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Global well-posedness and scattering for the derivative nonlinear Schrödinger equation with small rough data

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Abstract

We study the Cauchy problem for the generalized elliptical and non-elliptical derivative nonlinear Schrödinger equations (DNLS) and get the global well posedness of solutions with small data in modulation spaces $M^s_{2,1}(\mathbb{R}^n)$. Noticing that $B^{s+n/2}_{2,1} \subset M^s_{2,1} \subset B^s_{2,1}$ are optimal inclusions, we have shown the global well posedness of DNLS with a class of rough data. As a by-product, the existence of the scattering operators in modulation spaces with small data is also obtained.

MSC: 35Q55; 46E35; 47D08

Keywords: Derivative nonlinear Schrödinger equation; Scattering; Global well posedness; Small rough data; Modulation spaces

1. Introduction

This paper is a continuation of our earlier work [33] and we study the Cauchy problem for the generalized derivative nonlinear Schrödinger equation (gDNLS)

$$\mathrm{i}u_t + \Delta_\pm u = F(u, \bar{u}, \nabla u, \nabla \bar{u}), \qquad u(0,x) = u_0(x), \tag{1.1}$$

where u is a complex valued function of $(t,x) \in \mathbb{R} \times \mathbb{R}^n$,

$$\Delta_\pm u = \sum_{i=1}^n \varepsilon_i \partial^2_{x_i}, \quad \varepsilon_i \in \{1, -1\},\ i = 1, \ldots, n, \tag{1.2}$$

$\nabla = (\partial_{x_1}, \ldots, \partial_{x_n})$, $F : \mathbb{C}^{2n+2} \to \mathbb{C}$ is a polynomial series,

$$F(z) = F(z_1, \ldots, z_{2n+2}) = \sum_{m+1<|\beta|<\infty} c_\beta z^\beta, \quad c_\beta \in \mathbb{C}, \tag{1.3}$$

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$2 \leqslant m < \infty$, $m \in \mathbb{N}$, $\sup_\beta |c_\beta| < \infty$.[1] A typical nonlinear term is the following

$$F(u, \bar{u}, \nabla u, \nabla \bar{u}) = |u|^2 \vec{\lambda} \cdot \nabla u + u^2 \vec{\mu} \cdot \nabla \bar{u} + |u|^2 u,$$

which is a model equation in the strongly interacting many-body systems near criticality as recently described in terms of nonlinear dynamics [27,10,8]. Another typical nonlinearity is

$$F(u, \bar{u}, \nabla u, \nabla \bar{u}) = \left(1 \mp |u|^2\right)^{-1} |\nabla u|^2 u = \sum_{k=0}^{\infty} \pm |u|^{2k} |\nabla u|^2 u, \quad |u| < 1,$$

which is a deformation of the Schrödinger map equation [9].

A large amount of work has been devoted to the study of the local and global well posedness of (1.1), see Bejenaru and Tataru [2], Chihara [3,4], Kenig, Ponce and Vega [14,15], Klainerman [18], Klainerman and Ponce [19], Ozawa and Zhai [22], Shatah [23], B.X. Wang and Y.Z. Wang [33]. When the nonlinear term F satisfies some energy structure conditions, or the initial data suitably decay, the energy method, which went back to the work of Klainerman [18] and was developed in [3,4,19,22,23], yields the global existence of (1.1) in the elliptical case $\Delta_\pm = \Delta$. Recently, Ozawa and Zhai obtained the global well posedness in $H^s(\mathbb{R}^n)$ ($n \geqslant 3$, $s > 2 + n/2$, $m \geqslant 2$) with small data for (1.1) in the elliptical case, where an energy structure condition on F is still required.

By setting up the local smooth effects for the solutions of the linear Schrödinger equation, Kenig, Ponce and Vega [14,15] were able to deal with the non-elliptical case and they established the local well posedness of Eq. (1.1) in H^s with $s \gg n/2$. Recently, the local well posedness results have been generalized to the quasi-linear (ultrahyperbolic) Schrödinger equations, see [16,17].

In one spatial dimension, B.X. Wang and Y.Z. Wang [33] showed the global well posedness of gDNLS (1.1) for small data in critical Besov spaces $\dot{B}^{1+n/2-2/m}_{2,1} \cap \dot{B}^{1+n/2-1/M}_{2,1}(\mathbb{R})$, $m \geqslant 4$. In higher spatial dimensions $n \geqslant 2$, by using Kenig, Ponce and Vega's local smooth effects and establishing time-global maximal function estimates in space-local Lebesgue spaces, B.X. Wang and Y.Z. Wang [33] showed the global well posedness of gDNLS (1.1) for small data in Besov spaces $B^s_{2,1}(\mathbb{R}^n)$ with $s > n/2 + 3/2$, $m \geqslant 2 + 4/n$.

Wang and Huang [32] obtained the global well posedness of (1.1) in one spatial dimension with initial data in $M^{1+1/m}_{2,1}$, $m \geqslant 4$. In this paper, we will use a new way to study the global well posedness and scattering of (1.1) and show that (1.1) is globally well posed in $M^s_{2,1}(\mathbb{R}^n)$ for the small Cauchy data. Our starting point is the smooth effect estimates for the linear Schrödinger equation in one spatial dimension (cf. [7,13,14,24,34]), from which we get a series of linear estimates in higher dimensional anisotropic Lebesgue spaces, including the global smooth effect estimates, the maximal function estimates and their relations to the Strichartz estimates. The maximal function estimates follows an idea as in Ionescu and Kenig [12]. These estimates together with the frequency-uniform decomposition method yield the global well posedness and scattering of solutions in modulation spaces $M^s_{2,1}$, $s \geqslant 5/2$ ($s \geqslant 3/2$ if $m \geqslant 3$).

1.1. Modulation spaces $M^s_{2,1}$

In this paper, we are mainly interested in the cases that the initial data u_0 belongs to the modulation space $M^s_{2,1}$ for which the norm can be equivalently defined in the following way (cf. [11,30–32]):

$$\|f\|_{M^s_{2,1}} = \sum_{k \in \mathbb{Z}^n} \langle k \rangle^s \|\mathscr{F} f\|_{L^2(Q_k)}, \tag{1.4}$$

where $\langle k \rangle = 1 + |k|$, $Q_k = \{\xi\colon -1/2 \leqslant \xi_i - k_i < 1/2,\ i = 1, \ldots, n\}$. Modulation spaces $M^s_{2,1}$ are related to the Besov spaces and there holds the optimal inclusions $B^{n/2+s}_{2,1} \subset M^s_{2,1} \subset B^s_{2,1}$ (cf. [28,26,32]). So, comparing $M^s_{2,1}$ with $B^{s+n/2}_{2,1}$, we see that $M^s_{2,1}$ contains a class of functions u satisfying $\|u\|_{M^s_{2,1}} \ll 1$ but $\|u\|_{B^{s+n/2}_{2,1}} = \infty$ (and hence $\|u\|_{H^{s+n/2+\varepsilon}} = \infty$, for any $\varepsilon > 0$).

[1] In fact, c_β is not necessarily bounded, condition $\sup_\beta |c_\beta| < \infty$ can be replaced by $|c_\beta| \leqslant C^{|\beta|}$.

1.2. Main results

For the definitions of the anisotropic Lebesgue spaces $L^{p_1}_{x_i} L^{p_2}_{(x_j)_{j\neq i}} L^{p_2}_t(\mathbb{R}^{1+n})$ and the frequency-uniform decomposition operators $\{\Box_k\}_{k\in\mathbb{Z}^n}$, one can refer to Section 1.3. Denote for $k=(k_1,\dots,k_n)$,

$$\|u\|_{X^s_\alpha} = \sum_{i,\ell=1}^n \sum_{k\in\mathbb{Z}^n,\,|k_i|>4} \langle k_i\rangle^{s-1/2} \big\|\partial^\alpha_{x_\ell}\Box_k u\big\|_{L^\infty_{x_i} L^2_{(x_j)_{j\neq i}} L^2_t(\mathbb{R}^{1+n})} + \sum_{i,\ell=1}^n \sum_{k\in\mathbb{Z}^n} \big\|\partial^\alpha_{x_\ell}\Box_k u\big\|_{L^m_{x_i} L^\infty_{(x_j)_{j\neq i}} L^\infty_t(\mathbb{R}^{1+n})}, \tag{1.5}$$

$$\|u\|_{S^s_\alpha} = \sum_{\ell=1}^n \sum_{k\in\mathbb{Z}^n} \langle k\rangle^{s-1} \big\|\partial^\alpha_{x_\ell}\Box_k u\big\|_{L^\infty_t L^2_x \cap L^{2+m}_{x,t}(\mathbb{R}^{1+n})}, \tag{1.6}$$

$$\|u\|_{\tilde S^s_\alpha} = \sum_{\ell=1}^n \sum_{k\in\mathbb{Z}^n} \langle k\rangle^{s-1} \big\|\partial^\alpha_{x_\ell}\Box_k u\big\|_{L^\infty_t L^2_x \cap L^3_t L^6_x(\mathbb{R}^{1+n})}, \tag{1.7}$$

$$\|u\|_{X^s} = \sum_{\alpha=0,1} \|u\|_{X^s_\alpha}, \qquad \|u\|_{S^s} = \sum_{\alpha=0,1} \|u\|_{S^s_\alpha}, \qquad \|u\|_{\tilde S^s} = \sum_{\alpha=0,1} \|u\|_{\tilde S^s_\alpha}. \tag{1.8}$$

In the sequel we will always assume that X^s_α, S^s_α, $\tilde S^s_\alpha$ and X^s, S^s, $\tilde S^s$ are as in (1.5)–(1.8), respectively. The following are our main results.

