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# Exact boundary controllability for quasilinear wave equations in a planar tree-like network of strings

Qilong Gu [a,b], Tatsien Li [a,*]

[a] *School of Mathematical Sciences, Fudan University, Shanghai 200433, China*
[b] *Laboratoire de Mathématiques, CNRS UMR 6620, Université Blaise Pascal (Clermont-Ferrand 2), 63177 Aubière cedex, France*

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**Abstract**

In this paper the local exact boundary controllability for quasilinear wave equations on a planar tree-like network of strings is established and the number of boundary controls is equal to the number of simple nodes minus 1.


*MSC:* 35L05; 93B05

*Keywords:* Exact boundary controllability; Quasilinear wave equation; Planar tree-like network of strings

## 1. Introduction

There are many publications concerning the exact controllability for linear hyperbolic systems (see [14,15,17] and the references therein). In the semilinear case, some results on the exact boundary controllability for semilinear wave equations are obtained by Zuazua [19–21], Emanuilov [5] and Lasiecka and Triggiani [7], etc.

On the other hand, the exact boundary controllability for linear wave equations with Dirichlet boundary conditions on a planar tree-like network has been studied. The first result of this type was given in [18], in which the exact controllability for certain specific networks is obtained by means of boundary controls acting on all but one simple nodes. This result was later greatly extended in books [6] and [4]. Thus, for a planar tree-like network of linear strings with Dirichlet boundary conditions, if the network has $k$ simple nodes, then the number of controls is equal to $k - 1$.

Moreover, some related studies on the stabilization for linear wave equations with Dirichlet boundary conditions can be found in [1–4,16].

In recent years, based on the result on the semi-global classical solution (see [8]), the exact boundary controllability for general first order quasilinear hyperbolic systems has been established (see [9,10]), then this result has been applied to get the exact boundary controllability for 1-D quasilinear wave equations (see [11,12]).

* Corresponding author.
*E-mail addresses:* ql_gu@yahoo.com.cn (Q. Gu), dqli@fudan.edu.cn (T. Li).

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In this paper, by establishing the semi-global piecewise $C^2$ solution for quasilinear wave equations with various boundary conditions on a planar star-like network of strings, we get its local exact boundary controllability by means of a constructive method. Then, the exact boundary controllability previously obtained in [4,6,18] for linear wave equations with Dirichlet boundary conditions on a planar tree-like network of strings can be generalized to the quasilinear case with a totally different method.

This paper is organized as follows. The exact boundary controllability for a quasilinear wave equation on a single string will be presented in Section 2. Then, in Section 3, the existence and uniqueness of semi-global piecewise $C^2$ solution on a planar star-like network of strings with general boundary conditions will be established, and based on this, we get the local exact boundary controllability for quasilinear wave equations on a planar star-like network of strings. With a similar method, the local exact boundary controllability for quasilinear wave equations on a planar tree-like network of strings will be presented in Section 4.

## 2. Exact boundary controllability for quasilinear wave equations

For the purpose of this paper, in this section we recall the results about the exact boundary controllability for quasilinear wave equations on a single string given in [11,12]. Consider the following 1-D quasilinear wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(K\left(u, \frac{\partial u}{\partial x}\right)\right) = F\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right), \tag{1}$$

where $K = K(u, v)$ is a given $C^2$ function of $u$ and $v$, such that

$$K_v(u, v) > 0, \tag{2}$$

and $F = F(u, v, w)$ is a given $C^1$ function of $u$, $v$ and $w$, satisfying

$$F(0, 0, 0) = 0. \tag{3}$$

Moreover, without loss of generality, we may assume that

$$K(0, 0) = 0. \tag{4}$$

On one end $x = 0$, we give any one of the following physically meaningful boundary conditions:

$$u = h(t) \quad \text{(Dirichlet type)}, \tag{5a}$$

$$u_x = h(t) \quad \text{(Neumann type)}, \tag{5b}$$

$$u_x - \alpha u = h(t) \quad \text{(Third type)}, \tag{5c}$$

$$u_x - \beta u_t = h(t) \quad \text{(Dissipative type)}, \tag{5d}$$

where $\alpha$ and $\beta$ are given positive constants, $h(t)$ is a $C^2$ function (in case (5a)) or a $C^1$ function (in cases (5b)–(5d)).

Similarly, on another end $x = L$, the boundary condition is any one of the following conditions:

$$u = \bar{h}(t) \quad \text{(Dirichlet type)}, \tag{6a}$$

$$u_x = \bar{h}(t) \quad \text{(Neumann type)}, \tag{6b}$$

$$u_x + \bar{\alpha} u = \bar{h}(t) \quad \text{(Third type)}, \tag{6c}$$

$$u_x + \bar{\beta} u_t = \bar{h}(t) \quad \text{(Dissipative type)}, \tag{6d}$$

where $\bar{\alpha}$ and $\bar{\beta}$ are given positive constants, $\bar{h}(t)$ is a $C^2$ function (in case (6a)) or a $C^1$ function (in cases (6b)–(6d)).

