M^\alpha_{ij,k}\varphi^4_{,ij}, u^\alpha_{,k}\,dx = -4\int_\Omega \varphi^3 u^\alpha_{r,k} E_2^\alpha\left(\tau^\alpha_{r,k}, \nabla^2\varphi\right)dx - 12\int_\Omega \varphi^2 u^\alpha_{r,k} E_2^\alpha\left(\tau^\alpha_{r,k}, \varphi\otimes\varphi\right)dx \\ &\leqslant \frac{1}{N}\int_\Omega \varphi^4 E_2^\alpha\left(\tau^\alpha_{r,k}, \tau^\alpha_{r,k}\right)dx + CN\int_\Omega \psi^2\left|\nabla u_r^\alpha\right|^2 dx, \end{aligned} \tag{7.12}$$

with ψ chosen above, using the fact that

$$\varphi\left(E_2^\alpha\left(\nabla^2\varphi, \nabla^2\varphi\right) + E_2^\alpha(\nabla\varphi\otimes\nabla\varphi, \nabla\varphi\otimes\nabla\varphi)\right) \leqslant C\psi^2.$$

Thus, by (6.6), (7.11), (7.12) and (7.7) we obtain (7.10). □

Lemma 7.5. *The following "iterative" estimate holds true*:

$$\begin{aligned} &\left(1-\frac{2}{N}\right)\int_\Omega \varphi^4 E_2^\alpha\left(\tau^\alpha_{r,k}, \tau^\alpha_{r,k}\right)dx \\ &\quad\leqslant \frac{100}{99}\int_\Omega M^\alpha_{r,k} : M^N_{r-1,k}\,dx + \sum_{s=1}^{9N-1}\frac{1}{s+10}\int_{F_s}\varphi^4 M^\alpha_{r,k} : M^\alpha_{r,k}\,dx - \frac{1}{99}\int_{|\tau_r^\alpha|\leqslant 10}\varphi^4 E_2^\alpha\left(\tau^\alpha_{r,k}, \tau^\alpha_{r,k}\right)dx \\ &\qquad + \frac{C}{N}\int_\Omega \varphi^4 M^\alpha_{r,k} : M^\alpha_{r,k}\,dx + \frac{C}{N}\int_\Omega \varphi^4 M^N_{r-1,k} : M^N_{r-1,k}\,dx + \frac{C}{N} + \omega_r(\alpha), \end{aligned} \tag{7.13}$$

where F_s, $s = 1,\dots,9N-1$ is defined by $F_s = \{1+\frac{9}{s+1} < |\tau_r^\alpha| \leqslant 1+\frac{9}{s}\}$.

Proof. By (5.25), (5.22), (6.5) and (7.10)

$$\begin{aligned} \left(1-\frac{1}{N}\right)\int_\Omega \varphi^4 E_2^\alpha\left(\tau^\alpha_{r,k}, \tau^\alpha_{r,k}\right)dx &\leqslant -2\int_\Omega M^\alpha_{ij,k}\varphi^4_{,j}u^\alpha_{,ki}\,dx + \int_\Omega \varphi^4 M^\alpha_{r,k} : M^N_{r-1,k}\,dx + \frac{C}{N} + \omega_r(\alpha) \\ &= B_1^\alpha + B_2^\alpha + B_3^\alpha + B_4^\alpha + \int_\Omega \varphi^4 M^\alpha_{r,k} : M^N_{r-1,k}\,dx + \frac{C}{N} + \omega_r(\alpha), \end{aligned} \tag{7.14}$$

where

$$B_i^\alpha := 8\int_{\Omega_i}\varphi^3\, M^\alpha_{r,k} : S^{(k)}\,dx, \quad i = 1,\dots,4,$$

with $S^{(k)}$ defined in (6.9) and

$$\begin{aligned} &\Omega_1 = \left\{\left|\tau_r^\alpha\right| \leqslant 1\right\}, \qquad \Omega_2 = \left\{1 < \left|\tau_r^\alpha\right| \leqslant 1+\frac{1}{N}\right\}, \\ &\Omega_3 = \left\{1+\frac{1}{N} < \left|\tau_r^\alpha\right| \leqslant 10\right\}, \qquad \Omega_4 = \left\{10 < \left|\tau_r^\alpha\right|\right\}. \end{aligned} \tag{7.15}$$

