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The lifespan of spherically symmetric solutions of the compressible Euler equations outside an impermeable sphere

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Abstract

We consider smooth three-dimensional spherically symmetric Eulerian flows of ideal polytropic gases outside an impermeable sphere, with initial data equal to the sum of a constant flow with zero velocity and a smooth perturbation with compact support. Under a natural assumption on the form of the perturbation, we obtain precise information on the asymptotic behavior of the lifespan as the size of the perturbation tends to 0. When there is no sphere, so that the flow is defined in all space, corresponding results have been obtained in [P. Godin, The lifespan of a class of smooth spherically symmetric solutions of the compressible Euler equations with variable entropy in three space dimensions, Arch. Ration. Mech. Anal. 177 (2005) 479–511].

Résumé

Nous considérons des écoulements eulériens tridimensionnels lisses à symétrie sphérique de gaz parfaits polytropiques à l'extérieur d'une sphère imperméable, avec des données initiales somme d'un écoulement constant de vitesse nulle et d'une perturbation lisse à support compact. Sous une hypothèse naturelle sur la forme de la perturbation, nous obtenons une information précise sur le comportement asymptotique de la durée de vie quand la taille de la perturbation tend vers 0. S'il n'y a pas de sphère, de sorte que l'écoulement est défini dans tout l'espace, des résultats correspondants ont été obtenus dans [P. Godin, The lifespan of a class of smooth spherically symmetric solutions of the compressible Euler equations with variable entropy in three space dimensions, Arch. Ration. Mech. Anal. 177 (2005) 479–511].

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1. Introduction

For Eulerian compressible flows in all space with initial data suitably close to a constant flow with zero velocity, precise estimates for the lifespan have been obtained in the 2D axisymmetric isentropic case [1,12], and in the 3D spherically symmetric case for ideal polytropic gases with variable entropy [3]. The purpose of the present paper is to

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obtain results analogous with those of [3] outside an impermeable sphere. To achieve this, we shall adapt the methods and proofs of [3,4] to the needs of the present paper.

As in the boundaryless case [3], we shall introduce a suitable approximate flow, satisfying here the impermeability boundary condition, and obtained by combining a suitable isentropic flow with a suitable steady flow with constant pressure. Because of the boundary condition, some extra care will be needed to ensure that the approximate flow satisfies suitable compatibility conditions. Estimates for the approximate flow will follow from the results of [4]. To study the error (that is, the difference between the actual flow and the approximate flow), we shall adapt arguments of [3,4]. We shall rely heavily (as in [3,4]) on Sobolev type estimates with decay from [9].

Our paper is organized as follows. In Section 2, we state our results precisely and give some indication on notations to be used in this paper. In Section 3, we treat short time existence and introduce our approximate solution, which is studied in Section 4 with the help of results of [4]. An asymptotic lower bound for the lifespan is obtained in Sections 5 and 6. In Section 7 it is shown that this asymptotic lower bound actually is an asymptotic upper bound.

2. Statement of the results

Set $\mathcal{D}_R = \{x \in \mathbb{R}^3,\ |x| > R\}$, write $\partial_j = \partial/\partial x_j$, $\nabla = (\partial_1, \partial_2, \partial_3)$, and denote by $\cdot$ the Euclidean scalar product in $\mathbb{R}^3$. We consider the compressible Euler equations

$$\partial_t \rho + u \cdot \nabla \rho + \rho \nabla \cdot u = 0 \quad \text{if } 0 < t < T \text{ and } x \in \mathcal{D}_R, \tag{2.1}$$

$$\partial_t u + u \cdot \nabla u + \frac{1}{\rho} \nabla P = 0 \quad \text{if } 0 < t < T \text{ and } x \in \mathcal{D}_R, \tag{2.2}$$

$$\partial_t S + u \cdot \nabla S = 0 \quad \text{if } 0 < t < T \text{ and } x \in \mathcal{D}_R, \tag{2.3}$$

where t is the time variable, ρ the density, u the velocity, P the pressure, S the entropy. If $u^{(1)}$, $u^{(2)}$, $u^{(3)}$ are the components of u, $\nabla \cdot u$ of course means $\sum_{1 \leqslant j \leqslant 3} \partial_j u^{(j)}$, $u \cdot \nabla$ is $\sum_{1 \leqslant j \leqslant 3} u^{(j)} \partial_j$ and $u \cdot \nabla u = (u \cdot \nabla) u$. Throughout this paper we shall deal with ideal polytropic gases, namely we shall assume that $P = P(\rho, S) = K_1 \rho^\gamma e^{K_2 S}$, where K_1, γ, K_2 are strictly positive constants with $\gamma > 1$. We shall also set $r = |x|$. Throughout this paper, a function of (t, x) will be called radial if it depends only on (t, r). We shall say that (ρ, u, S) is spherically symmetric if $u = Ux/r$ with U real-valued and ρ, U, S are radial functions. Fix $M > 0$, $\bar{\rho} > 0$, $\bar{S} > 0$, and let $\varepsilon > 0$ be a small parameter, always assumed to belong to $(0, \varepsilon_0]$ for some small $\varepsilon_0 > 0$ throughout this paper. We shall consider the impermeability boundary condition

$$u \cdot x = 0 \quad \text{if } 0 < t < T \text{ and } r = R \tag{2.4}$$

(which of course will take a much simpler form for the spherically symmetric solutions to be considered in the present paper) and the family of spherically symmetric initial conditions

$$\rho(0, x) = \bar{\rho} + \varepsilon \rho_0(x, \varepsilon) \quad \text{if } x \in \mathcal{D}_R, \tag{2.5}$$

$$u(0, x) = \varepsilon u_0(x, \varepsilon) \quad \text{if } x \in \mathcal{D}_R, \tag{2.6}$$

$$S(0, x) = \bar{S} + \varepsilon S_0(x, \varepsilon) \quad \text{if } x \in \mathcal{D}_R, \tag{2.7}$$

where $\rho(0, x) > 0$, $\rho_0(x, \varepsilon) = \rho^0(r) + \varepsilon \rho^1(r, \varepsilon)$, $u_0(x, \varepsilon) = (u^0(r) + \varepsilon u^1(r, \varepsilon))x/r$, $S_0(x, \varepsilon) = S^0(r) + \varepsilon S^1(r, \varepsilon)$ for some functions ρ^j, u^j, S^j (fixed throughout this paper) which are $C^\infty([R, +\infty))$ functions of r and vanish when $r \geqslant R + M$, with ρ^0, u^0, S^0 independent of ε. We shall assume that the initial data (2.5)–(2.7) satisfy the usual compatibility conditions of all orders when $t = 0$, $r = R$, for the boundary value problem (2.1)–(2.4). Recall that this means that if (ρ, u, S) is a smooth solution of (2.1)–(2.3) for some $T > 0$ such that (2.5)–(2.7) hold, then $\partial_t^k u \cdot x = 0$ if $k \in \mathbb{N}$, $t = 0$ and $r = R$. We shall also assume that $|\partial_x^\alpha \rho^1| + |\partial_x^\alpha u^1| + |\partial_x^\alpha S^1| \leqslant \hat{C}_\alpha$ if $\alpha \in \mathbb{N}^3$, with $\hat{C}_\alpha$ independent of ε (and of $x \in \mathcal{D}_R$) and fixed throughout the paper. Set $I_\varepsilon = \{T > 0$, (2.1)–(2.7) has a unique $C^\infty([0, T) \times \overline{\mathcal{D}_R})$ solution (ρ, u, S) (hence $\rho > 0$)$\}$. In Section 3, the following theorem will be proved easily with the help of the results of [5].

Theorem 2.1. *If ε is small, then $I_\varepsilon \neq \emptyset$. Moreover, if $T \in I_\varepsilon$, then (ρ, u, S) is spherically symmetric if $0 \leqslant t < T$.*

Set $T_\varepsilon = \sup I_\varepsilon$. The purpose of this paper is to obtain precise information on the behavior of T_ε as $\varepsilon \overset{>}{\to} 0$. To describe our results, we need to introduce certain quantities. Set $\bar{c} = (\frac{\partial P}{\partial \rho}(\bar{\rho}, \bar{S}))^{1/2}$, $\overline{c_\rho} = \frac{\partial}{\partial \rho}(\frac{\partial P}{\partial \rho})^{1/2}(\bar{\rho}, \bar{S})$. Notice that $\bar{\rho}\overline{c_\rho} + \bar{c} > 0$. Define for $r \geqslant R$: $f_0(r) = -\int_r^{R+M} u^0(y)\,dy$, $f_1(r) = -\bar{c}(\frac{\rho^0}{\bar{\rho}} + K_2\frac{S^0}{\gamma})(r)$, and, for $q \in \mathbb{R}$: $F_0(q) = \frac{1}{2}(R + |q-R|) f_0(R + |q-R|) + \frac{1}{2}\int_{R+|q-R|}^{R+M} y f_1(y)\,dy + e^{q/R-2}\int_R^{R+(R-q)_+} e^{y/R} y(f_1(y) - \frac{1}{R} f_0(y))\,dy$; throughout this paper, $\int_a^b$ means $-\int_b^a$ if $b < a$, and $a_+ = \max(a, 0)$ if $a \in \mathbb{R}$. It follows from results of [4], recalled in Section 4 below, that $F_0 \in C^\infty(\mathbb{R})$, F_0 and all its derivatives are bounded, and $\max_{q\in\mathbb{R}}(-F_0''(q)) \geqslant 0$. Set $\tau^* = \bar{c}^2(\bar{c} + \bar{\rho}\overline{c_\rho})^{-1}(\max_{q\in\mathbb{R}}(-F_0''(q)))^{-1}$ if $\max_{q\in\mathbb{R}}(-F_0''(q)) > 0$, $\tau^* = +\infty$ otherwise. So $\tau^* \in (0, +\infty]$ and it follows from results of [4] recalled in Section 4 below that $\tau^* < +\infty$ if and only if $|u^0| + |\frac{\rho^0}{\bar{\rho}} + K_2\frac{S^0}{\gamma}| \not\equiv 0$. The purpose of the present paper is to prove the following long time existence result, whose boundaryless analogue has been obtained in [3].

Theorem 2.2. $\lim_{\varepsilon \overset{>}{\to} 0}(\varepsilon \ln T_\varepsilon) = \tau^*$.

As announced in the introduction, Theorem 2.2 will be proved by adapting the method used in [3]. We shall construct an approximate solution of (2.1)–(2.7), from which we shall obtain the following result, which is the first half of Theorem 2.2.

Theorem 2.3. $\liminf_{\varepsilon \overset{>}{\to} 0}(\varepsilon \ln T_\varepsilon) \geqslant \tau^*$.

Then we shall show how to modify the arguments of [3] to prove the next result:

Theorem 2.4. $\limsup_{\varepsilon \overset{>}{\to} 0}(\varepsilon \ln T_\varepsilon) \leqslant \tau^*$.

Theorem 2.2 follows at once from Theorem 2.3 and Theorem 2.4.

In the following sections we shall introduce a number of useful notations which will be used throughout this paper. Functions θ, w, z are defined at the beginning of Section 3 and corresponding functions Θ, W, Z just before (3.9). Functions θ_1, w_1, z_1, vector fields Γ_j, X, and some norms are introduced after the proof of Lemma 3.1, just before the statement of Theorem 3.1. Functions θ_2, w_2, z_2 are defined just before (3.20) and norms $E_m^{1/2}(t)$ just before Theorem 3.2. Norms $E_{m,l}^{1/2}(t)$ and (semi-)norms $Q_{m,l}(t)$ are introduced at the beginning of Section 5. W_2 appears in the proof of Proposition 5.1(6), Θ_j, W_1, Z_j are defined just before Lemma 5.3 and norms $\tilde{E}_{m,l}^{1/2}(t)$ just before Proposition 5.3. In Section 6, useful notations are grouped just before (6.30).

3. Proof of Theorem 2.1. The approximate solution

In this section we shall show that Theorem 2.1 follows easily from the results of [5], and we shall describe our approximate solution.

As in the boundaryless case [3], it is convenient to introduce new dependent variables θ, w, z defined by $\theta(t,x) = \frac{2}{\gamma-1}((P(t/\bar{c},x)/\bar{P})^{(\gamma-1)/2\gamma} - 1)$, $w(t,x) = u(t/\bar{c},x)/\bar{c}$, $z(t,x) = e^{K_2(S-\bar{S})(t/\bar{c},x)/\gamma} - 1$, where $\bar{P} = K_1\bar{\rho}^\gamma e^{K_2\bar{S}}$. Set $C_1 = (\gamma-1)/2$. Then, from (2.1)–(2.7), we obtain with a new T:

$$\partial_t\theta + w\cdot\nabla\theta + (1 + C_1\theta)\nabla\cdot w = 0 \quad \text{if } 0 < t < T \text{ and } x \in \mathcal{D}_R, \tag{3.1}$$

$$\partial_t w + w\cdot\nabla w + (1 + C_1\theta)(1+z)\nabla\theta = 0 \quad \text{if } 0 < t < T \text{ and } x \in \mathcal{D}_R, \tag{3.2}$$

$$\partial_t z + w\cdot\nabla z = 0 \quad \text{if } 0 < t < T \text{ and } x \in \mathcal{D}_R. \tag{3.3}$$

$$w\cdot x = 0 \quad \text{if } 0 < t < T \text{ and } |x| = R, \tag{3.4}$$

$$\theta(0,x) = \varepsilon\theta_0(x,\varepsilon) \quad \text{if } x \in \mathcal{D}_R, \tag{3.5}$$

$$w(0,x) = \varepsilon w_0(x,\varepsilon) \quad \text{if } x \in \mathcal{D}_R, \tag{3.6}$$

$$z(0,x) = \varepsilon z_0(x,\varepsilon) \quad \text{if } x \in \mathcal{D}_R, \tag{3.7}$$

where $\theta_0(x,\varepsilon) = ((P(0,x)/\bar{P})^{(\gamma-1)/2\gamma} - 1)/C_1\varepsilon$, $w_0 = u_0/\bar{c}$, $z_0(x,\varepsilon) = (e^{\varepsilon K_2 S_0(x,\varepsilon)/\gamma} - 1)/\varepsilon$. Notice for later use that $\theta_0(x,\varepsilon) = \theta^0(r) + \varepsilon\theta^1(r,\varepsilon)$, $w_0(x,\varepsilon) = (w^0(r) + \varepsilon w^1(r,\varepsilon))x/r$, $z_0(x,\varepsilon) = z^0(r) + \varepsilon z^1(r,\varepsilon)$, where θ^j, w^j, z^j are $C^\infty([R,+\infty))$ functions of r vanishing when $r \geqslant R+M$ with $|\partial_x^\alpha \theta^1| + |\partial_x^\alpha w^1| + |\partial_x^\alpha z^1| \leqslant \tilde{C}_\alpha$ for all $x \in \mathcal{D}_R$ and ε small. Now the operator on the left-hand side of (3.1)–(3.3) is symmetrizable hyperbolic (cf. [3]). In fact let $w^{(1)}, w^{(2)}, w^{(3)}$ be the components of w in the canonical basis of $\mathbb{R}^3$ and consider the 5×1 matrix $\phi = tr(\theta\ (w^{(i)})_{1\leqslant i\leqslant 3}\ z)$, where tr means transpose. Define the 5×5 matrices $A_j(\phi)$, $0 \leqslant j \leqslant 3$, with elements $A_j^{k,l}(\phi)$, $1 \leqslant k,l \leqslant 5$, in the following way: $A_0^{1,1}(\phi) = A_0^{5,5}(\phi) = 1$, $A_0^{k,k}(\phi) = 1/(1+z)$ if $2 \leqslant k \leqslant 4$, $A_0^{k,l}(\phi) = 0$ otherwise; if $1 \leqslant j \leqslant 3$, $A_j^{1,1}(\phi) = A_j^{5,5}(\phi) = w^{(j)}$, $A_j^{k,k}(\phi) = w^{(j)}/(1+z)$ if $2 \leqslant k \leqslant 4$, $A_j^{1,j+1}(\phi) = A_j^{j+1,1}(\phi) = 1 + C_1\theta$, $A_j^{k,l}(\phi) = 0$ otherwise. (3.1)–(3.3) read

$$A_0(\phi)\partial_t\phi + \sum_{1\leqslant j\leqslant 3} A_j(\phi)\partial_j\phi = 0 \quad \text{if } 0 < t < T \text{ and } x \in \mathcal{D}_R; \tag{3.8}$$

each $A_j(\phi)$, $0 \leqslant j \leqslant 3$, is (real) symmetric and $A_0(\phi) > 0$.

The formulation (3.8), (3.4)–(3.7) will be suitable to obtain L^2 estimates, but in our case (θ, w, z) will be spherically symmetric, which means that $\theta(t,x) = \Theta(t,r)$, $w(t,x) = W(t,r)x/r$, $z(t,x) = Z(t,r)$, with Θ, W, Z real-valued. So (3.1)–(3.7) will yield

$$\partial_t\Theta + W\partial_r\Theta + (1 + C_1\Theta)\left(\partial_r W + \frac{2}{r}W\right) = 0 \quad \text{if } 0 < t < T \text{ and } r > R, \tag{3.9}$$

$$\partial_t W + W\partial_r W + (1 + C_1\Theta)(1+Z)\partial_r\Theta = 0 \quad \text{if } 0 < t < T \text{ and } r > R, \tag{3.10}$$

$$\partial_t Z + W\partial_r Z = 0 \quad \text{if } 0 < t < T \text{ and } r > R, \tag{3.11}$$

$$W(t,R) = 0 \quad \text{if } 0 < t < T, \tag{3.12}$$

$$\Theta(0,r) = \varepsilon\Theta_0(r,\varepsilon) \quad \text{if } r > R, \tag{3.13}$$

$$W(0,r) = \varepsilon W_0(r,\varepsilon) \quad \text{if } r > R, \tag{3.14}$$

$$Z(0,r) = \varepsilon Z_0(r,\varepsilon) \quad \text{if } r > R, \tag{3.15}$$

where $\Theta_0(r,\varepsilon) = \theta_0(x,\varepsilon)$, $W_0(r,\varepsilon)x/r = w_0(x,\varepsilon)$, $Z_0(r,\varepsilon) = z_0(x,\varepsilon)$.

It is now easy to prove Theorem 2.1.

Proof of Theorem 2.1. Assume that ε is small enough. (3.9)–(3.15) is a mixed problem with characteristic boundary of constant multiplicity for the symmetrizable hyperbolic system on the left-hand side of (3.9)–(3.11). It follows with the help of Theorems 2 and 4 of [5] and a standard uniqueness argument that (3.9)–(3.15) has a unique C^∞ solution which gives rise to the only solution of (3.8), (3.4)–(3.7), so that $I_\varepsilon \neq \emptyset$ if ε is small enough. (For short time existence results, see also [10,11] and the references given in [10,5,11].) □

We now turn to the construction of the approximate solution of (3.1)–(3.7). As a first step, we want to approximate θ, w, replacing z by 0 (i.e. taking an isentropic flow). However the initial data $(\varepsilon\theta_0, \varepsilon w_0, 0)$ need not satisfy the compatibility conditions for (3.1)–(3.4) when $t = 0$ and $r = R$ when $(\varepsilon\theta_0, \varepsilon w_0, \varepsilon z_0)$ do. So we first have to modify θ_0, w_0 and this is the purpose of the next lemma.

Lemma 3.1. *If ε is small, one can find $\tilde{\theta}_0(x,\varepsilon) = \tilde{\Theta}_0(r,\varepsilon)$, $\tilde{w}_0(x,\varepsilon) = \tilde{W}_0(r,\varepsilon)x/r$ (where $\tilde{\Theta}_0$ and $\tilde{W}_0$ are real-valued and C^∞ with respect to $r \in [R,+\infty)$) which satisfy the following conditions*:

(1) $|\partial_x^\alpha\tilde{\theta}_0| + |\partial_x^\alpha\tilde{w}_0| \leqslant C_\alpha$ *if* $\alpha \in \mathbb{N}^3$ (*with C_α independent of ε*),
(2) $\tilde{\theta}_0 = 0$ *and* $\tilde{w}_0 = 0$ *if* $r \geqslant R+M$,
(3) $(\varepsilon\theta_0 + \varepsilon^2\tilde{\theta}_0, \varepsilon w_0 + \varepsilon^2\tilde{w}_0, 0)$ *are infinitely compatible initial data for* (3.1)–(3.4).

