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On the uniqueness of weak solutions for the 3D Navier–Stokes equations

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Abstract

In this paper, we improve some known uniqueness results of weak solutions for the 3D Navier–Stokes equations. The proof uses the Fourier localization technique and the losing derivative estimates.

Keywords: Navier–Stokes equations; Uniqueness; Weak solution; Fourier localization; Losing derivative estimates

1. Introduction

We consider the three-dimensional Navier–Stokes equations in $\mathbb{R}^3$

$$\begin{cases} u_t - \Delta u + u \cdot \nabla u + \nabla p = 0, \\ \operatorname{div} u = 0, \\ u(0) = u_0(x), \end{cases} \tag{1.1}$$

where $u = (u^1(t,x), u^2(t,x), u^3(t,x))$ and $p = p(t,x)$ denote the unknown velocity vector and the unknown scalar pressure of the fluid respectively, while $u_0(x)$ is a given initial velocity vector satisfying $\operatorname{div} u_0 = 0$.

In a seminal paper [21], J. Leray proved the global existence of weak solution with finite energy, that is,

$$u(t,x) \in \mathcal{L}_T \stackrel{\text{def}}{=} L^\infty\big(0,T;L^2\big) \cap L^2\big(0,T;H^1\big) \quad \text{for any } T > 0.$$

It is well known that weak solution is unique and regular in two spatial dimensions. In three dimensions, however, the question of regularity and uniqueness of weak solution is an outstanding open problem in mathematical fluid mechanics. In this paper, we are interested in the classical problem of finding sufficient conditions for weak solutions of (1.1) such that they become regular and unique. Let us firstly recall the definition of weak solution.

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Definition 1.1. Let $u_0 \in L^2(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$. A measurable function u is called a weak solution of (1.1) on $(0,T) \times \mathbb{R}^3$ if it satisfies the following conditions:

(1) $u \in \mathcal{L}_T \cap C_w([0,T]; L^2)$, where $C_w([0,T]; L^2)$ consists of all weak continuous functions with respect to time in $L^2(\mathbb{R}^3)$;
(2) $\operatorname{div} u = 0$ in the sense of distribution;
(3) For any function $\psi \in C_0^\infty([s,t] \times \mathbb{R}^3)$ with $\operatorname{div} \psi = 0$, there holds

$$\int_s^t \int_{\mathbb{R}^3} \bigl\{u \cdot \psi_t - \nabla u \cdot \nabla \psi + \nabla \psi \colon (u \otimes u)\bigr\}(t', x)\, dx\, dt' = \int_{\mathbb{R}^3} u(t,x) \cdot \psi(t,x)\, dx - \int_{\mathbb{R}^3} u(s,x) \cdot \psi(s,x)\, dx.$$

In addition, if u satisfies the energy inequality

$$\bigl\|u(t)\bigr\|_2^2 + 2\int_0^t \bigl\|\nabla u(t')\bigr\|_2^2\, dt' \leqslant \|u_0\|_2^2,$$

it is also called a Leray–Hopf weak solution.

The Leray–Hopf weak solutions are unique and regular in the class

$$\mathcal{P} = L^q\bigl(0,T; L^r\bigr) \quad \text{with } \frac{2}{q} + \frac{3}{r} = 1,\ 3 \leqslant r \leqslant \infty \quad [11,14,15,25,27],$$

$$\text{or} \quad \mathcal{P} = L^q\bigl(0,T; W^{1,r}\bigr) \quad \text{with } \frac{2}{q} + \frac{3}{r} = 2,\ \frac{3}{2} < r \leqslant \infty \quad [1],$$

$$\text{or} \quad \mathcal{P} = L^q\bigl(0,T; W^{s,r}\bigr) \quad \text{with } \frac{2}{q} + \frac{3}{r} = 1 + s,\ \frac{3}{1+s} < r \leqslant \infty,\ s \geqslant 0 \quad [26].$$

Recently, there are many researches devoted to refine the above results. First of all, we have the following refined regularity criterion in the framework of Besov spaces: the weak solutions are regular in the class

$$\mathcal{P} = C\bigl([0,T]; B^{-1}_{\infty,\infty}\bigr) \quad \text{or} \quad \mathcal{P} = L^q\bigl(0,T; B^r_{p,\infty}\bigr),$$

with $\frac{2}{q} + \frac{3}{p} = 1 + r$, $\frac{3}{1+r} < p \leqslant \infty$, and $-1 < r \leqslant 1$, see [4,8,17,18]. Concerning the refined uniqueness criterion of weak solutions, Kozono and Taniuchi [16] proved the uniqueness of the Leray–Hopf weak solutions in the class

$$\mathcal{P} = L^2(0,T; BMO).$$

Gallagher and Planchon [12] proved the uniqueness in the class

$$\mathcal{P} = L^q\bigl(0,T; \dot{B}_{p,q}^{-1+\frac{3}{p}+\frac{2}{q}}\bigr) \quad \text{with } \frac{3}{p} + \frac{2}{q} > 1.$$

Lemarié-Rieusset [19] proved the uniqueness in the class

$$\mathcal{P} = C\bigl([0,T]; X_1^{(0)}\bigr) \quad \text{or} \quad \mathcal{P} = L^{\frac{2}{1-r}}(0,T; X_r) \quad \text{with } r \in [0,1).$$

Finally, Germain [13] proved the uniqueness in the class

$$\mathcal{P} = C\bigl([0,T]; X_1^{(0)}\bigr) \quad \text{or} \quad \mathcal{P} = L^{\frac{2}{1-r}}(0,T; X_r) \quad \text{with } r \in [-1,1).$$

Here $B^s_{p,q}$ denotes the Besov space and

$$X_s := \begin{cases} \mathbf{M}\bigl(\dot{H}^s, L^2\bigr), & \text{if } s \in (0,1], \\ \Lambda^s BMO, & \text{if } s \in (-1,0], \\ \mathbf{Lip}, & \text{if } s = -1, \end{cases}$$

where $\mathbf{M}(\dot{H}^s, L^2)$ is the space of distributions such that their pointwise product with a function in $\dot{H}^s$ belongs to L^2, $\Lambda^s = (1-\Delta)^{\frac{s}{2}}$. $X_s^{(0)}$ denotes the closure of the Schwartz class in X_s. We want to point out that

$$X_s \hookrightarrow \Lambda^s BMO, \quad \text{if } s \in (0, 1]. \tag{1.2}$$

We refer to [13] for more properties about X_s. The key step of their proofs is to find a path space $\mathcal{P}$ so that the trilinear form

$$F(u, v, w) := \int_0^T \int_{\mathbb{R}^3} u \cdot \nabla v \cdot w \, dx \, dt$$

is continuous from $(\mathcal{L}_T)^2 \times \mathcal{P}$ to $\mathbb{R}$. Germain also pointed out that the path space $\mathcal{P}$ he found is optimal in some sense (see [13, p. 400] for precise meaning).

The purpose of this paper is to improve the above uniqueness results.

Theorem 1.2. *Let $u_0, v_0 \in L^2(\mathbb{R}^3)$ with $\operatorname{div} u_0 = \operatorname{div} v_0 = 0$. Let u and v be two Leray–Hopf weak solutions of (1.1) on $(0, T)$ with the initial data u_0 and v_0 respectively. Assume that*

$$u \in L^q\big(0, T; B^r_{p,\infty}\big),$$

with $\frac{2}{q} + \frac{3}{p} = 1 + r$, $\frac{3}{1+r} < p \leqslant \infty$, $r \in (0, 1]$, and $(p, r) \neq (\infty, 1)$. Then there holds

$$\big\|u(t) - v(t)\big\|_2^2 + \int_0^t \big\|\nabla(u - v)(t')\big\|_2^2 \, dt' \leqslant \|u_0 - v_0\|_2^2 \exp\left\{C \int_0^t \big(e + \big\|u(t')\big\|_{B^r_{p,\infty}}\big)^q \, dt'\right\}.$$

In particular, if $u_0 = v_0$, then $u = v$ a.e. on $(0, T) \times \mathbb{R}^3$.