Estimate of B_1^α. According to (5.20) and (5.22), in the region Ω_1 the following identity holds:

$$D^2 u_r^\alpha = M_r^\alpha - M_{r-1}^N - \alpha\,\tau_r^\alpha.$$

Hence, by (6.9)

$$\left|S^{(k)}\right|^2 \leqslant C\left|D^2 u_r^\alpha\right|^2 \leqslant C\left(\left|M_r^\alpha - M_{r-1}^N\right|^2 + \alpha^2\left|\tau_r^\alpha\right|^2\right),$$

and we have

$$\int\limits_{\Omega_1} \varphi^2\left|S^{(k)}\right|^2 dx \leqslant C\alpha^2 + C\left\|M_r^\alpha - M_{r-1}^N\right\|_{2;\Omega''}^2.$$

Thus, from the convergence (6.13) and the increment estimate (3.12), it follows that

$$B_1^\alpha \leqslant \frac{1}{N}\int\limits_{\Omega_1} \varphi^4\, M_{r,k}^\alpha : M_{r,k}^\alpha\, dx + CN\int\limits_{\Omega_1} \varphi^2\left|S^{(k)}\right|^2 dx \leqslant \frac{1}{N}\int\limits_{\Omega_1} \varphi^4\, M_{r,k}^\alpha : M_{r,k}^\alpha\, dx + \frac{C}{N} + \omega_r(\alpha). \tag{7.16}$$

Estimate of B_2^α. We remark, that (5.20) and (5.22) yield that for $|\tau_r^\alpha| \geqslant 1$ one has

$$D^2 u_r^\alpha = M_r^\alpha\left(\left|\tau_r^\alpha\right| - 1\right) - \left(M_{r-1}^N - M_r^\alpha\right) - \alpha\tau_r^\alpha\left|\tau_r^\alpha\right|, \tag{7.17}$$

so that in the region Ω_2 we have

$$\left|D^2 u_r^\alpha\right|^2 \leqslant \frac{C}{N^2}\left|M_r^\alpha\right|^2 + C\left|M_r^\alpha - M_{r-1}^N\right|^2 + C\alpha^2\left|\tau_r^\alpha\right|^4.$$

By the inequality $|S^{(k)}| \leqslant C|D^2 u_r^\alpha|$, see (6.9),

$$8\varphi^3 M_{r,k}^\alpha : S^{(k)} \leqslant \frac{1}{N}\varphi^4 M_{r,k}^\alpha : M_{r,k}^\alpha + CN\varphi^2\left|D^2 u_r^\alpha\right|^2,$$

so that by the former estimate, the boundedness of τ_r^α and M_r^α on Ω_2 (see (5.20)), (6.13) and (3.12) we have

$$B_2^\alpha \leqslant \frac{1}{N}\int\limits_{\Omega_2} \varphi^4 M_{r,k}^\alpha : M_{r,k}^\alpha\, dx + \frac{C}{N} + \omega_r(\alpha). \tag{7.18}$$

Estimate of B_3^α. Using the notation $F_s = \{1 + \frac{9}{s+1} < |\tau_r^\alpha| \leqslant 1 + \frac{9}{s}\}$ for $s = 1, \dots, 9N-1$, we write

$$\begin{aligned} B_3^\alpha &= 8\sum_{s=1}^{9N-1}\int\limits_{F_s} \varphi^3\, M_{r,k}^\alpha : S^{(k)}\, dx \\ &\leqslant \sum_{s=1}^{9N-1}\left[\frac{1}{2(s+10)}\int\limits_{F_s} \varphi^4 M_{r,k}^\alpha : M_{r,k}^\alpha\, dx + C(s+10)\int\limits_{F_s} \varphi^2\left|D^2 u_r^\alpha\right|^2 dx\right]. \end{aligned} \tag{7.19}$$

Now we show, that the last summand can be bounded by $\frac{C}{N} + \omega_r(\alpha)$.