By [11,12], we have

**Theorem 2.1.** *Let*

$$T > \frac{2L}{\sqrt{K_v(0,0)}}. \tag{7}$$

*Suppose that*

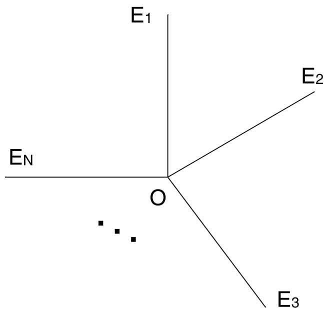


Fig. 1. A planar star-like network of strings.

$$\beta \neq \frac{1}{\sqrt{K_v(0,0)}}, \tag{8}$$

*where* $\beta$ *is given in* (5d). *For any given initial data* $(\varphi,\psi)$ *and final data* $(\Phi,\Psi)$ *with small norms* $\|(\varphi,\psi)\|_{C^2[0,L]\times C^1[0,L]}$ *and* $\|(\Phi,\Psi)\|_{C^2[0,L]\times C^1[0,L]}$, *and for any given function* $h(t)$ *with small norm* $\|h\|_{C^2[0,T]}$ (*in case* (5a)) *or* $\|h\|_{C^1[0,T]}$ (*in cases* (5b)–(5d)), *such that the conditions of* $C^2$ *compatibility are satisfied at the points* $(t,x)=(0,0)$ *and* $(T,0)$ *respectively, there exists a boundary control* $\bar{h}(t)$ *with small norm* $\|\bar{h}\|_{C^2[0,T]}$ (*in case* (6a)) *or* $\|\bar{h}\|_{C^1[0,T]}$ (*in cases* (6b)–(6d)), *such that the mixed initial–boundary value problem for Eq.* (1) *with the initial condition*

$$t=0:\quad u=\varphi(x),\quad u_t=\psi(x),\quad 0\leqslant x\leqslant L, \tag{9}$$

*one of the boundary conditions* (5) *on* $x=0$ *and one of the boundary conditions* (6) *on* $x=L$ *admits a unique* $C^2$ *solution* $u=u(t,x)$ *with small* $C^2$ *norm on the domain*

$$R(T)=\bigl\{(t,x)\bigm| 0\leqslant t\leqslant T,\ 0\leqslant x\leqslant L\bigr\}, \tag{10}$$

*which exactly satisfies the final condition*

$$t=T:\quad u=\Phi(x),\quad u_t=\Psi(x),\quad 0\leqslant x\leqslant L. \tag{11}$$

## 3. Exact boundary controllability for quasilinear wave equations on a planar star-like network of strings

In this section, we consider a planar star-like network which is composed of $N$ strings with a common joint point $O$. Take the joint point $O$ as $x=0$. Let $E_i$ and $L_i$ be another node and the length of the $i$-th string ($i=1,\dots,N$), respectively (see Fig. 1).

We consider the following quasilinear wave equation on the $i$-th string

$$\frac{\partial^2 u^i}{\partial t^2}-\frac{\partial}{\partial x}\left(K_i\left(u^i,\frac{\partial u^i}{\partial x}\right)\right)=F_i\left(u^i,\frac{\partial u^i}{\partial x},\frac{\partial u^i}{\partial t}\right)\quad (i=1,\dots,N), \tag{12}$$

where, for $i=1,\dots,N$, $K_i=K_i(u,v)$ is a given $C^2$ function of $u$ and $v$, such that

$$K_{iv}(u,v)>0, \tag{13}$$

and $F_i=F_i(u,v,w)$ is a given $C^1$ function of $u$, $v$ and $w$, satisfying

$$F_i(0,0,0)=0. \tag{14}$$

Moreover, without loss of generality, we assume that

$$K_i(0,0)=0. \tag{15}$$

The initial condition is given by

$$t=0:\quad u^i=\varphi_i(x),\quad u^i_t=\psi_i(x),\quad 0\leqslant x\leqslant L_i\ (i=1,\dots,N). \tag{16}$$

For $i = 1, \dots, N$, on the simple node $E_i$, we give any one of the following boundary conditions:

$$u^i = h_i(t) \quad \text{(Dirichlet type)}, \tag{17a}$$

$$u^i_x = h_i(t) \quad \text{(Neumann type)}, \tag{17b}$$

$$u^i_x + \alpha_i u^i = h_i(t) \quad \text{(Third type)}, \tag{17c}$$

$$u^i_x + \beta_i u^i_t = h_i(t) \quad \text{(Dissipative type)}, \tag{17d}$$

where $\alpha_i$ and $\beta_i$ are given positive constants, $h_i(t)$ is a $C^2$ function (in case (17a)) or a $C^1$ function (in cases (17b)–(17d)) and the conditions of $C^2$ compatibility are satisfied at $(0, L_i)$ $(i = 1, \dots, N)$. While, on the multiple node $O$, we have the interface conditions

$$\begin{cases} \displaystyle\sum_{i=1}^{N} K_i\big(u^i, u^i_x\big) = 0, \\ u^i = u^1 \quad (i = 2, \dots, N). \end{cases} \tag{18}$$

The first condition in (18) simply means that the total stress at $O$ is equal to zero, while, the second part of conditions in (18) shows the continuity of displacements at $O$.

For the purpose of getting the exact boundary controllability on the planar star-like network of strings, we need the existence and uniqueness of semi-global piecewise $C^2$ solution on it. In order to get it in a unified way, we first reduce each quasilinear wave equation to a first order quasilinear hyperbolic system.

For $i = 1, \dots, N$, setting

$$v^i = u^i_x, \qquad w^i = u^i_t, \tag{19}$$

Eq. (12) can be reduced to

$$\begin{cases} \dfrac{\partial u^i}{\partial t} = w^i, \\ \dfrac{\partial v^i}{\partial t} - \dfrac{\partial w^i}{\partial x} = 0, \\ \dfrac{\partial w^i}{\partial t} - K_{iv}\big(u^i, v^i\big)\dfrac{\partial v^i}{\partial x} \\ \quad = F_i\big(u^i, v^i, w^i\big) + K_{iu}\big(u^i, v^i\big)v^i \stackrel{\text{def}}{=} \tilde{F}_i\big(u^i, v^i, w^i\big) \end{cases} \qquad (i = 1, \dots, N), \tag{20}$$

where $\tilde{F}_i(u^i, v^i, w^i)$ is still a $C^1$ function of $u^i$, $v^i$ and $w^i$, satisfying

$$\tilde{F}_i(0, 0, 0) = 0. \tag{21}$$

For $i = 1, \dots, N$, noting (13), (20) is a strictly hyperbolic system with three distinct real eigenvalues $\lambda^i_j$ $(j = 1, 2, 3)$:

$$\lambda^i_1 = -\sqrt{K_{iv}\big(u^i, v^i\big)} < \lambda^i_2 = 0 < \lambda^i_3 = \sqrt{K_{iv}\big(u^i, v^i\big)}. \tag{22}$$