Remark 1.3. Due to the embedding relation

$$B^s_{p,q} \subsetneq B^s_{p,\infty}, \quad q < +\infty \quad \text{and} \quad \Lambda^{-r} BMO \subsetneq B^r_{\infty,\infty},$$

Theorem 1.2 is an improvement of the corresponding results given by Gallagher and Planchon [12] and Germain [13]. The proof only uses an important observation that if $u \in L^q(0, T; B^r_{p,\infty})$ with (p, q, r) as in Theorem 1.2, then u can be decomposed as

$$u = u^l + u^h \quad \text{with } u^l \in L^1(0, T; \mathbf{Lip}) \text{ and } u^h \in L^{\tilde{q}}\big(0, T; L^{\tilde{p}}\big)$$

for some $\tilde{p}, \tilde{q}$ satisfying $\frac{2}{\tilde{q}} + \frac{3}{\tilde{p}} = 1$, $\tilde{p} > 3$, see Lemma 3.1.

In the case where either $r \leqslant 0$ or $(p, r) = (\infty, 1)$, using Bony's decomposition and the losing derivative estimates, we prove

Theorem 1.4. *Let $u_0 \in L^2(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$. Let u and v be two weak solutions of (1.1) on $(0, T)$ with the same initial data u_0. Assume that u and v satisfy one of the following two conditions*:

(a) *$u \in L^{q_1}(0, T; B^{r_1}_{p_1,\infty})$ and $v \in L^{q_2}(0, T; B^{r_2}_{p_2,\infty})$, where*

$$\frac{2}{q_1} + \frac{3}{p_1} = 1 + r_1, \qquad \frac{2}{q_2} + \frac{3}{p_2} = 1 + r_2,$$

with $r_1, r_2 \in (-1, 0]$, $r_1 + r_2 > -1$, $\frac{3}{1+r_1} < p_1 \leqslant \infty$, $\frac{3}{1+r_2} < p_2 \leqslant \infty$.

(b) *$u, v \in L^1(0, T; B^1_{\infty,\infty})$.*

Then $u = v$ a.e. on $(0, T) \times \mathbb{R}^3$.

Remark 1.5. Due to the embedding relation

$$X_s \subsetneq \Lambda^s BMO \subsetneq B^{-s}_{\infty,\infty}, \quad s \in (0,1],$$

the condition imposed on weak solution in Theorem 1.4 is weaker than that of Germain [13] and Lemarié-Rieusset [19]. However, the price to pay is to impose the conditions on both weak solutions.

Remark 1.6. The main novelty of Theorem 1.4 is that weak solutions are unique in the class $L^1(0,T;B^1_{\infty,\infty})$. In particular, from the inequality

$$\|u\|_{B^1_{\infty,\infty}} \leqslant C\big(\|u\|_2 + \|\operatorname{curl} u\|_{B^0_{\infty,\infty}}\big) \quad \text{(see Section 2 for its proof)}, \tag{1.3}$$

we can obtain the *Beale–Kato–Majda* type uniqueness criterion: if weak solutions u and v with the same initial data satisfy

$$\operatorname{curl} u, \operatorname{curl} v \in L^1\big(0,T;B^0_{\infty,\infty}\big),$$

then $u = v$ on $(0,T)\times\mathbb{R}^3$. Secondly, Theorem 1.4 allows us to impose different conditions on both weak solutions. Thirdly, we do not impose the energy inequality on weak solutions.

Remark 1.7. Chemin and Lemarié-Rieusset [6,20] proved the uniqueness of weak solutions in the class $C([0,T];B^{-1}_{\infty,\infty})$. While, Theorem 1.4 gives the uniqueness in the class $L^1(0,T;B^1_{\infty,\infty})$. It is natural to expect that the uniqueness also holds in the class $L^{\frac{2}{1+r}}(0,T;B^r_{\infty,\infty})$ for $r \in (-1,1)$ from the viewpoint of interpolation. This problem remains unknown for the case of $r \in (-1,-\frac{1}{2}]$.

Remark 1.8. The result (b) in Theorem 1.4 is also valid for the Euler equation. In detail, let $u, v \in C_\omega([0,T);L^2(\mathbb{R}^3)) \cap L^1([0,T);B^1_{\infty,\infty}(\mathbb{R}^3))$ be two weak solutions of the Euler equation with the same initial data, then $u = v$ a.e. on $[0,T)$.

Notation. Throughout the paper, C stands for a generic constant. We will use the notation $A \lesssim B$ to denote the relation $A \leqslant CB$, and $\|\cdot\|_p$ denotes the norm of the Lebesgue space L^p.

2. Preliminaries

Let us firstly recall some basic facts on the Littlewood–Paley decomposition, one may check [5] for more details. Choose two nonnegative radial functions $\chi, \varphi \in \mathcal{S}(\mathbb{R}^3)$ supported respectively in $\mathcal{B} = \{\xi \in \mathbb{R}^3, |\xi| \leqslant \frac{4}{3}\}$ and $\mathcal{C} = \{\xi \in \mathbb{R}^3, \frac{3}{4} \leqslant |\xi| \leqslant \frac{8}{3}\}$ such that for any $\xi \in \mathbb{R}^3$,

$$\chi(\xi) + \sum_{j\geqslant 0}\varphi\big(2^{-j}\xi\big) = 1. \tag{2.1}$$

Let $h = \mathcal{F}^{-1}\varphi$ and $\tilde{h} = \mathcal{F}^{-1}\chi$, the frequency localization operator Δ_j and S_j are defined by

$$\Delta_j f = \varphi\big(2^{-j}D\big)f = 2^{3j}\int_{\mathbb{R}^3} h\big(2^j y\big) f(x-y)\,dy, \quad \text{for } j \geqslant 0,$$

$$S_j f = \chi\big(2^{-j}D\big)f = \sum_{-1\leqslant k\leqslant j-1}\Delta_k f = 2^{3j}\int_{\mathbb{R}^3}\tilde{h}\big(2^j y\big) f(x-y)\,dy,$$

and

$$\Delta_{-1} f = S_0 f, \qquad \Delta_j f = 0 \quad \text{for } j \leqslant -2.$$

With our choice of φ, one can easily verify that

$$\Delta_j \Delta_k f = 0 \quad \text{if } |j-k| \geqslant 2 \quad \text{and}$$
$$\Delta_j (S_{k-1} f \Delta_k f) = 0 \quad \text{if } |j-k| \geqslant 5. \tag{2.2}$$

For any $f \in \mathcal{S}'(\mathbb{R}^3)$, we have by (2.1) that

$$f = S_0(f) + \sum_{j \geqslant 0} \Delta_j f, \tag{2.3}$$

which is called the Littlewood–Paley decomposition. In the sequel, we will constantly use the Bony's decomposition from [2] that

$$uv = T_u v + T_v u + R(u, v), \tag{2.4}$$

with

$$T_u v = \sum_j S_{j-1} u \Delta_j v, \qquad R(u, v) = \sum_{|j'-j| \leqslant 1} \Delta_j u \Delta_{j'} v,$$

and we also denote

$$T'_u v = T_u v + R(u, v).$$

With the introduction of Δ_j, let us recall the definition of the inhomogenous Besov space from [29]:

Definition 2.1. Let $s \in \mathbb{R}$, $1 \leqslant p, q \leqslant \infty$, the inhomogenous Besov space $B^s_{p,q}$ is defined by

$$B^s_{p,q} = \left\{ f \in \mathcal{S}'\left(\mathbb{R}^3\right);\ \|f\|_{B^s_{p,q}} < \infty \right\},$$

where

$$\|f\|_{B^s_{p,q}} := \begin{cases} \left(\sum_{j=-1}^{\infty} 2^{jsq} \|\Delta_j f\|_p^q\right)^{\frac{1}{q}}, & \text{for } q < \infty, \\ \sup_{j \geqslant -1} 2^{js} \|\Delta_j f\|_p, & \text{for } \quad q = \infty. \end{cases}$$

Let us point out that $B^s_{\infty,\infty}$ is the usual Hölder space C^s for $s \in \mathbb{R} \setminus \mathbb{Z}$ and the following inclusion relations hold

$$\mathbf{Lip} \subsetneq B^1_{\infty,\infty}, \qquad \Lambda^{-s} BMO \subsetneq B^s_{\infty,\infty} \quad \text{for } s \in \mathbb{R}.$$

We refer to [13,29] for more properties.