Thanks to (7.17) on F_s we have

$$\left|D^2 u_r^\alpha\right|^2 \leqslant \frac{9}{s}\left|M_r^\alpha\right|^2\left(\left|\tau_r^\alpha\right| - 1\right) + C\left|M_r^\alpha - M_{r-1}^N\right|^2 + \alpha^2\left|\tau_r^\alpha\right|^4,$$

so that by (6.13), (3.8), (3.12), and the boundedness of M_r^α and τ_r^α on F_s (see (5.20)) we have

$$\begin{aligned} &\sum_{s=1}^{9N-1}(s+10)\int\limits_{F_s} \varphi^2\left|D^2 u_r^\alpha\right|^2 dx \\ &\quad\leqslant \sum_{s=1}^{9N-1}\int\limits_{F_s} \varphi^2\left(\frac{9(s+10)}{s}\left|M_r^\alpha\right|^2\left(\left|\tau_r^\alpha\right| - 1\right) + CN\left|M_r^\alpha - M_{r-1}^N\right|^2 + CN\alpha^2\right) dx \end{aligned}$$

$$\leqslant C\int\limits_{\Omega_3}\varphi^2\big|M_r^\alpha\big|\big(\big|\tau_r^\alpha\big|-1\big)\,dx+CN\big\|M_r^\alpha-M_{r-1}^N\big\|_{2;\Omega''}^2+CN\alpha^2. \tag{7.20}$$

By (5.20) and (5.22) we have $|M_r^\alpha||\tau_r^\alpha|\leqslant\alpha|\tau_r^\alpha|^2+|D^2u_r^\alpha+M_{r-1}^N|$, and by the triangle inequality $|M_r^\alpha|\geqslant|M_{r-1}^N|-|M_{r-1}^N-M_r^\alpha|$ we have also $-|M_r^\alpha|\leqslant-|M_{r-1}^N|+|M_{r-1}^N-M_r^\alpha|$. Therefore using (6.13) and (3.12) we can bound the right-hand side of (7.20) by

$$C\int\limits_{\Omega_3}\varphi^2\big(\big|D^2u_r^\alpha+M_{r-1}^N\big|-\big|M_{r-1}^N\big|\big)+\frac{C}{N}+\omega_r(\alpha).$$

Thus, by (7.1) we conclude that

$$B_3^\alpha\leqslant\sum_{s=1}^{9N-1}\frac{1}{2(s+10)}\int\limits_{F_s}\varphi^4M_{r,k}^\alpha:M_{r,k}^\alpha\,dx+\frac{C}{N}+\omega_r(\alpha). \qquad\square \tag{7.21}$$

Estimate of B_4^α. Applying the Cauchy inequality

$$\varphi^3E_2^\alpha\big(\tau_{r,k}^\alpha,S^{(k)}\big)\leqslant\frac{1}{100}\varphi^4E_2^\alpha\big(\tau_{r,k}^\alpha,\tau_{r,k}^\alpha\big)+C\varphi^2E_2^\alpha\big(S^{(k)},S^{(k)}\big),$$

and using (5.25), (5.26), (6.9) and (5.10) we obtain

$$B_4^\alpha=\int\limits_{\Omega_4}\varphi^4E_2^\alpha\big(\tau_{r,k}^\alpha,S^{(k)}\big)\,dx\leqslant\frac{1}{100}\int\limits_{\Omega_4}\varphi^4E_2^\alpha\big(\tau_{r,k}^\alpha,\tau_{r,k}^\alpha\big)\,dx+C\int\limits_{\Omega_4}\varphi^2\frac{|D^2u_r^\alpha|^2}{|\tau_r^\alpha|}\,dx+\omega_r(\alpha). \tag{7.22}$$

To show that the last summand is of order $\frac{1}{N}$, we first note that on the set Ω_4 the inequality

$$\frac{|D^2u_r^\alpha|^2}{|\tau_r^\alpha|}<10\big(\big|D^2u_r^\alpha+M_{r-1}^N\big|-\big|M_{r-1}^N\big|\big) \tag{7.23}$$

holds. To prove it, we multiply both sides by $|\tau_r^\alpha|=|D^2u_r^\alpha+M_{r-1}^N|$. Using the inequality $|M_{r-1}^N|\leqslant1$, which follows from $M_{r-1}^N\in\mathbb{K}$, the right-hand side of (7.23) can be bounded from below by

$$\begin{aligned}&10\big(\big|D^2u_r^\alpha\big|^2+2D^2u_r^\alpha:M_{r-1}^N+\big|M_{r-1}^N\big|^2-\big|M_{r-1}^N\big|\big|D^2u_r^\alpha+M_{r-1}^N\big|\big)\\&\quad\geqslant10\big(\big|D^2u_r^\alpha\big|^2-3\big|M_{r-1}^N\big|\big|D^2u_r^\alpha\big|\big)\geqslant10\big|D^2u_r^\alpha\big|^2-30\big|D^2u_r^\alpha\big|.\end{aligned}$$

Using again $|M_{r-1}^N|\leqslant1$, in the region Ω_4 we have that $|D^2u_r^\alpha|>9$, which yields that

$$10\big|D^2u_r^\alpha\big|^2-30\big|D^2u_r^\alpha\big|\geqslant\big|D^2u_r^\alpha\big|^2+51\big|D^2u_r^\alpha\big|>\big|D^2u_r^\alpha\big|^2,$$

and (7.23) is proved.