Theorem 1.1. *Let $n \geqslant 2$, $2 < m < \infty$, $m > 4/n$. Assume that $u_0 \in M^{3/2}_{2,1}$ and $\|u_0\|_{M^{3/2}_{2,1}} \leqslant \delta$ for some small $\delta > 0$. Then* (1.1) *has a unique global solution $u \in C(\mathbb{R}, M^{3/2}_{2,1}) \cap X^{3/2} \cap S^{3/2}$ and $\|u\|_{X^{3/2}\cap S^{3/2}} \lesssim \delta$. Moreover, the scattering operator of* (1.1) *carries a zero neighborhood in $C(\mathbb{R}, M^{3/2}_{2,1})$ into $C(\mathbb{R}, M^{3/2}_{2,1})$.*

In Theorem 1.1, if $u_0 \in M^s_{2,1}$ with $s > 3/2$, then we have $u \in C(\mathbb{R}, M^s_{2,1})$. If $m = 2$, we need to assume the initial data have stronger regularity:

Theorem 1.2. *Let $n \geqslant 3$, $m = 2$. Assume that $u_0 \in M^{5/2}_{2,1}$ and $\|u_0\|_{M^{5/2}_{2,1}} \leqslant \delta$ for some small $\delta > 0$. Then* (1.1) *has a unique global solution $u \in C(\mathbb{R}, M^{5/2}_{2,1}) \cap X^{5/2} \cap \tilde S^{5/2}$, and $\|u\|_{X^{5/2}\cap \tilde S^{5/2}} \lesssim \delta$. Moreover, the scattering operator of* (1.1) *carries a zero neighborhood in $C(\mathbb{R}, M^{5/2}_{2,1})$ into $C(\mathbb{R}, M^{5/2}_{2,1})$.*

When the nonlinearity F has a simple form, say,

$$\mathrm{i}u_t + \Delta_\pm u = \sum_{i=1}^n \lambda_i \partial_{x_i}\big(u^{\kappa_i+1}\big), \qquad u(0,x) = u_0(x), \tag{1.9}$$

we obtained in [32] the global well posedness of the DNLS (1.9) for the small data in modulation spaces $M^{1/\kappa_1}_{2,1}$ in one spatial dimension. In higher spatial dimensions $n \geqslant 2$, we have

Theorem 1.3. *Let $n \geqslant 2$, $\kappa_i > 2$, $\kappa_i > 4/n$, $\kappa_i \in \mathbb{N}$, $\lambda_i \in \mathbb{C}$, $m = \min_{1\leqslant i\leqslant n} \kappa_i$. Assume that $u_0 \in M^{1/2}_{2,1}$ and $\|u_0\|_{M^{1/2}_{2,1}} \leqslant \delta$ for some small $\delta > 0$. Then* (1.9) *has a unique global solution $u \in C(\mathbb{R}, M^{1/2}_{2,1}) \cap X^{3/2}_0 \cap S^{3/2}_0$ and $\|u\|_{X^{3/2}_0\cap S^{3/2}_0} \lesssim \delta$. Moreover, the scattering operator of* (1.9) *carries a zero neighborhood in $C(\mathbb{R}, M^{1/2}_{2,1})$ into $C(\mathbb{R}, M^{1/2}_{2,1})$.*

We remark that in Theorem 1.3, the same result holds if the nonlinear term $\partial_{x_i}(u^{\kappa_i+1})$ is replaced by $\partial_{x_i}(|u|^{\kappa_i}u)$ ($\kappa_i \in 2\mathbb{N}$).

Theorem 1.4. *Let $n \geqslant 3$, $\kappa_i \in \mathbb{N}$, $\lambda_i \in \mathbb{C}$, $m = \min_{1\leqslant i\leqslant n} \kappa_i = 2$. Assume that $u_0 \in M^{3/2}_{2,1}$ and $\|u_0\|_{M^{3/2}_{2,1}} \leqslant \delta$ for some small $\delta > 0$. Then* (1.9) *has a unique global solution $u \in C(\mathbb{R}, M^{3/2}_{2,1}) \cap X^{5/2}_0 \cap \tilde{S}^{5/2}_0$ and $\|u\|_{X^{5/2}_0 \cap \tilde{S}^{5/2}_0} \lesssim \delta$. Moreover, the scattering operator of* (1.9) *carries a zero neighborhood in $C(\mathbb{R}, M^{3/2}_{2,1})$ into $C(\mathbb{R}, M^{3/2}_{2,1})$.*

Using the embedding $H^{s+\varepsilon+n/2} \subset M^s_{2,1}$, we see that if $u_0 \in H^{s+n/2}$, then the same results hold in Theorems 1.1–1.4 for $s > 3/2$, $s > 5/2$, $s > 1/2$ and $s > 3/2$, respectively. When $m = 1$, Christ [5] showed the ill posedness of (1.9) in any H^s for one spatial dimension case. For general nonlinearity in (1.1), we do not know what happens in the case $m = 1$ in higher spatial dimensions.

1.3. Notations

The following are some notations which will be frequently used in this paper: $\mathbb{C}, \mathbb{R}, \mathbb{N}$ and $\mathbb{Z}$ will stand for the sets of complex number, reals, positive integers and integers, respectively. $c \leqslant 1$, $C > 1$ will denote positive universal constants, which can be different at different places. $a \lesssim b$ stands for $a \leqslant Cb$ for some constant $C > 1$, $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$. We write $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$. We denote by p' the dual number of $p \in [1, \infty]$, i.e., $1/p + 1/p' = 1$. We will use Lebesgue spaces $L^p := L^p(\mathbb{R}^n)$, $\|\cdot\|_p := \|\cdot\|_{L^p}$, Sobolev spaces $H^s = (I - \Delta)^{-s/2} L^2$. Some properties of these function spaces can be found in [1,29]. If there is no explanation, we always assume that spatial dimensions $n \geqslant 2$. We will use the function spaces $L^q_t L^p_x(\mathbb{R}^{n+1})$ and $L^p_x L^q_t(\mathbb{R}^{n+1})$ for which the norms are defined by

$$\|f\|_{L^q_t L^p_x(\mathbb{R}^{n+1})} = \big\| \|f\|_{L^p_x(\mathbb{R}^n)} \big\|_{L^q_t(\mathbb{R})}, \qquad \|f\|_{L^p_x L^q_t(\mathbb{R}^{n+1})} = \big\| \|f\|_{L^q_t(\mathbb{R})} \big\|_{L^p_x(\mathbb{R}^n)},$$

$L^p_{x,t}(\mathbb{R}^{n+1}) := L^p_x L^p_t(\mathbb{R}^{n+1})$. We denote by $L^{p_1}_{x_i} L^{p_2}_{(x_j)_{j\neq i}} L^{p_2}_t := L^{p_1}_{x_i} L^{p_2}_{(x_j)_{j\neq i}} L^{p_2}_t(\mathbb{R}^{1+n})$ the anisotropic Lebesgue space for which the norm is defined by

$$\|f\|_{L^{p_1}_{x_i} L^{p_2}_{(x_j)_{j\neq i}} L^{p_2}_t} = \big\| \|f\|_{L^{p_2}_{x_1,\dots,x_{i-1},x_{i+1},\dots,x_n} L^{p_2}_t(\mathbb{R}\times\mathbb{R}^{n-1})} \big\|_{L^{p_1}_{x_i}(\mathbb{R})}. \tag{1.10}$$