Proof. Write $a_1 = 1 + C_1\Theta$, $a_2 = (1 + C_1\Theta)(1+Z)$, $\Phi = (\Theta, W, Z)$. From (3.9)–(3.11) it follows by induction that

$$\partial_t^j\Phi = P_j\left(\frac{1}{r}, \left(\partial_r^k\Phi\right)_{0\leqslant k\leqslant j}\right) \quad \text{if } j \geqslant 1, \tag{3.16}$$

where P_j is an $\mathbb{R}^3$-valued polynomial function with $P_j(1/r, 0) = 0$. Write $\mathcal{L} = \partial_r(\partial_r + 2/r)$. By induction, we obtain with the help of (3.10), (3.16) that

$$\partial_t^j W = -a_2 \partial_t^{j-1} \partial_r \Theta - W \partial_t^{j-1} \partial_r W + P_{j1}\left(\frac{1}{r}, \left(\partial_r^k \Phi\right)_{0 \leqslant k \leqslant j-1}\right) \quad \text{if } j \geqslant 1, \tag{3.17}$$

and with the help of (3.9), (3.16) that

$$\partial_t^{j-1} \partial_r \Theta = -a_1 \partial_t^{j-2} \mathcal{L} W - W \partial_t^{j-2} \partial_r^2 \Theta + P_{j2}\left(\frac{1}{r}, \left(\partial_r^k \Phi\right)_{0 \leqslant k \leqslant j-1}\right) \quad \text{if } j \geqslant 2, \tag{3.18}$$

where P_{jl}, $l = 1, 2$, are polynomial functions and $\Lambda \mapsto P_{jl}(1/r, \Lambda)$ vanish of order 2 at 0. Taking (3.18) into account in (3.17) and using also (3.16), we easily find that

$$\partial_t^{2p} W = (a_1 a_2)^p \mathcal{L}^p W + \mathcal{M}_{2p}\left(\frac{1}{r}, \left(\partial_r^k \Phi\right)_{0 \leqslant k \leqslant 2p}\right) \quad \text{if } p \geqslant 1,$$
$$\partial_t^{2p+1} W = -(a_1 a_2)^p a_2 \mathcal{L}^p \partial_r \Theta + \mathcal{M}_{2p+1}\left(\frac{1}{r}, \left(\partial_r^k \Phi\right)_{0 \leqslant k \leqslant 2p+1}\right) \quad \text{if } p \geqslant 0,$$

where $\mathcal{M}_j(\frac{1}{r}, (\partial_r^k \Phi)_{0 \leqslant k \leqslant j}) = W \tilde{P}_{j1}(\frac{1}{r}, (\partial_r^k \Phi)_{0 \leqslant k \leqslant j}) + \tilde{P}_{j2}(\frac{1}{r}, (\partial_r^k \Phi)_{0 \leqslant k \leqslant j-1})$ in which $\tilde{P}_{jn}$ are polynomial functions and $\Lambda \mapsto \tilde{P}_{jn}(1/r, \Lambda)$ vanish of order n at 0. We easily conclude that the compatibility conditions when $t = 0$ and $r = R$ (for the mixed problem (3.9)–(3.15)) read

$$W = 0, \qquad \partial_r \Theta = 0, \qquad \mathcal{L}^p W = -(a_1 a_2)^{-p} \hat{P}_{2p}\left(\left(\partial_r^k \Phi\right)_{0 \leqslant k \leqslant 2p-1}\right) \quad \text{if } p \geqslant 1,$$
$$\mathcal{L}^p \partial_r \Theta = -(a_1 a_2)^{-p} a_2^{-1} \hat{P}_{2p+1}\left(\left(\partial_r^k \Phi\right)_{0 \leqslant k \leqslant 2p}\right) \quad \text{if } p \geqslant 0,$$

where $\hat{P}_j$ are polynomial functions which vanish of order 2 at 0. If we set $\Phi^k = (\theta^k, w^k, z^k)$, $k = 0, 1$, this can be rewritten as

$$\begin{aligned}
&\mathcal{L}^p w^0(R) = \mathcal{L}^p \partial_r \theta^0(R) = 0 \quad \text{if } p \geqslant 0, \\
&w^1(R, \varepsilon) = \partial_r \theta^1(R, \varepsilon) = 0, \\
&\mathcal{L}^p w^1(R, \varepsilon) = -\varepsilon^{-2}(a_1 a_2)^{-p}(0, R) \hat{P}_{2p}\left(\left(\varepsilon \partial_r^k \Phi^0(R) + \varepsilon^2 \partial_r^k \Phi^1(R, \varepsilon)\right)_{0 \leqslant k \leqslant 2p-1}\right) \quad \text{if } p \geqslant 1, \\
&\mathcal{L}^p \partial_r \theta^1(R, \varepsilon) = -\varepsilon^{-2}(a_1 a_2)^{-p}(0, R) a_2^{-1}(0, R) \hat{P}_{2p+1}\left(\left(\varepsilon \partial_r^k \Phi^0(R) + \varepsilon^2 \partial_r^k \Phi^1(R, \varepsilon)\right)_{0 \leqslant k \leqslant 2p}\right) \quad \text{if } p \geqslant 1.
\end{aligned} \tag{3.19}$$

Notice that if (3.19) holds and z^0, z^1 are replaced by new ones, we may keep the same θ^0, w^0 and find new θ^1, w^1 vanishing for $r \geqslant R + M$ and with each r-derivative uniformly bounded in r and ε, such that (3.19) still holds; indeed, the proof of the classical Borel theorem in Theorem 1.2.6 of [6] can readily be adapted to give such new θ^1, w^1. Lemma 3.1 follows easily. □

By the arguments of the proof of Theorem 2.1, there exists T such that, for ε small, (3.1)–(3.4) has a unique $C^\infty([0, T] \times \overline{\mathcal{D}_R})$ solution $(\theta_1, w_1, 0)$ with initial data $(\varepsilon\theta_0 + \varepsilon^2\tilde{\theta}_0, \varepsilon w_0 + \varepsilon^2 \tilde{w}_0, 0)$ when $t = 0$. Actually the situation now is even simpler than in Theorem 2.1: it suffices to consider (3.1), (3.2) with $z \equiv 0$, (3.4), with initial data $(\varepsilon\theta_0 + \varepsilon^2\tilde{\theta}_0, \varepsilon w_0 + \varepsilon^2 \tilde{w}_0, 0)$; in the formulation (3.9), (3.10) with $Z \equiv 0$, (3.12), the sphere $r = R$ is now noncharacteristic as long as $1 + C_1\Theta$ does not vanish. In fact it is convenient to reduce to a potential equation, which we shall do a little later for obtaining long time estimates.

Now set $z_1(t, x) = \varepsilon z_0(x, \varepsilon)$. Notice that $(0, 0, z_1)$ is also a solution of (3.1)–(3.4) (which corresponds to a solution of (2.1)–(2.4) with zero velocity and constant pressure). (θ_1, w_1, z_1) will be our approximate solution.

In order to describe important estimates, let us introduce some notations. Define $(\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4) = (\partial_t, \partial_1, \partial_2, \partial_3, X)$, where $X = t\partial_t + \sum_{1 \leqslant j \leqslant 3} x_j \partial_j$. If $a = (a_0, a_1, a_2, a_3, a_4) \in \mathbb{N}^5$, set $\Gamma^a = \partial_t^{a_0} \partial_1^{a_1} \partial_2^{a_2} \partial_3^{a_3} X^{a_4}$. When convenient, we shall write $f' = \partial f = (\partial_t f, \partial_1 f, \partial_2 f, \partial_3 f)$, ∂_0 instead of ∂_t, and $\partial_x^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3}$ if $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$. For an $\mathbb{R}^N$-valued function $f(t, x)$, we shall denote by $f(t)$ the function $x \mapsto f(t, x)$ and set $|f(t)| = \sup_{x \in \mathcal{D}_R} |f(t, x)|$, $\|f(t)\| = (\int_{\mathcal{D}_R} |f(t, x)|^2 \, dx)^{1/2}$, where $|\ |$ is the usual Euclidean norm on $\mathbb{R}^N$. For functions of x only, $\|\ \|$ will be the standard $L^2(\mathcal{D}_R)$ norm. Identifying $p \times q$ real matrices with elements of $\mathbb{R}^{pq}$, we shall also use the notations $|f(t,x)|$ (or $|f(t,r)|$ if f is a function of (t,r)), $|f|$, $|f(t)|$ when f is valued in the set of $p \times q$ real matrices. Finally we shall set $\langle y \rangle = 1 + |y|$ if $y \in \mathbb{R}$.

With τ^* as in Section 2, we have the following result, which will be proved in the next section.

Theorem 3.1. *Fix $a \in \mathbb{N}^5$ and $\tau \in (0, \tau^*)$. One can find $\varepsilon_0 > 0$, $C > 0$, and if $\bar{R} > R$, also $\bar{C} > 0$ such that the following holds: if $0 < \varepsilon \leqslant \varepsilon_0$ and $\varepsilon \ln T \leqslant \tau$, then θ_1, w_1 exist, belong to $C^\infty([0,T] \times \overline{\mathcal{D}_R})$, vanish when $r \geqslant t + R + M$, and satisfy the following estimates for all $t \in [0,T]$:*

(1) $\|\Gamma^a \theta_1(t)\| + \|\Gamma^a w_1(t)\| \leqslant C\varepsilon$,
(2) $|\Gamma^a \theta_1(t)| + |\Gamma^a w_1(t)| \leqslant C\varepsilon \langle t \rangle^{-1}$,
(3) $\langle t \rangle |\nabla \Gamma^a \theta_1(t,x)| + |\partial_t \Gamma^a \theta_1(t,x)| + |\Gamma^a w_1(t,x)| \leqslant \bar{C}\varepsilon \langle t \rangle^{-2-a_0}$ *if* $R \leqslant r \leqslant \bar{R}$.

Suppose now that $\tau \in (0, \tau^*)$ is fixed and let (θ, w, z) be a $C^\infty([0,T] \times \overline{\mathcal{D}_R})$ solution of (3.1)–(3.7), where $\varepsilon \ln T \leqslant \tau$. Assume that ε is small, and let (θ_1, w_1, z_1) be as above. Set $\theta_2 = \theta - \theta_1$, $w_2 = w - w_1$, $z_2 = z - z_1$. Then of course

$$\partial_t \theta_2 + \nabla \cdot w_2 = -(w \cdot \nabla \theta - w_1 \cdot \nabla \theta_1) - C_1(\theta \nabla \cdot w - \theta_1 \nabla \cdot w_1) \quad \text{if } 0 < t < T \text{ and } x \in \mathcal{D}_R, \tag{3.20}$$

$$\partial_t w_2 + \nabla \theta_2 = -(w \cdot \nabla w - w_1 \cdot \nabla w_1) - C_1(\theta \nabla \theta - \theta_1 \nabla \theta_1) - (1 + C_1\theta) z \nabla \theta \quad \text{if } 0 < t < T \text{ and } x \in \mathcal{D}_R, \tag{3.21}$$

$$\partial_t z_2 = -(w_1 + w_2) \cdot \nabla(z_1 + z_2) \quad \text{if } 0 < t < T \text{ and } x \in \mathcal{D}_R, \tag{3.22}$$

$$w_2 \cdot x = 0 \quad \text{if } 0 < t < T \text{ and } r = R, \tag{3.23}$$

$$\theta_2(0,x) = -\varepsilon^2 \tilde{\theta}_0(x, \varepsilon) \quad \text{if } x \in \mathcal{D}_R, \tag{3.24}$$

$$w_2(0,x) = -\varepsilon^2 \tilde{w}_0(x, \varepsilon) \quad \text{if } x \in \mathcal{D}_R, \tag{3.25}$$

$$z_2(0,x) = 0 \quad \text{if } x \in \mathcal{D}_R, \tag{3.26}$$

with $\tilde{\theta}_0$, $\tilde{w}_0$ as in Lemma 3.1. Now, if $x \in \mathcal{D}_R$, it is easily checked that $|\partial_t^k \partial_x^\alpha (\theta, w, z)(0,x)| \leqslant C_{k\alpha}\varepsilon$ and $|\partial_t^k \partial_x^\alpha (\theta_1, w_1, z_1)(0,x)| \leqslant C_{k\alpha}\varepsilon$, so by (3.24)–(3.26) and (3.20)–(3.22) we obtain that

$$\left|\partial_t^k \partial_x^\alpha (\theta_2, w_2, z_2)(0,x)\right| \leqslant C_{k\alpha}\varepsilon^2. \tag{3.27}$$

We shall use a fixed function $\psi : (0,1) \to (0, +\infty)$ such that $\varepsilon\psi(\varepsilon)$ is bounded and $\psi(\varepsilon) \to +\infty$ as $\varepsilon \overset{>}{\to} 0$. (Replacing ψ by another function ψ_1 of the same type such that $(\psi_1/\psi)(\varepsilon) \to 0$ as $\varepsilon \overset{>}{\to} 0$ will improve estimates.) Set $E_m(t) = \sum_{|a| \leqslant m} (\|\Gamma^a \theta_2(t)\|^2 + \|\Gamma^a w_2(t)\|^2 + \|\Gamma^a z_2(t)\|^2)$. With the help of Theorem 3.1 we shall prove in Sections 5 and 6 the following result, which measures the quality of the approximation of (θ, w, z) by (θ_1, w_1, z_1).

Theorem 3.2. *Fix $m \in \mathbb{N}$. If $\tau \in (0, \tau^*)$, one can find $C > 0$, $\varepsilon_0 > 0$ such that the following holds: if $0 < \varepsilon \leqslant \varepsilon_0$, $\varepsilon \ln T \leqslant \tau$ and (θ_2, w_2, z_2) is a $C^\infty([0,T] \times \overline{\mathcal{D}_R})$ solution of* (3.20)–(3.26)*, then $E_m^{1/2}(t) \leqslant C\varepsilon^2\psi(\varepsilon)$ if $t \in [0,T]$.*

As a system with unknown *tr*$(\theta_2\ w_2\ z_2)$, (3.20)–(3.22) is symmetrizable hyperbolic (cf. (3.8)). Using Theorem 4 of [5] and Theorem 3.2, we find that (3.20)–(3.26) has a $C^\infty([0, e^{\tau/\varepsilon}] \times \overline{\mathcal{D}_R})$ solution when $\tau \in (0, \tau^*)$ is fixed and ε is small. Theorem 2.3 then follows easily.

4. Proof of Theorem 3.1

In this section we shall study the approximate solution introduced in Section 3 and we shall prove Theorem 3.1.

As already said after (3.19), there exists $T > 0$ such that (3.1)–(3.4) has a $C^\infty([0,T] \times \overline{\mathcal{D}_R})$ solution $(\theta_1, w_1, 0)$ with initial data $(\varepsilon\theta_0 + \varepsilon^2\tilde{\theta}_0, \varepsilon w_0 + \varepsilon^2 \tilde{w}_0, 0)$. Assuming as we may that $1 + C_1\theta_1 > 0$, we may consider the corresponding solution $(\rho_1, u_1, \bar{S})$ of (2.1)–(2.7). We now recall some facts on the potential associated with $(\rho_1, u_1, \bar{S})$ (cf. [3]). Consider the potential function $v_1(t,x)$ vanishing for large $|x|$ and defined by the relations $\nabla v_1 = u_1$,

$\partial_t v_1 = -|u_1|^2/2 - h(\rho_1)$, where $h(s) = \int_{\bar\rho}^s \frac{\partial P}{\partial \sigma}(\sigma, \bar S)\sigma^{-1}\, d\sigma$ if $s > 0$. Then $v_1 \in C^\infty([0, T/\bar c] \times \overline{\mathcal{D}_R})$ and v_1 is radial. Set $\mathcal{H}(s) = h^{-1}(s)$, $\mathcal{M}(s) = \mathcal{H}(s)/\mathcal{H}'(s)$; then $\mathcal{M}(0) = \bar c^2$. If we set $v(t, x) = v_1(t/\bar c, x)$, it is easily checked that

$$\partial_t^2 v + \frac{2}{\bar c}\sum_{1\leqslant j\leqslant 3} \partial_j v \partial_t \partial_j v + \frac{1}{\bar c^2}\sum_{1\leqslant j,k\leqslant 3} \partial_j v \partial_k v \partial_{jk}^2 v - \frac{1}{\bar c^2}\mathcal{M}\left(-\bar c\partial_t v - \frac{1}{2}|\nabla v|^2\right)\Delta v = 0$$
$$\text{if } 0 < t < T \text{ and } x \in \mathcal{D}_R, \tag{4.1}$$

$$\partial_r v = 0 \quad \text{if } 0 < t < T \text{ and } r = R, \tag{4.2}$$

$$\partial_t^j v(0, x) = \varepsilon f_j(r) + \varepsilon^2 \tilde v_j(x, \varepsilon) \quad \text{if } x \in \mathcal{D}_R \text{ and } j = 0, 1, \tag{4.3}$$

where f_j are as in the definition of F_0 in Section 2, the functions $x \mapsto \tilde v_j(x, \varepsilon)$ are radial and in $C^\infty(\overline{\mathcal{D}_R})$, $|\partial_x^\alpha \tilde v_j| \leqslant C_\alpha$ with C_α independent of ε and of $x \in \mathcal{D}_R$, and $\tilde v_j(x, \varepsilon) = 0$ if $|x| \geqslant R + M$. Moreover the initial data in (4.3) satisfy compatibility conditions of all orders when $t = 0$ and $r = R$. So as proved in Section 3 of [4], (f_0, f_1) are infinitely compatible initial data for the boundary value problem $\Box v_0 = 0$ if $t > 0$ and $x \in \mathcal{D}_R$, $\partial_r v_0 = 0$ if $t > 0$ and $r = R$, where $\Box = \partial_t^2 - \sum_{1\leqslant j\leqslant 3} \partial_j^2$, and therefore, by Lemma 3.3 of [4], $F_0 \in C^\infty(\mathbb{R})$, and F_0 and all its derivatives are bounded. We may write (4.1) in the form $\Box v - \sum_{0\leqslant i,j\leqslant 3} f^{ij}(\partial v)\partial_{ij}^2 v = 0$ with $f^{ij} \in C^\infty$ in an open neighborhood of 0, $f^{ij}(0) = 0$ and $f^{ij} = f^{ji}$. Denote by f^{ijk} the partial derivative $\frac{\partial f^{ij}}{\partial \xi_k}(0)$ of the function $(\xi_0, \xi_1, \xi_2, \xi_3) = \xi \mapsto f^{ij}(\xi)$ at 0. Set $g = -\sum_{0\leqslant i,j,k\leqslant 3} f^{ijk}\hat\omega_i\hat\omega_j\hat\omega_k$, where $\hat\omega_0 = -1$ and $(\hat\omega_1, \hat\omega_2, \hat\omega_3) \in S^2$. We have $g = -2(\bar\rho\overline{c_\rho} + \bar c)/\bar c^2 < 0$, and, again by Lemma 3.3 of [4], $\max_{q\in\mathbb{R}}(-F_0''(q)) \geqslant 0$ and $\max_{q\in\mathbb{R}}(-F_0''(q)) = 0$ if and only if $|f_0| + |f_1| \equiv 0$. This gives at once the properties of F_0 and τ^* stated in Section 2, just before Theorem 2.2. Moreover, we have the following result, which follows from Theorem 3.1 of [4] and the continuation argument yielding Theorem 2.2 of [4] from Theorem 3.1 of [4].

Theorem 4.1. *Fix $\tau \in (0, \tau^*)$ and $a \in \mathbb{N}^5$. One can find $\varepsilon_0 > 0$, $C > 0$, and for each $\bar R > R$, also $\bar C > 0$, such that the following holds: if $0 < \varepsilon \leqslant \varepsilon_0$ and $\varepsilon \ln T \leqslant \tau$, then* (4.1)–(4.3) *has a unique $C^\infty([0, T] \times \overline{\mathcal{D}_R})$ solution v, which satisfies the following estimates for all $t \in [0, T]$:*

(1) $\|\Gamma^a v'(t)\| \leqslant C\varepsilon$,
(2) $|\Gamma^a v'(t, x)| \leqslant C\varepsilon\langle t\rangle^{-1}$,
(3) $|\Gamma^a v'(t, x)| \leqslant \bar C\varepsilon\langle t\rangle^{-a_0-1}$ *if* $R \leqslant r \leqslant \bar R$,
(4) $|\Gamma^a v(t, x)| \leqslant \bar C\varepsilon\langle t\rangle^{-a_0-2}$ *if* $a_1 + a_2 + a_3 > 0$ *and* $R \leqslant r \leqslant \bar R$.

With the help of Theorem 4.1, we are able to prove Theorem 3.1.

Proof of Theorem 3.1. We have $w_1(t, x) = \nabla v(t, x)/\bar c$, $\theta_1(t, x) = ((\rho_1(t/\bar c, x)/\bar\rho)^{(\gamma-1)/2} - 1)/C_1 = F((\bar c\partial_t v + |\nabla v|^2/2)(t, x))$, where $F \in C^\infty$ in an open neighborhood of 0 and $F(0) = 0$. Moreover, $\partial_t\theta_1 = -w_1 \cdot \nabla\theta_1 - (1 + C_1\theta_1)\nabla \cdot w_1$. Therefore Theorem 3.1 follows easily from Theorem 4.1 with the help of standard estimates of nonlinear functions. □

5. Proof of Theorem 3.2: Reduction to Proposition 5.4

In this section and the next one we shall prove Theorem 3.2 by an energy method. To treat the boundary terms, it is convenient to introduce intermediate norms (which was done in [4] for second order quasilinear wave equations) and therefore the energy method will be somewhat more complicated than the one used for the boundaryless case in [3].

Let ψ be as in the statement of Theorem 3.2. We are going to prove the following result, which is the analogue of Theorem 5 of [3] for our exterior mixed problem.

Theorem 5.1. *Fix $m \in \mathbb{N}$ with $m \geqslant 2$. If $\tau \in (0, \tau^*)$, one can find $\varepsilon_0 > 0$ such that the following holds: if $0 < \varepsilon \leqslant \varepsilon_0$, $\varepsilon \ln T \leqslant \tau$, and (θ_2, w_2, z_2) is a $C^\infty([0, T] \times \overline{\mathcal{D}_R})$ solution of* (3.20)–(3.26) *with $E_m^{1/2}(t) \leqslant \varepsilon^2\psi(\varepsilon)$ if $t \in [0, T]$, then $E_m^{1/2}(t) \leqslant \varepsilon^2\psi(\varepsilon)/2$ if $t \in [0, T]$.*

Since $E_m^{1/2}(0) \leqslant \bar{C}_0\varepsilon^2$ for some $\bar{C}_0 > 0$, Theorem 3.2 follows at once from Theorem 5.1; hence the proof of Theorem 5.1 will complete the proof of Theorem 2.3.

The rest of this section and the next section are devoted to the proof of Theorem 5.1.

Henceforth we shall set $\sigma_\pm(t,x) = (1+(t\pm r)^2)^{1/2}$. We start with two useful lemmas.

Lemma 5.1. *If $F \in C^\infty([R,+\infty))$ is such that $F(r)=0$ when r is large, and if $f(x)=F(r)x/r$, then* $\|\sigma_-(t)r^{-1}f\| \leqslant 2(\|\sigma_-(t)\partial_r f\| + \|f\|)$.

Proof. Set $G(t,r)=\sigma_-(t,x)F(r)$. If we integrate the identity $G^2(t,r) = \partial_r(rG^2(t,r)) - 2rG(t,r)\partial_r G(t,r)$ over $[R,+\infty)$ with respect to dr, Lemma 5.1 follows easily. □

Lemma 5.2. *If $j,l \in \mathbb{N}$, one can find $C>0$ such that the following holds: if $F \in C^\infty([0,T]\times[R,+\infty))$ and if, for each $t\in[0,T]$, the function $r \mapsto F(t,r)$ vanishes when $r=R$ and also when r is large, we have if $f(t,x) = F(t,r)x/r$:*

(1) $\|(\sigma_-\nabla\cdot\partial_r^j X^l f)(t)\| \leqslant C\mathcal{E}_{jl}(t)$,
(2) $\|(\sigma_-\partial_r^{j+1}X^l f)(t)\| \leqslant C\mathcal{E}_{jl}(t)$,

where $\mathcal{E}_{jl}(t) = \sum_{|\alpha|+k\leqslant j+l;\, k\leqslant l} \|(\sigma_-\nabla\cdot\partial_x^\alpha X^k f)(t)\| + \sum_{|\beta|+n\leqslant l}\|\partial_x^\beta X^n f(t)\|$.