From (7.22), (7.23), and (7.1) we have

$$B_4^\alpha\leqslant\frac{1}{100}\int\limits_{\Omega_4}\varphi^4E_2^\alpha\big(\tau_{r,k}^\alpha,\tau_{r,k}^\alpha\big)\,dx+\frac{C}{N}+\omega_r(\alpha). \tag{7.24}$$

Collecting (7.14), (7.16), (7.18), (7.21), and (7.24) we obtain

$$\begin{aligned}\Big(1-\frac{1}{N}\Big)\int\limits_{\Omega}\varphi^4E_2^\alpha\big(\tau_{r,k}^\alpha,\tau_{r,k}^\alpha\big)\,dx&\leqslant\int\limits_{\Omega}\varphi^4M_{r,k}^\alpha:M_{r-1,k}^N\,dx+\sum_{s=1}^{9N-1}\frac{1}{2(s+10)}\int\limits_{F_s}\varphi^4\,M_{r,k}^\alpha:M_{r,k}^\alpha\,dx\\&\quad+\frac{1}{100}\int\limits_{\Omega_4}\varphi^4E_2^\alpha\big(\tau_{r,k}^\alpha,\tau_{r,k}^\alpha\big)\,dx+\frac{1}{N}\int\limits_{\Omega}\varphi^4\,M_{r,k}^\alpha:M_{r,k}^\alpha\,dx+\frac{C}{N}+\omega_r(\alpha),\end{aligned}$$

or, by easy transformations,

$$\left(\frac{99}{100}-\frac{1}{N}\right)\int\limits_{\Omega}\varphi^4 E_2^\alpha\left(\tau_{r,k}^\alpha,\tau_{r,k}^\alpha\right)dx$$

$$\leqslant\int\limits_{\Omega}\varphi^4 M_{r,k}^\alpha : M_{r-1,k}^N\,dx+\sum_{s=1}^{9N-1}\frac{1}{2(s+10)}\int\limits_{F_s}\varphi^4 M_{r,k}^\alpha : M_{r,k}^\alpha\,dx-\frac{1}{100}\int\limits_{\Omega_1\cup\Omega_2\cup\Omega_3}\varphi^4 E_2^\alpha\left(\tau_{r,k}^\alpha,\tau_{r,k}^\alpha\right)dx$$

$$+\frac{C}{N}\int\limits_{\Omega}\varphi^4 M_{r,k}^\alpha : M_{r,k}^\alpha\,dx+\frac{C}{N}+\omega_r(\alpha).$$

The claim (7.13) now follows by multiplying the last inequality by $\frac{100}{99}$.

By using Lemmas 5.4 and 5.6 we can express (7.13) in a different form, which is more suitable for our uniform estimates of $M_{r,k}^\alpha$.

Corollary 7.6. *The following estimate holds*

$$\frac{1}{2}\int\limits_{\Omega}\varphi^4 E_2^\alpha\left(\tau_{r,k}^\alpha,\tau_{r,k}^\alpha\right)dx+\frac{1}{2}\int\limits_{\Omega}\varphi^4 E_2^\alpha\left(M_{r,k}^\alpha,M_{r,k}^\alpha\right)dx$$

$$\leqslant\left(\frac{1}{4}\cdot\frac{296}{99}+\frac{C}{N}+\omega_r(\alpha)\right)\int\limits_{\Omega}\varphi^4 M_{r,k}^\alpha : M_{r,k}^\alpha\,dx+\left(\frac{1}{4}\cdot\frac{100}{99}+\frac{C}{N}\right)\int\limits_{\Omega}\varphi^4 M_{r-1,k}^N : M_{r-1,k}^N\,dx$$

$$+\frac{C}{N}+\omega_r(\alpha). \tag{7.25}$$

Proof. We consider each of the domains Ω_i, $i=1,\dots,4$, defined in (7.15). First, remark, that (5.29) yields

$$-E_2^\alpha\left(\tau_{r,k}^\alpha,\tau_{r,k}^\alpha\right)\leqslant-\left(1+\omega_r(\alpha)\right)M_{r,k}^\alpha : M_{r,k}^\alpha \tag{7.26}$$

on $\Omega_1\cup\Omega_2$. We apply (7.13), dividing the integral over Ω into three integrals over the domains just defined.