It is also convenient to use the notation $L^{p_1}_{x_1} L^{p_2}_{x_2,\dots,x_n} L^{p_2}_t := L^{p_1}_{x_1} L^{p_2}_{(x_j)_{j\neq 1}} L^{p_2}_t$. For any $1 < k < n$, we denote by $\mathscr{F}_{x_1,\dots,x_k}$ the partial Fourier transform:

$$(\mathscr{F}_{x_1,\dots,x_k} f)(\xi_1, \dots, \xi_k, x_{k+1}, \dots, x_n) = \int_{\mathbb{R}^k} e^{-\mathrm{i}(x_1\xi_1+\dots+x_k\xi_k)} f(x)\, dx_1 \dots dx_k \tag{1.11}$$

and by $\mathscr{F}^{-1}_{\xi_1,\dots,\xi_k}$ the partial inverse Fourier transform, similarly for $\mathscr{F}_{t,x}$ and $\mathscr{F}^{-1}_{\tau,\xi}$. $\mathscr{F} := \mathscr{F}_{x_1,\dots,x_n}$, $\mathscr{F}^{-1} := \mathscr{F}^{-1}_{\xi_1,\dots,\xi_n}$. $D^s_{x_i} = (-\partial^2_{x_i})^{s/2} = \mathscr{F}^{-1}_{\xi_i} |\xi_i|^s \mathscr{F}_{x_i}$ expresses the partial Riesz potential in the x_i direction. $\partial^{-1}_{x_i} = \mathscr{F}^{-1}_{\xi_i} (\mathrm{i}\xi_i)^{-1} \mathscr{F}_{x_i}$. We will use the Bernstein multiplier estimate; cf. [1,29]. For any $r \in [1, \infty]$,

$$\big\|\mathscr{F}^{-1}\varphi\mathscr{F} f\big\|_r \leqslant C\|\varphi\|_{H^s}\|f\|_r, \quad s > n/2. \tag{1.12}$$

We will use the frequency-uniform decomposition operators (cf. [30–32]). Let $\{\sigma_k\}_{k\in\mathbb{Z}^n}$ be a function sequence satisfying

$$\begin{cases} \sigma_k(\xi) \geqslant c, & \forall \xi \in Q_k, \\ \operatorname{supp} \sigma_k \subset \{\xi\colon |\xi - k| \leqslant \sqrt{n}\}, \\ \sum_{k\in\mathbb{Z}^n} \sigma_k(\xi) \equiv 1, & \forall \xi \in \mathbb{R}^n, \\ |D^\alpha \sigma_k(\xi)| \leqslant C_m, & \forall \xi \in \mathbb{R}^n,\ |\alpha| \leqslant m \in \mathbb{N}. \end{cases} \tag{1.13}$$

Denote

$$\Upsilon = \big\{\{\sigma_k\}_{k\in\mathbb{Z}^n}\colon \{\sigma_k\}_{k\in\mathbb{Z}^n} \text{ satisfies (1.13)}\big\}. \tag{1.14}$$

Let $\{\sigma_k\}_{k\in\mathbb{Z}^n} \in \Upsilon$ be a function sequence.

$$\Box_k := \mathscr{F}^{-1}\sigma_k\mathscr{F}, \quad k \in \mathbb{Z}^n, \tag{1.15}$$

are said to be the frequency-uniform decomposition operators. One may ask the existence of the frequency-uniform decomposition operators. Indeed, let $\rho \in \mathscr{S}(\mathbb{R}^n)$ and $\rho : \mathbb{R}^n \to [0,1]$ be a smooth radial bump function adapted to the ball $B(0,\sqrt{n})$, say $\rho(\xi) = 1$ as $|\xi| \leqslant \sqrt{n}/2$, and $\rho(\xi) = 0$ as $|\xi| \geqslant \sqrt{n}$. Let ρ_k be a translation of ρ: $\rho_k(\xi) = \rho(\xi - k)$, $k \in \mathbb{Z}^n$. We write

$$\eta_k(\xi) = \rho_k(\xi)\Big(\sum_{k\in\mathbb{Z}^n}\rho_k(\xi)\Big)^{-1}, \quad k \in \mathbb{Z}^n. \tag{1.16}$$

We have $\{\eta_k\}_{k\in\mathbb{Z}^n} \in \Upsilon$. It is easy to see that for any $\{\eta_k\}_{k\in\mathbb{Z}^n} \in \Upsilon$,

$$\|f\|_{M^s_{2,1}} \sim \sum_{k\in\mathbb{Z}^n}\langle k\rangle^s\|\Box_k f\|_{L^2(\mathbb{R}^n)}.$$

We will use the function space $\ell^{1,s}_\Box(L^p_t L^r_x(I\times\mathbb{R}^n))$ which contains all of the functions $f(t,x)$ so that the following norm is finite:

$$\|f\|_{\ell^{1,s}_\Box(L^p_t L^r_x(I\times\mathbb{R}^n))} := \sum_{k\in\mathbb{Z}^n}\langle k\rangle^s\|\Box_k f\|_{L^p_t L^r_x(I\times\mathbb{R}^n)}. \tag{1.17}$$

For simplicity, we write $\ell^1_\Box(L^p_t L^r_x(I\times\mathbb{R}^n)) = \ell^{1,0}_\Box(L^p_t L^r_x(I\times\mathbb{R}^n))$.

This paper is organized as follows. In Section 2 we state the global smooth effect estimates of the solutions of the linear Schrödinger equation in anisotropic Lebesgue spaces, whose proofs will be left to Appendix A. In Sections 3 and 4 we consider the frequency-uniform localized versions for the global maximal function estimates, the global smooth effects, together with their relations to the Strichartz estimates. In Sections 5, 6 and 7 we prove our Theorem 1.3, Theorem 1.1 and Theorems 1.4 and 1.2, respectively. In Appendix B we generalize the Christ–Kiselev Lemma to the anisotropic Lebesgue spaces in higher dimensions.

2. Anisotropic global smooth effects

Recall that in [20], Linares and Ponce considered the linear estimates for the following problem:

$$\mathrm{i}\partial_t u - \partial^2_{xy}u = F(t,x,y), \qquad u(0,x) = 0. \tag{2.1}$$

Let u be the solution of the above question. Linares and Ponce obtained that[2]

$$\big\|D_x^{1/2}e^{\mathrm{i}t\partial^2_{xy}}\phi\big\|_{L^\infty_y L^2_t L^2_x(\mathbb{R}^3)} \lesssim \|\phi\|_{L^2(\mathbb{R}^2)}, \qquad \|\partial_x u\|_{L^\infty_y L^2_t L^2_x(\mathbb{R}^3)} \lesssim \|F\|_{L^1_y L^2_t L^2_x(\mathbb{R}^3)}. \tag{2.2}$$

Denote

$$S(t) = e^{\mathrm{i}t\Delta_\pm} = \mathscr{F}^{-1}e^{\mathrm{i}t\sum_{j=1}^n\varepsilon_j\xi_j^2}\mathscr{F}, \qquad \mathscr{A}f(t,x) = \int_0^t S(t-\tau)f(\tau,x)\,d\tau.$$

In two spatial dimensions, $S(t)$ and $\mathscr{A}f$ can be reduced to the semigroup $e^{\mathrm{i}t\partial^2_{xy}}$ and the solution of (2.1) by a simple transform, respectively. Using the one order smooth effect estimates in one spatial dimension as in [14], in higher spatial dimensions, we have

Proposition 2.1. *For any $i = 1,\dots,n$, we have the following estimate*:

$$\big\|D_{x_i}^{1/2}S(t)u_0\big\|_{L^\infty_{x_i}L^2_{(x_j)_{j\neq i}}L^2_t(\mathbb{R}^{1+n})} \lesssim \|u_0\|_2. \tag{2.3}$$

$$\|\partial_{x_i}\mathscr{A}f\|_{L^\infty_{x_i}L^2_{(x_j)_{j\neq i}}L^2_t(\mathbb{R}^{1+n})} \lesssim \|f\|_{L^1_{x_i}L^2_{(x_j)_{j\neq i}}L^2_t(\mathbb{R}^{1+n})}. \tag{2.4}$$

[2] The authors are grateful to the referee for his/her pointing out the work [20] after they finished the first version of the paper.

We leave the proof of Proposition 2.1 to Appendix A. The dual version of (2.3) is

Proposition 2.2. *For any $i = 1, \dots, n$, we have the following estimate*:

$$\|\partial_{x_i} \mathscr{A} f\|_{L_t^\infty L_x^2(\mathbb{R}^{1+n})} \lesssim \big\| D_{x_i}^{1/2} f \big\|_{L^1_{x_i} L^2_{(x_j)_{j\neq i}} L^2_t(\mathbb{R}^{1+n})}. \tag{2.5}$$

3. Linear estimates with $\Box_k$-decomposition

In this section we consider the smooth effect estimates, the maximal function estimates, the Strichartz estimates and their interaction estimates for the solutions of the linear Schrödinger equations by using the frequency-uniform decomposition operators. For convenience, we will use the following function sequence $\{\sigma_k\}_{k\in\mathbb{Z}^n}$:

Lemma 3.1. *Let $\eta_k : \mathbb{R} \to [0, 1]$ $(k \in \mathbb{Z})$ be a smooth-function sequence satisfying condition* (1.13). *Denote*

$$\sigma_k(\xi) := \eta_{k_1}(\xi_1)\dots\eta_{k_n}(\xi_n), \quad k = (k_1, \dots, k_n). \tag{3.1}$$

Then we have $\{\sigma_k\}_{k\in\mathbb{Z}^n} \in \Upsilon$.