Thus, the characteristics for system (20) are given by

$$\frac{dx}{dt} = \lambda^i_j \quad (j = 1, 2, 3). \tag{23}$$

Moreover, the corresponding left eigenvectors can be taken as

$$l^i_1 = (0, \sqrt{K_{iv}}, 1), \qquad l^i_2 = (1, 0, 0), \qquad l^i_3 = (0, -\sqrt{K_{iv}}, 1). \tag{24}$$

Let

$$U^i = \big(u^i, v^i, w^i\big)^T \tag{25}$$

and

$$v_j^i = l_j^i(U^i)U^i \quad (j = 1, 2, 3), \tag{26}$$

namely,

$$v_1^i = \sqrt{K_{iv}(u^i, v^i)}v^i + w^i, \qquad v_2^i = u^i, \qquad v_3^i = -\sqrt{K_{iv}(u^i, v^i)}v^i + w^i. \tag{27}$$

We have

$$\begin{cases} v_1^i + v_3^i = 2w^i, \\ v_1^i - v_3^i = 2\sqrt{K_{iv}(u^i, v^i)}v^i. \end{cases} \tag{28}$$

With this reduction, the initial condition (16) now becomes

$$t = 0: \quad U^i = \left(\varphi_i(x), \varphi_i'(x), \psi_i(x)\right)^T, \quad 0 \leqslant x \leqslant L_i. \tag{29}$$

For $i = 1, \dots, N$, noting the condition of $C^0$ compatibility: $h_i(0) = \varphi_i(L_i)$, the boundary condition (17) on the $i$-th simple node will be correspondingly replaced by

$$w^i = h_i'(t), \tag{30a}$$

$$v^i = h_i(t), \tag{30b}$$

$$v^i + \alpha_i u^i = h_i(t), \tag{30c}$$

$$v^i + \beta_i w^i = h_i(t). \tag{30d}$$

It is easy to see that, at least in a neighborhood of $U = 0$, the boundary condition (30) can be rewritten as

$$v_1^i + v_3^i = 2h_i'(t), \tag{31a}$$

$$v_1^i - v_3^i = 2\sqrt{K_{iv}(v_2^i, h_i(t))}h_i(t), \tag{31b}$$

$$v_1^i - v_3^i = 2\sqrt{K_{iv}(v_2^i, h_i(t) - \alpha_i v_2^i)}\left(h_i(t) - \alpha_i v_2^i\right), \tag{31c}$$

$$v_1^i - v_3^i = \sqrt{K_{iv}\left(v_2^i, h_i(t) - \frac{1}{2}\beta_i\left(v_1^i + v_3^i\right)\right)}\left(2h_i(t) - \beta_i\left(v_1^i + v_3^i\right)\right). \tag{31d}$$

Then, it can be rewritten as

$$v_1^i = G_{i1}(t, v_2^i, v_3^i) + H_{i1}(t), \tag{32}$$

or, when

$$\beta_i \neq \frac{1}{\sqrt{K_{iv}(0,0)}}, \tag{33}$$

as

$$v_3^i = \bar{G}_{i3}(t, v_1^i, v_2^i) + \bar{H}_{i3}(t). \tag{34}$$

On the other hand, noting the conditions of $C^0$ compatibility at $O$, the interface condition (18) can be correspondingly replaced by

$$\begin{cases} \sum_{i=1}^{N} K_i(u^i, v^i) = 0, \\ w^i = w^1 \quad (i = 2, \dots, N), \end{cases} \tag{35}$$

and then it can be rewritten as

$$\begin{cases} P_1 \stackrel{\text{def}}{=} \sum_{i=1}^{N} K_i(u^i, v^i) = 0, \\ P_i \stackrel{\text{def}}{=} v_1^i + v_3^i - v_1^1 - v_3^1 = 0 \quad (i = 2, \dots, N). \end{cases} \tag{36}$$

Since, noting (13) and (27), in a neighborhood of $U = 0$ we have

$$\det\left|\frac{\partial(P_1,\dots,P_N)}{\partial(v_3^1,\dots,v_3^N)}\right| = \sum_{i=1}^N K_{iv}\frac{\partial v^i}{\partial v_3^i} < 0, \tag{37}$$

$$\det\left|\frac{\partial(P_1,\dots,P_N)}{\partial(v_1^1,\dots,v_1^N)}\right| = \sum_{i=1}^N K_{iv}\frac{\partial v^i}{\partial v_1^i} > 0, \tag{38}$$

the interface condition (18) on the multiple node $O$ can be rewritten as

$$v_3^i = G_{i3}\big(t, v_1^1,\dots,v_1^N, v_2^1,\dots,v_2^N\big) + H_{i3}(t) \quad (i = 1,\dots,N) \tag{39}$$

or

$$v_1^i = \bar{G}_{i1}\big(t, v_2^1,\dots,v_2^N, v_3^1,\dots,v_3^N\big) + \bar{H}_{i1}(t) \quad (i = 1,\dots,N). \tag{40}$$

Then, by means of the results on the existence and uniqueness of semi-global $C^1$ solution given in [8], it is easy to get the following lemmas.