The following Bernstein's inequalities will be frequently used throughout the paper.

Lemma 2.2. *(See [5].) Let $1 \leqslant p \leqslant q \leqslant \infty$. Assume that $f \in L^p$, then there hold*

$$\operatorname{supp} \hat{f} \subset \left\{ |\xi| \leqslant C 2^j \right\} \quad \Rightarrow \quad \left\| \partial^\alpha f \right\|_q \leqslant C 2^{j|\alpha| + 3j(\frac{1}{p} - \frac{1}{q})} \|f\|_p,$$
$$\operatorname{supp} \hat{f} \subset \left\{ \frac{1}{C} 2^j \leqslant |\xi| \leqslant C 2^j \right\} \quad \Rightarrow \quad \|f\|_p \leqslant C 2^{-j|\alpha|} \sup_{|\beta| = |\alpha|} \left\| \partial^\beta f \right\|_p.$$

Here the constant C is independent of f and j.

We conclude this section by a proof of the inequality (1.3). Using Lemma 2.2, we have

$$\|\Delta_{-1} u\|_\infty \leqslant C \|u\|_2,$$

and for $j \geqslant 0$,

$$2^j \|\Delta_j u\|_\infty \leqslant C \|\Delta_j \nabla u\|_\infty.$$

Due to the Biot–Savart law [23], ∇u can be written as

$$\nabla u(x) = C w(x) + K * w(x), \quad w = \operatorname{curl} u,$$

where C is a constant matrix, and K is a matrix valued function with homogeneous of degree -3. So, we get that for $j \geqslant 0$,

$$2^j \|\Delta_j u\|_\infty \leqslant C \|\Delta_j w\|_\infty,$$

where we used the fact that

$$\|\Delta_j(Tf)\|_p \leqslant C\|\Delta_j f\|_p, \quad \text{for } j \geqslant 0,\ 1 \leqslant p \leqslant \infty,$$

if T is a singular integral operator of convolution type with smooth kernel [28]. Then the inequality (1.3) is concluded from the definition of Besov space.

3. Proofs of theorems

This section is devoted to the proof of Theorems 1.2 and 1.4.

3.1. The proof of Theorem 1.2

The proof is based on the following decomposition lemma which may be independent of interest.

Lemma 3.1. *Assume that $u \in L^q(0,T; B^r_{p,\infty})$ with $\frac{2}{q} + \frac{3}{p} = 1 + r$, $\frac{3}{1+r} < p \leqslant \infty$, $r \in (0,1]$, and $(p,r) \neq (\infty, 1)$. Then u can be decomposed as*

$$u = u^l + u^h \quad \text{with } u^l \in L^1(0,T; \mathbf{Lip}) \text{ and } u^h \in L^{\tilde q}\big(0,T; L^{\tilde p}\big)$$

for some $\tilde p$, $\tilde q$ satisfying $\frac{2}{\tilde q} + \frac{3}{\tilde p} = 1$, $\tilde p > 3$.

Proof. Fix $N \in \mathbb{N}$ to be determined later on. We set

$$u^l = S_N u, \qquad u^h = u - u^l.$$

By the definition of S_N and Lemma 2.2, we have

$$\|\nabla u^l\|_\infty \leqslant C \sum_{j \leqslant N-1} 2^{j(1+\frac{3}{p})} \|\Delta_j u\|_p \leqslant C 2^{2(1-\frac{1}{q})N} \|u\|_{B^r_{p,\infty}}. \tag{3.1}$$

Due to the conditions on (p,q,r), we can choose $\tilde p$ such that

$$\tilde p > \max(3, p) \quad \text{and} \quad \frac{3}{p} - \frac{3}{\tilde p} - r < 0.$$

Thus, by Lemma 2.2

$$\|u^h\|_{\tilde p} \leqslant \sum_{j \geqslant N} 2^{(\frac{3}{p} - \frac{3}{\tilde p})j} \|\Delta_j u\|_p \leqslant C 2^{(\frac{3}{p} - \frac{3}{\tilde p} - r)N} \|u\|_{B^r_{p,\infty}}. \tag{3.2}$$

Now we choose

$$N = \left[\frac{q}{2} \log_2\big(e + \|u\|_{B^r_{p,\infty}}\big)\right] + 1.$$

Then by (3.1), we have

$$\int_0^T \|\nabla u^l(t)\|_\infty \, dt \leqslant C \int_0^T \big(e + \|u(t)\|_{B^r_{p,\infty}}\big)^q \, dt < +\infty. \tag{3.3}$$

On the other hand, from (3.2) we get that

$$\int_0^T \left\|u^h(t)\right\|_{\tilde{p}}^{\tilde{q}}\,dt \leqslant C\int_0^T \left(e+\left\|u(t)\right\|_{B^r_{p,\infty}}\right)^q\,dt < +\infty. \tag{3.4}$$

Hence, we complete the proof of Lemma 3.1 by (3.3) and (3.4). □

Lemma 3.2. *Let u, v be as in Theorem* 1.2*. Set $w = u - v$. Then for any $t \in [0, T]$, there holds*

$$\langle u(t), v(t)\rangle + 2\int_0^t \langle \nabla u, \nabla v\rangle\,dt' = \langle u_0, v_0\rangle + \int_0^t \langle w\cdot\nabla u, w\rangle\,dt'.$$

Proof. Lemma 3.1 ensures that the trilinear form

$$F(u, v, w) := \int_0^T\int_{\mathbb{R}^3} u\cdot\nabla w\cdot v\,dx\,dt$$

is continuous from $(\mathcal{L}_T)^2 \times L^q(0, T; B^r_{p,\infty})$ to $\mathbb{R}$. Then the lemma can be proved by following the argument of Lemma 4.4 in [13]. Here we omit the details. □

Now we are in position to prove Theorem 1.2. Since u and v are Leray–Hopf weak solutions, there hold

$$\left\|u(t)\right\|_2^2 + 2\int_0^t \left\|\nabla u(t')\right\|_2^2\,dt' \leqslant \|u_0\|_2^2,$$

$$\left\|v(t)\right\|_2^2 + 2\int_0^t \left\|\nabla v(t')\right\|_2^2\,dt' \leqslant \|v_0\|_2^2.$$

On the other hand, Lemma 3.2 yields that

$$\langle u(t), v(t)\rangle + 2\int_0^t \langle \nabla u, \nabla v\rangle\,dt' = \langle u_0, v_0\rangle + \int_0^t \langle w\cdot\nabla u, w\rangle\,dt'.$$

Combining the above inequalities, we obtain

$$\begin{aligned}\left\|w(t)\right\|_2^2 + 2\int_0^t \left\|\nabla w(t')\right\|_2^2\,dt' &= \left\|u(t)\right\|_2^2 + \left\|v(t)\right\|_2^2 - 2\langle u, v\rangle(t) + 2\int_0^t \left\|\nabla u(t')\right\|_2^2\,dt' \\ &\quad + 2\int_0^t \left\|\nabla v(t')\right\|_2^2\,dt' - 4\int_0^t \langle\nabla u, \nabla v\rangle(t')\,dt' \\ &\leqslant \|u_0 - v_0\|_2^2 - 2\int_0^t \langle w\cdot\nabla u, w\rangle\,dt'. \end{aligned} \tag{3.5}$$