Recall that in [31], we established the following Strichartz estimates in a class of function spaces by using the frequency-uniform decomposition operators.

Lemma 3.2. *Let $2 \leqslant p < \infty$, $\gamma \geqslant 2 \vee \gamma(p)$,*

$$\frac{2}{\gamma(p)} = n\left(\frac{1}{2} - \frac{1}{p}\right).$$

Then we have

$$\big\|S(t)\varphi\big\|_{\ell^1_\Box(L^\gamma(\mathbb{R},L^p(\mathbb{R}^n)))} \lesssim \|\varphi\|_{M_{2,1}(\mathbb{R}^n)},$$
$$\|\mathscr{A} f\|_{\ell^1_\Box(L^\gamma(\mathbb{R},L^p(\mathbb{R}^n)))\cap \ell^1_\Box(L^\infty(\mathbb{R},L^2(\mathbb{R}^n)))} \lesssim \|f\|_{\ell^1_\Box(L^{\gamma'}(\mathbb{R},L^{p'}(\mathbb{R}^n)))}.$$

In particular, if $2 + 4/n \leqslant p < \infty$, then we have

$$\big\|S(t)\varphi\big\|_{\ell^1_\Box(L^p_{t,x}(\mathbb{R}^{1+n}))} \lesssim \|\varphi\|_{M_{2,1}(\mathbb{R}^n)},$$
$$\|\mathscr{A} f\|_{\ell^1_\Box(L^p_{t,x}(\mathbb{R}^{1+n}))\cap \ell^1_\Box(L^\infty_t L^2_x(\mathbb{R}^{1+n}))} \lesssim \|f\|_{\ell^1_\Box(L^{p'}_{t,x}(\mathbb{R}^{1+n}))}.$$

The next lemma is essentially known, see [29,30].

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a compact set with* $\operatorname{diam}\Omega < 2R$, $0 < p \leqslant q \leqslant \infty$. *Then there exists a constant $C > 0$, which depends only on p, q such that*

$$\|f\|_q \leqslant C R^{n(1/p-1/q)} \|f\|_p, \quad \forall f \in L^p_\Omega,$$

where $L^p_\Omega = \{f \in \mathcal{S}'(\mathbb{R}^n)\colon \operatorname{supp}\hat f \subset \Omega,\ \|f\|_p < \infty\}$.

In Lemma 3.3 we emphasize that the constant $C > 0$ is independent of the position of Ω in frequency spaces, say, in the case $\Omega = B(k, \sqrt{n}\,)$, $k \in \mathbb{Z}^n$, Lemma 3.3 uniformly holds for all $k \in \mathbb{Z}^n$.

Lemma 3.4. *We have for any $\sigma \in \mathbb{R}$ and $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ with $|k_i| \geqslant 4$,*

$$\big\|\Box_k D^\sigma_{x_i} u\big\|_{L^{p_1}_{x_1} L^{p_2}_{x_2,\dots,x_n} L^{p_2}_t(\mathbb{R}^{1+n})} \lesssim \langle k_i\rangle^\sigma \|\Box_k u\|_{L^{p_1}_{x_1} L^{p_2}_{x_2,\dots,x_n} L^{p_2}_t(\mathbb{R}^{1+n})}.$$

Replacing $D^\sigma_{x_i}$ by $\partial^\sigma_{x_i}$ $(\sigma \in \mathbb{N})$, the above inequality holds for all $k \in \mathbb{Z}^n$.

Proof. Using Lemma 3.1, one has that

$$\Box_k D^{\sigma}_{x_i} u = \sum_{\ell=-1}^{1} \int_{\mathbb{R}} \big(\mathscr{F}^{-1}_{\xi_i}\big(\eta_{k_i+\ell}(\xi_i)|\xi_i|^{\sigma}\big)\big)(y_i)(\Box_k u)(x_i - y_i)\,dy_i.$$

Using Young's inequality and noticing that $\|\mathscr{F}^{-1}_{\xi_i}(\eta_{k_i+\ell}(\xi_i)|\xi_i|^{\sigma})\|_{L^1(\mathbb{R})} \lesssim \langle k_i\rangle^{\sigma}$, one has the result. □

Ionescu and Kenig [12] showed the following maximal function estimates in higher spatial dimensions $n \geqslant 3$:

$$\big\|\triangle_k S(t)u_0\big\|_{L^2_{x_i} L^{\infty}_{(x_j)_{j\neq i}} L^{\infty}_t(\mathbb{R}^{1+n})} \lesssim 2^{(n-1)k/2}\|\triangle_k u_0\|_{L^2(\mathbb{R}^n)}. \tag{3.2}$$

We partially resort to their idea to obtain the following

Proposition 3.5. *Let $4/n < q \leqslant \infty$, $q \geqslant 2$. Then we have*

$$\big\|\Box_k S(t)u_0\big\|_{L^q_{x_i} L^{\infty}_{(x_j)_{j\neq i}} L^{\infty}_t(\mathbb{R}^{1+n})} \lesssim \langle k_i\rangle^{1/q}\|\Box_k u_0\|_{L^2(\mathbb{R}^n)}. \tag{3.3}$$

Proof. For convenience, we write $\bar{x} = (x_1, \ldots, x_{n-1})$. By standard dual estimate method, it suffices to show that (see [12])

$$\big\|\mathscr{F}^{-1} e^{\mathrm{i}t|\xi|^2_{\pm}}\eta_{k_1}(\xi_1)\eta_{\bar{k}}(\bar{\xi})\big\|_{L^{q/2}_{x_1} L^{\infty}_{\bar{x},t}(\mathbb{R}^n)} \lesssim \langle k_1\rangle^{2/q}.$$

In view of the decay of $\Box_k S(t)$, we see that (cf. [31])

$$\big\|\mathscr{F}^{-1}_{\bar{\xi}} e^{\mathrm{i}t|\bar{\xi}|^2_{\pm}}\eta_{\bar{k}}(\bar{\xi})\big\|_{L^{\infty}_{\bar{x}}(\mathbb{R}^{n-1})} \lesssim \big(1+|t|\big)^{-(n-1)/2},$$
$$\big\|\mathscr{F}^{-1}_{\xi_1} e^{\mathrm{i}t\xi_1^2}\eta_{k_1}(\xi_1)\big\|_{L^{\infty}_{x_1}(\mathbb{R})} \lesssim \big(1+|t|\big)^{-1/2}.$$

On the other hand, integrating by part, one has that for $|x_1| > 4|t|\langle k_1\rangle$,

$$\big|\mathscr{F}^{-1}_{\xi_1} e^{\mathrm{i}t\xi_1^2}\eta_{k_1}(\xi_1)\big| \lesssim |x_1|^{-2}.$$

Hence, for $|x_1| > 1$,

$$\big|\mathscr{F}^{-1} e^{\mathrm{i}t|\xi|^2_{\pm}}\eta_{k_1}(\xi_1)\eta_{\bar{k}}(\bar{\xi})\big| \lesssim \big(1+|x_1|\big)^{-2} + \langle k_1\rangle^{n/2}\big(\langle k_1\rangle + |x_1|\big)^{-n/2}.$$

So, we have

$$\big\|\mathscr{F}^{-1} e^{\mathrm{i}t|\xi|^2_{\pm}}\eta_{k_1}(\xi_1)\eta_{\bar{k}}(\bar{\xi})\big\|_{L^{q/2}_{x_1} L^{\infty}_{\bar{x},t}(\mathbb{R}^n)} \lesssim 1 + \langle k_1\rangle^{n/2}\big\|\big(\langle k_1\rangle + |x_1|\big)^{-n/2}\big\|_{L^{q/2}_{x_1}(\mathbb{R})} \lesssim \langle k_1\rangle^{2/q}.$$

This finishes the proof of (3.3). □

Remark 3.6. We conjecture that (3.3) also holds in the case $p = 4/n$ if $n = 2$. Using similar way as in [33], one can show that (3.3) is sharp.