Proof. We first prove (1) and (2) when $l=0$. Let us check first that

$$\big(\nabla\cdot\partial_r^j f\big)(t,x) = \sum_{|\alpha|\leqslant j}\big(c_{j\alpha}\nabla\cdot\partial_x^\alpha f\big)(t,x) + c_j r^{-j-1}F(t,r), \tag{5.1}$$

where $c_{j\alpha}\in C^\infty(\overline{\mathcal{D}_R})$ and are positively homogeneous of degree $|\alpha|-j$, and $c_j\in\mathbb{R}$ with $c_0=0$. If $j=0$, (5.1) is obvious and it is easily proved by induction on j if we use the relations $(\nabla\cdot\partial_r^{J+1}f)(t,x) = \sum_{1\leqslant k\leqslant 3} r^{-1}x_k\partial_k\nabla\cdot\partial_r^J f(t,x) + 2r^{-2}\partial_r^J F(t,r)$, $\partial_r^{J+1}F(t,r) = (\nabla\cdot\partial_r^J f)(t,x) - 2r^{-1}\partial_r^J F(t,r)$. On the other hand, since $F(t,R)=0$, it follows from the arguments of the proof of (6.7) of [12] that

$$\big\|(\sigma_-\nabla f)(t)\big\| \leqslant C\big(\big\|(\sigma_-\nabla\cdot f)(t)\big\| + \big\|f(t)\big\|\big), \tag{5.2}$$

where $\nabla f = (\partial_i f_j)_{1\leqslant i,j\leqslant 3}$. Since $|\partial_r f| \leqslant |\nabla f|$, this gives (2) when $j=l=0$. Using (5.1), (5.2), and Lemma 5.1, we obtain (1) when $l=0<j$. Of course (1) is obvious when $l=j=0$. To prove (2) when $l=0<j$, we write $\partial_r^{j+1}F = \nabla\cdot\partial_r^j f - 2r^{-1}\partial_r^j F$ and just apply (1) (with $l=0$) and the case $j=l=0$ of (2). Now assume that (1), (2) have been proved for all j when $l\leqslant L$, and let us show that they still hold for all j if $l=L+1$. Write $Xf = g_1+g_2$, where $g_1 = (X-R\partial_r)f$, $g_2 = R\partial_r f$. With the help of the induction hypothesis on (1), (2), applied to g_1 and f with $l=L$, and of Lemma 5.1, we obtain (1), (2) for $l=L+1$. The proof of Lemma 5.2 is complete. □

Let us define $E_{m,l}^{1/2}(t)$, $m,l\in\mathbb{N}$, $0\leqslant l\leqslant m$, and $Q_{m,l}(t)$, $m,l\in\mathbb{N}$, $0\leqslant l\leqslant m-1$, in the following way:

$$E_{m,l}(t) = \sum_{|a|\leqslant m;\, a_4\leqslant l}\big(\big\|\Gamma^a\theta_2(t)\big\|^2 + \big\|\Gamma^a w_2(t)\big\|^2 + \big\|\Gamma^a z_2(t)\big\|^2\big),$$

$$Q_{m,l}(t) = \sum_{|a|\leqslant m-1;\, a_4\leqslant l}\big(\big\|\big(\sigma_-\nabla\Gamma^a\theta_2\big)(t)\big\| + \big\|\big(\sigma_-\partial_t\Gamma^a w_2\big)(t)\big\| + \big\|\big(\sigma_-\nabla\cdot\Gamma^a w_2\big)(t)\big\|\big).$$

Notice that $E_{m,m}=E_m$. To economize notations in the sequel, it is also convenient to write $H_{m,l} = Q_{m,l} + E_{m-1,l}^{1/2}$ when $0\leqslant l\leqslant m-1$. Sometimes it will be convenient to set $E_{m,-1} = Q_{m,-1} = H_{m,-1} = 0$ if $m\in\mathbb{N}$. Useful estimates (mostly of Sobolev type) are contained in the following proposition (where the estimates are slightly better, regarding the number of consumed derivatives, than in Lemma 2 of [3], because now $r\geqslant R$). We set $\sigma_*(t,x) = r\sigma_-^{1/2}(t,x) + r^{1/2}\sigma_-(t,x)$.

Proposition 5.1. *The following estimates hold*:

(1) $|r\Gamma^a\theta_2(t)| + |r\Gamma^a w_2(t)| + |r\Gamma^a z_2(t)| \leqslant CE^{1/2}_{|a|+1,a_4}(t)$,
(2) $|r(\sigma_-\nabla\Gamma^a\theta_2)(t)| \leqslant CQ_{|a|+2,a_4}(t)$,
(3) $|(\sigma_*\partial_t^k X^l\theta_2)(t)| + |(\sigma_*\partial_t^k X^l w_2)(t)| \leqslant CH_{k+l+1,l}(t)$,
(4) $|r(\sigma_-\nabla\Gamma^a w_2)(t)| \leqslant C(Q_{|a|+2,a_4}(t) + E^{1/2}_{a_0+a_4,a_4}(t))$,
(5) $\|(\sigma_-\nabla\Gamma^a w_2)(t)\| \leqslant C(Q_{|a|+1,a_4}(t) + E^{1/2}_{a_0+a_4,a_4}(t))$,
(6) $\langle t\rangle|w_2(t,x)| \leqslant C_{\bar R}Q_{1,0}(t)$ *if* $R \leqslant r \leqslant \bar R$.

Proof. (1) and (2) follow easily by the proof of Lemma 4.2 of [9]. (3) is a consequence of Lemma 4.1 of [4] and of Lemma 5.2(2). To prove (4), write $\Gamma^a = \partial_t^k\partial_x^\alpha X^l$ and observe that

$$\left|\nabla\Gamma^a w_2\right| \leqslant C\left(\sum_{1\leqslant p\leqslant|\alpha|+1}\left|\partial_t^k\partial_r^p X^l w_2\right| + r^{-|\alpha|-1}\left|\partial_t^k X^l w_2\right|\right). \tag{5.3}$$

Now by the proof of Lemma 4.2 of [9], $|r(\sigma_-\partial_t^k\partial_r^p X^l w_2)(t)| \leqslant C\sum_{0\leqslant j\leqslant 1}\|(\sigma_-\partial_t^k\partial_r^{p+j}X^l w_2)(t)\|$, so by Lemma 5.2(2), we find that $|r(\sigma_-\partial_t^k\partial_r^p X^l w_2)(t)| \leqslant C(Q_{k+p+l+1,l}(t) + E^{1/2}_{k+l,l}(t))$ if $p > 0$. If we then estimate $|(\sigma_-\partial_t^k X^l w_2)(t)|$ by (3) and use (5.3), (4) follows. To prove (5), observe first that

$$\left\|r^{-1}\left(\sigma_-\partial_t^k X^l w_2\right)(t)\right\| \leqslant C\left(\left\|\left(\sigma_-\partial_r\partial_t^k X^l w_2\right)(t)\right\| + \left\|\partial_t^k X^l w_2(t)\right\|\right), \tag{5.4}$$

by Lemma 5.1. Multiplying (5.3) by σ_- and taking $L^2(\mathcal{D}_R)$ norms, we obtain (5) if we make use of Lemma 5.2(2) and of (5.4). Finally (6) follows if we define $W_2(t,r)$ by the relation $w_2(t,x) = W_2(t,r)x/r$ and write that $|W_2(t,r)| = r^{-2}|\int_R^r \partial_s(W_2(t,s)s^2)\,ds| \leqslant Cr^{-2}\int_{R\leqslant|y|\leqslant r}|(\nabla\cdot w_2)(t,y)|\,dy$. The proof of Proposition 5.1 is complete. □

For the sequel, let us introduce some useful notation already used in [3] in the boundaryless case. If $\Gamma^a = \partial^\alpha X^k$ and we apply $\partial^\alpha(X+1)^k$ to (3.20), (3.21), we obtain that

$$\partial_t\Gamma^a\theta_2 + \nabla\cdot\Gamma^a w_2 = h_0^a,$$
$$\partial_t\Gamma^a w_2 + \nabla\Gamma^a\theta_2 = h^a,$$

where $h_0^a = \sum_{1\leqslant j\leqslant 6}\tau_j^a$, $h^a = \sum_{7\leqslant j\leqslant 13}\tau_j^a$, with

$$\tau_1^a = -\sum_{b\leqslant a}\binom{a}{b}\Gamma^b w_1\cdot\nabla\Gamma^{a-b}\theta_2, \qquad \tau_2^a = -\sum_{b\leqslant a}\binom{a}{b}\Gamma^b w_2\cdot\nabla\Gamma^{a-b}\theta_1,$$

$$\tau_3^a = -\sum_{b\leqslant a}\binom{a}{b}\Gamma^b w_2\cdot\nabla\Gamma^{a-b}\theta_2, \qquad \tau_4^a = -C_1\sum_{b\leqslant a}\binom{a}{b}\Gamma^b\theta_1\nabla\cdot\Gamma^{a-b}w_2,$$

$$\tau_5^a = -C_1\sum_{b\leqslant a}\binom{a}{b}\Gamma^b\theta_2\nabla\cdot\Gamma^{a-b}w_1, \qquad \tau_6^a = -C_1\sum_{b\leqslant a}\binom{a}{b}\Gamma^b\theta_2\nabla\cdot\Gamma^{a-b}w_2,$$

$$\tau_7^a = -\sum_{b\leqslant a}\binom{a}{b}\Gamma^b w_1\cdot\nabla\Gamma^{a-b}w_2, \qquad \tau_8^a = -\sum_{b\leqslant a}\binom{a}{b}\Gamma^b w_2\cdot\nabla\Gamma^{a-b}w_1,$$

$$\tau_9^a = -\sum_{b\leqslant a}\binom{a}{b}\Gamma^b w_2\cdot\nabla\Gamma^{a-b}w_2, \qquad \tau_{10}^a = -C_1\sum_{b\leqslant a}\binom{a}{b}\Gamma^b\theta_1\nabla\Gamma^{a-b}\theta_2,$$

$$\tau_{11}^a = -C_1\sum_{b\leqslant a}\binom{a}{b}\Gamma^b\theta_2\nabla\Gamma^{a-b}\theta_1, \qquad \tau_{12}^a = -C_1\sum_{b\leqslant a}\binom{a}{b}\Gamma^b\theta_2\nabla\Gamma^{a-b}\theta_2,$$

$$\tau_{13}^a = -\partial^\alpha(X+1)^k((1+C_1\theta)z\nabla\theta).$$

We are going to prove the following result.

Proposition 5.2. *Fix m, $l \in \mathbb{N}$ with $m \geqslant 2$ and $1 \leqslant l \leqslant m$. One can find η, $C > 0$, and, for each $\tau \in (0, \tau^*)$, also C_0, $\varepsilon_0 > 0$ such that the following holds: if $0 < \varepsilon \leqslant \varepsilon_0$, and, for some $T \in (0, e^{\tau/\varepsilon}]$, (θ_2, w_2, z_2) is a $C^\infty([0,T] \times \overline{\mathcal{D}_R})$ solution of* (3.20)–(3.26) *with $E^{1/2}_{m-1,l-1}(t) \leqslant \eta$ for all $t \in [0,T]$, then $Q_{m,l-1}(t) \leqslant C E^{1/2}_{m,l}(t) + C_0 \varepsilon^2 \langle t\rangle^{-2}$ if $t \in [0,T]$.*

Proof. We adapt the proof of Proposition 3 of [3] (where $l = m$ and there was no boundary). First we obtain, if $\Gamma^a = \partial^\alpha X^k$, $|a| \leqslant m-1$, $k \leqslant l-1$:

$$Q_{m,l-1}(t) \leqslant C E^{1/2}_{m,l}(t) + C \sum_{|a| \leqslant m-1;\, a_4 \leqslant l-1} \bigl(t \|h^a_0(t)\| + \|(\sigma_+ h^a)(t)\|\bigr). \tag{5.5}$$

Indeed, (5.5) is proved exactly as (37) of [3]. Then we easily obtain with the help of Theorem 3.1(1), (2) and of Proposition 5.1(1), (2), (4), by elementary modifications of corresponding arguments of the proof of Proposition 3 of [3]:

$$\|(\sigma_+ \tau^a_j)(t)\| \leqslant C_\tau \varepsilon E^{1/2}_{m,l-1}(t) \quad \text{if } j \in \{1,2,4,5,7,8,10,11\}, \tag{5.6}$$

$$\|(\sigma_+ \tau^a_j)(t)\| \leqslant C E^{1/2}_{m-1,l-1}(t) H_{m,l-1}(t) \quad \text{if } j \in \{3,6,9,12\}, \tag{5.7}$$

$$\|(\sigma_+ \tau^a_{13})(t)\| \leqslant C_\tau \left(\frac{\varepsilon^2}{\langle t\rangle^2} + \varepsilon E^{1/2}_{m-1,l-1}(t)\right) + C Q_{m,l-1}(t)\bigl(E^{1/2}_{m-1,l-1}(t) + \varepsilon\bigr). \tag{5.8}$$

Proposition 5.2 follows from (5.5)–(5.8). □

Until the end of Section 6, we shall make the assumption that $E^{1/2}_{m,\max(l,1)}(t) \leqslant \varepsilon^2 \psi(\varepsilon)$ (this assumption could be relaxed at some places, but we keep it for simplicity and it is sufficient for our purposes). When convenient, we shall write $\theta_j(t,x) = \Theta_j(t,r)$, $w_j(t,x) = W_j(t,r)x/r$ (W_2 has already been defined in the proof of Proposition 5.1(6)), $z_j(t,x) = Z_j(t,r)$. The next lemma will be useful.

Lemma 5.3. *Fix m, $l \in \mathbb{N}$ with $m \geqslant 2$ and $0 \leqslant l \leqslant m$, and $\tau \in (0, \tau^*)$. One can find $\varepsilon_0 > 0$, $R_0 \geqslant R$ such that the following holds: if $0 < \varepsilon \leqslant \varepsilon_0$, $\varepsilon \ln T \leqslant \tau$ and (θ_2, w_2, z_2) is a $C^\infty([0,T] \times \overline{\mathcal{D}_R})$ solution of* (3.20)–(3.26) *with $E^{1/2}_{m,\max(l,1)}(t) \leqslant \varepsilon^2 \psi(\varepsilon)$ if $t \in [0,T]$, then $z_1(t,x) = z_2(t,x) = 0$ if $t \in [0,T]$ and $r \geqslant R_0$.*

Proof. We have $|W_j(t,r)| \leqslant \int_R^r |\partial_s W_j(t,s)|\, ds$, so with the help of Theorem 3.1(2), Proposition 5.1(4) and Proposition 5.2, we obtain that $|W(t,r)| \leqslant C_\tau \varepsilon (r-R)\langle t\rangle^{-1}$ since $\varepsilon\psi(\varepsilon)$ is bounded. Now $z(t,x) = 0$ if $r \geqslant \tilde{R}(t)$ where $\tilde{R}'(t) = W(t, \tilde{R}(t))$ and $\tilde{R}(0) = R + M$. A standard comparison argument shows that, for some $\tilde{R}_0 \geqslant R$, $\tilde{R}(t) \leqslant \tilde{R}_0$ if $\varepsilon \ln t \leqslant \tau$, so Lemma 5.3 follows since $z_1(t,x) = 0$ when $r \geqslant R + M$. □

Since we suppose until the end of Section 6 that ε is small and that $E^{1/2}_{m,\max(l,1)}(t) \leqslant \varepsilon^2 \psi(\varepsilon)$, we shall assume at the same time that $|z| \leqslant 1/2$, as we may thanks to Proposition 5.1(1).

Henceforth we shall set $a = (k, 0, 0, 0, \lambda)$ so that $\Gamma^a = \partial_t^k X^\lambda$. If $c \in \mathbb{N}^5$, it follows from (3.20)–(3.22) that

$$(\partial_t + w \cdot \nabla)\Gamma^a \theta_2 + (1 + C_1\theta)\nabla \cdot \Gamma^a w_2 = \hat{h}^a_0, \tag{5.9}$$

$$\left(\frac{1}{1+z}\partial_t + \frac{w}{1+z} \cdot \nabla\right)\Gamma^a w_2 + (1 + C_1\theta)\nabla \Gamma^a \theta_2 = \frac{\hat{h}^a}{1+z}, \tag{5.10}$$

$$(\partial_t + w \cdot \nabla)\Gamma^c z_2 = \hat{g}^c, \tag{5.11}$$

where $\hat{h}^a_0$, $\hat{h}^a$, $\hat{g}^c$ are defined as follows: $\hat{h}^a_0 = \sum_{j \in \{2,5\}} \tau^a_j + \sum_{j \in \{1,3,4,6\}} \hat{\tau}^a_j$, $\hat{h}^a = \sum_{j \in \{8,11\}} \tau^a_j + \sum_{j \in \{7,9,10,12,13\}} \hat{\tau}^a_j$, where τ^a_j are the same as in h^a_0, h^a, but now with $\Gamma^a = \partial_t^k X^\lambda$ and $|a| \leqslant m$. If $a \neq 0$, $\hat{\tau}^a_j$ ($j \neq 13$) are defined as τ^a_j (with $\Gamma^a = \partial_t^k X^\lambda$ and $|a| \leqslant m$) but with the supplementary condition that $b \neq 0$ in the sum; $\hat{\tau}^0_j = 0$ if $j \neq 13$; finally $\hat{\tau}^a_{13} = \tau^a_{13} + (1 + C_1\theta) z \nabla \Gamma^a \theta_2$. Also $\hat{g}^c = \sum_{1 \leqslant i,j \leqslant 2} \hat{g}^c_{ij}$, with $\hat{g}^c_{i1} = -\sum_{d \leqslant c} \binom{c}{d} \Gamma^d w_i \cdot \nabla \Gamma^{c-d} z_1$; when $c \neq 0$, $\hat{g}^c_{i2} = -\sum_{0 \neq d \leqslant c} \binom{c}{d} \Gamma^d w_i \cdot \nabla \Gamma^{c-d} z_2$, and $\hat{g}^0_{i2} = 0$.

If $\xi = (\xi^{(1)}, \xi^{(2)}, \xi^{(3)}) \in \mathbb{R}^3$, let $(\xi^{(i)})_{1\leqslant i\leqslant 3}$ be the row matrix of the components of ξ. Define the 5×1 matrices $\phi^{ac} = tr(\Gamma^a\theta_2\ (\Gamma^a w_2)^{(i)}_{1\leqslant i\leqslant 3}\ \Gamma^c z_2)$, $F^{ac} = tr(\hat{h}^a_0\ (\hat{h}^a/(1+z))^{(i)}_{1\leqslant i\leqslant 3}\ \hat{g}^c)$. With $A_j(\phi)$, $0\leqslant j\leqslant 3$, as in (3.8), (5.9)–(5.11) can be written

$$A_0(\phi)\partial_t\phi^{ac} + \sum_{1\leqslant j\leqslant 3} A_j(\phi)\partial_j\phi^{ac} = F^{ac}. \tag{5.12}$$

Taking the pointwise Euclidean scalar product (in $\mathbb{R}^5$) of (5.12) with ϕ^{ac}, integrating over $\mathcal{D}_R$ and using the symmetry of $A_j(\phi)$, we obtain, writing dS for the canonical surface measure on $\partial\mathcal{D}_R$:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\langle A_0(\phi)\phi^{ac},\phi^{ac}\rangle(t) - \frac{1}{2}\int_{\partial\mathcal{D}_R}\sum_{1\leqslant j\leqslant 3}\frac{x_j}{r}\big(A_j(\phi)\phi^{ac}\cdot\phi^{ac}\big)(t,x)\,dS\\ &\quad = \langle F^{ac},\phi^{ac}\rangle(t) + \frac{1}{2}\sum_{0\leqslant j\leqslant 3}\langle\big(\partial_j A_j(\phi)\big)\phi^{ac},\phi^{ac}\rangle(t);\end{aligned} \tag{5.13}$$

here and in the sequel, we also denote by $\cdot$ the standard Euclidean scalar product of $\nu\times 1$ real matrices (whatever ν) and, for two functions $f(t,x)$, $\tilde{f}(t,x)$ valued in the set of $\nu\times 1$ real matrices, we write $\langle f,\tilde{f}\rangle(t) = \int_{\mathcal{D}_R} f(t,x)\cdot\tilde{f}(t,x)\,dx$ ($\nu = 5$ in (5.13)). It is easy to check that

$$\int_{\partial\mathcal{D}_R}\sum_{1\leqslant j\leqslant 3}\frac{x_j}{r}\big(A_j(\phi)\phi^{ac}\cdot\phi^{ac}\big)(t,x)\,dS = 8\pi R^2\big((1+C_1\Theta)\Gamma^a\Theta_2\Gamma^a W_2\big)(t,R). \tag{5.14}$$

Set $\tilde{E}_{m,l}(t) = \sum(\|\Gamma^a\theta_2(t)\|^2 + \|\Gamma^a w_2(t)\|^2 + \|\Gamma^c z_2(t)\|^2)$, where the sum is taken over all $a = (k,0,0,0,\lambda)\in\mathbb{N}^5$, $c = (c_0,c_1,c_2,c_3,c_4)\in\mathbb{N}^5$ with λ, $c_4\leqslant l$, $|a|$, $|c|\leqslant m$. The next proposition justifies the choice of a in (5.9)–(5.11).