We decompose $u = u^l + u^h$ as in Lemma 3.1 and rewrite

$$\int_0^t \langle w\cdot\nabla u, w\rangle\,dt' = \int_0^t \langle w\cdot\nabla u^l, w\rangle\,dt' + \int_0^t \langle w\cdot\nabla u^h, w\rangle\,dt'.$$

We get by Hölder inequality that

$$\left|\int_0^t \langle w\cdot\nabla u^l, w\rangle\,dt'\right| \leqslant \int_0^t \|w(t')\|_2^2\|\nabla u^l(t')\|_\infty\,dt'. \tag{3.6}$$

Integration by parts, we get

$$\int_0^t \langle w\cdot\nabla u^h, w\rangle\,dt' = -\int_0^t \langle w\cdot\nabla w, u^h\rangle\,dt',$$

from which and the Gagliardo–Nirenberg inequality, it follows that

$$\begin{aligned}\left|\int_0^t \langle w\cdot\nabla u^h, w\rangle\,dt'\right| &\leqslant \int_0^t \|\nabla w\|_2\|w\|_{\frac{2\tilde p}{\tilde p-2}}\|u^h\|_{\tilde p}\,dt'\\ &\leqslant C\int_0^t \|\nabla w\|_2\|w\|_2^{1-\frac{3}{\tilde p}}\|\nabla w\|_2^{\frac{3}{\tilde p}}\|u^h\|_{\tilde p}\,dt'\\ &\leqslant C\left(\int_0^t \|w(t')\|_2^2\|u^h(t')\|_{\tilde p}^{\tilde q}\,dt'\right)^{\frac{1}{\tilde q}}\left(\int_0^t \|\nabla w(t')\|_2^2\,dt'\right)^{1-\frac{1}{\tilde q}}\\ &\leqslant C\int_0^t \|w(t')\|_2^2\|u^h(t')\|_{\tilde p}^{\tilde q}\,dt' + \int_0^t \|\nabla w(t')\|_2^2\,dt'.\end{aligned}$$

This together with (3.5) and (3.6) gives

$$\|w(t)\|_2^2 + \int_0^t \|\nabla w(t')\|_2^2\,dt' \leqslant \|u_0-v_0\|^2 + C\int_0^t \|w(t')\|_2^2\big(\|\nabla u^l(t')\|_\infty + \|u^h(t')\|_{\tilde p}^{\tilde q}\big)\,dt'.$$

This jointed with the Gronwall inequality produces that

$$\begin{aligned}\|w(t)\|_2^2 + \int_0^t \|\nabla w(t')\|_2^2\,dt' &\leqslant \|u_0-v_0\|^2\exp\left\{C\int_0^t \big(\|\nabla u^l(t')\|_\infty + \|u^h(t')\|_{\tilde p}^{\tilde q}\big)\,dt'\right\}\\ &\leqslant \|u_0-v_0\|_2^2\exp\left\{C\int_0^t \big(e+\|u(t')\|_{B^r_{p,\infty}}\big)^q\,dt'\right\}.\end{aligned}$$

This finishes the proof of Theorem 1.2.

3.2. The proof of Theorem 1.4

Assume that u and v are two weak solutions of (1.1) on $(0,T)$ with the initial data u_0. Let $w=u-v$, w satisfies the equation in the sense of distribution

$$w_t - \Delta w + w\cdot\nabla u + v\cdot\nabla w + \nabla\tilde p = 0, \tag{3.7}$$

for some pressure $\tilde p$. We get by taking the operation Δ_j on both sides of (3.7) that

$$\partial_t\Delta_j w - \Delta\Delta_j w + \Delta_j(w\cdot\nabla u) + \Delta_j(v\cdot\nabla w) + \nabla\Delta_j\tilde p = 0. \tag{3.8}$$

Multiplying (3.8) by $\Delta_j w$, we get by Lemma 2.2 for $j\geqslant -1$ that

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j w(t)\|_2^2 + ca_j2^{2j}\|\Delta_j w(t)\|_2^2 \leqslant -\langle\Delta_j(w\cdot\nabla u),\Delta_j w\rangle - \langle\Delta_j(v\cdot\nabla w) - v\cdot\nabla\Delta_j w, \Delta_j w\rangle, \tag{3.9}$$

with $a_{-1}=0$ and $a_j=1$ for $j\geqslant 0$. Here we used the fact that

$$\langle \Delta_j(v\cdot\nabla w),\Delta_j w\rangle=\langle \Delta_j(v\cdot\nabla w)-v\cdot\nabla\Delta_j w,\Delta_j w\rangle.$$

Case 1. u and v satisfy the assumption (a).

Due to $r_1+r_2>-1$, one of r_1 and r_2 must be bigger than $-\frac{1}{2}$. Without loss of generality, we assume that $r_1>-\frac{1}{2}$.

Step 1. Estimate of $\langle \Delta_j(w\cdot\nabla u),\Delta_j w\rangle$.

Using the Bony's decomposition (2.4), we have

$$\Delta_j(w\cdot\nabla u)=\Delta_j\big(T_{w^i}\partial_i u\big)+\Delta_j\big(T_{\partial_i u}w^i\big)+\Delta_j R\big(w^i,\partial_i u\big).$$

Considering the support of the Fourier transform of the term $T_{w^i}\partial_i u$, we have

$$\Delta_j\big(T_{w^i}\partial_i u\big)=\sum_{|j'-j|\leqslant 4}\Delta_j\big(S_{j'-1}w^i\partial_i\Delta_{j'}u\big). \tag{3.10}$$

This gives by Lemma 2.2 that

$$\begin{aligned}\big\|\Delta_j\big(T_{w^i}\partial_i u\big)\big\|_2&\lesssim\sum_{|j'-j|\leqslant 4}2^{j'}\sum_{k\leqslant j'-2}\|\Delta_k w\|_{\frac{2p_1}{p_1-2}}\|\Delta_{j'}u\|_{p_1}\lesssim\sum_{|j'-j|\leqslant 4}2^{j'}\sum_{k\leqslant j'-2}2^{k\frac{3}{p_1}}\|\Delta_k w\|_2\|\Delta_{j'}u\|_{p_1}\\&\lesssim 2^{j(1-r_1)}\|u\|_{B^{r_1}_{p_1,\infty}}\sum_{j'\leqslant j+2}2^{j'\frac{3}{p_1}}\|\Delta_{j'}w\|_2.\end{aligned} \tag{3.11}$$

Similarly, we have

$$\Delta_j\big(T_{\partial_i u}w^i\big)=\sum_{|j'-j|\leqslant 4}\Delta_j\big(S_{j'-1}(\partial_i u)\Delta_{j'}w^i\big). \tag{3.12}$$

Applying Lemma 2.2 to (3.12) yields that

$$\begin{aligned}\big\|\Delta_j\big(T_{\partial_i u}w^i\big)\big\|_2&\lesssim\sum_{|j'-j|\leqslant 4}\sum_{k\leqslant j'-2}2^k\|\Delta_k u\|_\infty\|\Delta_{j'}w\|_2\\&\lesssim 2^{j(1-r_1+\frac{3}{p_1})}\|u\|_{B^{r_1}_{p_1,\infty}}\sum_{|j'-j|\leqslant 4}\|\Delta_{j'}w\|_2.\end{aligned} \tag{3.13}$$