**Lemma 3.1.** *Under the assumptions given at the beginning of this section, suppose furthermore that the conditions of piecewise $C^2$ compatibility or $C^2$ compatibility are satisfied at the points $(t,x) = (0,0)$ and $(0, L_i)$ $(i = 1,\dots,N)$, respectively. For any given $T_0 > 0$, the forward mixed initial–boundary value problem* (12) *and* (16)–(18) *admits a unique semi-global piecewise $C^2$ solution $u^i = u^i(t,x)$ $(i = 1,\dots,N)$ with small piecewise $C^2$ norm on the domain $R(T_0) = \bigcup_{i=1}^N R_i(T_0)$, where*

$$R_i(T_0) = \big\{(t,x) \,\big|\, 0 \leqslant t \leqslant T_0,\ 0 \leqslant x \leqslant L_i\big\}, \tag{41}$$

*provided that, for $i = 1,\dots,N$, the norms $\|(\varphi_i,\psi_i)\|_{C^2[0,L_i]\times C^1[0,L_i]}$ and $\|h_i\|_{C^2[0,T_0]}$ (for* (17a)*) or $\|h_i\|_{C^1[0,T_0]}$ (for* (17b)–(17d)*) are small enough.*

**Lemma 3.2.** *Under the assumptions given at the beginning of this section, and suppose that* (33) *holds for $i = 1,\dots,N$. For any given $T_0 > 0$, suppose furthermore that the conditions of piecewise $C^2$ compatibility or $C^2$ compatibility are satisfied at the points $(t,x) = (T_0, 0)$ and $(T_0, L_i)$ $(i = 1,\dots,N)$, respectively. Then the backward mixed initial–boundary value problem* (12), (17)–(18) *with the final condition*

$$t = T_0:\quad u^i = \Phi_i(x),\quad u_t^i = \Psi_i(x),\quad 0 \leqslant x \leqslant L_i\ (i = 1,\dots,N) \tag{42}$$

*admits a unique semi-global piecewise $C^2$ solution $u^i = u^i(t,x)$ $(i = 1,\dots,N)$ with small piecewise $C^2$ norm on the domain $R(T_0)$, provided that, for $i = 1,\dots,N$, the norms $\|(\Phi_i,\Psi_i)\|_{C^2[0,L_i]\times C^1[0,L_i]}$ and $\|h_i\|_{C^2[0,T_0]}$ (for* (17a)*) or $\|h_i\|_{C^1[0,T_0]}$ (for* (17b)–(17d)*) are small enough.*

Based on these two lemmas, we have

**Theorem 3.1.** *Let*

$$T > \frac{2L_1}{\sqrt{K_{1v}(0,0)}} + \max_{i=2,\dots,N}\frac{2L_i}{\sqrt{K_{iv}(0,0)}}. \tag{43}$$

*Suppose that*

$$\beta_1 \neq \frac{1}{\sqrt{K_{1v}(0,0)}}, \tag{44}$$

*where $\beta_1$ is given in* (17d) *for $i = 1$. For any given initial data $(\varphi_i,\psi_i)$ $(i = 1,\dots,N)$ and final data $(\Phi_i,\Psi_i)$ $(i = 1,\dots,N)$ with small norms $\sum_{i=1}^N \|(\varphi_i,\psi_i)\|_{C^2[0,L_i]\times C^1[0,L_i]}$ and $\sum_{i=1}^N \|(\Phi,\Psi)\|_{C^2[0,L_i]\times C^1[0,L_i]}$, and for any given function $h_1(t)$ with small norm $\|h_1\|_{C^2[0,T]}$ (in case* (17a)*) or $\|h_1\|_{C^1[0,T]}$ (in cases* (17b)–(17d)*), such that the conditions of $C^2$ compatibility or piecewise $C^2$ compatibility are satisfied at the points $(t,x) = (0, L_1), (T, L_1)$ and $(0,0), (T,0)$, respectively, there exist boundary controls $h_i(t)$ $(i = 2,\dots,N)$ with small norms $\|h_i\|_{C^2[0,T]}$*

*$(i = 2, \dots, N)$ (in case (17a)) or $\|h_i\|_{C^1[0,T]}$ $(i = 2, \dots, N)$ (in cases (17b)–(17d)), such that the mixed initial–boundary value problem for system (12) with the initial condition (16), the boundary condition (17) on $x = L_i$ $(i = 1, \dots, N)$ and the interface condition (18) on $x = 0$ admits a unique piecewise $C^2$ solution $u^i = u^i(t, x)$ $(i = 1, \dots, N)$ with small piecewise $C^2$ norm on the domain $R(T) = \bigcup_{i=1}^N R_i(T)$, where*

$$R_i(T) = \{(t, x) \mid 0 \leqslant t \leqslant T,\ 0 \leqslant x \leqslant L_i\}, \tag{45}$$

*which exactly satisfies the final condition*

$$t = T:\quad u^i = \Phi_i(x), \quad u^i_t = \Psi_i(x), \quad 0 \leqslant x \leqslant L_i\ (i = 1, \dots, N). \tag{46}$$

In order to get Theorem 3.1, it suffices to prove the following lemma.

**Lemma 3.3.** *Under the assumptions of Theorem 3.1, system (12) admits a piecewise $C^2$ solution $u^i(t, x)$ $(i = 1, \dots, N)$ with small norm $\sum_{i=1}^N \|u^i\|_{C^2[R_i(T)]}$ on the domain $R(T) = \bigcup_{i=1}^N R_i(T)$, which satisfies simultaneously the boundary condition (17) for $i = 1$ on $x = L_1$, the interface condition (18) on $x = 0$, the initial condition (16) and the finial condition (46).*

**Proof.** Noting (43), there exists an $\epsilon_0 > 0$ so small that

$$T > \max_{|(u^1,v^1)| \leqslant \epsilon_0} \frac{2L_1}{\sqrt{K_{1v}(u^1, v^1)}} + \max_{i=2,\dots,N} \max_{|(u^i,v^i)| \leqslant \epsilon_0} \frac{2L_i}{\sqrt{K_{iv}(u^i, v^i)}}. \tag{47}$$