Proposition 5.3. *Fix $m, l\in\mathbb{N}$ with $m\geqslant 2$ and $0\leqslant l\leqslant m$, and $\tau\in(0,\tau^*)$. One can find ε_0, $C_0>0$ such that the following holds: if (θ_2,w_2,z_2) is a $C^\infty([0,T]\times\overline{\mathcal{D}_R})$ solution of* (3.20)–(3.26) *with $0<\varepsilon\leqslant\varepsilon_0$, $\varepsilon\ln T\leqslant\tau$, and $E^{1/2}_{m,\max(l,1)}(t)\leqslant\varepsilon^2\psi(\varepsilon)$ when $t\in[0,T]$, then $E^{1/2}_{m,l}(t)\leqslant C_0(\tilde{E}^{1/2}_{m,l}(t)+\varepsilon^2\langle t\rangle^{-3})$ if $t\in[0,T]$.*

Proof. We may assume that $l\leqslant m-1$ since the case $l=m$ follows at once if we know that the proposition holds when $l\leqslant m-1$. Set $\Phi_{p,k,\lambda}(t) = \sum_{n\leqslant p}(\|\partial_r^n\Gamma^a\theta_2(t)\| + \|\partial_r^n\Gamma^a w_2(t)\|)$, $\xi_{p,k,\lambda}(t) = \sum_{n\leqslant p}(\|\partial_r^n\hat{h}^a_0(t)\| + \|\partial_r^n\hat{h}^a(t)\|)$, where $\Gamma^a = \partial_t^k X^\lambda$ with $\lambda\leqslant l$ and $k+\lambda\leqslant m$. Taking ε_0 small and using Theorem 3.1(2) and Proposition 5.1(1), we may and shall assume that $C_1|\theta|\leqslant 1/2$. Recall that we also assume that $|z|\leqslant 1/2$. We obtain from (5.9), (5.10) that

$$tr\big(\partial_r\Gamma^a\Theta_2\ \partial_r\Gamma^a W_2\big) = B(\Theta,W,Z)tr\left(H^a_0\ H^a\ \partial_t\Gamma^a\Theta_2\ \left(\frac{\partial_t^j\Gamma^a W_2}{r^{1-j}}\right)_{0\leqslant j\leqslant 1}\right), \tag{5.15}$$

where B is a C^∞ 2×5 matrix, $H^a_0(t,r) = \hat{h}^a_0(t,x)$, $H^a(t,r) = \hat{h}^a(t,x)\cdot x/r$. Applying ∂_r^p to (5.15), with $p\leqslant m-1$, we obtain with the help of Theorem 3.1(2) and Proposition 5.1(1):

$$\Phi_{p+1,k,\lambda}(t)\leqslant C\big(\Phi_{p,k+1,\lambda}(t)+\Phi_{p,k,\lambda}(t)+\xi_{p,k,\lambda}(t)\big). \tag{5.16}$$

It is not hard to check with the help of (5.16) that

$$E^{1/2}_{m,l}(t)\leqslant C\tilde{E}^{1/2}_{m,l}(t) + C\sum_{p+|d|\leqslant m-1;\ d=(\bar{k},0,0,0,\bar{\lambda});\ \bar{\lambda}\leqslant l}\big(\big\|\partial_r^p\hat{h}^d_0(t)\big\| + \big\|\partial_r^p\hat{h}^d(t)\big\|\big), \tag{5.17}$$

so it remains to bound the factor of the second C on the right-hand side of (5.17). Until the end of the proof of Proposition 5.3, we assume that p, d are as in (5.17). With the help of Theorem 3.1(2), we obtain

$$\big\|\partial_r^p\tau^d_j(t)\big\|\leqslant C\frac{\varepsilon}{\langle t\rangle}E^{1/2}_{m-1,l}(t)\quad\text{if } j\in\{2,5,8,11\}, \tag{5.18}$$

$$\big\|\partial_r^p\hat{\tau}^d_j(t)\big\|\leqslant C\frac{\varepsilon}{\langle t\rangle}E^{1/2}_{m-1,l}(t)\quad\text{if } j\in\{1,4,7,10\}. \tag{5.19}$$

Since $m \geqslant 2$, we have

$$\|\partial_r^p \hat{\tau}_j^d(t)\| \leqslant \frac{C}{\langle t\rangle}\big(E_{m,l}^{1/2}(t)H_{m-1,l-1}(t) + E_{m,1}^{1/2}(t)E_{m-1,l}^{1/2}(t)\big) \quad \text{if } j \in \{3,6,9,12\}. \tag{5.20}$$

Let us check (5.20) when $j=9$ since the other cases are similar. Write $A_{nbpd} = \partial_r^n \Gamma^b w_2 \cdot \partial_r^{p-n}\nabla\Gamma^{d-b}w_2$; here and in the sequel $P_1 f \cdot P_2 \nabla g$ means of course the vector function $\sum_{1\leqslant j\leqslant 3} P_1 f^{(j)} P_2 \partial_j g$ if P_1, P_2 are scalar differential operators, f, g are $\mathbb{R}^3$-valued functions, and $f^{(j)}$ are the components of f in the canonical basis. We have $\partial_r^p \hat{\tau}_9^d = -C_1 \sum_{0\neq b\leqslant d,\, 0\leqslant n\leqslant p} \binom{p}{n}\binom{d}{b} A_{nbpd}$. If $d_4 - b_4 \leqslant l-1$, we can use Proposition 5.1(1), (5) to bound $|r\partial_r^n \Gamma^b w_2(t)| \|(\sigma_- \partial_r^{p-n}\nabla\Gamma^{d-b}w_2)(t)\|$ and obtain

$$\|(\sigma_+ A_{nbpd})(t)\| \leqslant C E_{m,l}^{1/2}(t) H_{m-1,l-1}(t). \tag{5.21}$$

If now $d_4 - b_4 = l$, then $\Gamma^b = \partial_t^{|b|}$ and $d_4 = l$. Set $\chi = \partial_t^{|b|-1}\partial_r^n w_2$ (recall that $b \neq 0$). Writing $\partial_t \chi = t^{-1}X\chi - t^{-1}r\partial_r\chi$ and using Proposition 5.1(1), we obtain

$$\|(\sigma_+ A_{nbpd})(t)\| \leqslant C E_{m,1}^{1/2}(t) E_{m-1,l}^{1/2}(t) \quad \text{if } t \geqslant 1. \tag{5.22}$$

On the other hand it is easy to check with the help of Proposition 5.1(1) that

$$\|(\sigma_+ A_{nbpd})(t)\| \leqslant C E_{m,0}^{1/2}(t) E_{m-1,0}^{1/2}(t) \quad \text{if } t \leqslant 1. \tag{5.23}$$

(5.20) for $j=9$ follows from (5.21)–(5.23). Finally we also have

$$\|\partial_r^p \hat{\tau}_{13}^d(t)\| \leqslant \frac{C}{\langle t\rangle}\left(\left(E_{m,1}^{1/2}(t) + \frac{\varepsilon^2}{\langle t\rangle}\right)E_{m-1,l}^{1/2}(t) + \varepsilon Q_{m-1,l-1}(t) + \frac{\varepsilon^2}{\langle t\rangle^2}\right). \tag{5.24}$$

Let us check (5.24). If $\Gamma^d = \partial_t^{\bar{k}} X^{\bar{\lambda}}$, $\bar{\lambda} \leqslant l$, we have $\hat{\tau}_{13}^d = \sum_{1\leqslant j\leqslant 3} B_j$, where $B_1 = -\partial_t^{\bar{k}}(X+1)^{\bar{\lambda}}(z\nabla\theta) + z\nabla\Gamma^d\theta$, $B_2 = -\partial_t^{\bar{k}}(X+1)^{\bar{\lambda}}(C_1\theta z\nabla\theta) + C_1\theta z\nabla\Gamma^d\theta$, $B_3 = -(1+C_1\theta)z\nabla\Gamma^d\theta_1$. If $0 \neq b \leqslant d$ and $n \leqslant p$, set $D_{ijbn} = \partial_r^n \Gamma^b z_i \partial_r^{p-n}\nabla\Gamma^{d-b}\theta_j$. With the help of Lemma 5.3 and of Theorem 3.1(3), we easily obtain that

$$\|(\sigma_+ D_{11bn})(t)\| \leqslant C\frac{\varepsilon^2}{\langle t\rangle^2}, \qquad \|(\sigma_+ D_{21bn})(t)\| \leqslant C\frac{\varepsilon}{\langle t\rangle^2}E_{m-1,l}^{1/2}(t). \tag{5.25}$$

Now set $\kappa_1 = \varepsilon$, $\kappa_2 = E_{m,l}^{1/2}$. We readily obtain with the help of Proposition 5.1(1) that

$$\|(\sigma_+ D_{i2bn})(t)\| \leqslant C\kappa_i(t) Q_{m-1,l-1}(t) \quad \text{if } d_4 - b_4 \leqslant l-1. \tag{5.26}$$

If $d_4 - b_4 \geqslant l$, then $\Gamma^b = \partial_t^{|b|}$ and so $D_{12bn} \equiv 0$. Set now $\chi = \partial_t^{|b|-1}\partial_r^n z_2$. Writing again $\partial_t\chi = t^{-1}X\chi - t^{-1}r\partial_r\chi$, we easily obtain with the help of Proposition 5.1(1):

$$\|\big(\sigma_+ \partial_t\chi\, \partial_t^{\bar{k}-|b|}\partial_r^{p-n+1}X^l\theta_2\big)(t)\| \leqslant C E_{m,1}^{1/2}(t) E_{m-1,l}^{1/2}(t) \quad \text{if } t \geqslant 1 \text{ and } d_4 - b_4 = l. \tag{5.27}$$

If $t<1$ and $d_4 - b_4 = l$, a still better bound clearly holds; combining this with (5.26), (5.27) and the fact that $D_{12bn} \equiv 0$ if $d_4 - b_4 = l$ (recall that $b \neq 0$), we obtain in particular that

$$\|(\sigma_+ D_{i2bn})(t)\| \leqslant C\big(E_{m,1}^{1/2}(t)E_{m-1,l}^{1/2}(t) + \kappa_i(t) Q_{m-1,l-1}(t)\big). \tag{5.28}$$

Using (5.25), (5.28), we find that

$$\|\big(\sigma_+ \partial_r^p B_1\big)(t)\| \leqslant C\left(E_{m,1}^{1/2}(t)E_{m-1,l}^{1/2}(t) + \varepsilon Q_{m-1,l-1}(t) + \frac{\varepsilon^2}{\langle t\rangle^2}\right).$$

Replacing z_i by θz_i in B_1, we obtain B_2. Set $F_{ijbn} = \partial_r^n \Gamma^b(\theta z_i)\partial_r^{p-n}\nabla\Gamma^{d-b}\theta_j$. We can repeat the arguments leading to (5.25), (5.26), with z_i replaced by θz_i now, using furthermore Lemma 3.1(2), to obtain that $\|(\sigma_+ F_{11bn})(t)\| \leqslant C\varepsilon^3\langle t\rangle^{-2}$, $\|(\sigma_+ F_{21bn})(t)\| \leqslant C\varepsilon^2 E_{m-1,l}^{1/2}(t)\langle t\rangle^{-2}$; and $\|(\sigma_+ F_{i2bn})(t)\| \leqslant C\varepsilon\kappa_i(t)Q_{m-1,l-1}(t)$ if $d_4 - b_4 \leqslant l-1$. On the other hand, to estimate $\|(\sigma_+ F_{i2bn})(t)\|$ if $d_4 - b_4 = l$, we notice that $|(\sigma_+\partial_r^\nu\theta_1)(t)| \leqslant C\varepsilon$ by Theorem 3.1(2), that $|(\sigma_+\partial_r^\nu\partial_t^\mu\theta_1)(t)| \leqslant C\varepsilon\langle t\rangle^{-1}$ if $\mu > 0$ and r has a fixed bound, by Theorem 3.1(3), and that, for $\nu + \mu \leqslant m-1$

(and ε small), $|(\sigma_+\partial_r^\nu\partial_t^\mu\theta_2)(t)| \leqslant C(E_{m,1}^{1/2}(t)+\varepsilon^2\langle t\rangle^{-2})$ by Proposition 5.1(2), (3) and Proposition 5.2. Making use of Lemma 5.3 and of Proposition 5.1(1), we obtain that $\|(\sigma_+F_{i2bn})(t)\| \leqslant C\varepsilon(E_{m,1}^{1/2}(t)+\varepsilon\langle t\rangle^{-1})E_{m-1,l}^{1/2}(t)$. Collecting estimates, we find that

$$\big\|\big(\sigma_+\partial_r^p B_2\big)(t)\big\| \leqslant C\varepsilon\bigg(\bigg(E_{m,1}^{1/2}(t)+\frac{\varepsilon}{\langle t\rangle}\bigg)E_{m-1,l}^{1/2}(t)+\varepsilon Q_{m-1,l-1}(t)+\frac{\varepsilon^2}{\langle t\rangle^2}\bigg).$$

Moreover, with the help of Proposition 5.1(1), Lemma 5.3 and Theorem 3.1(3), recalling that $C_1|\theta| \leqslant 1/2$, we also find that

$$\big\|\big(\sigma_+\partial_r^p B_3\big)(t)\big\| \leqslant C\frac{\varepsilon^2}{\langle t\rangle^2}.$$

(5.24) follows from the estimates of $\|(\sigma_+\partial_r^p B_j)(t)\|$, $1 \leqslant j \leqslant 3$. Finally (5.17)–(5.20) and (5.24) yield Proposition 5.3 if we make use of Proposition 5.2. $\square$

Theorem 5.1 will readily follow from the next proposition.

Proposition 5.4. *Fix $m, l \in \mathbb{N}$ with $m \geqslant \max(l,2)$, and $\tau \in (0,\tau^*)$. One can find ε_0, $C>0$, and for each $\bar{R}>R$, also $\bar{C}>0$, such that the following holds. If $0<\varepsilon \leqslant \varepsilon_0$ and if (θ_2, w_2, z_2) is a $C^\infty([0,T]\times\overline{\mathcal{D}_R})$ solution of (3.20)–(3.26) with $\varepsilon \ln T \leqslant \tau$, such that $E_{m,\max(l,1)}^{1/2}(t) \leqslant \varepsilon^2\psi(\varepsilon)$ when $t \in [0,T]$, then we have*:

$(1,l)$ *if $a=(k,0,0,0,\lambda) \in \mathbb{N}^5$, $c \in \mathbb{N}^5$ with $|a|$, $|c| \leqslant m$ and λ, $c_4 \leqslant l$, one can write $\langle F^{ac}, \phi^{ac}\rangle = \dfrac{dH_1^c}{dt} + H_2^{ac}$ in $[0,T]$, where H_1^c, $H_2^{ac} \in C^\infty([0,T])$, $H_1^c(0)=0$, and*

$$\big|H_1^c(t)\big| \leqslant C\varepsilon^5\psi^2(\varepsilon), \qquad \big|H_2^{ac}(t)\big| \leqslant C\bigg(\frac{\varepsilon}{\langle t\rangle}E_{m,l}^{1/2}(t)E_{m,\max(l,1)}^{1/2}(t)+\frac{\varepsilon^4}{\langle t\rangle^2}\psi(\varepsilon)\bigg);$$

(2) $|\sum_{0\leqslant j\leqslant 3}(\partial_j A_j(\phi))(t)| \leqslant C\frac{\varepsilon}{\langle t\rangle}$ *if $t \in [0,T]$*;

$(3,l)$ *if $a=(k,0,0,0,l) \in \mathbb{N}^5$ with $l \geqslant 1$ and $k+l \leqslant m$, then*

$$-\int_0^T \big((1+C_1\Theta)\Gamma^a\Theta_2\Gamma^a W_2\big)(t,R)\,dt \geqslant C^{-1}\int_0^T t^{2l-1}\big(\partial_t^{k+l}\Theta_2\big)^2(t,R)\,dt - C\varepsilon^4\psi(\varepsilon);$$

$(4,l)$ $E_{m,l}^{1/2}(t) \leqslant C\varepsilon^2\psi^{1/2}(\varepsilon)$ *when $t \in [0,T]$, and $\int_0^T t^{2l-1}(\partial_t^{k+l}\Theta_2)^2(t,R)\,dt \leqslant C\varepsilon^4\psi(\varepsilon)$, if $l \geqslant 1$ and $k+l \leqslant m$*;

$(5,l)$ $\langle t\rangle^{k+1}(|\partial_t^k\partial^\alpha\theta_2(t,x)|+|\partial_t^k\partial^\alpha w_2(t,x)|) \leqslant \bar{C}\varepsilon^2\psi^{1/2}(\varepsilon)$ *if $k+|\alpha| \leqslant m-1$, $l \geqslant 1$, $k \leqslant l-1$, $t \in [0,T]$, and $R \leqslant r \leqslant \bar{R}$*;

$(6,l)$ $\langle t\rangle^{k+1}|\partial_t^{k+1}\partial^\alpha z_2(t)| \leqslant C\varepsilon^2$ *if $k+|\alpha| \leqslant m-2$, $l \geqslant 1$, $k \leqslant l-1$, and $t \in [0,T]$.*

The first half of $(4,0)$ also holds and is of course a consequence of $(4,1)$; we have chosen the above formulation since, for each fixed $l \geqslant 1$, we shall obtain all of $(4,l)$ at the same time. $(6,m)$ is identical with $(6,m-1)$ and is introduced for notational convenience. The proof will show that $(5,1)$ still holds with $[R,\bar{R}]$ replaced by $[R,+\infty)$.

Notice that Theorem 5.1 follows at once from the first inequality of Proposition 5.4$(4,m)$. So the proof of Theorem 3.2 (and therefore also that of Theorem 2.3) will be completed if Proposition 5.4 is proved. The next section is devoted to the (fairly long) proof of Proposition 5.4.

6. Proof of Proposition 5.4

In this section we shall prove Proposition 5.4.

Proposition 5.4$(1,l)$ will be proved by adapting the arguments of the boundaryless case (see [3], where $l=m$); in the present situation, we have to keep in mind that we cannot consume more than l derivatives with respect to X in L^2, which will somewhat complicate the proof. Proposition 5.4$(3,l)$–$(6,l)$ will be proved by induction on l.

A corresponding induction procedure was used in [4] for solutions of quasilinear wave equations satisfying Neumann boundary condition.

Throughout this section, we shall suppose that θ_2, w_2, z_2 satisfy the assumptions of Proposition 5.4, and that $0 < \varepsilon \leqslant \varepsilon_0$, with ε_0 small allowed to depend on τ; in particular, throughout this section, we shall suppose that ε_0 is so small that (i) Theorem 3.1, Propositions 5.2 and 5.3, and Lemma 5.3 can be applied, and (ii) $C_1|\theta| \leqslant 1/2$ and $|z| \leqslant 1/2$. We shall denote by C various strictly positive constants (which might depend on τ) but are independent of ε, T when $\varepsilon \ln T \leqslant \tau$.

Proof of Proposition 5.4(1, l). Set $P_j^a = \langle \tau_j^a, \Gamma^a\theta_2\rangle$ if $j \in \{2,5\}$, $P_j^a = \langle \hat\tau_j^a, \Gamma^a\theta_2\rangle$ if $j \in \{1,3,4,6\}$, $P_j^a = \langle (1+z)^{-1}\tau_j^a, \Gamma^a w_2\rangle$ if $j \in \{8,11\}$, $P_j^a = \langle (1+z)^{-1}\hat\tau_j^a, \Gamma^a w_2\rangle$ if $j \in \{7,9,10,12,13\}$, $\hat P_{ij}^c = \langle \hat g_{ij}^c, \Gamma^c z_2\rangle$ if $i, j \in \{1,2\}$. Then $\langle F^{ac}, \phi^{ac}\rangle = \sum_{1\leqslant j\leqslant 13} P_j^a + \sum_{1\leqslant i,j\leqslant 2} \hat P_{ij}^c$. With the help of Theorem 3.1(2), we readily obtain that

$$\left|P_j^a(t)\right| \leqslant C\frac{\varepsilon}{\langle t\rangle}E_{m,l}(t) \quad \text{if } j \in \{1,2,4,5,7,8,10,11\}. \tag{6.1}$$

We have

$$\left|P_j^a(t)\right| \leqslant \frac{C}{\langle t\rangle}E_{m,l}(t)H_{m,(l-1)_+}(t) \quad \text{if } j \in \{3,6,9,12\}. \tag{6.2}$$

Let us prove (6.2) for $j = 9$; the other cases can be handled in the same way. Set $\mathcal{T} = \|(\sigma_+\Gamma^b w_2 \cdot \nabla\Gamma^{a-b}w_2)(t)\|$, where $0 \neq b \leqslant a$. If $|b| \leqslant m-1$ and $a_4 - b_4 \leqslant l-1$, then $\mathcal{T} \leqslant C|r\Gamma^b w_2(t)|\|(\sigma_-\nabla\Gamma^{a-b}w_2)(t)\|$; hence we obtain with the help of Proposition 5.1(1), (5) that

$$\mathcal{T} \leqslant CE_{m,l}^{1/2}(t)H_{m,l-1}(t). \tag{6.3}$$

If $|b| \leqslant m-1$ and $a_4 - b_4 \geqslant l$ (so $a_4 = l$, $b_4 = 0$), we write $\mathcal{T} \leqslant C|(\sigma_+\partial_t^{|b|}w_2)(t)|\|\nabla\partial_t^{|a-b|-l}X^l w_2(t)\|$, hence we obtain with the help of Proposition 5.1(3) that

$$\mathcal{T} \leqslant CE_{m,l}^{1/2}(t)H_{m,0}(t). \tag{6.4}$$

Finally, if $|b| = m$ and $b_4 \leqslant l$, then $a = b$ and $\mathcal{T} \leqslant C\|\Gamma^b w_2(t)\||r(\sigma_-\nabla w_2)(t)|$, whence we obtain with the help of Proposition 5.1(4) that

$$\mathcal{T} \leqslant CE_{m,l}^{1/2}(t)\bigl(Q_{2,0}(t) + E_{0,0}^{1/2}(t)\bigr). \tag{6.5}$$

(6.3)–(6.5) give (6.2) when $j = 9$. Let us show that

$$\left|P_{13}^a(t)\right| \leqslant \frac{C}{\langle t\rangle}E_{m,l}^{1/2}(t)\left(\Psi_{m,l}(t) + \frac{\varepsilon^4\psi(\varepsilon)}{\langle t\rangle}\right), \tag{6.6}$$

where $\Psi_{m,l}(t) = \varepsilon^2\psi(\varepsilon)E_{m,l}^{1/2}(t) + \varepsilon Q_{m,l-1}(t) + \varepsilon^2\langle t\rangle^{-2}$. To prove (6.6), we are going to adapt arguments which led to (5.24). Let B_1, B_2, B_3 be as in the proof of (5.24), but now with d replaced by $a = (k,0,0,0,\lambda)$ with $|a| \leqslant m$ and $\lambda \leqslant l$. Set $D_{ijb} = \Gamma^b z_i \nabla\Gamma^{a-b}\theta_j$ if $0 \neq b \leqslant a$. With the help of Lemma 5.3 and Theorem 3.1(3), we obtain $\|(\sigma_+D_{11b})(t)\| \leqslant C\varepsilon^2\langle t\rangle^{-2}$, $\|(\sigma_+D_{21b})(t)\| \leqslant C\varepsilon\langle t\rangle^{-2}E_{m,l}^{1/2}(t)$. If $a_4 - b_4 \geqslant l$, then $\Gamma^b = \partial_t^{|b|}$, so $D_{12b} \equiv 0$ since $b \neq 0$. If $a_4 - b_4 \leqslant l-1$, then $\|(\sigma_+D_{12b})(t)\| \leqslant C\varepsilon Q_{m,l-1}(t)$. To estimate D_{22b}, assume first that $a_4 - b_4 \leqslant l-1$. If $|a-b| \leqslant m-2$, we can write with the help of Lemma 5.3 that $\|(\sigma_+D_{22b})(t)\| \leqslant C\|\Gamma^b z_2(t)\||(\sigma_-\nabla\Gamma^{a-b}\theta_2)(t)|$, which can be bounded above by $CE_{m,l}^{1/2}(t)Q_{m,l-1}(t)$ with the help of Proposition 5.1(2). If $|a-b| = m-1$, then $|a| = m$ and $|b| = 1$, so we obtain with the help of Lemma 5.3 that $\|(\sigma_+D_{22b})(t)\| \leqslant C|\Gamma^b z_2(t)|\|(\sigma_-\nabla\Gamma^{a-b}\theta_2)(t)\|$, which can be bounded above by $CE_{2,1}^{1/2}(t)Q_{m,l-1}(t)$ with the help of Proposition 5.1(1). If now $a_4 - b_4 \geqslant l$, then $\Gamma^b = \partial_t^{|b|}$ and $a_4 = l$. Set $\chi = \partial_t^{|b|-1}z_2$. Assume first that $|b| \leqslant m-1$. If $t \geqslant 1$, we argue as we did for (5.27); we write $\partial_t\chi = t^{-1}X\chi - t^{-1}r\partial_r\chi$, and using Proposition 5.1(1), we obtain that $\|(\sigma_+D_{22b})(t)\| \leqslant CE_{m,1}^{1/2}(t)E_{m,l}^{1/2}(t)$, which clearly still holds if $t < 1$. If $|b| = m$, then $l = 0$ and $a = b$; in that case $\|(\sigma_+D_{22b})(t)\| \leqslant \|\partial_t^m z_2(t)\||(\sigma_+\nabla\theta_2)(t)|$ which can be bounded above by $CE_{m,0}^{1/2}(t)Q_{2,0}(t)$ with the help of Proposition 5.1(2). Summing up, we obtain that