Since $\operatorname{div} w=0$, we have

$$\Delta_j R\big(w^i,\partial_i u\big)=\sum_{j',j''\geqslant j-3;\,|j'-j''|\leqslant 1}\partial_i\Delta_j\big(\Delta_{j'}w^i\Delta_{j''}u\big), \tag{3.14}$$

from which and Lemma 2.2, it follows that

$$\begin{aligned}\big\|\Delta_j R\big(w^i,\partial_i u\big)\big\|_{\frac{2p_1}{p_1+2}}&\lesssim\sum_{j',j''\geqslant j-3;\,|j'-j''|\leqslant 1}2^j\|\Delta_{j'}w\|_2\|\Delta_{j''}u\|_{p_1}\\&\lesssim 2^j\|u\|_{B^{r_1}_{p_1,\infty}}\sum_{j'\geqslant j-3}2^{-j'r_1}\|\Delta_{j'}w\|_2.\end{aligned} \tag{3.15}$$

Summing up (3.11)–(3.15), we obtain

$$\begin{aligned}\big|\langle \Delta_j(w\cdot\nabla u),\Delta_j w\rangle\big|&\lesssim 2^{j(1-r_1)}\|u\|_{B^{r_1}_{p_1,\infty}}\sum_{j'\leqslant j}2^{j'\frac{3}{p_1}}\|\Delta_{j'}w\|_2\|\Delta_j w\|_2\\&\quad+2^{j(1+\frac{3}{p_1})}\|u\|_{B^{r_1}_{p_1,\infty}}\sum_{j'\geqslant j}2^{-j'r_1}\|\Delta_{j'}w\|_2\|\Delta_j w\|_2.\end{aligned} \tag{3.16}$$

Step 2. Estimate of $\langle \Delta_j(v\cdot\nabla w) - v\cdot\nabla\Delta_j w, \Delta_j w\rangle$.

Using the Bony's decomposition (2.4), we write

$$\Delta_j(v\cdot\nabla w) = \Delta_j\big(T_{v^i}\partial_i w\big) + \Delta_j\big(T_{\partial_i w}v^i\big) + \Delta_j R\big(v^i,\partial_i w\big),$$
$$v\cdot\nabla\Delta_j w = T_{v^i}\partial_i\Delta_j w + T'_{\partial_i\Delta_j w}v^i.$$

Then we have

$$\Delta_j(v\cdot\nabla w) - v\cdot\nabla\Delta_j w = \big[\Delta_j, T_{v^i}\big]\partial_i w + \Delta_j\big(T_{\partial_i w}v^i\big) + \Delta_j R\big(v^i,\partial_i w\big) - T'_{\partial_i\Delta_j w}v^i.$$

Similar arguments as in deriving (3.11) and (3.15), we have

$$\big\|\Delta_j\big(T_{\partial_i w}v^i\big)\big\|_2 \lesssim 2^{-jr_2}\|v\|_{B^{r_2}_{p_2,\infty}} \sum_{j'\leqslant j+2} 2^{j'(1+\frac{3}{p_2})}\|\Delta_{j'}w\|_2, \tag{3.17}$$

$$\big\|\Delta_j R\big(v^i,\partial_i w\big)\big\|_{\frac{2p_2}{p_2+2}} \lesssim 2^j\|v\|_{B^{r_2}_{p_2,\infty}} \sum_{j'\geqslant j-3} 2^{-j'r_2}\|\Delta_{j'}w\|_2. \tag{3.18}$$

In view of the definition of $T'_{\partial_i\Delta_j w}v^i$,

$$T'_{\partial_i\Delta_j w}v^i = \sum_{j'\geqslant j-2} S_{j'+2}\Delta_j\partial_i w\Delta_{j'}v^i,$$

and note that $S_{j'+2}\Delta_j w = \Delta_j w$ for $j' > j$, we get

$$\big\langle T'_{\partial_i\Delta_j w}v^i, \Delta_j w\big\rangle = \sum_{j-2\leqslant j'\leqslant j} \big\langle S_{j'+2}\Delta_j\partial_i w\Delta_{j'}v^i, \Delta_j w\big\rangle,$$

from which and Lemma 2.2, it follows that

$$\big|\big\langle T'_{\partial_i\Delta_j w}v^i, \Delta_j w\big\rangle\big| \lesssim 2^{j(1+\frac{3}{p_2}-r_2)}\|v\|_{B^{r_2}_{p_2,\infty}}\|\Delta_j w\|_2^2. \tag{3.19}$$

Now, we turn to estimate $[T_{v^i},\Delta_j]\partial_i w$. In view of the definition of Δ_j, we write

$$\begin{aligned}\big[T_{v^i},\Delta_j\big]\partial_i w &= \sum_{|j'-j|\leqslant 4}\big[S_{j'-1}v^i,\Delta_j\big]\partial_i\Delta_{j'}w\\ &= \sum_{|j'-j|\leqslant 4} 2^{3j}\int_{\mathbb{R}^3} h\big(2^j(x-y)\big)\big(S_{j'-1}v^i(x) - S_{j'-1}v^i(y)\big)\partial_i\Delta_{j'}w(y)\,dy\\ &= \sum_{|j'-j|\leqslant 4} 2^{4j}\int_{\mathbb{R}^3}\int_0^1 y\cdot\nabla S_{j'-1}v^i(x-\tau y)\,d\tau\,\partial_i h\big(2^j y\big)\Delta_{j'}w(x-y)\,dy,\end{aligned} \tag{3.20}$$

from which and the Minkowski inequality, we deduce that

$$\begin{aligned}\big\|\big[T_{v^i},\Delta_j\big]\partial_i w\big\|_2 &\lesssim \sum_{|j'-j|\leqslant 4}\|\nabla S_{j'-1}v\|_\infty\|\Delta_{j'}w\|_2\\ &\lesssim 2^{j(1+\frac{3}{p_2}-r_2)}\|v\|_{B^{r_2}_{p_2,\infty}}\sum_{|j'-j|\leqslant 4}\|\Delta_{j'}w\|_2.\end{aligned} \tag{3.21}$$

Summing up (3.17)–(3.21), we obtain

$$\begin{aligned}\big|\big\langle\Delta_j(v\cdot\nabla w) - v\cdot\nabla\Delta_j w, \Delta_j w\big\rangle\big| &\lesssim 2^{-jr_2}\|v\|_{B^{r_2}_{p_2,\infty}}\sum_{j'\leqslant j} 2^{j'(1+\frac{3}{p_2})}\|\Delta_{j'}w\|_2\|\Delta_j w\|_2\\ &\quad + 2^{j(1+\frac{3}{p_2})}\|v\|_{B^{r_2}_{p_2,\infty}}\sum_{j'\geqslant j} 2^{-j'r_2}\|\Delta_{j'}w\|_2\|\Delta_j w\|_2.\end{aligned} \tag{3.22}$$

Under the assumption (a), we can choose s such that

$$-r_1 < s < \min(1 + r_1, 1 + r_2). \tag{3.23}$$

From (3.9), (3.16) and (3.22), it follows that

$$\begin{aligned}
&2^{-2js}\|\Delta_j w(t)\|_2^2 + a_j 2^{2j(1-s)} \int_0^t \|\Delta_j w(t')\|_2^2\,dt' \\
&\lesssim \int_0^t \|u\|_{B^{r_1}_{p_1,\infty}} 2^{j(1-r_1-2s)} \sum_{j'\leqslant j} 2^{j'\frac{3}{p_1}} \|\Delta_{j'} w\|_2 \|\Delta_j w\|_2\,dt' \\
&\quad + \int_0^t \|u\|_{B^{r_1}_{p_1,\infty}} 2^{j(1+\frac{3}{p_1}-2s)} \sum_{j'\geqslant j} 2^{-j'r_1} \|\Delta_{j'} w\|_2 \|\Delta_j w\|_2\,dt' \\
&\quad + \int_0^t \|v\|_{B^{r_2}_{p_2,\infty}} 2^{-j(r_2+2s)} \sum_{j'\leqslant j} 2^{j'(1+\frac{3}{p_2})} \|\Delta_{j'} w\|_2 \|\Delta_j w\|_2\,dt' \\
&\quad + \int_0^t \|v\|_{B^{r_2}_{p_2,\infty}} 2^{j(1+\frac{3}{p_2}-2s)} \sum_{j'\geqslant j} 2^{-j'r_2} \|\Delta_{j'} w\|_2 \|\Delta_j w\|_2\,dt' \\
&:= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.
\end{aligned} \tag{3.24}$$