Let

$$T_1 = \max_{|(u^1,v^1)| \leqslant \epsilon_0} \frac{L_1}{\sqrt{K_{1v}(u^1, v^1)}} + \max_{i=2,\dots,N} \max_{|(u^i,v^i)| \leqslant \epsilon_0} \frac{L_i}{\sqrt{K_{iv}(u^i, v^i)}} \tag{48}$$

and

$$T_2 = \max_{i=2,\dots,N} \max_{|(u^i,v^i)| \leqslant \epsilon_0} \frac{L_i}{\sqrt{K_{iv}(u^i, v^i)}}. \tag{49}$$

(i) We first consider the following forward mixed initial–boundary value problem for system (12) with the initial condition (16), the interface condition (18), the boundary condition (17) for $i = 1$ on $x = L_1$, and the following artificial boundary conditions

$$x = L_i:\quad u^i = f_i(t) \quad (i = 2, \dots, N), \tag{50}$$

where $f_i$ $(i = 2, \dots, N)$ are any given $C^2$ functions of $t$ with small $C^2[0, T_1]$ norm and the conditions of $C^2$ compatibility at the point $(t, x) = (0, L_i)$ $(i = 2, \dots, N)$ are assumed to be satisfied, respectively. Then, by Lemma 3.1, there exists a unique semi-global piecewise $C^2$ solution $u = u_I(t, x) = (u^1_I(t, x), \dots, u^N_I(t, x))$ on the domain $R^I = \bigcup_{i=1}^N R^I_i$, where

$$R^I_i = \{(t, x) \mid 0 \leqslant t \leqslant T_1,\ 0 \leqslant x \leqslant L_i\} \quad (i = 1, \dots, N), \tag{51}$$

which has a small piecewise $C^2$ norm, in particular,

$$\left|\left(u_I, \frac{\partial u_I}{\partial x}\right)\right| \leqslant \epsilon_0, \quad \forall (t, x) \in R^I. \tag{52}$$

Thus, we can determine the corresponding value of $(u^1_I, u^1_{Ix})$ at $x = L_1$ as

$$x = L_1:\quad (u^1_I, u^1_{Ix}) = (a_1(t), a_2(t)), \quad 0 \leqslant t \leqslant T_1, \tag{53}$$

the $C^2[0, T_1]$ norm of $a_1(t)$ and the $C^1[0, T_1]$ norm of $a_2(t)$ are small and $(a_1(t), a_2(t))$ satisfies the boundary condition (17) for $i = 1$ at $x = L_1$ on the interval $0 \leqslant t \leqslant T_1$. Similarly, we can also determine the values of $(u_I, u_{Ix})$ at $x = 0$ as

$$x = 0:\quad (u^i_I, u^i_{Ix}) = (b_{i1}(t), b_{i2}(t)), \quad 0 \leqslant t \leqslant T_1\ (i = 1, \dots, N), \tag{54}$$

the $C^2[0,T_1]$ norm of $b_{i1}(t)$ $(i=1,\dots,N)$ and the $C^1[0,T_1]$ norm of $b_{i2}(t)$ $(i=1,\dots,N)$ are small and $(b_{i1}(t),b_{i2}(t))$ $(i=1,\dots,N)$ satisfy the interface condition (18) at $x=0$ on the interval $0\leqslant t\leqslant T_1$.

(ii) We next consider the following backward mixed initial–boundary value problem for system (12) with the final condition (46), the interface condition (18), the boundary condition (17) for $i=1$, and the following artificial boundary conditions

$$x=L_i:\quad u^i=g_i(t)\quad (i=2,\dots,N), \tag{55}$$

where $g_i$ $(i=2,\dots,N)$ are any given $C^2$ functions of $t$ with small $C^2[T-T_1,T]$ norms and the conditions of $C^2$ compatibility at the point $(t,x)=(T,L_i)$ $(i=2,\dots,N)$ are assumed to be satisfied, respectively. Then, by Lemma 3.2, there exists a unique semi-global piecewise $C^2$ solution $u=u_{II}(t,x)=(u^1_{II}(t,x),\dots,u^N_{II}(t,x))$ on the domain $R^{II}=\bigcup_{i=1}^N R_i^{II}$, where

$$R_i^{II}=\{(t,x)\mid T-T_1\leqslant t\leqslant T,\ 0\leqslant x\leqslant L_i\}\quad (i=1,\dots,N), \tag{56}$$

which has a small piecewise $C^2$ norm, in particular,

$$\left|\left(u_{II},\frac{\partial u_{II}}{\partial x}\right)\right|\leqslant \epsilon_0,\quad \forall (t,x)\in R^{II}. \tag{57}$$

Thus, we can determine the corresponding value of $(u^1_{II},u^1_{IIx})$ at $x=L_1$ as

$$x=L_1:\quad (u^1_{II},u^1_{IIx})=(\bar a_1(t),\bar a_2(t)),\quad T-T_1\leqslant t\leqslant T, \tag{58}$$

the $C^2[T-T_1,T]$ norm of $\bar a_1(t)$ and the $C^1[T-T_1,T]$ norm of $\bar a_2(t)$ are small and $(\bar a_1(t),\bar a_2(t))$ satisfies the boundary condition (17) for $i=1$ at $x=L_1$ on the interval $T-T_1\leqslant t\leqslant T$. Similarly, we can determine the values of $(u_{II},u_{IIx})$ at $x=0$ as

$$x=0:\quad (u^i_{II},u^i_{IIx})=(\bar b_{i1}(t),\bar b_{i2}(t)),\quad T-T_1\leqslant t\leqslant T\ (i=1,\dots,N), \tag{59}$$

the $C^2[T-T_1,T]$ norm of $\bar b_{i1}(t)$ $(i=1,\dots,N)$ and the $C^1[T-T_1,T]$ norm of $\bar b_{i2}(t)$ $(i=1,\dots,N)$ are small and $(\bar b_{i1}(t),\bar b_{i2}(t))$ $(i=1,\dots,N)$ satisfy the interface condition (18) at $x=0$ on the interval $T-T_1\leqslant t\leqslant T$.