The dual version of Proposition 3.5 is the following

Proposition 3.7. *Let $2 \leqslant q \leqslant \infty$, $q > 4/n$. Then we have for any $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$,*

$$\bigg\|\Box_k \int_{\mathbb{R}} S(t-\tau)f(\tau)\,d\tau\bigg\|_{L^{\infty}_t L^2_x(\mathbb{R}^{1+n})} \lesssim \langle k_i\rangle^{1/q}\|\Box_k f\|_{L^{q'}_{x_i} L^1_{(x_j)_{j\neq i}} L^1_t(\mathbb{R}^{1+n})}. \tag{3.4}$$

Proof. Denote

$$\tilde{\Box}_k = \sum_{\ell\in\Lambda} \Box_{k+\ell}, \quad \Lambda = \{\ell \in \mathbb{Z}^n\colon\ \operatorname{supp}\sigma_k \cap \operatorname{supp}\sigma_{k+\ell} \neq \varnothing\}. \tag{3.5}$$

Write

$$\mathcal{L}_k(f,\psi) := \left|\int_{\mathbb{R}} \left(\Box_k \int_{\mathbb{R}} S(t-\tau) f(\tau)\,d\tau,\ \psi(t)\right) dt\right| \tag{3.6}$$

By Proposition 3.5,

$$\begin{aligned}\mathcal{L}_k(f,\psi) &\leqslant \|\Box_k f\|_{L^{q'}_{x_i} L^1_{(x_j)_{j\neq i}} L^1_t(\mathbb{R}^{1+n})} \left\|\tilde{\Box}_k \int_{\mathbb{R}} S(\tau-t)\psi(t)\,dt\right\|_{L^q_{x_i} L^\infty_{(x_j)_{j\neq i}} L^\infty_t(\mathbb{R}^{1+n})} \\ &\leqslant \|\Box_k f\|_{L^{q'}_{x_i} L^1_{(x_j)_{j\neq i}} L^1_t(\mathbb{R}^{1+n})} \langle k_i\rangle^{1/q} \|\tilde{\Box}_k \psi\|_{L^1_t L^2_x(\mathbb{R}^{1+n})}. \end{aligned} \tag{3.7}$$

By duality, we have the result, as desired. □

In view of Propositions 2.1 and 2.2, we have

Proposition 3.8. *We have for any $k = (k_1,\dots,k_n) \in \mathbb{Z}^n$,*

$$\|\Box_k \mathscr{A} \partial_{x_i} f\|_{L^\infty_{x_i} L^2_{(x_j)_{j\neq i}} L^2_t(\mathbb{R}^{1+n})} \lesssim \|\Box_k f\|_{L^1_{x_i} L^2_{(x_j)_{j\neq i}} L^2_t(\mathbb{R}^{1+n})}, \tag{3.8}$$

$$\|\Box_k \mathscr{A} \partial_{x_i} f\|_{L^\infty_t L^2_x(\mathbb{R}^{1+n})} \lesssim \langle k_i\rangle^{1/2} \|\Box_k f\|_{L^1_{x_i} L^2_{(x_j)_{j\neq i}} L^2_t(\mathbb{R}^{1+n})}. \tag{3.9}$$

Proof. By Proposition 2.1, we immediately have (3.8). In view of Proposition 2.2 and Lemma 3.4, we have (3.9) in the case $|k_i| \geqslant 3$. If $|k_i| \leqslant 2$, in view of Proposition 2.2,

$$\|\Box_k \mathscr{A} \partial_{x_i} f\|_{L^\infty_t L^2_x(\mathbb{R}^{1+n})} \lesssim \left\|D^{-1/2}_{x_i} \Box_k \mathscr{A} \partial_{x_i} f\right\|_{L^\infty_t L^2_x(\mathbb{R}^{1+n})} \lesssim \|\Box_k f\|_{L^1_{x_i} L^2_{(x_j)_{j\neq i}} L^2_t(\mathbb{R}^{1+n})},$$

which implies the result, as desired. □

By the duality, we also have the following

Proposition 3.9. *Let $2 < q \leqslant \infty$ $q > 4/n$. Then we have*

$$\|\Box_k \mathscr{A} \partial_{x_i} f\|_{L^q_{x_i} L^\infty_{(x_j)_{j\neq i}} L^\infty_t(\mathbb{R}^{1+n})} \lesssim \langle k_i\rangle^{1/2+1/q} \|\Box_k f\|_{L^1_{x_i} L^2_{(x_j)_{j\neq i}} L^2_t(\mathbb{R}^{1+n})}. \tag{3.10}$$

Proof. By Propositions 3.7, 3.8 and Lemma 3.4,

$$\begin{aligned}\mathcal{L}_k(\partial_{x_1} f,\psi) &\leqslant \left\|\Box_k \int_{\mathbb{R}} S(-\tau)\partial_{x_1} f(\tau)\,d\tau\right\|_{L^2(\mathbb{R}^n)} \left\|\tilde{\Box}_k \int_{\mathbb{R}} S(-t)\psi(t)\,dt\right\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \langle k_1\rangle^{1/2} \|\Box_k f\|_{L^1_{x_1} L^2_{x_2,\dots,x_n} L^2_t(\mathbb{R}^{1+n})} \langle k_1\rangle^{1/q} \|\tilde{\Box}_k \psi\|_{L^{q'}_{x_1} L^1_{x_2,\dots,x_n} L^1_t(\mathbb{R}^{1+n})} \\ &\lesssim \langle k_1\rangle^{1/2+1/q} \|\Box_k f\|_{L^1_{x_1} L^2_{x_2,\dots,x_n} L^2_t(\mathbb{R}^{1+n})} \|\psi\|_{L^{q'}_{x_1} L^1_{x_2,\dots,x_n} L^1_t(\mathbb{R}^{1+n})}. \end{aligned} \tag{3.11}$$

Again, by duality, it follows from (3.11) and Christ–Kiselev's Lemma that (3.10) holds. □

Proposition 3.10. *Let $2 \leqslant r < \infty$, $2/\gamma(r) = n(1/2 - 1/r)$ and $\gamma > \gamma(r) \vee 2$. We have*

$$\big\|\Box_k S(t)u_0\big\|_{L_t^\gamma L_x^r(\mathbb{R}^{1+n})} \lesssim \|\Box_k u_0\|_{L^2(\mathbb{R}^n)}, \tag{3.12}$$

$$\|\Box_k \mathscr{A} f\|_{L_t^\infty L_x^2 \cap L_t^\gamma L_x^r(\mathbb{R}^{1+n})} \lesssim \|\Box_k f\|_{L_t^{\gamma'} L_x^{r'}(\mathbb{R}^{1+n})}, \tag{3.13}$$

$$\|\Box_k \mathscr{A} \partial_{x_i} f\|_{L_t^\gamma L_x^r(\mathbb{R}^{1+n})} \lesssim \langle k_i\rangle^{1/2} \|\Box_k f\|_{L^1_{x_i} L^2_{(x_j)_{j\neq i}} L^2_t(\mathbb{R}^{1+n})}, \tag{3.14}$$

$$\|\Box_k \mathscr{A} \partial_{x_i} f\|_{L^\infty_{x_i} L^2_{(x_j)_{j\neq i}} L^2_t(\mathbb{R}^{1+n})} \lesssim \langle k_i\rangle^{1/2} \|\Box_k f\|_{L_t^{\gamma'} L_x^{r'}(\mathbb{R}^{1+n})}, \tag{3.15}$$

and for $2 \leqslant q < \infty$, $q > 4/n$, $\alpha = 0, 1$,

$$\big\|\Box_k \mathscr{A} \partial^\alpha_{x_i} f\big\|_{L^q_{x_i} L^\infty_{(x_j)_{j\neq i}} L^\infty_t(\mathbb{R}^{1+n})} \lesssim \langle k_i\rangle^{\alpha+1/q} \|\Box_k f\|_{L_t^{\gamma'} L_x^{r'}(\mathbb{R}^{1+n})}, \tag{3.16}$$

Proof. From Lemma 3.2 it follows that (3.12) and (3.13) hold. We now show (3.14). We use the same notations as in Proposition 3.9. By Lemmas 3.2, 3.4 and Proposition 3.8,

$$\begin{aligned}\mathcal{L}_k(\partial_{x_1} f, \psi) &\lesssim \langle k_i\rangle^{1/2} \|\Box_k f\|_{L^1_{x_1} L^2_{x_2,\dots,x_n} L^2_t(\mathbb{R}^{1+n})} \|\tilde{\Box}_k \psi\|_{L_t^{\gamma'} L_x^{r'}(\mathbb{R}^{1+n})} \\ &\lesssim \langle k_i\rangle^{1/2} \|\Box_k f\|_{L^1_{x_1} L^2_{x_2,\dots,x_n} L^2_t(\mathbb{R}^{1+n})} \|\psi\|_{L_t^{\gamma'} L_x^{r'}(\mathbb{R}^{1+n})}.\end{aligned} \tag{3.17}$$

By duality, it follows from (3.24) and Christ–Kiselev's Lemma that (3.14) holds. Exchanging the roles of f and ψ, we immediately have (3.15) in the case $r > 2$. If $r = 2$, (3.15) is a straightforward consequence of the $1/2$-order smooth effect of $S(t)$. By Lemmas 3.2, 3.4, Proposition 3.7, and Christ–Kiselev's Lemma that we have (3.16) in the case $q > 2$, or $q = 2$ and $r > 2$. In the case $q = r = 2$, in view of the maximal function estimate, we see that (3.16) also holds. □