We set

$$W(t) = \sup_{j\geqslant -1} 2^{-js}\|\Delta_j w(t)\|_2.$$

Using (3.23) and the Young's inequality, we obtain

$$\begin{aligned}
\mathrm{I} &\leqslant \sum_{j'\leqslant j} 2^{(j'-j)(r_1+s)} \int_0^t \|u\|_{B^{r_1}_{p_1,\infty}} W(t')^{\frac{2}{q_1}} \big(2^{j'(1-s)}\|\Delta_{j'} w\|_2\big)^{1-\frac{2}{q_1}} 2^{j(1-s)} \|\Delta_j w\|_2\,dt' \\
&\leqslant C\bigg(\int_0^t \|u\|^{q_1}_{B^{r_1}_{p_1,\infty}} W(t')^2\,dt'\bigg)^{\frac{1}{q_1}} \bigg(\sup_{j\geqslant -1} 2^{2j(1-s)} \int_0^t \|\Delta_j w(t')\|_2^2\,dt'\bigg)^{\frac{1}{q_1'}} \\
&\leqslant C\int_0^t \|u\|^{q_1}_{B^{r_1}_{p_1,\infty}} W(t')^2\,dt' + \delta \sup_{j\geqslant -1} 2^{2j(1-s)} \int_0^t \|\Delta_j w(t')\|_2^2\,dt',
\end{aligned}$$

and for II, we have

$$\begin{aligned}
\mathrm{II} &\leqslant \sum_{j'\geqslant j} 2^{(j'-j)(s-1-r_1)} \int_0^t \|u\|_{B^{r_1}_{p_1,\infty}} W(t')^{\frac{2}{q_1}} 2^{j'(1-s)} \|\Delta_{j'} w\|_2 \big(2^{j(1-s)}\|\Delta_j w\|_2\big)^{1-\frac{2}{q_1}}\,dt' \\
&\leqslant C\int_0^t \|u\|^{q_1}_{B^{r_1}_{p_1,\infty}} W(t')^2\,dt' + \delta \sup_{j\geqslant -1} 2^{2j(1-s)} \int_0^t \|\Delta_j w(t')\|_2^2\,dt',
\end{aligned}$$

and similarly for IV,

$$\mathrm{IV} \leqslant C\int_0^t \|v\|^{q_2}_{B^{r_2}_{p_2,\infty}} W(t')^2\,dt' + \delta \sup_{j\geqslant -1} 2^{2j(1-s)} \int_0^t \|\Delta_j w(t')\|_2^2\,dt',$$

and for III,

$$\mathrm{III} \leqslant \sum_{j'\leqslant j} 2^{(j'-j)(1+r_2+s)} \int_0^t \|v\|_{B^{r_2}_{p_2,\infty}} W(t')^{\frac{2}{q_2}} \left(2^{j'(1-s)}\|\Delta_{j'}w\|_2\right)^{1-\frac{2}{q_2}} 2^{j(1-s)}\|\Delta_j w\|_2\,dt'$$

$$\leqslant C\int_0^t \|v\|^{q_2}_{B^{r_2}_{p_2,\infty}} W(t')^2\,dt' + \delta \sup_{j\geqslant -1} 2^{2j(1-s)} \int_0^t \left\|\Delta_j w(t')\right\|_2^2\,dt'.$$

Collecting these estimates with (3.24) implies that

$$W(t)^2 \leqslant C\int_0^t \left(\left\|u(t')\right\|^{q_1}_{B^{r_1}_{p_1,\infty}} + \left\|v(t')\right\|^{q_2}_{B^{r_2}_{p_2,\infty}}\right) W(t')^2\,dt'.$$

This together with the Gronwall inequality shows that

$$W(t)=0, \quad \text{i.e. } u=v=0.$$

This completes the proof of case (a).

Case 2. u and v satisfy the assumption (b).

Since u and v are non-Lipschitz vectors, we will use the idea of the losing derivative estimate which was firstly introduced by Chemin and Lerner [7]. We can refer to [9] for a systematic study. Recently, Danchin and Paicu [10] applied this idea to prove the uniqueness of weak solution for the 2-D Boussinesq equations with partial viscosity. The present proof is motivated by [10]. We also refer to [3,22,24] for the other applications about the losing derivative estimate.

Let $s\in(0,1)$. For $\lambda>0$, we set

$$W_j^\lambda(t) = 2^{-js} e^{-\lambda\varepsilon_j(t)} \left\|\Delta_j w(t)\right\|_2,$$

where $\varepsilon_j(t)$ is defined by

$$\varepsilon_j(t) = \int_0^t 2^{j'} \sum_{j'\leqslant j+4} \left(\left\|\Delta_{j'}u(t')\right\|_\infty + \left\|\Delta_{j'}v(t')\right\|_\infty\right) dt'.$$

We get by (3.9) that

$$\frac{d}{dt}W_j^\lambda(t) + \lambda\varepsilon_j'(t)W_j^\lambda(t) + a_j 2^{2j} W_j^\lambda(t) \lesssim 2^{-js}e^{-\lambda\varepsilon_j(t)}\left(\left\|\Delta_j(w\cdot\nabla u)\right\|_2 + \left\|\Delta_j(v\cdot\nabla w) - v\cdot\nabla\Delta_j w + \sum_{j'>j}\partial_i\Delta_j w\Delta_{j'}v^i\right\|_2\right). \tag{3.25}$$

Here we used the fact that

$$\left\langle \partial_i\Delta_j w\Delta_{j'}v^i, \Delta_j w\right\rangle = -\left\langle \Delta_{j'}\partial_i v^i\Delta_j w, \Delta_j w\right\rangle = 0.$$

Since $W_j^\lambda(0)=0$, we get by integrating (3.25) on $[0,t]$ that

$$
\begin{aligned}
&W_j^\lambda(t)+\lambda\int_0^t \varepsilon_j'(t')W_j^\lambda(t')\,dt'+a_j2^{2j}\int_0^t W_j^\lambda(t')\,dt'\\
&\quad\lesssim 2^{-js}\int_0^t e^{-\lambda\varepsilon_j(t')}\big\|\Delta_j(w\cdot\nabla u)(t')\big\|_2\,dt'\\
&\qquad+2^{-js}\int_0^t e^{-\lambda\varepsilon_j(t')}\Big\|\Delta_j(v\cdot\nabla w)-v\cdot\nabla\Delta_j w+\sum_{j'>j}\partial_i\Delta_j w\Delta_{j'}v^i\Big\|_2(t')\,dt'.
\end{aligned}\tag{3.26}
$$

Step 1. Estimate of $\|\Delta_j(w\cdot\nabla u)\|_2$.