(iii) We now construct $\tilde a_1(t)\in C^2[0,T]$ with small $C^2$ norm and $\tilde a_2(t)\in C^1[0,T]$ with small $C^1$ norm, such that

$$(\tilde a_1(t),\tilde a_2(t))=\begin{cases}(a_1(t),a_2(t)), & 0\leqslant t\leqslant T_1,\\ (\bar a_1(t),\bar a_2(t)), & T-T_1\leqslant t\leqslant T\end{cases} \tag{60}$$

and $(\tilde a_1(t),\tilde a_2(t))$ satisfies the boundary condition (17) for $i=1$ at $x=L_1$ on the whole interval $0\leqslant t\leqslant T$.

Noting (13), we now change the status of $t$ and $x$ and consider the following leftward mixed initial–boundary value problem for system (12) for $i=1$ with the initial condition

$$x=L_1:\quad u^1=\tilde a_1(t),\quad u^1_x=\tilde a_2(t),\quad 0\leqslant t\leqslant T \tag{61}$$

and the boundary conditions

$$t=0:\quad u^1=\varphi_1(x),\quad 0\leqslant x\leqslant L_1, \tag{62}$$

$$t=T:\quad u^1=\Phi_1(x),\quad 0\leqslant x\leqslant L_1, \tag{63}$$

where $\varphi^1(x)$ and $\Phi^1(x)$ are given by (16) and (46) respectively.

Obviously, the conditions of $C^2$ compatibility at the points $(t,x)=(0,L_1)$ and $(T,L_1)$ are satisfied respectively. Then, by Lemma 3.1, there exists a unique semi-global $C^2$ solution $u^1=u^1(t,x)$ with small $C^2$ norm on the domain $R_1(T)=\{(t,x)\mid 0\leqslant t\leqslant T,\ 0\leqslant x\leqslant L_1\}$ and

$$\left|\left(u^1,\frac{\partial u^1}{\partial x}\right)\right|\leqslant \epsilon_0,\quad \forall (t,x)\in R_1(T). \tag{64}$$

Since both $u^1(t,x)$ and $u^1_I(t,x)$ satisfy system (12) for $i=1$, the initial condition (61) on $0\leqslant t\leqslant T_1$ and the boundary condition (62), it is easy to see that

$$u^1(t,x)\equiv u^1_I(t,x) \tag{65}$$

on the domain

$$\left\{(t,x)\;\Big|\;0\leqslant t\leqslant T_2+\frac{(T_1-T_2)x}{L_1},\ 0\leqslant x\leqslant L_1\right\}. \tag{66}$$

Thus, in particular, we get

$$t=0:\quad u^1=\varphi_1(x),\quad u^1_t=\psi_1(x),\quad 0\leqslant x\leqslant L_1 \tag{67}$$

and

$$x=0:\quad u^1=b_{11}(t),\quad u^1_x=b_{12}(t),\quad 0\leqslant t\leqslant T_2, \tag{68}$$

where $\varphi_1(x)$ and $\psi_1(x)$ are given by (16), $b_{11}(t)$ and $b_{12}(t)$ are given by (54).

In a similar way we get

$$t=T:\quad u^1=\Phi_1(x),\quad u^1_t=\Psi_1(x),\quad 0\leqslant x\leqslant L_1 \tag{69}$$

and

$$x=0:\quad u^1=\bar b_{11}(t),\quad u^1_x=\bar b_{12}(t),\quad T-T_2\leqslant t\leqslant T, \tag{70}$$

where $\Phi_1(x)$ and $\Psi_1(x)$ are given by (46), $\bar b_{11}(t)$ and $\bar b_{12}(t)$ are given by (59).

(iv) Let $(\tilde b_{11}(t),\tilde b_{12}(t))$ be the value of $(u^1,u^1_x)$ on $x=0$. The $C^2[0,T]$ norm of $\tilde b_{11}(t)$ and the $C^1[0,T]$ norm of $\tilde b_{12}(t)$ are small and

$$\big(\tilde b_{11}(t),\tilde b_{12}(t)\big)=\begin{cases}(b_{11}(t),b_{12}(t)), & 0\leqslant t\leqslant T_2,\\ (\bar b_{11}(t),\bar b_{12}(t)), & T-T_2\leqslant t\leqslant T.\end{cases} \tag{71}$$

We now construct $\tilde b_{i2}(t)\in C^1[0,T]$ $(i=2,\dots,N-1)$ with small $C^1$ norm, such that

$$\tilde b_{i2}(t)=\begin{cases}b_{i2}(t), & 0\leqslant t\leqslant T_2,\\ \bar b_{i2}(t), & T-T_2\leqslant t\leqslant T,\end{cases}\qquad (i=2,\dots,N-1), \tag{72}$$

where $b_{i2}(t)$ and $\bar b_{i2}(t)$ $(i=2,\dots,N-1)$ are given by (54) and (59) respectively. Noting (13), the interface condition (18) together with $u^1=\tilde b_{11}(t)$ and $u^i_x=\tilde b_{i2}(t)$ $(i=1,\dots,N-1)$ can uniquely determine the value of $u^i$ $(i=2,\dots,N)$ and $u^N_x$ at $x=0$ on the interval $[0,T]$. Let $\tilde b_{i1}(t)=u^i$ $(i=2,\dots,N)$ and $\tilde b_{N2}(t)=u^N_x$ at $x=0$ on the interval $[0,T]$. It is easy to see that $\tilde b_{i1}(t)$ $(i=2,\dots,N)$ have small $C^2[0,T]$ norms, $\tilde b_{N2}(t)$ has a small $C^1[0,T]$ norm and

$$\big(\tilde b_{i1}(t),\tilde b_{i2}(t)\big)=\begin{cases}(b_{i1}(t),b_{i2}(t)), & 0\leqslant t\leqslant T_2,\\ (\bar b_{i1}(t),\bar b_{i2}(t)), & T-T_2\leqslant t\leqslant T,\end{cases}\qquad (i=2,\dots,N), \tag{73}$$

where $(b_{i1}(t),b_{i2}(t))$ and $(\bar b_{i1}(t),\bar b_{i2}(t))$ are given by (54) and (59) respectively. Obviously, $(\tilde b_{i1}(t),\tilde b_{i2}(t))$ $(i=1,\dots,N)$ satisfy the interface condition (18).