Corollary 3.11. *Let $4/n \leqslant p < \infty$, $2 \leqslant q < \infty$, $q > 4/n$. We have*

$$\big\|D^{1/2}_{x_1} \Box_k S(t)u_0\big\|_{L^\infty_{x_1} L^2_{x_2,\dots,x_n} L^2_t(\mathbb{R}^{1+n})} \lesssim \|\Box_k u_0\|_{L^2(\mathbb{R}^n)}, \tag{3.18}$$

$$\big\|\Box_k S(t)u_0\big\|_{L^q_{x_1} L^\infty_{x_2,\dots,x_n} L^\infty_t(\mathbb{R}^{1+n})} \lesssim \langle k_i\rangle^{1/q} \|\Box_k u_0\|_{L^2(\mathbb{R}^n)}, \tag{3.19}$$

$$\big\|\Box_k S(t)u_0\big\|_{L^{2+p}_{t,x} \cap L^\infty_t L^2_x(\mathbb{R}^{1+n})} \lesssim \|\Box_k u_0\|_{L^2(\mathbb{R}^n)}, \tag{3.20}$$

$$\|\Box_k \mathscr{A} \partial_{x_1} f\|_{L^\infty_{x_1} L^2_{x_2,\dots,x_n} L^2_t(\mathbb{R}^{1+n})} \lesssim \|\Box_k f\|_{L^1_{x_1} L^2_{x_2,\dots,x_n} L^2_t(\mathbb{R}^{1+n})}, \tag{3.21}$$

$$\|\Box_k \mathscr{A} \partial_{x_1} f\|_{L^q_{x_1} L^\infty_{x_2,\dots,x_n} L^\infty_t(\mathbb{R}^{1+n})} \lesssim \langle k_1\rangle^{1/2+1/q} \|\Box_k f\|_{L^1_{x_1} L^2_{x_2,\dots,x_n} L^2_t(\mathbb{R}^{1+n})}, \tag{3.22}$$

$$\|\Box_k \mathscr{A} f\|_{L^\infty_t L^2_x \cap L^{2+p}_{t,x}(\mathbb{R}^{1+n})} \lesssim \langle k_1\rangle^{1/2} \|\Box_k f\|_{L^1_{x_1} L^2_{x_2,\dots,x_n} L^2_t(\mathbb{R}^{1+n})}. \tag{3.23}$$

$$\|\Box_k \mathscr{A} \partial_{x_1} f\|_{L^\infty_{x_1} L^2_{x_2,\dots,x_n} L^2_t(\mathbb{R}^{1+n})} \lesssim \langle k_1\rangle^{1/2} \|\Box_k f\|_{L^{(2+p)/(1+p)}_{t,x}(\mathbb{R}^{1+n})}, \tag{3.24}$$

$$\|\Box_k \mathscr{A} \partial_{x_1} f\|_{L^q_{x_1} L^\infty_{x_2,\dots,x_n} L^\infty_t(\mathbb{R}^{1+n})} \lesssim \langle k_1\rangle^{1+1/q} \|\Box_k f\|_{L^{(2+p)/(1+p)}_{t,x}(\mathbb{R}^{1+n})}, \tag{3.25}$$

$$\|\Box_k \mathscr{A} f\|_{L^\infty_t L^2_x \cap L^{2+p}_{t,x}(\mathbb{R}^{1+n})} \lesssim \|\Box_k f\|_{L^{(2+p)/(1+p)}_{t,x}(\mathbb{R}^{1+n})}. \tag{3.26}$$

In (3.22), $q > 2$ is required. Moreover, replacing $L^{2+p}_{x,t}$ by $L^3_t L^6_x$, the results also hold.

4. Linear estimates with derivative interaction

In view of (3.21) in Corollary 3.11, the operator $\mathscr{A}$ in the space $L^\infty_{x_1} L^2_{x_2,\dots,x_n} L^2_t(\mathbb{R}^{1+n})$ has succeed in absorbing the partial derivative ∂_{x_1}. However, it seem that $\mathscr{A}$ can not deal with the partial derivative ∂_{x_2} in the space $L^\infty_{x_1} L^2_{x_2,\dots,x_n} L^2_t(\mathbb{R}^{1+n})$. So, we need a new way to handle the interaction between $L^\infty_{x_1} L^2_{x_2,\dots,x_n} L^2_t(\mathbb{R}^{1+n})$ and ∂_{x_2}. We have the following

Proposition 4.1. *Let $i=2,\dots,n$, $2\leqslant q\leqslant\infty$, $q>4/n$. Let $2\leqslant r<\infty$, $2/\gamma(r)=n(1/2-1/r)$, $\gamma\geqslant\gamma(r)$, $\gamma>2$. Then we have*

$$\|\Box_k\partial_{x_i}\mathscr{A}f\|_{L^\infty_{x_1}L^2_{x_2,\dots,x_n}L^2_t(\mathbb{R}^{1+n})}\lesssim\big\|\partial_{x_i}\partial_{x_1}^{-1}\Box_k f\big\|_{L^1_{x_1}L^2_{x_2,\dots,x_n}L^2_t(\mathbb{R}^{1+n})},\tag{4.1}$$

$$\|\Box_k\partial_{x_i}\mathscr{A}f\|_{L^\infty_{x_1}L^2_{x_2,\dots,x_n}L^2_t(\mathbb{R}^{1+n})}\lesssim\big\|\partial_{x_i}D_{x_1}^{-1/2}\Box_k f\big\|_{L^{\gamma'}L^{r'}_x(\mathbb{R}^{1+n})},\tag{4.2}$$

$$\|\Box_k\partial_{x_i}\mathscr{A}f\|_{L^q_{x_1}L^\infty_{x_2,\dots,x_n}L^\infty_t(\mathbb{R}^{1+n})}\lesssim\langle k_i\rangle^{1/2}\langle k_1\rangle^{1/q}\|\Box_k f\|_{L^1_{x_i}L^2_{(x_j)_{j\neq i}}L^2_t(\mathbb{R}^{1+n})},\tag{4.3}$$

$$\|\Box_k\partial_{x_i}\mathscr{A}f\|_{L^q_{x_1}L^\infty_{x_2,\dots,x_n}L^\infty_t(\mathbb{R}^{1+n})}\lesssim\langle k_i\rangle\langle k_1\rangle^{1/q}\|\Box_k f\|_{L^{\gamma'}_tL^{r'}_x(\mathbb{R}^{1+n})}.\tag{4.4}$$

In (4.3)*, $q>2$ is required.*

Proof. (4.1) is a straightforward consequence of Proposition 2.1. We have

$$\begin{aligned}\mathcal{L}(\partial_{x_2}f,\psi)&:=\left|\int_{\mathbb{R}}\left(\int_{\mathbb{R}}S(t-\tau)\partial_{x_2}f(\tau)\,d\tau,\psi(t)\right)dt\right|\\&\leqslant\left\|\int_{\mathbb{R}}S(-\tau)\partial_{x_2}D_{x_1}^{-1/2}f(\tau)\,d\tau\right\|_{L^2(\mathbb{R}^n)}\left\|D_{x_1}^{1/2}\int_{\mathbb{R}}S(-t)\psi(t)\,dt\right\|_{L^2(\mathbb{R}^n)}.\end{aligned}\tag{4.5}$$

By the Strichartz inequality and Proposition 2.2,

$$\mathcal{L}(\partial_{x_2}f,\psi)\lesssim\big\|\partial_{x_2}D_{x_1}^{-1/2}f\big\|_{L^{\gamma'}_tL^{r'}_x(\mathbb{R}^{1+n})}\|\psi\|_{L^1_{x_1}L^2_{x_2,\dots,x_n}L^2_t(\mathbb{R}^{1+n})}.\tag{4.6}$$

By duality, (4.6) implies (4.2) in the case $r>2$. In the case $r=2$, in view of the $1/2$-order smooth effect of $S(t)$, we see that (4.2) also holds true. Similarly, in view of Propositions 2.2, 3.7 and Lemma 3.4,

$$\begin{aligned}\mathcal{L}(\partial_{x_2}\Box_k f,\psi)&\leqslant\left\|\int_{\mathbb{R}}S(-\tau)D^1_{x_2}\Box_k f(\tau)\,d\tau\right\|_{L^2(\mathbb{R}^n)}\left\|\tilde{\Box}_k\int_{\mathbb{R}}S(-t)\psi(t)\,dt\right\|_{L^2(\mathbb{R}^n)}\\&\lesssim\langle k_2\rangle^{1/2}\|\Box_k f\|_{L^1_{x_2}L^2_{(x_j)_{j\neq2}}L^2_t(\mathbb{R}^{1+n})}\langle k_1\rangle^{1/q}\|\tilde{\Box}_k\psi\|_{L^{q'}_{x_1}L^1_{x_2,\dots,x_n}L^1_t(\mathbb{R}^{1+n})}\\&\lesssim\langle k_2\rangle^{1/2}\langle k_1\rangle^{1/q}\|\Box_k f\|_{L^1_{x_2}L^2_{(x_j)_{j\neq2}}L^2_t(\mathbb{R}^{1+n})}\|\psi\|_{L^{q'}_{x_1}L^1_{x_2,\dots,x_n}L^1_t(\mathbb{R}^{1+n})}.\end{aligned}\tag{4.7}$$