$\|(\sigma_+ D_{22b})(t)\| \leqslant C(E^{1/2}_{m,l}(t)E^{1/2}_{m,1}(t) + E^{1/2}_{m,0}(t)Q_{2,0}(t))$ if $a_4 - b_4 \geqslant l$. From the estimates of the various $\sigma_+ D_{ijb}$, we conclude that

$$\big\|(\sigma_+ B_1)(t)\big\| \leqslant C\Psi_{m,l}(t). \tag{6.7}$$

Let us handle B_2. Set $F_{ijb} = \Gamma^b(\theta z_i)\nabla\Gamma^{a-b}\theta_j$, where $0 \neq b \leqslant a$. We can check with the help of Theorem 3.1(1), (2) and Proposition 5.1(1) that $\|\Gamma^b(\theta z_i)(t)\| \leqslant C\varepsilon\kappa_i(t)$, whence it follows with the help of Lemma 5.3 and Theorem 3.1(3) that $\|(\sigma_+ F_{i1b})(t)\| \leqslant C\varepsilon^2\langle t\rangle^{-2}\kappa_i(t)$. Now F_{i2b} is a linear combination of terms of the form $\mathcal{E}_{ikbd} = \Gamma^d z_i \Gamma^{b-d}\theta_k \nabla\Gamma^{a-b}\theta_2$, where $d \leqslant b$. With the help of Lemma 5.3, Theorem 3.1(2), (3) and Proposition 5.1(1), (2), (3), it is not hard to check that $\|(\sigma_+\mathcal{E}_{11bd})(t)\| \leqslant C\varepsilon^2\langle t\rangle^{-1}(E^{1/2}_{m,l}(t) + Q_{m,l-1}(t))$, $\|(\sigma_+\mathcal{E}_{21bd})(t)\| \leqslant C\varepsilon E^{1/2}_{m,l}(t)(E^{1/2}_{m,l}(t) + Q_{2,0}(t)\langle t\rangle^{-1})$, $\|(\sigma_+\mathcal{E}_{i2bd})(t)\| \leqslant CE^{1/2}_{m,l}(t)(H_{m,l-1}(t) + Q_{m,0}(t))\kappa_i(t)$. This yields an estimate of $\|(\sigma_+ F_{i2b})(t)\|$. Collecting the estimates of $\|(\sigma_+ F_{ijb})(t)\|$, we find that

$$\big\|(\sigma_+ B_2)(t)\big\| \leqslant C\left(\frac{\varepsilon^4\psi(\varepsilon)}{\langle t\rangle} + \varepsilon\Psi_{m,l}(t)\right). \tag{6.8}$$

Finally, with the help of Lemma 5.3 and Theorem 3.1(3), recalling that $C_1|\theta| \leqslant 1/2$, we find that

$$\big\|(\sigma_+ B_3)(t)\big\| \leqslant C\frac{\varepsilon}{\langle t\rangle^2}\big(\varepsilon + E^{1/2}_{1,0}(t)\big). \tag{6.9}$$

(6.6) immediately follows from (6.7)–(6.9). Now we have

$$\big|\hat{P}^c_{1j}(t)\big| \leqslant C\frac{\varepsilon}{\langle t\rangle^2}\kappa_j(t)E^{1/2}_{m,l}(t). \tag{6.10}$$

Indeed, with the help of Lemma 5.3 and Theorem 3.1(3), we see that $\|(\Gamma^d w_1 \cdot \nabla\Gamma^{c-d}z_j)(t)\| \leqslant C\varepsilon\langle t\rangle^{-2}\kappa_j(t)$ when $d \leqslant c$ if $d \neq 0$ when $j = 2$; (6.10) follows at once.

We now start with the long estimate of $\hat{P}^c_{2j}$. Set $\nu^{cd}_j = \Gamma^d w_2 \cdot \nabla\Gamma^{c-d}z_j$ where $d \leqslant c$, and where also $d \neq 0$ if $j = 2$. We claim that

$$\big\|\nu^{cd}_j(t)\big\| \leqslant \frac{C}{\langle t\rangle}H_{m,(l-1)_+}(t)\kappa_j(t) \quad \text{if } d_4 \leqslant (l-1)_+. \tag{6.11}$$

Indeed, let $R_0 \geqslant R$ be such that $z_1(t,x) = z_2(t,x) = 0$ when $r \geqslant R_0$ (cf. Lemma 5.3). Define the function χ_{R_0} by the relations $\chi_{R_0}(x) = 1$ if $R \leqslant r \leqslant R_0$, $\chi_{R_0}(x) = 0$ if $r > R_0$. Assume that $d_4 \leqslant (l-1)_+$. If $\chi = \chi_{R_0}$ in the case that $d_0 + d_1 + d_2 + d_3 = 0$ and $\chi \equiv 1$ otherwise, we obtain with the help of Proposition 5.1(3) (or (6) if $d = 0$), (5) that $\|\chi(\sigma_-\Gamma^d w_2)(t)\| \leqslant CH_{m,(l-1)_+}(t)$, which gives (6.11) in view of Proposition 5.1(1) if we assume also that $|c-d| \leqslant m-2$ in the case that $j = 2$. If now $|c-d| = m-1$ and $j = 2$, we have $|c| = m$, $|d| = 1$ since $d \neq 0$, and (6.11) is easily obtained with the help of Proposition 5.1(3), (4).

In order to be able to control $\hat{P}^c_{2j}$, we are going to study $\nu^{cd}_j\Gamma^c z_2$ when $d_4 = l \geqslant 1$. So from now on and until this task is completed, we assume that $d_4 = l \geqslant 1$, so that $\Gamma^d = \partial^\beta X^l$, $\Gamma^c = \partial^{\beta+\mu}X^l$. Then $\nu^{cd}_j = \partial^\beta X^l w_2 \cdot \nabla\partial^\mu z_j$, where $|\beta| + |\mu| + l \leqslant m$. If we set $\tilde{\nu}^{cd}_j = X\partial^\beta X^{l-1}w_2 \cdot \nabla\partial^\mu z_j$, it follows with the help of Proposition 5.1(3), (4) that

$$\big\|\big(\nu^{cd}_j - \tilde{\nu}^{cd}_j\big)(t)\big\| \leqslant \frac{C}{\langle t\rangle}H_{m,l-1}(t)\kappa_j(t). \tag{6.12}$$

Henceforth, let $R_1 \geqslant R$ be such that $z_2(t,x) = 0$ if $r \geqslant R_1$ (one could take e.g. $R_1 = R_0$ with R_0 as in Lemma 5.3). Set $\varphi^{cd}_j(t) = \int_{\mathcal{D}_{RR_1}}\tilde{\nu}^{cd}_j(t,x)\Gamma^c z_2(t,x)\,dx$, where $\mathcal{D}_{RR_1} = \{x \in \mathbb{R}^3,\ R < r < R_1\}$. We have written $\varphi^{cd}_j(t)$ as an integral over $\mathcal{D}_{RR_1}$ (and not over $\mathcal{D}_R$) because this will allow a somewhat more concise description of some long estimates. We can write $\varphi^{cd}_j(t) = d(tI^{cd}_j(t))/dt - I^{cd}_j(t) + J^{cd}_j(t) + N^{cd}_j(t)$, where

$$I^{cd}_j(t) = \int_{\mathcal{D}_{RR_1}}\big(\partial^\beta X^{l-1}w_2 \cdot \nabla\partial^\mu z_j\big)(t,x)\Gamma^c z_2(t,x)\,dx,$$

$$J^{cd}_j(t) = \int_{\mathcal{D}_{RR_1}}\big(r\partial_r\partial^\beta X^{l-1}w_2 \cdot \nabla\partial^\mu z_j\big)(t,x)\Gamma^c z_2(t,x)\,dx,$$

$$N_j^{cd}(t) = -t \int\limits_{\mathcal{D}_{RR_1}} \partial^\beta X^{l-1} w_2(t,x) \cdot \partial_t\big(\Gamma^c z_2 \nabla \partial^\mu z_j\big)(t,x)\,dx.$$

It is convenient to set $\tilde{\kappa}_1 = \varepsilon$, $\tilde{\kappa}_2 = E_{m,0}^{1/2}$. Let us handle I_j^{cd}. If $j=1$ or $|\mu| \leqslant m-2$, we estimate the second factor under the integral sign in L^∞ and each other factor in L^2, using Proposition 5.1(1), (3), (5). We obtain that

$$\big|I_j^{cd}(t)\big| \leqslant \frac{C}{\langle t\rangle} H_{m,l-1}(t)\tilde{\kappa}_j(t) E_{m,l}^{1/2}(t) \tag{6.13}$$

if $j=1$ or $|\mu| \leqslant m-2$. If now $j=2$ and $|\mu| = m-1$, we have $l=1$, $\beta=0$, and we estimate the first factor under the integral sign in L^∞ and each other factor in L^2, and we find with the help of Proposition 5.1(6) that (6.13) still holds. Using similar arguments for J_j^{cd}, we easily obtain that also

$$\big|J_j^{cd}(t)\big| \leqslant \frac{C}{\langle t\rangle} H_{m,l-1}(t)\tilde{\kappa}_j(t) E_{m,l}^{1/2}(t). \tag{6.14}$$

Write $N_j^{cd} = N_{j1}^{cd} + N_{j2}^{cd}$, where

$$N_{j1}^{cd}(t) = -t \int\limits_{\mathcal{D}_{RR_1}} \big(\partial^\beta X^{l-1} w_2 \cdot \partial_t \nabla \partial^\mu z_j\big)(t,x) \Gamma^c z_2(t,x)\,dx,$$

$$N_{j2}^{cd}(t) = -t \int\limits_{\mathcal{D}_{RR_1}} \big(\partial^\beta X^{l-1} w_2 \cdot \nabla \partial^\mu z_j\big)(t,x) \partial_t \Gamma^c z_2(t,x)\,dx.$$

Notice that $N_{11}^{cd} \equiv 0$. As for N_{21}^{cd}, if $|\mu| \leqslant m-2$, we may estimate the first factor under the integral sign in L^∞ and each other factor in L^2, writing that $t\|\partial_t \nabla \partial^\mu z_2(t)\| \leqslant \|X \nabla \partial^\mu z_2(t)\| + \|r\partial_r \nabla \partial^\mu z_2(t)\|$. We readily obtain with the help of Proposition 5.1(3), (4):

$$\big|N_{21}^{cd}(t)\big| \leqslant \frac{C}{\langle t\rangle} H_{m,l-1}(t) E_{m,1}^{1/2}(t) E_{m,l}^{1/2}(t) \quad \text{if } |\mu| \leqslant m-2. \tag{6.15}$$

If now $|\mu| = m-1$, then $l=1$ and $\beta=0$, so $\Gamma^c = \partial^\mu X$. Set $g_1^{cd} = t\partial_t \partial^\mu z_2$, $g_2^{cd} = r\partial_r \partial^\mu z_2$, $g_3^{cd} = (m-1)\partial^\mu z_2$, $G_j^{cd}(t) = -t\int_{\mathcal{D}_{RR_1}} W_2(t,r)(\partial_t \partial_r \partial^\mu z_2 g_j^{cd})(t,x)\,dx$, so that $N_{21}^{cd} = \sum_{1\leqslant j\leqslant 3} G_j^{cd}$. Now observe that if $r \mapsto g(r)$ and $x \mapsto F(x)$ are smooth scalar functions such that the function $x \mapsto g(r)F(x)$ vanishes when $r=R$ and when r is large, integration by parts readily shows that

$$\int\limits_{\mathcal{D}_R} g(r)\partial_r F(x)\,dx = -\int\limits_{\mathcal{D}_R} \nabla \cdot \left(g(r)\frac{x}{r}\right) F(x)\,dx. \tag{6.16}$$

Using (6.16), we obtain that $G_1^{cd}(t) = 2^{-1}t^2 \int_{\mathcal{D}_{RR_1}} (\partial_r + 2/r) W_2(t,r)(\partial_t \partial^\mu z_2)^2(t,x)\,dx$. Let us check that

$$\big\|\partial_t \partial^\mu z_2(t)\big\| \leqslant \frac{C}{\langle t\rangle}\left(\frac{\varepsilon}{\langle t\rangle} + H_{m,0}(t)\right)\big(\varepsilon + E_{m,0}^{1/2}(t)\big) \quad \text{if } |\mu| \leqslant m-1. \tag{6.17}$$

Indeed, it follows from (5.11) that $\partial_t \partial^\mu z_2 = -w \cdot \nabla \partial^\mu z_2 + \hat{g}^{\mu^*}$ if $\Gamma^{\mu^*} = \partial^\mu$. (6.17) easily follows with the help of Lemma 5.3, Theorem 3.1(3) and Proposition 5.1(3), (4). Now using Proposition 5.1(6) and (6.17), we find that $|G_1^{cd}(t)| \leqslant C\langle t\rangle^{-1}(Q_{2,0}(t) + E_{0,0}^{1/2}(t))(\varepsilon\langle t\rangle^{-1} + H_{m,0}(t))^2(\varepsilon + E_{m,0}^{1/2}(t))^2$. From (5.11) it also follows that $\partial_r \partial_t \partial^\mu z_2 = -W\partial_r^2 \partial^\mu z_2 + \sum_{1\leqslant j\leqslant 3} x_j \hat{g}^{\mu^j}/r$ if $\Gamma^{\mu^j} = \partial_j \partial^\mu$. Hence $G_2^{cd} = \sum_{1\leqslant k\leqslant 2} G_{2k}^{cd}$, where $G_{21}^{cd}(t) = 2^{-1}t\int_{\mathcal{D}_{RR_1}} r(W_2 W)(t,r)\partial_r(\partial_r \partial^\mu z_2)^2(t,x)\,dx$ and $G_{22}^{cd}(t) = -t\sum_{1\leqslant j\leqslant 3}\int_{\mathcal{D}_{RR_1}} W_2(t,r)x_j(\hat{g}^{\mu^j}\partial_r \partial^\mu z_2)(t,x)\,dx$. Applying (6.16) to $G_{21}^{cd}(t)$ and estimating $x \mapsto (\nabla \cdot (W_2 W x))(t,x)$ in L^∞, we obtain with the help of Theorem 3.1(3) and Proposition 5.1(4), (6) that $|G_{21}^{cd}(t)| \leqslant C\langle t\rangle^{-1}(\varepsilon\langle t\rangle^{-1} + Q_{1,0}(t))(Q_{2,0}(t) + E_{0,0}^{1/2}(t))E_{m,0}(t)$. Using Theorem 3.1(3) and Proposition 5.1(1), (3), (4), (5), (6), we estimate each of the last two factors under the integral sign of $G_{22}^{cd}(t)$ in L^2 and each other factor in L^∞, and we find that $|G_{22}^{cd}(t)| \leqslant C\langle t\rangle^{-1} Q_{1,0}(t)(\varepsilon\langle t\rangle^{-1} +$

$Q_{m,0}(t) + E_{m-1,0}^{1/2}(t))(\varepsilon + E_{m,0}^{1/2}(t))E_{m,0}^{1/2}(t)$. So we now have an upper bound for $|G_2^{cd}(t)|$ by adding the upper bounds we have just obtained for $|G_{21}^{cd}(t)|$ and $|G_{22}^{cd}(t)|$. Let us handle G_3^{cd}. Using (6.16), we obtain that $G_3^{cd} = \sum_{1\leqslant k\leqslant 2} G_{3k}^{cd}$, with $G_{31}^{cd}(t) = 2(m-1)t\int_{\mathcal{D}_{RR_1}} r^{-1} W_2(t,r)(\partial^\mu z_2 \partial_t \partial^\mu z_2)(t,x)\,dx$ and $G_{32}^{cd}(t) = (m-1)t\int_{\mathcal{D}_{RR_1}} \partial_r(W_2(t,r)\partial^\mu z_2(t,x))\partial_t\partial^\mu z_2(t,x)\,dx$. In $G_{31}^{cd}(t)$, we estimate each of the last two factors under the integral sign in L^2 and each other factor in L^∞. For $G_{32}^{cd}(t)$, we write $\partial_r(W_2(t,r)\partial^\mu z_2(t,x)) = \partial_r W_2(t,r)\partial^\mu z_2(t,x) + W_2(t,r)\partial_r\partial^\mu z_2(t,x)$ and estimate each of the factors $\partial^\mu z_2$, $\partial_r\partial^\mu z_2$, $\partial_t\partial^\mu z_2$ under the integral sign in L^2 and each other factor in L^∞. Using Proposition 5.1(4), (6) and (6.17), we obtain that $|G_3^{cd}(t)| \leqslant C\langle t\rangle^{-1}(Q_{2,0}(t) + E_{0,0}^{1/2}(t))E_{m,0}^{1/2}(t)(\varepsilon + E_{m,0}^{1/2}(t))(\varepsilon\langle t\rangle^{-1} + H_{m,0}(t))$. Collecting the estimates of the G_j^{cd}, we find that

$$\left|N_{21}^{cd}(t)\right| \leqslant \frac{C}{\langle t\rangle}\left(\frac{\varepsilon}{\langle t\rangle} + H_{m,0}(t)\right)^2 \left(\varepsilon + E_{m,0}^{1/2}(t)\right)\left(\varepsilon^2 + E_{m,0}^{1/2}(t)\right) \quad \text{if } |\mu| = m-1. \tag{6.18}$$

Let us pass to N_{j2}^{cd}. Using (5.11), we can write $N_{j2}^{cd} = \sum_{1\leqslant k\leqslant 5} N_{j2k}^{cd}$, where $N_{j21}^{cd}(t) = t\int_{\mathcal{D}_{RR_1}} (\partial^\beta X^{l-1} w_2 \cdot \nabla\partial^\mu z_j)(t,x)(w\cdot\nabla\partial^{\beta+\mu}X^l z_2)(t,x)\,dx$, $N_{j2k}^{cd}(t) = -t\int_{\mathcal{D}_{RR_1}} (\partial^\beta X^{l-1} w_2 \cdot \nabla\partial^\mu z_j)(t,x)\tilde{g}_{k-1}^c(t,x)\,dx$ if $k \geqslant 2$, where $\tilde{g}_1^c = \hat{g}_{11}^c$, $\tilde{g}_2^c = \hat{g}_{12}^c$, $\tilde{g}_3^c = \hat{g}_{21}^c$, $\tilde{g}_4^c = \hat{g}_{22}^c$. If $|c| \leqslant m-1$, we estimate the first and third factors under the integral sign in L^∞ and each other factor in L^2, and we find with the help of Theorem 3.1(3) and Proposition 5.1(3), (4), (6) that

$$\left|N_{j21}^{cd}(t)\right| \leqslant \frac{C}{\langle t\rangle} H_{m-1,l-1}(t)\tilde{\kappa}_j(t)\left(\frac{\varepsilon}{\langle t\rangle} + Q_{1,0}(t)\right)E_{m,l}^{1/2}(t) \quad \text{if } |c| \leqslant m-1. \tag{6.19}$$

If now $|c| = m$, we obtain by integration by parts that $N_{j21}^{cd} = \sum_{1\leqslant k\leqslant 3} N_{j21k}^{cd}$, where

$$N_{j211}^{cd}(t) = -t\int_{\mathcal{D}_{RR_1}} \left(\partial^{\beta+\mu}X^l z_2 (w\cdot\nabla)\partial^\beta X^{l-1} w_2 \cdot \nabla\partial^\mu z_j\right)(t,x)\,dx,$$

$$N_{j212}^{cd}(t) = -t\int_{\mathcal{D}_{RR_1}} \left(\partial^{\beta+\mu}X^l z_2 (w\cdot\nabla)\nabla\partial^\mu z_j \cdot \partial^\beta X^{l-1} w_2\right)(t,x)\,dx,$$

$$N_{j213}^{cd}(t) = -t\int_{\mathcal{D}_{RR_1}} \left(\partial^{\beta+\mu}X^l z_2 (\nabla\cdot w)\left(\partial^\beta X^{l-1} w_2 \cdot \nabla\partial^\mu z_j\right)\right)(t,x)\,dx.$$

But we have

$$\begin{gathered}\left|N_{j21k}^{cd}(t)\right| \leqslant \frac{C}{\langle t\rangle} E_{m,l}^{1/2}(t) H_{m,l-1}(t)\left(\frac{\varepsilon}{\langle t\rangle} + Q_{2,0}(t) + E_{0,0}^{1/2}(t)\right)\tilde{\kappa}_j(t) \\ \text{if } |c| = m, \text{ and if furthermore } |\mu| \leqslant m-2 \text{ when } j = k = 2.\end{gathered} \tag{6.20}$$