(v) Finally, for $i=2,\dots,N$, we solve the following rightward mixed initial–boundary value problem on the domain $R_i(T)=\{(t,x)\mid 0\leqslant t\leqslant T,\ 0\leqslant x\leqslant L_i\}$ for system (12) with the initial condition

$$x=0:\quad u^i=\tilde b_{i1}(t),\quad u^i_x=\tilde b_{i2}(t),\quad 0\leqslant t\leqslant T \tag{74}$$

and the boundary conditions

$$t=0:\quad u^i=\varphi_i(x),\quad 0\leqslant x\leqslant L_i, \tag{75}$$

$$t=T:\quad u^i=\Phi_i(x),\quad 0\leqslant x\leqslant L_i, \tag{76}$$

where $\varphi_i(x)$ and $\Phi_i(x)$ are given by (16) and (46) respectively.

For each $i=2,\dots,N$, the conditions of $C^2$ compatibility at the points $(t,x)=(0,0)$ and $(T,0)$ are satisfied respectively and there exists a unique semi-global $C^2$ solution $u^i=u^i(t,x)$ with small $C^2$ norm on each $R_i(T)$. In particular, we have

$$\left|\left(u^i,\frac{\partial u^i}{\partial x}\right)\right|\leqslant\epsilon_0,\quad \forall (t,x)\in R_i(T)\ (i=2,\dots,N). \tag{77}$$

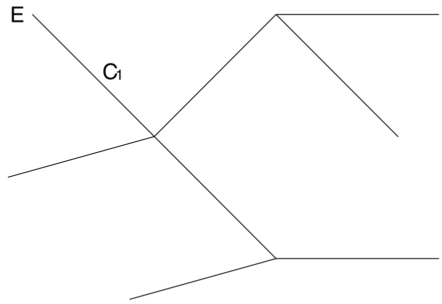


Fig. 2. A planar tree-like network of strings.

Since for each $i = 2, \dots, N$, both $u^i(t,x)$ and $u^i_I(t,x)$ satisfy system (12), the initial condition (74) for $0 \leqslant t \leqslant T_2$ and the boundary condition (75), by the uniqueness of $C^2$ solution to this one-sided mixed initial–boundary value problem (see [13]) it is easy to see that

$$u^i(t,x) \equiv u^i_I(t,x) \tag{78}$$

on the domain

$$\left\{(t,x) \,\middle|\, 0 \leqslant t \leqslant T_2\left(1 - \frac{x}{L_i}\right),\ 0 \leqslant x \leqslant L_i\right\}. \tag{79}$$

Then, in particular, we get

$$t = 0: \quad u^i = \varphi_i(x), \quad u^i_t = \psi_i(x), \quad 0 \leqslant x \leqslant L_i. \tag{80}$$

In a similar way we get

$$t = T: \quad u^i = \Phi_i(x), \quad u^i_t = \Psi_i(x), \quad 0 \leqslant x \leqslant L_i. \tag{81}$$

Thus, $(u^1(t,x), \dots, u^N(t,x))$ is a solution required by Lemma 3.3. □

**Remark 3.1.** From the proof of Lemma 3.3, the boundary controls which realize the exact boundary controllability are not unique.

## 4. Exact boundary controllability for quasilinear wave equations on a planar tree-like network of strings

Using a method similar to that in Section 3, in this section we consider the local exact boundary controllability for quasilinear wave equations in a planar tree-like network composed of $N$ strings: $C_1, \dots, C_N$. Without loss of generality, we suppose that one end of string $C_1$ is a simple node in the network. We take this simple node as the starting node $E$ (see Fig. 2).

For the $i$-th string, let $d_{i0}$ and $d_{i1}$ be the $x$-coordinates of its two ends and $L_i = d_{i1} - d_{i0}$ its length. For simplicity, in what follows we simply say node $d_{i0}$ (resp. $d_{i1}$) instead of the node corresponding to $d_{i0}$ (resp. $d_{i1}$). We always suppose that node $d_{i0}$ is closer to $E$ than node $d_{i1}$ in the network (node $d_{10}$ is just $E$).

For $i = 1, \dots, N$, we consider the following quasilinear wave equations on the string $C_i$

$$\frac{\partial^2 u^i}{\partial t^2} - \frac{\partial}{\partial x}\left(K_i\left(u^i, \frac{\partial u^i}{\partial x}\right)\right) = F_i\left(u^i, \frac{\partial u^i}{\partial x}, \frac{\partial u^i}{\partial t}\right), \quad d_{i0} \leqslant x \leqslant d_{i1}\ (i = 1, \dots, N), \tag{82}$$

where $K_i = K_i(u, v)$ is a given $C^2$ function of $u$ and $v$, such that

$$K_{iv}(u, v) > 0, \tag{83}$$

and $F_i = F_i(u, v, w)$ is a given $C^1$ function of $u$, $v$ and $w$, satisfying

$$F_i(0, 0, 0) = 0. \tag{84}$$

Moreover, without loss of generality, we assume that

$$K_i(0,0)=0. \tag{85}$$

The initial condition for system (82) is given by

$$t=0:\quad u^i=\varphi_i(x),\quad u^i_t=\psi_i(x),\quad d_{i0}\leqslant x\leqslant d_{i1}\ (i=1,\dots,N). \tag{86}$$

Let $\mathcal{M}$ and $\mathcal{S}$ be two subsets of $\{1,\dots,N\}$, such that $i\in\mathcal{M}$ if and only if $d_{i1}$ is a multiple node, while, $i\in\mathcal{S}$ if and only if $d_{i1}$ is a simple node.