By duality, (4.3) follows from (4.7). Finally,

$$\begin{aligned}\mathcal{L}(\partial_{x_2}\Box_k f,\psi)&\leqslant\left\|\int_{\mathbb{R}}S(-\tau)\partial_{x_2}\Box_k f(\tau)\,d\tau\right\|_{L^2(\mathbb{R}^n)}\left\|\tilde{\Box}_k\int_{\mathbb{R}}S(-t)\psi(t)\,dt\right\|_{L^2(\mathbb{R}^n)}\\&\lesssim\langle k_2\rangle\langle k_1\rangle^{1/q}\|\psi\|_{L^{q'}_{x_1}L^1_{x_2,\dots,x_n}L^1_t(\mathbb{R}^{1+n})}\|\Box_k f\|_{L^{\gamma'}_tL^{r'}_x(\mathbb{R}^{1+n})}.\end{aligned}\tag{4.8}$$

If $r>2$ or $q>2$, (4.8) and Christ–Kiselev's Lemma imply (4.4), as desired. If $r=q=2$, in view of Proposition 3.5, we have also (4.4). □

Lemma 4.2. *Let $\psi:[0,\infty)\to[0,1]$ be a smooth bump function satisfying $\psi(x)=1$ as $|x|\leqslant1$ and $\psi(x)=0$ if $|x|\geqslant2$. Denote $\psi_1(\xi)=\psi(\xi_2/2\xi_1)$, $\psi_2(\xi)=1-\psi(\xi_2/2\xi_1)$, $\xi\in\mathbb{R}^n$. Then we have for $\sigma\geqslant0$,*

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^n,\,|k_1|>4}\langle k_1\rangle^\sigma\big\|\mathscr{F}^{-1}_{\xi_1,\xi_2}\psi_1\mathscr{F}_{x_1,x_2}\Box_k\partial_{x_2}\mathscr{A}f\big\|_{L^\infty_{x_1}L^2_{x_2,\dots,x_n}L^2_t(\mathbb{R}^{1+n})}\\&\quad\lesssim\sum_{k\in\mathbb{Z}^n,\,|k_1|>4}\langle k_1\rangle^\sigma\|\Box_k f\|_{L^1_{x_1}L^2_{x_2,\dots,x_n}L^2_t(\mathbb{R}^{1+n})},\end{aligned}\tag{4.9}$$

and for $\sigma\geqslant1$,

$$\sum_{k\in\mathbb{Z}^n,\,|k_1|>4} \langle k_1\rangle^\sigma \big\| \mathscr{F}^{-1}_{\xi_1,\xi_2}\psi_2\mathscr{F}_{x_1,x_2}\Box_k\partial_{x_2}\mathscr{A}f\big\|_{L^\infty_{x_1}L^2_{x_2,\dots,x_n}L^2_t(\mathbb{R}^{1+n})}
\lesssim \sum_{k\in\mathbb{Z}^n,\,|k_2|>4}\langle k_2\rangle^\sigma \|\Box_k f\|_{L^1_{x_1}L^2_{x_2,\dots,x_n}L^2_t(\mathbb{R}^{1+n})}. \tag{4.10}$$

Proof. For simplicity, we denote

$$I = \big\| \mathscr{F}^{-1}_{\xi_1,\xi_2}\psi_1\mathscr{F}_{x_1,x_2}\Box_k\partial_{x_2}\mathscr{A}f\big\|_{L^\infty_{x_1}L^2_{x_2,\dots,x_n}L^2_t(\mathbb{R}^{1+n})},$$
$$II = \big\| \mathscr{F}^{-1}_{\xi_1,\xi_2}\psi_2\mathscr{F}_{x_1,x_2}\Box_k\partial_{x_2}\mathscr{A}f\big\|_{L^\infty_{x_1}L^2_{x_2,\dots,x_n}L^2_t(\mathbb{R}^{1+n})}.$$

Let η_k be as in Lemma 3.1. For $k\in\mathbb{Z}^n$, $|k_1|>4$, applying the almost orthogonality of $\Box_k$, we have

$$I \lesssim \sum_{|\ell_1|,|\ell_2|\leqslant 1} \left\| \mathscr{F}^{-1}_{\xi_1,\xi_2}\psi\left(\frac{\xi_2}{2\xi_1}\right)\frac{\xi_2}{\xi_1}\prod_{i=1,2}\eta_{k_i+\ell_i}(\xi_i)\mathscr{F}_{x_1,x_2}\Box_k\partial_{x_1}\mathscr{A}f\right\|_{L^\infty_{x_1}L^2_{x_2,\dots,x_n}L^2_t(\mathbb{R}^{1+n})}. \tag{4.11}$$

Denote

$$(f\circledast_{12} g)(x) = \int_{\mathbb{R}^2} f(t,x_1-y_1,x_2-y_2,x_3,\dots,x_n)g(t,y_1,y_2)\,dy_1\,dy_2. \tag{4.12}$$

We have for any Banach function space X defined on $\mathbb{R}^{1+n}$,

$$\|f\circledast_{12} g\|_X \leqslant \|g\|_{L^1_{y_1,y_2}(\mathbb{R}^2)}\sup_{y_1,y_2}\big\|f(\cdot,\cdot-y_1,\cdot-y_2,\cdot,\dots,\cdot)\big\|_X. \tag{4.13}$$

Hence, by (4.11) and (4.13),

$$I \lesssim \sum_{|\ell_1|,|\ell_2|\leqslant 1} \left\| \mathscr{F}^{-1}_{\xi_1,\xi_2}\psi\left(\frac{\xi_2}{2\xi_1}\right)\frac{\xi_2}{\xi_1}\prod_{i=1,2}\eta_{k_i+\ell_i}(\xi_i)\right\|_{L^1(\mathbb{R}^2)}\|\Box_k\partial_{x_1}\mathscr{A}f\|_{L^\infty_{x_1}L^2_{x_2,\dots,x_n}L^2_t(\mathbb{R}^{1+n})}. \tag{4.14}$$

Using Bernstein's multiplier estimate, for $|k_1|>4$, we have

$$\left\| \mathscr{F}^{-1}_{\xi_1,\xi_2}\psi\left(\frac{\xi_2}{2\xi_1}\right)\frac{\xi_2}{\xi_1}\prod_{i=1,2}\eta_{k_i+\ell_i}(\xi_i)\right\|_{L^1(\mathbb{R}^2)} \lesssim \sum_{|\alpha|\leqslant 2}\left\| D^\alpha\left[\psi\left(\frac{\xi_2}{2\xi_1}\right)\frac{\xi_2}{\xi_1}\prod_{i=1,2}\eta_{k_i+\ell_i}(\xi_i)\right]\right\|_{L^2(\mathbb{R}^2)}\lesssim 1. \tag{4.15}$$

By Proposition 3.8, (4.14) and (4.15), we have

$$I\lesssim \|\Box_k f\|_{L^1_{x_1}L^2_{x_2,\dots,x_n}L^2_t(\mathbb{R}^{1+n})},\quad |k_1|\geqslant 4. \tag{4.16}$$

Next, we consider the estimate of II. Using Proposition 4.1,

$$II \lesssim \big\|\mathscr{F}^{-1}_{\xi_1,\xi_2}(\xi_2/\xi_1)\psi_2\mathscr{F}_{x_1,x_2}\Box_k f\big\|_{L^1_{x_1}L^2_{x_2,\dots,x_n}L^2_t(\mathbb{R}^{1+n})}
\lesssim \sum_{|\ell_1|,|\ell_2|\leqslant 1}\left\| \mathscr{F}^{-1}_{\xi_1,\xi_2}\left(1-\psi\left(\frac{\xi_2}{2\xi_1}\right)\right)\frac{\xi_2}{\xi_1}\prod_{i=1,2}\eta_{k_i+\ell_i}(\xi_i)\right\|_{L^1(\mathbb{R}^2)}\|\Box_k f\|_{L^1_{x_1}L^2_{x_2,\dots,x_n}L^2_t(\mathbb{R}^{1+n})}. \tag{4.17}$$

Notice that $\operatorname{supp}\psi_2\subset\{\xi:\ |\xi_2|\geqslant 2|\xi_1|\}$. If $|k_1|\geqslant 4$, we have $|k_2|>6$ and $|k_2|\geqslant|k_1|$ in the summation of the left-hand side of (4.10). So, $\sum_{k\in\mathbb{Z}^n,\,|k_1|>4}\langle k_1\rangle^\sigma II\leqslant \sum_{k\in\mathbb{Z}^n,\,|k_1|>4}\langle k_2\rangle^{\sigma-1}\langle k_1\rangle II$.