Using the Bony's decomposition (2.4), we write

$$
\Delta_j(w\cdot\nabla u)=\Delta_j\big(T_{w^i}\partial_i u\big)+\Delta_j\big(T_{\partial_i u}w^i\big)+\Delta_j R\big(w^i,\partial_i u\big).
$$

By (3.10) and Lemma 2.2, we get

$$
\begin{aligned}
\big\|\Delta_j\big(T_{w^i}\partial_i u\big)\big\|_2&\lesssim\sum_{|j'-j|\leqslant 4}2^{j'}\sum_{k\leqslant j'-2}\|\Delta_k w\|_2\|\Delta_{j'}u\|_\infty\lesssim\sum_{|j'-j|\leqslant 4}2^{j'}\sum_{k\leqslant j'-2}2^{ks}e^{\lambda\varepsilon_k(t)}W_k^\lambda(t)\|\Delta_{j'}u\|_\infty\\
&\lesssim\sum_{j'\leqslant j+2}2^{j's}e^{\lambda\varepsilon_{j'}(t)}W_{j'}^\lambda(t)\varepsilon_j'(t).
\end{aligned}\tag{3.27}
$$

By (3.12), (3.14) and Lemma 2.2, we have

$$
\begin{aligned}
\big\|\Delta_j\big(T_{\partial_i u}w^i\big)\big\|_2&\lesssim\sum_{|j'-j|\leqslant 4}\sum_{k\leqslant j'-2}2^k\|\Delta_k u\|_\infty\|\Delta_{j'}w\|_2\\
&\lesssim\sum_{|j'-j|\leqslant 4}2^{j's}e^{\lambda\varepsilon_{j'}(t)}W_{j'}^\lambda(t)\sum_{k\leqslant j'-2}2^k\|\Delta_k u\|_\infty\\
&\lesssim\sum_{|j'-j|\leqslant 4}2^{j's}e^{\lambda\varepsilon_{j'}(t)}W_{j'}^\lambda(t)\varepsilon_j'(t),
\end{aligned}\tag{3.28}
$$

and

$$
\begin{aligned}
\big\|\Delta_j R\big(w^i,\partial_i u\big)\big\|_2&\lesssim\sum_{j',j''\geqslant j-3;\,|j'-j''|\leqslant 1}2^j\|\Delta_{j'}w\|_2\|\Delta_{j''}u\|_\infty\\
&\lesssim\sum_{j',j''\geqslant j-3;\,|j'-j''|\leqslant 1}2^{j's+j}e^{\lambda\varepsilon_{j'}(t)}W_{j'}^\lambda(t)\|\Delta_{j''}u\|_\infty\\
&\lesssim\sum_{j'\geqslant j-3}2^{j'(s-1)+j}e^{\lambda\varepsilon_{j'}(t)}W_{j'}^\lambda(t)\varepsilon_{j'}'(t).
\end{aligned}\tag{3.29}
$$

Summing up (3.27)–(3.29), we obtain

$$
\begin{aligned}
2^{-js}\int_0^t e^{-\lambda\varepsilon_j(t')}\big\|\Delta_j(w\cdot\nabla u)(t')\big\|_2\,dt'&\lesssim\sum_{j'\leqslant j}2^{(j'-j)s}\int_0^t e^{\lambda(\varepsilon_{j'}(t')-\varepsilon_j(t'))}W_{j'}^\lambda(t')\varepsilon_j'(t')\,dt'\\
&\quad+\sum_{j'\geqslant j}2^{-(j'-j)(1-s)}\int_0^t e^{\lambda(\varepsilon_{j'}(t')-\varepsilon_j(t'))}W_{j'}^\lambda(t')\varepsilon_j'(t')\,dt'.
\end{aligned}\tag{3.30}
$$

Step 2. Estimate of $\|\Delta_j(v\cdot\nabla w)-v\cdot\nabla\Delta_j w+\sum_{j'>j}\partial_i\Delta_j w\Delta_{j'}v^i\|_2$.

Using the Bony's decomposition (2.4), we write

$$\Delta_j(v\cdot\nabla w)-v\cdot\nabla\Delta_j w=\big[\Delta_j,T_{v^i}\big]\partial_i w+\Delta_j\big(T_{\partial_i w}v^i\big)+\Delta_j R\big(v^i,\partial_i w\big)-T'_{\partial_i\Delta_j w}v^i.$$

Similar to the proof of (3.27) and (3.29), we get

$$\big\|\Delta_j\big(T_{\partial_i w}v^i\big)\big\|_2\lesssim\sum_{j'\leqslant j+2}2^{j's}e^{\lambda\varepsilon_{j'}(t)}W^\lambda_{j'}(t)\varepsilon'_j(t),\tag{3.31}$$

$$\big\|\Delta_j R\big(v^i,\partial_i w\big)\big\|_2\lesssim\sum_{j'\geqslant j-3}2^{j'(s-1)+j}e^{\lambda\varepsilon_{j'}(t)}W^\lambda_{j'}(t)\varepsilon'_{j'}(t).\tag{3.32}$$

Using the formula (3.20) again, we have

$$\big\|\big[\Delta_j,T_{v^i}\big]\partial_i w\big\|_2\lesssim\sum_{|j'-j|\leqslant 4}2^{j's}e^{\lambda\varepsilon_{j'}(t)}W^\lambda_{j'}(t)\varepsilon'_j(t).\tag{3.33}$$

Note that

$$T'_{\partial_i\Delta_j w}v^i-\sum_{j'>j}\partial_i\Delta_j w\Delta_{j'}v^i=\sum_{j-2\leqslant j'\leqslant j}S_{j'+2}\Delta_j\partial_i w\Delta_{j'}v^i,$$

it gives by Lemma 2.2 that

$$\bigg\|T'_{\partial_i\Delta_j w}v^i-\sum_{j'>j}\partial_i\Delta_j w\Delta_{j'}v^i\bigg\|_2\lesssim 2^{js}e^{\lambda\varepsilon_j(t)}W^\lambda_j(t)\varepsilon'_j(t).\tag{3.34}$$

Summing up (3.31)–(3.34), we obtain

$$\begin{aligned}&2^{-js}\int_0^t e^{-\lambda\varepsilon_j(t')}\big\|\Delta_j(v\cdot\nabla w)-v\cdot\nabla\Delta_j w)(t')\big\|_2\,dt'\\&\quad\lesssim\sum_{j'\leqslant j}2^{(j'-j)s}\int_0^t e^{\lambda(\varepsilon_{j'}(t')-\varepsilon_j(t'))}W^\lambda_{j'}(t')\varepsilon'_j(t')\,dt'\\&\qquad+\sum_{j'\geqslant j}2^{-(j'-j)(1-s)}\int_0^t e^{\lambda(\varepsilon_{j'}(t')-\varepsilon_j(t'))}W^\lambda_{j'}(t')\varepsilon'_{j'}(t')\,dt'.\end{aligned}\tag{3.35}$$

From (3.26), (3.30) and (3.35), it follows that

$$\begin{aligned}&W^\lambda_j(t)+\lambda\int_0^t\varepsilon'_j(t')W^\lambda_j(t')\,dt'+a_j2^{2j}\int_0^t W^\lambda_j(t')\,dt'\\&\quad\lesssim\sum_{j'\leqslant j}2^{(j'-j)s}\int_0^t e^{\lambda(\varepsilon_{j'}(t')-\varepsilon_j(t'))}W^\lambda_{j'}(t')\varepsilon'_j(t')\,dt'\\&\qquad+\sum_{j'\geqslant j}2^{-(j'-j)(1-s)}\int_0^t e^{\lambda(\varepsilon_{j'}(t')-\varepsilon_j(t'))}W^\lambda_{j'}(t')\varepsilon'_{j'}(t')\,dt'\\&\quad:=\mathrm{I}+\mathrm{II}.\end{aligned}\tag{3.36}$$

Write

$$\varepsilon'_j(t')=\varepsilon'_{j'}(t')+\big(\varepsilon'_j(t')-\varepsilon'_{j'}(t')\big),$$

and note that $\varepsilon'_j(t')-\varepsilon'_{j'}(t')\geqslant 0$ for $j\geqslant j'$, we obtain

$$I \lesssim \sum_{j' \leqslant j} 2^{(j'-j)s} \int_0^t W_{j'}^{\lambda}(t')\varepsilon_{j'}'(t')\,dt' + \frac{1}{\lambda} \sum_{j' \leqslant j} 2^{(j'-j)s} \sup_{t' \in [0,t]} W_{j'}^{\lambda}(t'), \tag{3.37}$$

here we used the inequality

$$\int_0^t e^{\lambda(\varepsilon_{j'}(t') - \varepsilon_j(t'))}\big(\varepsilon_j'(t') - \varepsilon_{j'}'(t')\big)\,dt' \lesssim \frac{1}{\lambda}, \quad \text{for } j' \leqslant j.$$