$$\left(\frac{1}{2}-\frac{1}{N}\right)\int\limits_{\Omega}\varphi^4 E_2^\alpha\left(\tau_{r,k}^\alpha,\tau_{r,k}^\alpha\right)dx+\frac{1}{2}\int\limits_{\Omega}\varphi^4 E_2^\alpha\left(M_{r,k}^\alpha,M_{r,k}^\alpha\right)dx$$

$$\leqslant\frac{1}{2}\int\limits_{\Omega}\varphi^4 E_2^\alpha\left(M_{r,k}^\alpha,M_{r,k}^\alpha\right)dx+\frac{1}{4}\cdot\left(\frac{100}{99}+\frac{C}{N}\right)\int\limits_{\Omega}\varphi^4 M_{r,k}^\alpha : M_{r,k}^\alpha\,dx$$

$$-\frac{1}{2}\cdot\frac{1}{99}\int\limits_{\Omega_1\cup\Omega_2\cup\Omega_3}\varphi^4 E_2^\alpha\left(\tau_{r,k}^\alpha,\tau_{r,k}^\alpha\right)dx+\frac{1}{2}\sum_{s=1}^{9N-1}\frac{1}{s+10}\int\limits_{F_s}\varphi^4 M_{r,k}^\alpha : M_{r,k}^\alpha\,dx$$

$$+\frac{1}{4}\cdot\left(\frac{100}{99}+\frac{C}{N}\right)\int\limits_{\Omega}\varphi^4 M_{r-1,k}^N : M_{r-1,k}^N\,dx+\frac{C}{N}+\omega_r(\alpha). \tag{7.27}$$

Estimates over $\boldsymbol{\Omega_1\cup\Omega_2}$. By (5.28) and (7.26) the sum of the integrals over $\Omega_1\cup\Omega_2$ corresponding to the first three terms in (7.27) is bounded by

$$\left(\frac{1}{4}\cdot\frac{100}{99}+\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{99}+\frac{C}{N}+\omega_r(\alpha)\right)\int\limits_{\Omega_1\cup\Omega_2}\varphi^4 M_{r,k}^\alpha : M_{r,k}^\alpha\,dx$$

$$\leqslant\left(\frac{1}{4}\cdot\frac{296}{99}+\omega_r(\alpha)+\frac{C}{N}\right)\int\limits_{\Omega_1\cup\Omega_2}\varphi^4 M_{r,k}^\alpha : M_{r,k}^\alpha\,dx. \tag{7.28}$$

Estimates over Ω_3. The integral over Ω_3 is estimated by considering the integrals over the sets F_s, defined in (7.13). Using (5.26), (5.29), the bounds

$$\frac{s+10}{s+1} < \left|\tau_r^\alpha\right| \leqslant \frac{s+9}{s},$$

on each F_s, and the inequality

$$\frac{1}{2}\cdot\frac{s+1}{s+10} - \frac{1}{2}\cdot\frac{1}{99}\cdot\frac{s+10}{s+1} + \frac{1}{2}\cdot\frac{1}{s+10} < \frac{1}{2}\cdot\frac{98}{99}$$

valid for $s \in \mathbb{N}$, the sum of the integrals over F_s corresponding to the first four terms in (7.27) is bounded by

$$\sum_{s=1}^{9N-1}\left[\left(\frac{1}{4}\cdot\frac{100}{99} + \frac{1}{2}\cdot\frac{s+1}{s+10} - \frac{1}{2}\cdot\frac{1}{99}\cdot\frac{s+10}{s+1} + \frac{1}{2}\cdot\frac{1}{s+10} + \omega_r(\alpha) + \frac{C}{N}\right)\right]\cdot\int\limits_{F_s}\varphi^4 M_{r,k}^\alpha : M_{r,k}^\alpha\,dx$$
$$\leqslant \left(\frac{1}{4}\cdot\frac{296}{99} + \frac{C}{N} + \omega_r(\alpha)\right)\int\limits_{\Omega_3}\varphi^4 M_{r,k}^\alpha : M_{r,k}^\alpha\,dx. \tag{7.29}$$