Actually (6.20) is easily obtained with the help of Lemma 5.3, Theorem 3.1(3) and Proposition 5.1(1), (3), (4), (5), (6). Indeed, if furthermore $|\beta| + l \leqslant m-1$ when $k = 1$, we estimate the factors containing z_2 or z_j (under the integral sign in the definition of N_{j21k}^{cd}) in L^2 and each other factor in L^∞. If now $k = 1$ and $|\beta| + l = m$, then $\mu = 0$, and in this case we estimate the first and third factors (under the integral sign in the definition of N_{j211}^{cd}) in L^2 and each other factor in L^∞. (6.20) follows easily. (6.20) yields an upper bound of $|N_{j21}^{cd}(t)|$ if $|c| = m$ provided furthermore $|\mu| \leqslant m-2$ when $j = 2$. If now $|c| = m$ and $|\mu| = m-1$, we have $\beta = 0$, $l = 1$, and so $N_{221}^{cd}(t) = t\int_{\mathcal{D}_{RR_1}} (W_2 W)(t,r)(\partial_r\partial^\mu z_2 \partial_r\partial^\mu X z_2)(t,x)\,dx$. With G_{21}^{cd} as above, we have $N_{221}^{cd} = G_{21}^{cd} + G_4^{cd} + G_5^{cd}$, with $G_4^{cd}(t) = (|\mu|+1)t\int_{\mathcal{D}_{RR_1}} (W_2 W)(t,r)(\partial_r\partial^\mu z_2)^2(t,x)\,dx$, $G_5^{cd}(t) = t^2\int_{\mathcal{D}_{RR_1}} (W_2 W)(t,r)(\partial_r\partial^\mu z_2 \partial_t\partial_r\partial^\mu z_2)(t,x)\,dx$. In $G_4^{cd}(t)$, we estimate the first two factors under the integral sign in L^∞, and we obtain with the help of Theorem 3.1(3) and Proposition 5.1(6) that $|G_4^{cd}(t)| \leqslant C\langle t\rangle^{-1} Q_{1,0}(t)E_{m,0}(t)(\varepsilon\langle t\rangle^{-1} + Q_{1,0}(t))$. Now it follows from (5.11) that $G_5^{cd}(t) = \sum_{1\leqslant j\leqslant 2} G_{5j}^{cd}(t)$, where $G_{51}^{cd}(t) = -2^{-1}t^2\int_{\mathcal{D}_{RR_1}} (W_2 W^2)(t,r)\partial_r(\partial_r\partial^\mu z_2(t,x))^2\,dx$ and $G_{52}^{cd}(t) = t^2\sum_{1\leqslant i\leqslant 3}\int_{\mathcal{D}_{RR_1}} (W_2 W)(t,r)r^{-1}x_i(\partial_r\partial^\mu z_2 \hat{g}^{\mu^i})(t,x)\,dx$, in which $\Gamma^{\mu^i} = \partial_i\partial^\mu$. In $G_{51}^{cd}(t)$, we use (6.16), and then estimate the function $x \mapsto (\partial_r\partial^\mu z_2)^2(t,x)$ in L^1 and each other factor under the integral sign in L^∞ with the help of Theorem 3.1(3) and Proposition 5.1(4), (6). In $G_{52}^{cd}(t)$ we estimate each of the last two factors under the integral sign in L^2 and each other factor in L^∞, using Theorem 3.1(3) and Proposition 5.1(1), (5), (6). This gives that $|G_5^{cd}(t)| \leqslant C\langle t\rangle^{-1}(\varepsilon\langle t\rangle^{-1} + Q_{1,0}(t))E_{m,0}^{1/2}(t)((Q_{2,0}(t) + E_{0,0}^{1/2}(t))E_{m,0}^{1/2}(t)(\varepsilon\langle t\rangle^{-1} + Q_{1,0}(t)) + Q_{1,0}(t)(\varepsilon\langle t\rangle^{-1} + H_{m,0}(t))(\varepsilon + E_{m,0}^{1/2}(t)))$. From the estimates of G_{21}^{cd}, G_4^{cd}, G_5^{cd}, we obtain that

$$\begin{aligned}&\left|N_{221}^{cd}(t)\right| \leqslant \frac{C}{\langle t\rangle}\left(\frac{\varepsilon}{\langle t\rangle} + Q_{1,0}(t)\right)H_{2,0}(t)E_{m,0}^{1/2}(t)\left(E_{m,0}^{1/2}(t) + \left(\frac{\varepsilon}{\langle t\rangle} + H_{m,0}(t)\right)\left(\varepsilon + E_{m,0}^{1/2}(t)\right)\right)\\ &\quad \text{if } |c| = m \text{ and } |\mu| = m-1.\end{aligned} \tag{6.21}$$

Let us pass to N_{j2k}^{cd}, where $k \in \{2,3\}$. We estimate the first factor under the integral sign in L^∞ and each other factor in L^2, making use of Theorem 3.1(3) and Proposition 5.1(3), (4). We obtain

$$\left|N_{j2k}^{cd}(t)\right| \leqslant \frac{C}{\langle t\rangle^2}\varepsilon H_{m,l-1}(t)\tilde\kappa_j(t)\kappa_{k-1}(t) \quad \text{if } k \in \{2,3\}. \tag{6.22}$$

We now consider N_{j2k}^{cd} when $k \in \{4,5\}$. Recall that $\Gamma^c = \partial^{\beta+\mu}X^l$, $|c| \leqslant m$, and $\tilde g_{k-1}^c = \sum_{h\leqslant c}\binom{c}{h}\Gamma^h w_2 \cdot \nabla\Gamma^{c-h}z_{k-3}$ (so that $\Gamma^h = \partial^\nu X^\lambda$ with $\nu \leqslant \beta+\mu$ and $\lambda \leqslant l$), where furthermore $h \neq 0$ in the sum if $k=5$. For $k \in \{4,5\}$, set $\tilde g_{k-1}^{c,-} = -\sum_{h\leqslant c;\, \lambda\leqslant l-1}\binom{c}{h}\Gamma^h w_2 \cdot \nabla\Gamma^{c-h}z_{k-3}$, where furthermore $h \neq 0$ in the sum if $k=5$; set also $\tilde g_{k-1}^{c,+} = \tilde g_{k-1}^c - \tilde g_{k-1}^{c,-}$, and define $N_{j2k}^{cd,\pm}$ as N_{j2k}^{cd} ($k \in \{4,5\}$), but with $\tilde g_{k-1}^c$ replaced by $\tilde g_{k-1}^{c,\pm}$. If $\lambda \leqslant l-1$, we obtain with the help of Proposition 5.1(3), (4) that $\|(\Gamma^h w_2 \cdot \nabla\Gamma^{c-h}z_{k-3})(t)\| \leqslant C\langle t\rangle^{-1}H_{m,l-1}(t)\kappa_{k-3}(t)$, whence it follows, again with the help of Proposition 5.1(3), (4), that

$$\left|N_{j2k}^{cd,-}(t)\right| \leqslant \frac{C}{\langle t\rangle}H_{m,l-1}^2(t)\tilde\kappa_j(t)\kappa_{k-3}(t). \tag{6.23}$$

Now, if $\lambda\ (= h_4) = l$, we have $\Gamma^h = \partial^\nu X^l$ with $\nu \leqslant \beta+\mu$, and so $N_{j2k}^{cd,+}(t) = -t\sum_{h\leqslant c;\, h_4=l;\, 1\leqslant n\leqslant 3}\binom{c}{h}S_{jkn}^{cdh}(t)$, where

$$S_{jk1}^{cdh}(t) = t\int_{\mathcal{D}_{RR_1}}\left(\partial^\beta X^{l-1}w_2 \cdot \nabla\partial^\mu z_j\right)(t,x)\left(\partial_t\partial^\nu X^{l-1}w_2 \cdot \nabla\partial^{\beta+\mu-\nu}z_{k-3}\right)(t,x)\,dx,$$

$$S_{jk2}^{cdh}(t) = \int_{\mathcal{D}_{RR_1}}\left(\partial^\beta X^{l-1}w_2 \cdot \nabla\partial^\mu z_j\right)(t,x)r\left(\partial_r\partial^\nu X^{l-1}w_2 \cdot \nabla\partial^{\beta+\mu-\nu}z_{k-3}\right)(t,x)\,dx,$$

$$S_{jk3}^{cdh}(t) = |\nu|\int_{\mathcal{D}_{RR_1}}\left(\partial^\beta X^{l-1}w_2 \cdot \nabla\partial^\mu z_j\right)(t,x)\left(\partial^\nu X^{l-1}w_2 \cdot \nabla\partial^{\beta+\mu-\nu}z_{k-3}\right)(t,x)\,dx.$$

Let us start with the case $n=1$, in which finding a suitable bound is less simple because of the additional factor t in front of the integral in the definition of S_{jk1}^{cdh}. It is convenient to write $\Gamma^c = \partial^\xi X^l$ (recall that $\Gamma^d = \partial^\beta X^l$). Then we have (for each $j \in \{1,2\}$ and each $k \in \{4,5\}$): $t\sum_{d,h\leqslant c;\, d_4=h_4=l}\binom{c}{d}\binom{c}{h}S_{jk1}^{cdh}(t) = \sum_{\beta,\nu\leqslant\xi}\Phi_{j,k-3}^{\xi\beta\nu}(t)$, where $\Phi_{j,k-3}^{\xi\beta\nu}(t) = \binom{\xi}{\beta}\binom{\xi}{\nu}t^2\int_{\mathcal{D}_{RR_1}}(\partial^\beta X^{l-1}w_2 \cdot \nabla\partial^{\xi-\beta}z_j)(t,x)(\partial_t\partial^\nu X^{l-1}w_2 \cdot \nabla\partial^{\xi-\nu}z_{k-3})(t,x)\,dx$. Now $\Phi_{j,k-3}^{\xi\beta\nu} + \Phi_{k-3,j}^{\xi\nu\beta} = \binom{\xi}{\beta}\binom{\xi}{\nu}(dF_{jk1}^{\xi\beta\nu}/dt - 2t^{-1}F_{jk1}^{\xi\beta\nu} + F_{jk2}^{\xi\beta\nu} + F_{jk3}^{\xi\beta\nu})$, with

$$F_{jk1}^{\xi\beta\nu}(t) = t^2\int_{\mathcal{D}_{RR_1}}\left(\partial^\beta X^{l-1}w_2 \cdot \nabla\partial^{\xi-\beta}z_j\right)(t,x)\left(\partial^\nu X^{l-1}w_2 \cdot \nabla\partial^{\xi-\nu}z_{k-3}\right)(t,x)\,dx,$$

$$F_{jk2}^{\xi\beta\nu}(t) = -t^2\int_{\mathcal{D}_{RR_1}}\left(\partial^\beta X^{l-1}w_2 \cdot \partial_t\nabla\partial^{\xi-\beta}z_j\right)(t,x)\left(\partial^\nu X^{l-1}w_2 \cdot \nabla\partial^{\xi-\nu}z_{k-3}\right)(t,x)\,dx,$$

$$F_{jk3}^{\xi\beta\nu}(t) = -t^2\int_{\mathcal{D}_{RR_1}}\left(\partial^\beta X^{l-1}w_2 \cdot \nabla\partial^{\xi-\beta}z_j\right)(t,x)\left(\partial^\nu X^{l-1}w_2 \cdot \partial_t\nabla\partial^{\xi-\nu}z_{k-3}\right)(t,x)\,dx.$$

To handle $F^{\xi\beta\nu}_{jk1}$, we estimate the first and third factors under the integral sign in L^∞ and each other factor in L^2. With the help of Proposition 5.1(3), (4), we obtain

$$\left|F^{\xi\beta\nu}_{jk1}(t)\right| \leqslant \frac{C}{\langle t\rangle^2} t^2 H^2_{m,l-1}(t)\tilde\kappa_j(t)\tilde\kappa_{k-3}(t). \tag{6.24}$$

We have $F^{\xi\beta\nu}_{1k2} = 0$. On the other hand, if $|\xi-\beta| \leqslant m-2$, we can estimate in L^∞ the first and third factors under the integral sign of $F^{\xi\beta\nu}_{2k2}(t)$, making use of Proposition 5.1(3), (4), and each other factor in L^2, using (6.17) for the third factor. This gives

$$\left|F^{\xi\beta\nu}_{2k2}(t)\right| \leqslant \frac{C}{\langle t\rangle} H^2_{m,l-1}(t)\left(\frac{\varepsilon}{\langle t\rangle} + H_{m,0}(t)\right)\left(E^{1/2}_{m,0}(t)+\varepsilon\right)\tilde\kappa_{k-3}(t) \quad \text{if } |\xi-\beta| \leqslant m-2. \tag{6.25}$$

If now $|\xi-\beta| = m-1$, recall that $|\xi| \leqslant m-l$ and $l \geqslant 1$; so actually $|\xi| = m-1$, $\beta = 0$, $l = 1$. In that case we integrate by parts and write $F^{\xi 0\nu}_{2k2} = \sum_{1\leqslant i\leqslant 3} F^{\xi\nu}_{2k2i}$, where

$$F^{\xi\nu}_{2k21}(t) = t^2 \int_{\mathcal{D}_{RR_1}} \left((\nabla\cdot w_2)\partial_t\partial^\xi z_2\right)(t,x)\left(\partial^\nu w_2\cdot\nabla\partial^{\xi-\nu} z_{k-3}\right)(t,x)\,dx,$$

$$F^{\xi\nu}_{2k22}(t) = t^2 \int_{\mathcal{D}_{RR_1}} \left(\left(w_2\cdot\nabla\partial^\nu w_2\right)\cdot\nabla\partial^{\xi-\nu} z_{k-3}\right)(t,x)\partial_t\partial^\xi z_2(t,x)\,dx,$$

$$F^{\xi\nu}_{2k23}(t) = t^2 \int_{\mathcal{D}_{RR_1}} \left(\partial^\nu w_2\cdot\left((w_2\cdot\nabla)\nabla\partial^{\xi-\nu} z_{k-3}\right)\right)(t,x)\partial_t\partial^\xi z_2(t,x)\,dx.$$

To handle $F^{\xi\nu}_{2k21}$, we estimate the first and third factors under the integral sign in L^∞ and each other factor in L^2. Thanks to Proposition 5.1(3), (4) and (6.17), we obtain that $|F^{\xi\nu}_{2k21}(t)| \leqslant C\langle t\rangle^{-1}(Q_{2,0}(t) + E^{1/2}_{0,0}(t))(\varepsilon\langle t\rangle^{-1} + H_{m,0}(t))H_{|\nu|+1,0}(t)(E^{1/2}_{m,0}(t) + \varepsilon)\tilde\kappa_{k-3}(t)$. Let us consider now $F^{\xi\nu}_{2k22}$. If $|\nu| \leqslant m-2$, we estimate the first two factors under the integral sign in L^∞ and each other factor in L^2, and therefore we find that $|F^{\xi\nu}_{2k22}(t)| \leqslant C\langle t\rangle^{-1}Q_{1,0}(t)(\varepsilon\langle t\rangle^{-1} + H_{m,0}(t))H_{|\nu|+2,0}(t)(E^{1/2}_{m,0}(t)+\varepsilon)\tilde\kappa_{k-3}(t)$ if $|\nu| \leqslant m-2$. If now $|\nu| = m-1$ (hence $\nu = \xi$), we estimate the first and third factors under the integral sign in L^∞ and each other factor in L^2, using Proposition 5.1(1), (5), (6) and (6.17), and we find that $|F^{\xi\nu}_{2k22}(t)| \leqslant C\langle t\rangle^{-1}Q_{1,0}(t)H_{m,0}(t)(\varepsilon\langle t\rangle^{-1} + H_{m,0}(t))(E^{1/2}_{m,0}(t) + \varepsilon)\tilde\kappa_{k-3}(t)$ if $|\nu| = m-1$. To handle $F^{\xi\nu}_{2k23}$ (with $|\xi-\nu| \leqslant m-2$ if furthermore $k = 5$), we estimate the first and second factors under the integral sign in L^∞ and each other factor in L^2 with the help of Proposition 5.1(3), (4), (6) and (6.17), and we obtain that $|F^{\xi\nu}_{2k23}(t)| \leqslant C\langle t\rangle^{-1}Q_{1,0}(t)(\varepsilon\langle t\rangle^{-1} + H_{m,0}(t))(E^{1/2}_{m,0}(t)+\varepsilon)H_{|\nu|+1,0}(t)\tilde\kappa_{k-3}(t)$ if $k = 4$, or if $|\xi-\nu| \leqslant m-2$ and $k = 5$. Collecting estimates, we find that

$$\begin{aligned}&\left|F^{\xi 0\nu}_{2k2}(t)\right| \leqslant \frac{C}{\langle t\rangle}\left(Q_{2,0}(t) + E^{1/2}_{0,0}(t)\right)\left(\frac{\varepsilon}{\langle t\rangle} + H_{m,0}(t)\right)H_{m,0}(t)\left(E^{1/2}_{m,0}(t)+\varepsilon\right)\tilde\kappa_{k-3}(t),\\ &\quad\text{when } |\xi| = m-1 \text{ and } l = 1, \text{ if } k = 4 \text{ or if } k = 5 \text{ and } \nu \neq 0.\end{aligned} \tag{6.26}$$

Finally, if $|\xi| = m-1$ and $l = 1$, $F^{\xi 00}_{252}$ is equal to $-G^{cd}_5$ above (in the study of N^{cd}_{221}) with W replaced by W_2 and μ by ξ, so by arguments similar to those used above for G^{cd}_5, we find that

$$\begin{aligned}&\left|F^{\xi 00}_{252}(t)\right| \leqslant \frac{C}{\langle t\rangle} Q_{1,0}(t)\left(Q_{2,0}(t) + E^{1/2}_{0,0}(t)\right)E^{1/2}_{m,0}(t)\left(\frac{\varepsilon}{\langle t\rangle} + H_{m,0}(t)\right)\left(E^{1/2}_{m,0}(t)+\varepsilon\right)\\ &\quad\text{if } |\xi| = m-1 \text{ and } l = 1.\end{aligned} \tag{6.27}$$

Set $k_1 = k-3$, $j_1 = j+3$ (recall that $k \in \{4,5\}$ now and $j \in \{1,2\}$). Since $F^{\xi\beta\nu}_{jk3} = F^{\xi\nu\beta}_{k_1 j_1 2}$, (6.25)–(6.27) yield estimates for $F^{\xi\beta\nu}_{jk3}$. Summing up, if we set $\mathcal{A}^{\xi\beta\nu}_{jk} = \binom{\xi}{\beta}\binom{\xi}{\nu}F^{\xi\beta\nu}_{jk1}$, $\mathcal{B}^{\xi\beta\nu}_{jk} = \binom{\xi}{\beta}\binom{\xi}{\nu}(-2t^{-1}F^{\xi\beta\nu}_{jk1} + F^{\xi\beta\nu}_{jk2} + F^{\xi\beta\nu}_{jk3})$, then $\mathcal{A}^{\xi\beta\nu}_{jk}$, $\mathcal{B}^{\xi\beta\nu}_{jk}$ can be estimated by (6.24)–(6.27). Notice that $\mathcal{A}^{\xi\beta\nu}_{jk}(0) = 0$. But $\Phi^{\xi\beta\nu}_{j,k-3} + \Phi^{\xi\nu\beta}_{k-3,j} = d\mathcal{A}^{\xi\beta\nu}_{jk}/dt + \mathcal{B}^{\xi\beta\nu}_{jk}$, and $t\sum_{1\leqslant j,k-3\leqslant 2;\, d,h\leqslant c;\, d_4=h_4=l}\binom{c}{d}\binom{c}{h}S^{cdh}_{jk1}=\sum_{1\leqslant j,k-3\leqslant 2;\,\beta,\nu\leqslant\xi}(\Phi^{\xi\beta\nu}_{j,k-3}+\Phi^{\xi\nu\beta}_{k-3,j})/2$, so to complete the estimate of $\sum_{1\leqslant j,k-3\leqslant 2;\, d\leqslant c}\binom{c}{d}N^{cd,+}_{j2k}$, it remains to handle $-t\sum_{1\leqslant j,k-3\leqslant 2;\, d,h\leqslant c;\, d_4=h_4=l}\binom{c}{d}\binom{c}{h}S^{cdh}_{jkn}$ for $n\in\{2,3\}$. Let us start with S^{cdh}_{jk2}. If $|\nu|+l\leqslant m-1$, we estimate the first and fourth factors under the integral sign in L^∞ and the second and fifth factors in L^2, using Proposition 5.1(3), (4) (the third factor is r). If $|\nu|+l=m$, then $\nu=\beta+\mu$ and we estimate the first and fifth factors under the integral sign in L^∞ and the second and fourth factors in L^2, using Proposition 5.1(1), (3), (4), (5). Altogether we obtain that $|S^{cdh}_{jk2}(t)|\leqslant C\langle t\rangle^{-2}H^2_{m,l-1}(t)\tilde\kappa_j(t)\tilde\kappa_{k-3}(t)$. As for S^{cdh}_{jk3}, we estimate the first and third factors under the integral sign in L^∞ and each other factor in L^2, using Proposition 5.1(3), (4). We obtain that $|S^{cdh}_{jk3}(t)|\leqslant C\langle t\rangle^{-2}H^2_{m,l-1}(t)\tilde\kappa_j(t)\tilde\kappa_{k-3}(t)$. Using the estimates of $\mathcal{A}^{\xi\beta\nu}_{jk}$, $\mathcal{B}^{\xi\beta\nu}_{jk}$, S^{cdh}_{jkn}, we find that $\sum_{1\leqslant j,k-3\leqslant 2;\, d\leqslant c;\, d_4=l\geqslant 1}\binom{c}{d}N^{cd,+}_{j2k}(t)=dM^c_1(t)/dt+M^c_2(t)$, where $M^c_1, M^c_2\in C^\infty([0,T])$, $M^c_1(0)=0$, and

$$\left|M^c_1(t)\right|\leqslant C\varepsilon^6\psi^2(\varepsilon),\qquad \left|M^c_2(t)\right|\leqslant C\left(\frac{\varepsilon^2}{\langle t\rangle}E_{m,l}(t)+\frac{\varepsilon^6}{\langle t\rangle^5}\right). \tag{6.28}$$

By assumption, $E^{1/2}_{m,\max(l,1)}(t)\leqslant\varepsilon^2\psi(\varepsilon)$ if $0\leqslant t\leqslant T$. Hence it follows from Proposition 5.2 that

$$Q_{m,\lambda-1}(t)\leqslant C\big(E^{1/2}_{m,\lambda}(t)+\varepsilon^2\langle t\rangle^{-2}\big)\quad\text{if } 1\leqslant\lambda\leqslant l. \tag{6.29}$$

In order to complete the proof of Proposition 5.4(1, l), we are going to use (6.29) in the estimates we have proved in this section. From (6.1), (6.2), (6.29) we obtain that $\sum_{1\leqslant j\leqslant 12}|P^a_j(t)|\leqslant C\varepsilon E_{m,l}(t)\langle t\rangle^{-1}$, whereas (6.6), (6.29) yield that $|P^a_{13}(t)|\leqslant C(\varepsilon E_{m,l}(t)\langle t\rangle^{-1}+\varepsilon^4\psi(\varepsilon)\langle t\rangle^{-3}+\varepsilon^6\psi^2(\varepsilon)\langle t\rangle^{-2})$. (6.10) gives that $\sum_{1\leqslant j\leqslant 2}|\hat P^c_{1j}(t)|\leqslant C\varepsilon^4\psi(\varepsilon)\langle t\rangle^{-2}$. If $l=0$, we obtain with the help of (6.11), (6.29) that $|\sum_{1\leqslant j\leqslant 2}\hat P^c_{2j}(t)|\leqslant C(\varepsilon E^{1/2}_{m,0}(t)E^{1/2}_{m,1}(t)\langle t\rangle^{-1}+\varepsilon^5\psi(\varepsilon)\langle t\rangle^{-3})$. On the other hand, if $l\geqslant 1$, (6.11)–(6.15), (6.18)–(6.23), (6.28) and (6.29) yield that $\sum_{1\leqslant j\leqslant 2}\hat P^c_{2j}(t)=dA^c(t)/dt+B^c(t)$ with $A^c, B^c\in C^\infty([0,T])$, $A^c(0)=0$, and $|A^c(t)|\leqslant C\varepsilon^5\psi^2(\varepsilon)$, $|B^c(t)|\leqslant C(\varepsilon E_{m,l}(t)\langle t\rangle^{-1}+\varepsilon^5\psi(\varepsilon)\langle t\rangle^{-3})$. This proves Proposition 5.4(1, l), $0\leqslant l\leqslant m$. □

We can now prove Proposition 5.4(2).