Since $\varepsilon_{j'}(t') - \varepsilon_j(t')$ is an increasing function in t' for $j' \geqslant j$, we have

$$\mathrm{II} \lesssim \sum_{j' \geqslant j} 2^{-(j'-j)(1-s)} e^{\lambda(\varepsilon_{j'}(t) - \varepsilon_j(t))} \int_0^t W_{j'}^{\lambda}(t')\varepsilon_{j'}'(t')\,dt'. \tag{3.38}$$

Let us for the moment assume that

$$\lambda\big(\|u\|_{L^1(0,t;B^1_{\infty,\infty})} + \|v\|_{L^1(0,t;B^1_{\infty,\infty})}\big) < (1-s)\log 2. \tag{3.39}$$

Notice that

$$\varepsilon_{j'}(t) - \varepsilon_j(t) \leqslant (j'-j)\big(\|u\|_{L^1(0,t;B^1_{\infty,\infty})} + \|v\|_{L^1(0,t;B^1_{\infty,\infty})}\big),$$

which together with (3.38) ensures that

$$\mathrm{II} \lesssim \int_0^t W_{j'}^{\lambda}(t')\varepsilon_{j'}'(t')\,dt'. \tag{3.40}$$

Summing up (3.36), (3.37) and (3.40), we obtain

$$\begin{aligned} &\sup_{j \geqslant -1, t' \in [0,t]} W_j^{\lambda}(t') + \lambda \sup_{j \geqslant -1} \int_0^t \varepsilon_j'(t') W_j^{\lambda}(t')\,dt' + \sup_{j \geqslant -1} 2^{2j} \int_0^t W_j^{\lambda}(t')\,dt' \\ &\quad \leqslant C \sup_{j \geqslant -1} \int_0^t \varepsilon_j'(t') W_j^{\lambda}(t')\,dt' + \frac{C}{\lambda} \sup_{j \geqslant -1, t' \in [0,t]} W_j^{\lambda}(t'), \end{aligned}$$

from which, we get by taking λ big enough that

$$\sup_{j \geqslant -1, t' \in [0,t]} W_j^{\lambda}(t') = 0.$$

On the other hand, the assumption (b) ensures that we can choose $t > 0$ small enough such that (3.39) holds. Thus, $u = v$ on $[0, t]$, and then we can conclude that $u = v$ on $[0, T]$ by a standard continuity argument. The proof of case (b) is completed.

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References

[1] H. Beirão da Veiga, A new regularity class for the Navier–Stokes equations in $\mathbb{R}^n$, Chinese Ann. Math. Ser. B 16 (1995) 407–412.
[2] J.-M. Bony, Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires, Ann. Sci. École Norm. Sup. 14 (1981) 209–246.
[3] M. Cannone, Q. Chen, C. Miao, A losing estimate for the ideal MHD equations with application to blow-up criterion, SIAM J. Math. Anal. 38 (2007) 1847–1859.
[4] Q. Chen, Z. Zhang, Space–time estimates in the Besov spaces and the Navier–Stokes equations, Methods Appl. Anal. 13 (2006) 107–122.
[5] J.-Y. Chemin, Perfect Incompressible Fluids, Oxford University Press, New York, 1998.
[6] J.-Y. Chemin, Théorèmes d'unicité pour le système de Navier–Stokes tridimensionnel, J. Anal. Math. 77 (1999) 27–50.
[7] J.-Y. Chemin, N. Lerner, Flot de champs de vecteurs non lipschitziens et équations de Navier–Stokes, J. Differential Equations 121 (1995) 314–328.
[8] A. Cheskidov, R. Shvydkoy, On the regularity of weak solutions of the 3D Navier–Stokes equations in $B^{-1}_{\infty,\infty}$, arXiv:0708.3067v2[math.AP].
[9] R. Danchin, Estimates in Besov spaces for transport and transport–diffusion equations with almost Lipschitz coefficients, Rev. Mat. Iberoamericana 21 (2005) 863–888.
[10] R. Danchin, M. Paicu, Le Théorème de Leray et le Théorème de Fujita–Kato pour le système de Boussinesq partiellement visqueux, Bulletin de la Societe Mathematique de France 136 (2008) 261–309.
[11] L. Escauriaza, G. Seregin, V. Šverák, $L_{3,\infty}$-solutions to the Navier–Stokes equations and backward uniqueness, Russian Math. Surveys 58 (2003) 211–250.
[12] I. Gallagher, F. Planchon, On global infinite energy solutions to the Navier–Stokes equations in two dimensions , Arch. Ration. Mech. Anal. 161 (2002) 307–337.
[13] P. Germain, Multipliers, paramultipliers, and weak–strong uniqueness for the Navier–Stokes equations, J. Differential Equations 226 (2006) 373–428.
[14] Y. Giga, Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier–Stokes system, J. Differential Equations 62 (1986) 186–212.
[15] H. Kozono, H. Sohr, Remark on uniqueness of weak solutions to the Navier–Stokes equations, Analysis 16 (1996) 255–271.
[16] H. Kozono, Y. Taniuchi, Bilinear estimates in BMO and the Navier–Stokes equations, Math. Z. 235 (2000) 173–194.
[17] H. Kozono, T. Ogawa, Y. Taniuchi, The critical Sobolev inequalities in Besov spaces and regularity criterion to some semi-linear evolution equations, Math. Z. 242 (2002) 251–278.
[18] H. Kozono, Y. Shimada, Bilinear estimates in homogeneous Triebel–Lizorkin spaces and the Navier–Stokes equations, Math. Nachr. 276 (2004) 63–74.
[19] P.G. Lemarié-Rieusset, Recent Developments in the Navier–Stokes Problem, Res. Notes Math., vol. 43, Chapman and Hall/CRC, 2002.
[20] P.G. Lemarié-Rieusset, Uniqueness for the Navier–Stokes problem: Remarks on a theorem of Jean-Yves Chemin, Nonlinearity 20 (2007) 1475–1490.
[21] J. Leray, Sur le mouvement d'un liquids visqeux emplissant l'espace, Acta Math. 63 (1934) 193–248.
[22] F.H. Lin, P. Zhang, Z. Zhang, On the global existence of smooth solution to the 2-D FENE dumbbell model , Comm. Math. Phys. 277 (2008) 531–553.
[23] A.J. Majda, A.L. Bertozzi, Vorticity and Incompressible Flow, Cambridge University Press, Cambridge, 2002.
[24] N. Masmoudi, P. Zhang, Z. Zhang, Global well-posedness for 2D polymeric fluid models and growth estimate, Physica D 237 (2008) 1663–1675.
[25] G. Prodi, Un teorema di unicità per le equazioni di Navier–Stokes, Ann. Mat. Pura Appl. 48 (1959) 173–182.
[26] F. Ribaud, A remark on the uniqueness problem for the weak solutions of Navier–Stokes equations, Ann. Fac. Sci. Toulouse Math. 11 (2002) 225–238.
[27] J. Serrin, The initial value problem for the Navier–Stokes equations, in: R.E. Langer (Ed.), Nonlinear Problems, University of Wisconsin Press, Madison, 1963, pp. 69–98.
[28] E.M. Stein, Harmonic analysis: Real-variable methods, orthogonality, and oscillatory integrals, Princeton University Press, Princeton, NJ, 1993.
[29] H. Triebel, Theory of Function Spaces, Monogr. Math., vol. 78, Birkhäuser-Verlag, Basel, 1983.