Estimates over Ω_4. By (5.26) and the lower bound $|\tau_r^\alpha| > 10$, the sum of the integrals over Ω_4 corresponding to the first three terms in (7.27) is bounded by

$$\left(\frac{1}{4}\cdot\frac{100}{99} + \frac{1}{20} + \omega_r(\alpha) + \frac{C}{N}\right)\int\limits_{\Omega_4}\varphi^4 M_{r,k}^\alpha : M_{r,k}^\alpha\,dx \leqslant \left(\frac{1}{4}\cdot\frac{296}{99} + \omega_r(\alpha) + \frac{C}{N}\right)\int\limits_{\Omega_4}\varphi^4 M_{r,k}^\alpha : M_{r,k}^\alpha\,dx. \tag{7.30}$$

The claim now follows from (7.28)–(7.30). □

8. Uniform $W^{1,2}_{loc}$ estimates of approximate solutions

To carry out the proof of the uniform boundedness of $\|M_N\|_{L^\infty((0,T);W^{1,2}_{loc}(\Omega;\mathbb{M}^{2\times 2}_{sym}))}$ we will make use of the refined version of iterative estimate (6.4), deduced in the previous section, which results in a discrete analogue of Gronwall inequality. To this aim, we need to estimate the last term of (6.4). To make the estimates uniform, we will use the convergence of u_r^α to δu_r^N as in (5.7)–(5.9), and the convergence of M_r^α to M_r^N as in (6.13).

So, the goal of this section is to prove the following inequality first

$$\left(1 - \frac{C}{N}\right)\int\limits_\Omega \varphi^4 M_{r,l}^N : M_{r,l}^N\,dx \leqslant \left(1 + \frac{C}{N}\right)\int\limits_\Omega \varphi^4 M_{r-1,l}^N : M_{r-1,l}^N\,dx + \frac{C}{N}, \tag{8.1}$$

with C independent of N, and then to deduce Theorem 2.2.

We begin as in (6.3), using (5.22) and (5.25):

$$\int\limits_\Omega \varphi^4 M_{r,k}^\alpha : M_{r,k}^\alpha\,dx \leqslant \frac{1}{2}\int\limits_\Omega \varphi^4 E_2^\alpha\left(\tau_{r,k}^\alpha, \tau_{r,k}^\alpha\right)dx + \frac{1}{2}\int\limits_\Omega \varphi^4 E_2^\alpha\left(M_{r,k}^\alpha, M_{r,k}^\alpha\right)dx.$$

Thus, (7.25) yields

$$\left(\frac{1}{4}\cdot\frac{100}{96} - \frac{C}{N} + \omega_r(\alpha)\right)\int\limits_\Omega M_{r,k}^\alpha : M_{r,k}^\alpha\,dx$$
$$\leqslant \frac{1}{4}\cdot\frac{100}{96}\int\limits_\Omega \varphi^4 M_{r-1,k}^N : M_{r-1,k}^N\,dx + \frac{C}{N}\int\limits_\Omega \varphi^4 M_{r,k}^N : M_{r,k}^N\,dx + \frac{C}{N} + \omega_r(\alpha). \tag{8.2}$$

Now, to deduce (8.1) it remains to pass to the limit with respect to α in (8.2), to use (6.13) and the lower semicontinuity of the norm, and to sum the resulting expressions with respect to k.

Proof of Theorem 2.2. Iterating (8.1) we get the following for every $r = 1, \ldots, N$

$$\int_\Omega \varphi^4 M_{r,l}^N : M_{r,l}^N \, dx \leqslant \frac{(1+C/N)^N}{(1-C/N)^N} \int_\Omega \varphi^4 M_{0,l} : \sigma_{0,l} \, dx + \frac{2C}{N} \sum_{i=1}^N \frac{(1+C/N)^{i-1}}{(1-C/N)^i}$$
$$\leqslant e^{2C} \int_\Omega \varphi^4 M_{0,l} : M_{0,l} \, dx + 2Ce^{2C}. \tag{8.3}$$

Thus, we obtain

$$\sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \|M_N(t)\|_{1,2;\Omega'} \leqslant C(\Omega'),$$

and the conclusion follows from convergence of $M_N(t) \rightharpoonup M(t)$ in $L^2(\Omega; \mathbb{M}^{2\times2}_{sym})$ for every $t \in [0, T]$. $\square$

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