Proof of Proposition 5.4(2). Since we assume that ε is so small that $|z|\leqslant 1/2$ (cf. Section 5), an explicit computation shows that $|\sum_{0\leqslant j\leqslant 3}\partial_j A_j(\phi)|\leqslant C(|\nabla\cdot w|+|\nabla\theta|)$, so Proposition 5.4(2) easily follows with the help of Theorem 3.1(2), Proposition 5.2, and Proposition 5.1(4), (2). □

In the proof of Proposition 5.4(3, l) and (4, l), it will be convenient to make use of the following lemma.

Lemma 6.1. *Let m be as in Proposition* 5.4. *If ε is small,* (3.20)–(3.26) *has a unique $C^\infty([0,1]\times\overline{\mathcal{D}_R})$ solution such that $E^{1/2}_{m+1}(t)\leqslant C\varepsilon^2\psi^{1/2}(\varepsilon)$ and $\int_0^1|\partial_t^j\partial_r^k(\Theta_2,W_2,Z_2)|^2(t,R)\,dt\leqslant C\varepsilon^4\psi(\varepsilon)$ if $j+k\leqslant m$.*

Proof. Set $J_\varepsilon=\{s\in(0,1]$, (3.20)–(3.26) has a unique $C^\infty([0,s]\times\overline{\mathcal{D}_R})$ solution with $E^{1/2}_{m+1}(t)\leqslant\varepsilon^2\psi(\varepsilon)$ if $t\in[0,s]\}$. By (3.27), $E^{1/2}_{m+1}(0)\leqslant C\varepsilon^2$; hence, by the results of [5], $J_\varepsilon\neq\emptyset$ if ε is small; and J_ε is closed in (0, 1] thanks to Theorem 4 of [5]. Assume that $s_1\in J_\varepsilon$. We apply (5.13), (5.14), Proposition 5.3, with $l=0$ and m replaced by $m+1$, and Proposition 5.4(1, 0), with m replaced by $m+1$. With the help of the Gronwall inequality, we see that there exists $\varepsilon_0>0$ (independent of s_1) such that $E^{1/2}_{m+1}(t)\leqslant C\varepsilon^2\psi^{1/2}(\varepsilon)\leqslant\varepsilon^2\psi(\varepsilon)/2$ if $0<\varepsilon\leqslant\varepsilon_0$ and $t\in[0,s_1]$. Hence using Theorem 4 of [5], we conclude that J_ε is open in (0, 1]. So $J_\varepsilon=(0,1]$, and the estimate just used above to show that J_ε is open in (0, 1] also gives that $E^{1/2}_{m+1}(t)\leqslant C\varepsilon^2\psi^{1/2}(\varepsilon)$ if $t\in[0,1]$. Using a standard trace inequality on $r=R$, $0<t<1$, we complete the proof of Lemma 6.1. □

We now start with the proof of Proposition 5.4(3, l), (4, l), (5, l) and (6, l).

Proof of Proposition 5.4(3, 1), (4, 1), (5, 1) and (6, 1). If $r = R$, then $X\Theta_2 = t\partial_t\Theta_2$ and $XW_2 = -R(1 + C_1\Theta)^{-1}(\partial_t\Theta_2 + C_1\Theta_2\partial_r W_1)$. (Recall that we assume, as we may, that $C_1|\Theta| \leqslant 1/2$.) Hence we can write, when $r = R$: $\partial_t^k X\Theta_2 = t\partial_t^{k+1}\Theta_2 + k\partial_t^k\Theta_2$ and $\partial_t^k XW_2 = -R(1 + C_1\Theta)^{-1}\partial_t^{k+1}\Theta_2 + V_k$, where, since $E_{m,1}^{1/2}(t) \leqslant \varepsilon^2\psi(\varepsilon)$, we have $|\partial_t^k\Theta_2| \leqslant C\varepsilon^2\psi(\varepsilon)\langle t\rangle^{-1}$ and $|V_k| \leqslant C\varepsilon^3\psi(\varepsilon)\langle t\rangle^{-2}$ if $k \leqslant m-1$ and $r = R$, as one can easily verify with the help of Theorem 3.1(3), Proposition 5.1(3) and Proposition 5.2. If $\Gamma^a = \partial_t^k X$, $k \leqslant m-1$, and $r = R$, we therefore find that $-(1 + C_1\Theta)\Gamma^a\Theta_2\Gamma^a W_2 = Rt(\partial_t^{k+1}\Theta_2)^2 + Rk\partial_t(\partial_t^k\Theta_2)^2/2 - (1 + C_1\Theta)t\partial_t^{k+1}\Theta_2 V_k + \tilde{V}_k$, with $|\tilde{V}_k(t)| \leqslant C\varepsilon^5\psi^2(\varepsilon)\langle t\rangle^{-3}$. It follows from (3.27) that $|\partial_t^k\Theta_2(0, r)| \leqslant C_k\varepsilon^2$. So writing $-|\partial_t^{k+1}\Theta_2 V_k| \geqslant -\delta(\partial_t^{k+1}\Theta_2)^2 - C_\delta V_k^2$, we easily obtain that $-\int_0^T ((1 + C_1\Theta)\Gamma^a\Theta_2\Gamma^a W_2)(t, R)\,dt \geqslant C_0\int_0^T t(\partial_t^{k+1}\Theta_2)^2(t, R)\,dt - C\varepsilon^4\psi(\varepsilon)$, since $\varepsilon\psi(\varepsilon)$ is bounded. This proves Proposition 5.4(3, 1). Using (3.27), (5.13), (5.14), Proposition 5.4(1, 1), (2), (3, 1) and the Gronwall inequality, we obtain Proposition 5.4(4, 1). Proposition 5.4(5, 1) (even with $[R, \bar{R}]$ replaced by $[R, +\infty)$) follows with the help of Proposition 5.4(4, 1), Proposition 5.1(2), (3), (4), and Proposition 5.2. Finally we obtain Proposition 5.4(6, 1) by differentiating the relation $\partial_t z_2 = -(w_1 + w_2)\cdot\nabla(z_1 + z_2)$, using Lemma 5.3, Theorem 3.1(3), Proposition 5.1(1), and Proposition 5.4(4, 1), (5, 1). $\square$

Proceeding by induction, we shall assume until the end of this section that Proposition 5.4(3, λ)–(6, λ) has been proved if $1 \leqslant \lambda \leqslant l-1$ and we shall show that it still holds if $\lambda = l$ ($2 \leqslant l \leqslant m$). In order to be able to achieve this, we now proceed to prove boundary estimates.

We shall have to handle $\Gamma^a\Theta_2\Gamma^a W_2$ when $r = R$, if $\Gamma^a = \partial_t^k X^l$. It is convenient to introduce the following notations: $D_0 = \partial_t$, $D_1 = \partial_r$, $D^\alpha = \partial_t^{\alpha_0}\partial_r^{\alpha_1}$ if $\alpha = (\alpha_0, \alpha_1) \in \mathbb{N}^2$, $\zeta_1 = \Theta_2$, $\zeta_2 = W_2$, $\zeta = tr(\zeta_1\ \zeta_2)$, $\Phi_{(k)} = ((D^\alpha\Theta_1)_{|\alpha|\leqslant k}, (D^\alpha W_1)_{|\alpha|\leqslant k})$, $Y = (\Phi_{(0)}, \zeta_1, \zeta_2)$, $H = (\Theta, W, Z)$, $G(H) = W^2 - (1 + Z)(1 + C_1\Theta)^2$. Let $A(H)$ be the 2×2 matrix defined by $A_{11}(H) = A_{22}(H) = -W/G(H)$, $A_{12}(H) = (1 + C_1\Theta)/G(H)$, $A_{21}(H) = (1 + Z)(1 + C_1\Theta)/G(H)$. It is not difficult to check the following useful identity by induction over n if $n \geqslant 1$:

$$\partial_r^n\zeta = A^n(H)\partial_t^n\zeta + \mathcal{T}_n + B_{n0}(r)\zeta + \sum_{1\leqslant i\leqslant 3}\mathcal{T}_{ni}; \tag{6.30}$$

here $\mathcal{T}_1 = 0$, $\mathcal{T}_n = \sum_{1\leqslant\nu\leqslant n-1} B_{n\nu}(r, (D^\alpha Z)_{|\alpha|\leqslant n-\nu})\partial_t^\nu\zeta$ if $n \geqslant 2$, where $B_{n\nu}$ are C^∞ 2×2 matrices with $(B_{n0}(r))_{ij} = 0$ if $(i, j) \neq (2, 2)$, and the $\mathcal{T}_{ni}$ are defined as follows. $\mathcal{T}_{11} = 0$, and, for $n \geqslant 2$, each entry of $\mathcal{T}_{n1}$ is a sum of terms of the form $F(r, (D^\alpha(Y, Z))_{|\alpha|\leqslant j})M_p\partial_t^{q+1}\zeta_d$ with $j + p + q = n-1$, $1 \leqslant q+1 \leqslant n-1$, $d \in \{1, 2\}$, where $F \in C^\infty$ and M_p is some derivative D^β of order p of some component of Y. With H_0^0 as in (5.15) and τ_8^0, τ_{11}^0 as in Section 5, $\mathcal{T}_{12} = tr((WH_0^0 - (1 + C_1\Theta)(\tau_8^0 + \tau_{11}^0)\cdot x/r - 2r^{-1}(1 + C_1\Theta)WW_2)/G(H)\ (W(\tau_8^0 + \tau_{11}^0)\cdot x/r - (1 + Z)(1 + C_1\Theta)H_0^0 + 2r^{-1}W^2W_2)/G(H))$. In general, each entry of $\mathcal{T}_{n2}$ is a sum of terms of the form $F(r, (D^\alpha(Y, Z))_{|\alpha|\leqslant j})P_p\partial_t^q\zeta_d$ with $j + p + q \leqslant n-1$, where $F \in C^\infty$ and P_p is a derivative D^β of order p of some component of $(\Phi_{(1)}, \zeta_1, \zeta_2)$. Each entry of $\mathcal{T}_{n3}$ is a sum of terms of the form $F(r, (D^\alpha(Y, Z))_{|\alpha|\leqslant j})N_pS_q$ with $j + p + q \leqslant n-1$, where $F \in C^\infty$ and N_p is some derivative D^β of order p of Z and S_q is some derivative D^ϖ of order q of $\partial_r\Theta_1$. The representation (6.30) could be refined, but it will be sufficient for our purposes. Now it is easily checked that

$$\partial_t^k X^l\zeta = \sum_{\delta_{k0}\leqslant i+n\leqslant l} c_{klin}t^i r^n\partial_t^{i+k}\partial_r^n\zeta, \tag{6.31}$$

where c_{klin} are strictly positive constants and $\delta_{00} = 1$, $\delta_{k0} = 0$ if $k \neq 0$. (We have $l > 0$ since we are assuming that $l \geqslant 2$.) It follows from (6.30) that

$$\partial_t^{i+k}\partial_r^n\zeta = \sum_{1\leqslant\mu\leqslant 6} S_{i+k,n,\mu} \quad \text{if } n \geqslant 1, \tag{6.32}$$

with $S_{i+k,n,1} = \partial_t^{i+k}(A^n(H)\partial_t^n\zeta)$, $S_{i+k,n,2} = \partial_t^{i+k}\mathcal{T}_n$, $S_{i+k,n,3} = \partial_t^{i+k}(B_{n0}(r)\zeta)$, $S_{i+k,n,3+j} = \partial_t^{i+k}\mathcal{T}_{nj}$ if $j \in \{1, 2, 3\}$. Let $A_0(Z)$ be the 2×2 matrix with entries $(A_0(Z))_{11} = (A_0(Z))_{22} = 0$, $(A_0(Z))_{12} = -(1 + Z)^{-1}$, $(A_0(Z))_{21} = -1$, so that $A(H) = A_0(Z) + f(H)$, where $f \in C^\infty$ near 0 and vanishes if $\Theta = W = 0$. Estimates when $r = R$ will be obtained with the help of the following lemma.

Lemma 6.2. *If $i+n \leqslant l$, $k+l \leqslant m$, $l \geqslant 2$, $n \geqslant 1$, and $T \geqslant 1$, we have*:

(1) $S_{i+k,n,1} = A^n(H)\partial_t^{i+k+n}\zeta + K_{i+k,n}$ *when $r = R$, with*

$$\int_1^T t^{2i+1}|K_{i+k,n}|^2(t,R)\,dt \leqslant C\varepsilon^6\psi(\varepsilon),$$

(2) $\int_1^T t^{2i+1}|S_{i+k,n,2}|^2(t,R)\,dt \leqslant C\varepsilon^4\psi(\varepsilon)$,
(3) $S_{i+k,n,3} \equiv 0$ *when $r = R$,*
(4) $\int_1^T t^{2i+1}|S_{i+k,n,4}|^2(t,R)\,dt \leqslant C\varepsilon^6\psi(\varepsilon)$,
(5) $\int_1^T t^{2i+1}|S_{i+k,n,5}|^2(t,R)\,dt \leqslant C\varepsilon^6\psi(\varepsilon)$,
(6) $\int_1^T t^{2i+1}|S_{i+k,n,6}|^2(t,R)\,dt \leqslant C\varepsilon^4$.

Proof. Throughout the proof of Lemma 6.2, we assume that $r = R$.

(1) If $I > 0$, $\partial_t^I A^n(H)$ is a sum of terms of the form $\mathcal{A}(H)\prod_{1\leqslant i\leqslant I,1\leqslant j\leqslant 3}(\partial_t^i H_j)^{\beta_{ij}}$, where $\mathcal{A} \in C^\infty$, $\beta_{ij} \geqslant 0$, and $\sum_{1\leqslant i\leqslant I,\,1\leqslant j\leqslant 3} i\beta_{ij} = I$. Using Theorem 3.1(2), (3), Proposition 5.1(1), Proposition 5.4$(5, l-1)$ and $(6, l-1)$, and the fact that $\sum_{1\leqslant i\leqslant I,\,1\leqslant j\leqslant 3}\beta_{ij}\min(i, l-1) \geqslant \min(I, l-1)$, we obtain that

$$\bigl|\bigl(\partial_t^I A^n(H)\bigr)(t,R)\bigr| \leqslant C\frac{\varepsilon}{\langle t\rangle^{\min(I,l-1)}} \quad \text{if } 0 < I \leqslant m-1. \tag{6.33}$$

When proving (1), we may and shall suppose that $i+k > 0$; until the end of the proof of (1), we shall assume that $0 < I \leqslant i+k$. Set $\mathcal{E}_{i+k,n,I} = \partial_t^I A^n(H)\partial_t^{i+k+n-I}\zeta$, so that $t^{i+1/2}(S_{i+k,n,1} - A^n(H)\partial_t^{i+k+n}\zeta) = \sum_{0<I\leqslant i+k}\binom{i+k}{I}t^{i+1/2}\mathcal{E}_{i+k,n,I}$. Define $P_{\mu nI}(t) = t^{I-n-\mu+1}\partial_t^I tr((A^n(H))_{11}\,(A^n(H))_{21})(t,R)$, so that

$$t^{i+1/2}\mathcal{E}_{i+k,n,I}(t,R) = t^{i-I+n+\mu-1/2}\partial_t^{i-I+n+k}\Theta_2(t,R)P_{\mu nI}(t).$$

We have $|P_{\mu nI}(t)| \leqslant C\varepsilon t^{I-n-\mu+1-\min(I,l-1)}$ if $t \geqslant 1$, thanks to (6.33).

Assume first that $I \leqslant i+n-1$; hence $I \leqslant l-1$. Then we choose $\mu = 0$, so $|P_{\mu nI}(t)| \leqslant C\varepsilon t^{1-n}$ if $t \geqslant 1$. Now $i-I+n \geqslant 1$; also $i-I+n \leqslant l-1$ since $i+n \leqslant l$ and $I > 0$; and $i-I+n+k \leqslant m-1$. So using Proposition 5.4$(4, i-I+n)$ we obtain

$$\int_1^T t^{2i+1}|\mathcal{E}_{i+k,n,I}|^2(t,R)\,dt \leqslant C\varepsilon^6\psi(\varepsilon). \tag{6.34}$$

If now $i+n \leqslant I$, we choose $\mu = I-i-n+1$ (so $\mu \leqslant k$). Then $|P_{\mu nI}(t)| \leqslant C\varepsilon t^{1-n}$ if $t \geqslant 1$, and we have $i-I+n+k \leqslant m-2$. We obtain (6.34) by using Proposition 5.4$(4, 1)$. This completes the proof of Lemma 6.2(1).

(2) Set $\mathcal{F}_{i+k,n,\nu,I} = \partial_t^I B_{n\nu}(r, (D^\alpha Z)_{|\alpha|\leqslant n-\nu})\partial_t^{i+k-I+\nu}\zeta$ with $1 \leqslant \nu \leqslant n-1$ and $0 \leqslant I \leqslant i+k$, and define $P_{\mu n\nu I}(t) = t^{I-\nu+1-\mu}\partial_t^I tr((B_{n\nu}(r,(D^\alpha Z)_{|\alpha|\leqslant n-\nu}))_{11}(B_{n\nu}(r,(D^\alpha Z)_{|\alpha|\leqslant n-\nu}))_{21})(t,R)$, so that

$$t^{i+1/2}\mathcal{F}_{i+k,n,\nu,I}(t,R) = t^{i-I+\nu+\mu-1/2}\partial_t^{i-I+\nu+k}\Theta_2(t,R)P_{\mu n\nu I}(t).$$

Arguing as in the proof of (6.33) with the help of Proposition 5.1(1) and Proposition 5.4$(6, l-1)$, we obtain that, when $I > 0$ and $1 \leqslant \nu \leqslant n-1$:

$$\bigl|\bigl(\partial_t^I B_{n\nu}\bigl(r, \bigl(D^\alpha Z\bigr)_{|\alpha|\leqslant n-\nu}\bigr)\bigr)(t,R)\bigr| \leqslant \frac{C}{\langle t\rangle^{\min(I,l-1)}}\varepsilon^2. \tag{6.35}$$

Using (6.35) in the case that $I > 0$, we find that $|P_{\mu n\nu I}(t)| \leqslant Ct^{I-\nu-\mu+1-\min(I,l-1)}$ if $t \geqslant 1$. If $I \leqslant i+\nu-1$, we choose $\mu = 0$, notice that $1 \leqslant i-I+\nu \leqslant l-1$ and that $i-I+\nu+k \leqslant m-1$, and, since $|P_{\mu n\nu I}(t)| \leqslant Ct^{1-\nu}$ if $t \geqslant 1$, we obtain that

$$\int_1^T t^{2i+1}|\mathcal{F}_{i+k,n,\nu,I}|^2(t,R)\,dt \leqslant C\varepsilon^4\psi(\varepsilon) \tag{6.36}$$

with the help of Proposition 5.4$(4, i - I + \nu)$. If now $i + \nu \leqslant I$ (so $I > 0$), we choose $\mu = I - i - \nu + 1$ (so $\mu \leqslant k$). Since $i - I + \nu + k \leqslant m - 2$ and $|P_{\mu n \nu I}(t)| \leqslant C\varepsilon^2 t^{-\nu}$ if $t \geqslant 1$, we obtain (6.36) with the help of Proposition 5.4(4, 1). This proves (2).

(3) is obvious. As for (4), notice that, when $n \geqslant 2$, each entry of $\partial_t^{i+k}\mathcal{T}_{n1}$ is a linear combination of terms of the form $\mathcal{G}_{i+k,j,p,q,I_1,I_2} = \partial_t^{I_1} F(r, (D^\alpha(Y,Z))_{|\alpha|\leqslant j})\partial_t^{I_2} M_p \partial_t^{i+k-I_1-I_2+q+1}\zeta_d$ with F, M_p as in the definition of $\mathcal{T}_{n1}$, $j + p + q = n - 1$, $q + 1 \leqslant n - 1$, and $I_1 + I_2 \leqslant i + k$. Set $I = I_1 + I_2$, $P_{\mu j p q I_1 I_2}(t) = t^{I-q-\mu}(\partial_t^{I_1} F(r, (D^\alpha(Y,Z))_{|\alpha|\leqslant j})\partial_t^{I_2} M_p)(t, R)$, so that $t^{i+1/2}\mathcal{G}_{i+k,j,p,q,I_1,I_2}(t, R) = t^{i-I+q+1/2+\mu} \times \partial_t^{i-I+q+1+k}\Theta_2(t, R)P_{\mu j p q I_1 I_2}(t)\delta_{d1}$, with $\delta_{11} = 1$, $\delta_{21} = 0$. With the help of Theorem 3.1(2), (3), Proposition 5.1(1) and Proposition 5.4$(5, l - 1)$ and $(6, l - 1)$, and arguing as in the proof of (6.33) if $I_1 > 0$, we obtain that $|P_{\mu j p q I_1 I_2}(t)| \leqslant C\varepsilon t^{I-q-\mu-\min(I+1,l-1)}$ if $t \geqslant 1$. If we take $\mu = \min(l - 2 - i + I - q, k)$, then $\mu \leqslant k$, $i - I + q + 1 + \mu \in [1, l - 1]$, and $I - q - \mu \leqslant \min(I, l - 2)$. Also $i - I + q + 1 + k \leqslant m$. But then we obtain $\int_1^T t^{2i+1}\mathcal{G}^2_{i+k,j,p,q,I_1,I_2}(t, R)\,dt \leqslant C\varepsilon^6\psi(\varepsilon)$ with the help of Proposition 5.4$(4, i - I + q + 1 + \mu)$. This proves Lemma 6.2(4).