At any simple node $d_{10}$ or $d_{i1}$ $(i\in\mathcal{S})$, the boundary condition is given as any one of (17), while at any multiple node $d_{i1}$ $(i\in\mathcal{M})$, we have the interface condition

$$\begin{cases}\displaystyle\sum_{j\in\mathcal{J}_i}K_j\big(u^j,u^j_x\big)=K_i\big(u^i,u^i_x\big),\\ u^j=u^i,\quad \forall j\in\mathcal{J}_i,\end{cases} \tag{87}$$

where $\mathcal{J}_i$ denotes the set of all the indices $j$ such that node $d_{j0}$ is just node $d_{i1}$.

Similar to Theorem 3.1, we have

**Theorem 4.1.** *Let*

$$T>2\max_{i\in\mathcal{S}}\sum_{j\in\mathcal{D}_i}\frac{L_j}{\sqrt{K_{jv}(0,0)}}, \tag{88}$$

*where $\mathcal{D}_i$ stands for the set of indices corresponding to all the canals in the unique string-like subnetwork connecting nodes $d_{10}$ and $d_{i1}$. Suppose that*

$$\beta_1\neq\frac{1}{\sqrt{K_{1v}(0,0)}}, \tag{89}$$

*where $\beta_1$ is given in* (17d) *for $i=1$. For any given initial data $(\varphi_i,\psi_i)$ $(i=1,\dots,N)$ and final data $(\Phi_i,\Psi_i)$ $(i=1,\dots,N)$ with small norms $\sum_{i=1}^N\|(\varphi_i,\psi_i)\|_{C^2[0,L]\times C^1[0,L]}$ and $\sum_{i=1}^N\|(\Phi,\Psi)\|_{C^2[0,L]\times C^1[0,L]}$, and for any given function $h_1(t)$ with small norm $\|h_1\|_{C^2[0,T]}$ (in case* (17a)) *or $\|h_1\|_{C^1[0,T]}$ (in cases* (17b)–(17d)), *such that the conditions of $C^2$ compatibility or piecewise $C^2$ compatibility are satisfied at the points $(t,x)=(0,L_1),(T,L_1)$ and $(0,d_{i0}),(T,d_{i0})$ $(i\in\mathcal{M})$, respectively, there exist boundary controls $h_i(t)$ $(i\in\mathcal{S})$ with small norms $\|h_i\|_{C^2[0,T]}$ $(i\in\mathcal{S})$ (in case* (17a)) *or $\|h_i\|_{C^1[0,T]}$ $(i\in\mathcal{S})$ (in cases* (17b)–(17d)), *such that on the domain $R(T)=\bigcup_{i=1}^N R_i(T)$, where $R_i(T)$ is given by* (45), *the mixed initial–boundary value problem for system* (82) *with the initial condition* (86), *the boundary condition* (17) *on all simple nodes $d_{10}$ and $d_{i1}$ $(i\in\mathcal{S})$ and the interface condition* (87) *on all multiple nodes $d_{i1}$ $(i\in\mathcal{M})$ admits a unique piecewise $C^2$ solution $u^i=u^i(t,x)$ $(i=1,\dots,N)$ with small piecewise $C^2$ norm, which exactly satisfies the final condition*

$$t=T:\quad u^i=\Phi_i(x),\quad u^i_t=\Psi_i(x),\quad d_{i0}\leqslant x\leqslant d_{i1}\ (i=1,\dots,N). \tag{90}$$

**Proof.** This theorem can be proved in a completely similar way as in the proof of Theorem 3.1. Indeed, after having solved a forward problem and a backward problem on this tree-like network as in step (i) and step (ii) of the proof of Lemma 3.3, we can solve a rightward problem as in step (iii) and get $u^1$ on canal $C_1$. Then, as in step (iv), we can determine $u^j$ $(j\in\mathcal{J}_1)$ at node $d_{11}$ (in a non-unique way!) by $u^1$ and the interface condition (87) at $d_{11}$. Consider $d_{j0}$ $(j\in\mathcal{J}_1)$ as a new starting node and do step (iii) and step (iv) again. Noting (88), it is easy to see that we can continue this procedure until we get the solution $u^i$ $(i=1,\dots,N)$ on the whole network. This finishes the proof. □

**Remark 4.1.** In conclusion, for a tree-like network with $k$ simple nodes, we need only $k-1$ boundary controls. The controls are given on all the simple nodes except the starting one, and each simple node has one control on it.

**Remark 4.2.** If the boundary conditions (17b)–(17d) on the simple node $d_{10}$ or $d_{i1}$ $(i\in\mathcal{S})$ are replaced respectively by

$$K_i\left(u^i, u^i_x\right) = h_i(t), \tag{91}$$

$$K_i\left(u^i, u^i_x\right) + \alpha_i u^i = h_i(t) \tag{92}$$

and

$$K_i\left(u^i, u^i_x\right) + \beta_i u^i_t = h_i(t), \tag{93}$$

the conclusion of Theorem 4.1 is still valid, provided that (89) is replaced by

$$\beta_1 \neq \sqrt{K_{1v}(0,0)}. \tag{94}$$

**Remark 4.3.** For linear wave equations with Dirichlet boundary conditions on a planar tree-like network, as shown in [4,6,18], if we want to reduce the number of controlled simple nodes, then the problem on the exact boundary controllability becomes much more complicated and it depends very sensitively on both the topology of the network and the diophantine properties of the lengths of the strings involved. What should be the corresponding situation in the quasilinear case is still an open problem.

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