Let us prove (5). Each entry of $\partial_t^{i+k}\mathcal{T}_{n2}$ is a sum of terms of the form $\mathcal{H}_{i+k,j,p,q,I_1,I_2} = \partial_t^{I_1} F(r, (D^\alpha(Y,Z))_{|\alpha|\leqslant j}) \times \partial_t^{I_2} P_p \partial_t^{i+k-I_1-I_2+q}\zeta_d$ with $j + p + q \leqslant n - 1$ and $I_1 + I_2 \leqslant i + k$, where F, P_p are as in the definition of $\mathcal{T}_{n2}$. Define $I = I_1 + I_2$, $\mu = \min(l - 1 - i + I - q, k)$. Observe that if $q > 0$, or if $q = 0$ and $I < i + k$, then $i - I + q + \mu \in [1, l - 1]$; if $q = 0$ and $I = i + k$, then $i - I + q + \mu = 0$. Also $i - I + q + k \leqslant m - 1$. Set $N_{\mu j p q I_1 I_2}(t) = t^{I-q-\mu+1}(\partial_t^{I_1} F(r, (D^\alpha(Y,Z))_{|\alpha|\leqslant j})\partial_t^{I_2} P_p)(t, R)$, so that $t^{i+1/2}\mathcal{H}_{i+k,j,p,q,I_1,I_2}(t, R) = t^{i-I+q-1/2+\mu}\partial_t^{i-I+q+k}\Theta_2(t, R)N_{\mu j p q I_1 I_2}(t)\delta_{d1}$. With the help of Theorem 3.1(2), (3), Proposition 5.1(1) and Proposition 5.4$(5, l-1)$, $(6, l-1)$, and arguing as in the proof of (6.33) if $I_1 > 0$, we obtain that $|N_{\mu j p q I_1 I_2}(t)| \leqslant C\varepsilon t^{s_{\mu q I_1 I_2}}$, where $s_{\mu q I_1 I_2} = I_1 - \mu - 1 - \min(I_1, l - 1)$ if $n = 1$ (so in this case, $s_{\mu q I_1 I_2}$ does not depend on I_2), and $s_{\mu q I_1 I_2} = I - q - \mu + 1 - \min(I + 1, l - 1)$ if $n \geqslant 2$. It is not hard to check that $s_{\mu q I_1 I_2} \leqslant -1$ if $n = 1$, or if $n \geqslant 2$ and $q > 0$, whereas $s_{\mu q I_1 I_2} \leqslant 0$ if $n \geqslant 2$ and $q = 0$. We use Proposition 5.4$(4, i - I + q + \mu)$ if $q > 0$, or if $q = 0$ and $I < i + k$, and Proposition 5.4(5, 1) if $q = 0$ and $I = i + k$. We obtain that $\int_1^T t^{2i+1}\mathcal{H}^2_{i+k,j,p,q,I_1,I_2}(t, R)\,dt \leqslant C\varepsilon^6\psi(\varepsilon)$. Lemma 6.2(5) is proved.

Finally Lemma 6.2(6) follows easily with the help of Proposition 5.1(1), Proposition 5.4$(5, l - 1)$, $(6, l - 1)$ and Theorem 3.1(2), (3) if we use the arguments leading to (6.33) to estimate the derivatives of strictly positive order of $F(r, (D^\alpha(Y,Z))_{|\alpha|\leqslant j})$. The proof of Lemma 6.2 is complete. □

Notice that we have

$$A^n(\tilde{H})\partial_t^{i+k+n}\zeta = (-1)^n M_n(\tilde{H})\partial_t^{i+k+n}\Theta_2\delta_n \quad \text{if } r = R, \tag{6.37}$$

where $\tilde{H} = (\Theta, 0, Z)$ and M_n is a C^∞ scalar function in a neighborhood of $(0, 0, 0)$ with $M_n(0, 0, 0) = 1$, and where $\delta_n = tr(1\ 0)$ if n is even and $\delta_n = tr(0\ 1)$ if n is odd. Until the end of the proof of Proposition 5.4$(3, l)$, we shall still assume that $T \geqslant 1$ and that $i + n \leqslant l$, $k + l \leqslant m$. With the help of (6.32), Lemma 6.2, and (6.37), it follows at once that

$$\partial_t^{i+k}\partial_r^n\zeta = (-1)^n M_n(\tilde{H})\partial_t^{i+k+n}\Theta_2\delta_n + L_{i+k,n} \quad \text{if } r = R, \tag{6.38}$$

where $L_{i+k,n} = K_{i+k,n} + \sum_{2\leqslant\mu\leqslant 6} S_{i+k,n,\mu}$ and $\int_1^T t^{2i+1}|L_{i+k,n}|^2(t, R)\,dt \leqslant C\varepsilon^4\psi(\varepsilon)$, and of course $L_{i+k,0} = 0$. Combining (6.38) with Proposition 5.4$(4, l - 1)$, we easily obtain the following estimate:

$$\int_1^T t^{2i+1}\left|\partial_t^{i+k}\partial_r^n\zeta\right|^2(t, R)\,dt \leqslant C\varepsilon^4\psi(\varepsilon) \quad \text{if } i \leqslant l - 2 \text{ and } k + n \geqslant 1, \tag{6.39}$$

whereas Proposition 5.4$(5, l - 1)$ implies that

$$\int_1^T t^{2i}\left(\partial_t^i\Theta_2\right)^2(t, R)\,dt \leqslant C\varepsilon^4\psi(\varepsilon) \quad \text{if } i \leqslant l - 2. \tag{6.40}$$

With the boundary estimates we have obtained, we are now ready to complete the proof of Proposition 5.4. As already said above, we assume until the end of this section that $l \geqslant 2$ and that Proposition 5.4$(3, \lambda)$–$(6, \lambda)$ has already been proved if $1 \leqslant \lambda \leqslant l - 1$ (where l is fixed and $2 \leqslant l \leqslant m$), and we show that it still holds if $\lambda = l$.

Proof of Proposition 5.4(3, l), (4, l). We can write $-(1+C_1\Theta)\partial_t^k X^l\Theta_2\partial_t^k X^l W_2 = \sum c_I \mathcal{B}_I$, where $I=(i,j,n,p,k,l)$, all c_I are strictly positive constants, $\mathcal{B}_I = -(1+C_1\Theta)t^{i+j}\partial_t^{i+k}\partial_r^n\Theta_2\partial_t^{j+k}\partial_r^p W_2$ and the sum is taken over all I with k,l fixed such that $k+l\leqslant m$, $\delta_{k0}\leqslant i+n\leqslant l$, $\delta_{k0}\leqslant j+p\leqslant l$, where $\delta_{00}=1$ and $\delta_{k0}=0$ if $k\neq 0$ as before. Since $W_2=0$ if $r=R$, it follows that $j\leqslant l-1$ if $\mathcal{B}_I(t,R)\not\equiv 0$. Recall that $C_1|\Theta|\leqslant 1/2$. We have the following estimates if $T\geqslant 1$ and $\delta>0$:

$$\int_1^T \big|\mathcal{B}_I(t,R)\big|\,dt \leqslant C\varepsilon^4\psi(\varepsilon) \quad \text{if } i,j\leqslant l-2; \tag{6.41}$$

$$\int_1^T \big|\mathcal{B}_I(t,R)\big|\,dt \leqslant C\varepsilon^4\psi(\varepsilon) \quad \text{if } i=l-1 \text{ and } j\leqslant l-2, \text{ or if } j=l-1,\ i\leqslant l-2 \text{ and } k+n\geqslant 1; \tag{6.42}$$

$$\int_1^T \big|\mathcal{B}_I(t,R)\big|\,dt \leqslant \delta\int_1^T t^{2(l-1)}\big(\partial_t^l\Theta_2\big)^2(t,R)\,dt + C_\delta\varepsilon^4\psi(\varepsilon) \quad \text{if } j=l-1,\ i\leqslant l-2 \text{ and } k=n=0; \tag{6.43}$$

$$\int_1^T \mathcal{B}_I(t,R)\,dt \geqslant -C\varepsilon^4\psi(\varepsilon) \quad \text{if } i=j=l-1; \tag{6.44}$$

$$\int_1^T \big|\mathcal{B}_I(t,R)\big|\,dt \leqslant \delta\int_1^T t^{2l-1}\big(\partial_t^{l+k}\Theta_2\big)^2(t,R)\,dt + C_\delta\varepsilon^4\psi(\varepsilon) \quad \text{if } i=l \text{ and } j\leqslant l-2; \tag{6.45}$$

$$\int_1^T \mathcal{B}_I(t,R)\,dt \geqslant C^{-1}\int_1^T t^{2l-1}\big(\partial_t^{l+k}\Theta_2\big)^2(t,R)\,dt - C\varepsilon^4\psi(\varepsilon) \quad \text{if } i=l,\ j=l-1, \text{ and } p=1. \tag{6.46}$$

Indeed, (6.41) follows at once from (6.39) and (6.40).

To check (6.42) when $i=l-1$ and $j\leqslant l-2$, we write $|\mathcal{B}_I(t,R)|\leqslant C(t^{2l-3}(\partial_t^{l-1+k}\Theta_2)^2(t,R) + t^{2j+1}(\partial_t^{j+k}\partial_r^p W_2)^2(t,R))$ and apply Proposition 5.4(4, $l-1$) and (6.39). If now $j=l-1$ and $i\leqslant l-2$, assume first that $k+n\geqslant 1$. We then use (6.39) with $|\partial_t^{i+k}\partial_r^n\zeta|$ replaced by $|\partial_t^{i+k}\partial_r^n\Theta_2|$. But applying (6.38) and Proposition 5.4(4, $l-1$), we find that $\int_1^T t^{2l-3}(\partial_t^{l-1+k}\partial_r W_2)^2(t,R)\,dt\leqslant C\varepsilon^4\psi(\varepsilon)$. Altogether, we obtain the second part of (6.42). If $k+n=0$ (and still $j=l-1$, $i\leqslant l-2$), we write $|\mathcal{B}_I(t,R)|\leqslant \delta t^{2(l-1)}(\partial_t^{l-1}\partial_r W_2)^2(t,R) + C_\delta t^{2i}(\partial_t^i\Theta_2)^2(t,R)$, and using (6.38) and (6.40), we obtain (6.43).

If $i=j=l-1$, we may assume that $n=0$, $p=1$, and we write by (6.38):

$$\partial_t^{l-1+k}\partial_r W_2 = -M_1(\tilde H)\partial_t^{l+k}\Theta_2 + \hat L_{l-1+k,1} \quad \text{if } r=R, \tag{6.47}$$

where $\hat L_{l-1+k,1}$ is the second component of $L_{l-1+k,1}$, so that $\int_1^T t^{2l-1}\hat L_{l-1+k,1}^2(t,R)\,dt\leqslant C\varepsilon^4\psi(\varepsilon)$. So if $B(\tilde H) = M_1(\tilde H)(1+C_1\Theta)$, we obtain with the help of (6.47) that $\mathcal{B}_I = \partial_t(B(\tilde H)t^{2l-2}(\partial_t^{l-1+k}\Theta_2)^2/2) - \partial_t(B(\tilde H)t^{2l-2}/2)\times(\partial_t^{l-1+k}\Theta_2)^2 - (1+C_1\Theta)t^{2l-2}\partial_t^{l-1+k}\Theta_2\hat L_{l-1+k,1}$ when $r=R$. Observe that $B(\tilde H)(t,R)>0$ for ε small, and that $|\partial_t B(\tilde H)(t,R)|\leqslant C\varepsilon\langle t\rangle^{-1}$ by Theorem 3.1(3) (or (2)) and Proposition 5.4(5, 1) and (6, 1). (6.44) now follows easily with the help of Proposition 5.4(4, $l-1$) and (3.27).

Let us prove (6.45). Since $i=l$, we have $n=0$. But $|\mathcal{B}_I(t,R)|\leqslant \delta t^{2l-1}(\partial_t^{l+k}\Theta_2)^2(t,R) + C_\delta t^{2j+1}\times(\partial_t^{j+k}\partial_r^p W_2)^2(t,R)$ and (6.45) easily follows with the help of (6.39).

At last, let us handle (6.46). We have $i=l$, hence $n=0$, and $j=l-1$, so we may assume that $p=1$. With the help of (6.47) it follows that $\mathcal{B}_I = B(\tilde H)t^{2l-1}(\partial_t^{l+k}\Theta_2)^2 - (1+C_1\Theta)t^{2l-1}\partial_t^{l+k}\Theta_2\hat L_{l-1+k,1}$ when $r=R$, where $B(\tilde H)$ is as in the proof of (6.44), and (6.46) follows.

Proposition 5.4(3, l) follows easily from (6.41)–(6.46) and Lemma 6.1.

We now prove Proposition 5.4(4, l). Using (5.13), (5.14), Proposition 5.3, Proposition 5.4(2), (1, λ) for $0\leqslant\lambda\leqslant l$, (3, λ) for $1\leqslant\lambda\leqslant l$, and the Gronwall inequality, we obtain Proposition 5.4(4, l). $\square$

Proof of Proposition 5.4(5, l). Set $\xi = tr(\theta_2\, w_2)$. Since $E_{m,l}^{1/2}(t) \leqslant C\varepsilon^2\psi^{1/2}(\varepsilon)$ by Proposition 5.4(4, l), it follows from Proposition 5.1(2)–(4) and Proposition 5.2 that

$$\left|X^\lambda \partial^\alpha \xi(t)\right| \leqslant C\varepsilon^2\psi^{1/2}(\varepsilon)\langle t\rangle^{-1} \quad \text{if } \lambda \leqslant l-1 \text{ and } |\alpha| + \lambda \leqslant m-1. \tag{6.48}$$

But

$$t^{l-1}\partial_t^{l-1}\partial^\alpha \xi = X^{l-1}\partial^\alpha\xi - \sum_{i\leqslant l-2;\, 1\leqslant i+j\leqslant l-1} c_{l-1,i,j}\, t^i r^j \partial_t^i \partial_r^j \partial^\alpha \xi, \tag{6.49}$$

where $c_{l-1,i,j}$ are strictly positive constants. Using (6.48) to bound the first term on the right-hand side of (6.49), Proposition 5.4(5, $l-1$) to bound the other terms on the right-hand side of (6.49), and (6.48) with $\lambda = 0$, we obtain Proposition 5.4(5, l). □

Proof of Proposition 5.4(6, l). We may assume that $l \leqslant m-1$. Proposition 5.4(6, l) follows easily by applying $\partial_t^k \partial^\alpha$ to (3.22) if we make use of Theorem 3.1(3), Proposition 5.4(5, l), (4, 1), Proposition 5.1(1) and Proposition 5.4(6, $l-1$). □

The proof of Proposition 5.4 is complete.

7. Proof of Theorem 2.4

In this section we shall prove Theorem 2.4.

We shall use a method from [8,7,1] (cf. also [3,4]) to show that some derivative of any solution must blow up on a certain characteristic before some time close to τ^*. Actually we shall indicate how to adapt the proof of Theorem 3 of [3] to obtain Theorem 2.4 of the present paper. Many arguments are very similar; we concentrate on the modifications due to the presence of the boundary $r = R$. We may assume that $\tau^* < +\infty$; as recalled in Section 2, this is equivalent to saying that $|u^0| + |\frac{\rho^0}{\bar\rho} + K_2\frac{S^0}{\gamma}| \not\equiv 0$. We shall prove the following result, which is the analogue of Proposition 5 of [3] for the mixed problem considered in the present paper.

Proposition 7.1. *Assume that $\bar\tau \in (0, +\infty)$. Then, for each $\delta > 0$, there exists $\varepsilon_0 > 0$ such that the following holds: if $0 < \varepsilon \leqslant \varepsilon_0$ and (ρ, u, S) is a $C^\infty([0, e^{\bar\tau/\varepsilon}] \times \overline{\mathcal{D}_R})$ solution of* (2.1)–(2.7)*, then $\bar\tau \leqslant \tau^* + \delta$.*

Let us write $u(t,x) = U(t,r)x/r$, $c(\rho,S) = (\frac{\partial P}{\partial\rho}(\rho,S))^{1/2}$. ρ, S, c can be considered as functions of (t,r) which we shall also denote by ρ, S, c to simplify notations (we shall hardly use x-coordinates in this section); so we shall write $\rho(t,r)$, $S(t,r)$, $c(t,r)$.

If $q > R - \bar c e^{\bar\tau/\varepsilon}$, define $t \mapsto r_q^+(t)$, $t \in [((R-q)/\bar c)_+, e^{\bar\tau/\varepsilon}]$, as the maximal solution of $\frac{dr_q^+}{dt}(t) = (U+c)(t, r_q^+(t))$ which satisfies the following initial condition: if $q \geqslant R$, we ask that $r_q^+(0) = q$; if $q < R$, we ask that $r_q^+((R-q)/\bar c) = R$. In other words, the map $t \mapsto (t, r_q^+(t))$ parametrizes the 3-characteristic curve (associated with ρ, U, S) emanating from the only point (t_0, r_0) with $t_0(r_0 - R) = 0$, $t_0 \geqslant 0$, $r_0 \geqslant R$, such that $r_0 - \bar c t_0 = q$. Adapting arguments from Section 7 of [3] with the help of Theorem 3.1(2), Theorem 3.2 and Proposition 5.1(1), (3) of the present paper, we see that, if $\tau \in (0, \tau^*)$ and $q_0 > R - \bar c e^{\bar\tau/\varepsilon}$ are fixed, one can find $\varepsilon_0 > 0$ such that

$$\left|r_q^+(t) - \bar c t - q\right| \leqslant C \quad \text{if } 0 < \varepsilon \leqslant \varepsilon_0,\ \varepsilon \ln t \leqslant \tau \text{ and } q \geqslant q_0.$$

Henceforth we shall assume that ε is so small that $M/\bar c \leqslant 1/\varepsilon < e^{\bar\tau/\varepsilon}$. Set $D_\varepsilon = \{(t,r) \in \mathbb{R}^2,\ 1/\varepsilon \leqslant t \leqslant e^{\bar\tau/\varepsilon},\ r_{R-M}^+(t) < r < r_{R+M}^+(t)\}$. Then it is easily seen that, if ε is small, $C^{-1} \leqslant \langle t\rangle^{-1} r \leqslant C$ in D_ε and $S = \bar S$ there. As in Section 7 of [3], we introduce the following functions: $\mathcal{A} = r(\rho - \bar\rho)$, $\mathcal{B} = rU$, $\mathcal{Z}_1 = \frac12(\frac{\partial_r\mathcal{A}}{\rho} + \frac{\partial_r\mathcal{B}}{c})$, $\mathcal{Z}_2 = \frac12(-\frac{\partial_r\mathcal{A}}{\rho} + \frac{\partial_r\mathcal{B}}{c})$, and set, if $1/\varepsilon \leqslant t \leqslant e^{\bar\tau/\varepsilon}$:

$$J(t) = \sup_{\frac{1}{\varepsilon}\leqslant s\leqslant t} \int_{(s,r)\in D_\varepsilon} \left|\mathcal{Z}_1(s,r)\right| dr,$$

$$N(t) = \sup_{\frac{1}{\varepsilon} \leqslant s \leqslant t} \sup_{(s,r)\in D_\varepsilon} \left(\left|\mathcal{A}(s,r)\right| + \left|\mathcal{B}(s,r)\right|\right),$$

$$G(t) = \sup_{\frac{1}{\varepsilon} \leqslant s \leqslant t} \sup_{(s,r)\in D_\varepsilon} s\left|\mathcal{Z}_2(s,r)\right|.$$

As in [3] but using now the estimates provided by Proposition 5.1(1), Theorem 3.1(2) and Theorem 3.2 of the present paper, we can easily check that, if $\tau \in (0, \tau^*)$ is fixed,

$$J(t) \leqslant C\varepsilon \quad \text{and} \quad N(t) \leqslant C\varepsilon \quad \text{if } 0 < \varepsilon \leqslant \varepsilon_0 \text{ and } \tfrac{1}{\varepsilon} \leqslant t \leqslant e^{\tau/\varepsilon}. \tag{7.1}$$

Now fix $\Psi : (0, 1) \to [0, +\infty)$ such that $\Psi(\varepsilon)(\ln(1/\varepsilon))^{-1} \to +\infty$ as $\varepsilon \overset{>}{\to} 0$. We have

$$\left|\partial^\alpha(v - \varepsilon v_0)(t)\right| \leqslant C_\alpha \varepsilon^2 \Psi(\varepsilon)\langle t\rangle^{-1} \quad \text{if } \alpha \neq 0,\ 0 \leqslant t \leqslant \tfrac{\bar{c}}{\varepsilon} \text{ and } 0 < \varepsilon \leqslant \varepsilon_0, \tag{7.2}$$

where v is as in (4.1)–(4.3) and v_0 is the solution of the linear mixed problem $\Box v_0 = 0$ if $t > 0$, $x \in \mathcal{D}_R$, $\partial_r v_0 = 0$ if $t > 0$ and $r = R$, $\partial_t^j v_0(0, x) = f_j(r)$ if $x \in \mathcal{D}_R$ and $j = 0, 1$, with the same f_j as in (4.3). Indeed (7.2) follows at once from Theorem 3.5 of [4] and the proof of Theorem 3.1 of [4] if we perform the change of variables $(t, x) \mapsto (t/\bar{c}, x/\bar{c})$. (Actually, (7.2) still holds with $\Psi(\varepsilon) = \ln(1/\varepsilon)$, as follows from the estimates of Sections 5 and 6 of [2], but since the present paper is already very long, we shall ignore this fact.) With the help of (7.2) and of Theorem 3.1(2) and Theorem 3.2, we can duplicate the arguments which led to (89) of [3] and obtain that, for some $G_1 > 0$:

$$G\left(\frac{1}{\varepsilon}\right) \leqslant G_1 \varepsilon \Psi(\varepsilon) \quad \text{if } \varepsilon \text{ is small.} \tag{7.3}$$

Henceforth we fix Ψ such that furthermore $\varepsilon\Psi(\varepsilon) \to 0$ as $\varepsilon \overset{>}{\to} 0$. Using (7.1), (7.3) and arguing as in the proof of Lemma 5 of [3], we obtain the following result.

Lemma 7.1. *One can find J_1, N_1, G_1, $\varepsilon_0 > 0$ such that the following holds: if $0 < \varepsilon \leqslant \varepsilon_0$, then $J(t) \leqslant J_1\varepsilon$, $N(t) \leqslant N_1\varepsilon$, $G(t) \leqslant G_1\varepsilon\Psi(\varepsilon)$ and $r \geqslant \bar{c}t/2$ in $\overline{D_\varepsilon}$.*

By Lemma 3.3 of [4], one can find $q_0 \in [R - M, R + M]$ such that $-F_0''(q_0) = \max_{q\in\mathbb{R}}(-F_0''(q))$ (see also Section 4 of the present paper where other information about F_0 has also been recalled). Set $f(t) = -\mathcal{Z}_1(t, r_{q_0}^+(t))$. Repeating the arguments of Section 7 of [3], we easily obtain that $f'(t) = a_0(t)f^2(t) + a_1(t)f(t) + a_2(t)$, where it can be easily checked with the help of Lemma 7.1 above that $|a_0(t) - (\bar{\rho}\overline{c_\rho} + \bar{c})/\bar{c}t| \leqslant Ct^{-2}$ and $|a_1(t)| + |a_2(t)| \leqslant C\varepsilon\Psi(\varepsilon)t^{-2}$ if $1/\varepsilon \leqslant t \leqslant e^{\bar{\tau}/\varepsilon}$ and $0 < \varepsilon \leqslant \varepsilon_0$. On the other hand, by a straightforward adaptation of arguments used in [3], we obtain with the help of (7.2): $f(1/\varepsilon) = \bar{c}\varepsilon/((\bar{\rho}\overline{c_\rho} + \bar{c})\tau^*) + \mathcal{O}(\varepsilon^2\Psi(\varepsilon))$ as $\varepsilon \overset{>}{\to} 0$. Proposition 7.1 follows by the same arguments as in [3]. This completes the proof of Theorem 2.4.

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