$$\left\| \mathscr{F}^{-1}_{\xi_1,\xi_2}\left(1-\psi\left(\frac{\xi_2}{2\xi_1}\right)\right)\frac{\xi_2}{\xi_1}\prod_{i=1,2}\eta_{k_i+\ell_i}(\xi_i)\right\|_{L^1(\mathbb{R}^2)}
\lesssim \sum_{|\alpha|\leqslant 2}\left\| D^\alpha\left[\mathscr{F}^{-1}_{\xi_1,\xi_2}\left(1-\psi\left(\frac{\xi_2}{2\xi_1}\right)\right)\frac{\xi_2}{\xi_1}\prod_{i=1,2}\eta_{k_i+\ell_i}(\xi_i)\right]\right\|_{L^2(\mathbb{R}^2)}
\lesssim \langle k_2\rangle\langle k_1\rangle^{-1}. \tag{4.18}$$

(4.17) and (4.18) yield the estimate of II, as desired. □

Lemma 4.3. *Let $2 \leqslant q \leqslant \infty$, $q > 4/n$ and (γ, r) be as in Proposition* 4.1. *Let $k = (k_1, \dots, k_n)$, $k_{\max} := \max_{1\leqslant i\leqslant n} |k_i|$. Then we have*

$$\|\Box_k \partial_{x_i} \mathscr{A} f\|_{L^q_{x_1} L^\infty_{x_2,\dots,x_n} L^\infty_t(\mathbb{R}^{1+n})} \lesssim \langle k_{\max}\rangle^{1+1/q} \|\Box_k f\|_{L^{\gamma'}_t L^{r'}_x(\mathbb{R}^{1+n})}. \tag{4.19}$$

Proof. It follows from (4.4) that (4.19) holds. □

Lemma 4.4. *Let $k = (k_1, \dots, k_n)$, $k_{\max} := \max_{1\leqslant i\leqslant n} |k_i|$ and $q > 2 \vee 4/n$. Then we have for $\sigma \geqslant 0$ and $i, \alpha = 1, \dots, n$,*

$$\sum_{k\in\mathbb{Z}^n,\, |k_\alpha|=k_{\max}>4} \langle k\rangle^\sigma \|\Box_k \partial_{x_i} \mathscr{A} f\|_{L^q_{x_1} L^\infty_{x_2,\dots,x_n} L^\infty_t(\mathbb{R}^{1+n})} \lesssim \sum_{k\in\mathbb{Z}^n,\, |k_\alpha|>4} \langle k_\alpha\rangle^{\sigma+1/2+1/q} \|\Box_k f\|_{L^1_{x_\alpha} L^2_{(x_j)_{j\neq\alpha}} L^2_t(\mathbb{R}^{1+n})}. \tag{4.20}$$

Proof. First, we consider the case $\alpha = 1$. In view of (3.22) and $|k_1| = k_{\max} > 4$,

$$\begin{aligned}\|\Box_k \partial_{x_i} \mathscr{A} f\|_{L^q_{x_1} L^\infty_{x_2,\dots,x_n} L^\infty_t(\mathbb{R}^{1+n})} &\lesssim \sum_{|\ell_1|,|\ell_i|\leqslant 1} \left\| \mathscr{F}^{-1}_{\xi_1,\xi_i} \left(\frac{\xi_i}{\xi_1} \eta_{k_i+\ell_i}(\xi_i) \eta_{k_1+\ell_1}(\xi_1) \right) \right\|_{L^1(\mathbb{R}^2)} \\ &\quad \times \|\Box_k \partial_{x_1} \mathscr{A} f\|_{L^q_{x_1} L^\infty_{x_2,\dots,x_n} L^\infty_t(\mathbb{R}^{1+n})} \\ &\lesssim \langle k_i\rangle \langle k_1\rangle^{-1} \langle k_1\rangle^{1/2+1/q} \|\Box_k f\|_{L^1_{x_1} L^2_{x_2,\dots,x_n} L^2_t(\mathbb{R}^{1+n})} \\ &\lesssim \langle k_1\rangle^{1/2+1/q} \|\Box_k f\|_{L^1_{x_1} L^2_{x_2,\dots,x_n} L^2_t(\mathbb{R}^{1+n})}.\end{aligned} \tag{4.21}$$

(4.21) implies the result, as desired. Next, we consider the case $\alpha = 2$. Notice that $|k_2| = \max_{1\leqslant i\leqslant n} |k_i| > 4$. By (4.3),

$$\begin{aligned}\|\Box_k \partial_{x_i} \mathscr{A} f\|_{L^q_{x_1} L^\infty_{x_2,\dots,x_n} L^\infty_t(\mathbb{R}^{1+n})} &\lesssim \sum_{|\ell_2|,|\ell_i|\leqslant 1} \left\| \mathscr{F}^{-1}_{\xi_i,\xi_2} \left(\frac{\xi_i}{\xi_2} \eta_{k_i+\ell_i}(\xi_i) \eta_{k_2+\ell_2}(\xi_2) \right) \right\|_{L^1(\mathbb{R}^2)} \\ &\quad \times \|\Box_k \partial_{x_2} \mathscr{A} f\|_{L^q_{x_1} L^\infty_{x_2,\dots,x_n} L^\infty_t(\mathbb{R}^{1+n})} \\ &\lesssim \langle k_i\rangle \langle k_2\rangle^{-1} \langle k_2\rangle^{1/2+1/q} \|\Box_k f\|_{L^1_{x_2} L^2_{x_1,x_3,\dots,x_n} L^2_t(\mathbb{R}^{1+n})} \\ &\lesssim \langle k_2\rangle^{1/2+1/q} \|\Box_k f\|_{L^1_{x_2} L^2_{x_1,x_3,\dots,x_n} L^2_t(\mathbb{R}^{1+n})}.\end{aligned} \tag{4.22}$$

The other cases $\alpha = 3, \dots, n$ is analogous to the case $\alpha = 2$ and we omit the details of the proof. □

Remark 4.5. From the proof of Lemma 4.4, we easily see that

$$\sum_{k\in\mathbb{Z}^n,\, |k_\alpha|=k_{\max}>4} \langle k\rangle^\sigma \|\Box_k \partial_{x_i} \mathscr{A} f\|_{L^q_{x_\beta} L^\infty_{(x_j)_{j\neq\beta}} L^\infty_t(\mathbb{R}^{1+n})} \lesssim \sum_{k\in\mathbb{Z}^n,\, |k_\alpha|>4} \langle k_\alpha\rangle^{\sigma+1/2+1/q} \|\Box_k f\|_{L^1_{x_\alpha} L^2_{(x_j)_{j\neq\alpha}} L^2_t(\mathbb{R}^{1+n})}. \tag{4.23}$$

5. Proof of Theorem 1.3

Proof of Theorem 1.3. We now give the details of the proof of Theorem 1.3. Denote $\kappa = \min(\kappa_1, \dots, \kappa_n)$,

$$\rho_1(u) = \sum_{i=1}^n \sum_{k\in\mathbb{Z}^n,\, |k_i|>4} \langle k_i\rangle \|\Box_k u\|_{L^\infty_{x_i} L^2_{(x_j)_{j\neq i}} L^2_t(\mathbb{R}^{1+n})},$$

$$\rho_2(u)=\sum_{i=1}^{n}\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{1/2-1/\kappa}\|\Box_k u\|_{L^\kappa_{x_i}L^\infty_{(x_j)_{j\neq i}}L^\infty_t(\mathbb{R}^{1+n})},$$
$$\rho_3(u)=\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{1/2}\|\Box_k u\|_{L^\infty_t L^2_x\cap L^{2+\kappa}_{x,t}(\mathbb{R}^{1+n})}.$$

We give a brief explanation to $\rho_i(u)$. In view of the smooth effect as in Proposition 2.1, $\rho_1(u)$ is applied for handling the derivative in the nonlinearity. $\rho_2(u)$ arises from the nonlinear estimates $\|u^{\kappa+1}\|_{L^1_{x_1}L^2_{x_2,\dots,x_n}L^2_t}\leqslant \|u\|_{L^\infty_{x_1}L^2_{x_2,\dots,x_n}L^2_t}\|u\|^\kappa_{L^\kappa_{x_1}L^\infty_{x_2,\dots,x_n}L^\infty_t}$. Since the smooth effect is a worse estimate for the low frequency part, we need to use the Strichartz estimates to deal with the low frequency part and so, $\rho_3(u)$ is introduced. Put

$$X:=\left\{u\in\mathscr{S}'\big(\mathbb{R}^{1+n}\big):\ \|u\|_X:=\sum_{i=1}^{3}\rho_i(u)\leqslant\delta_0\right\}.$$

We consider the following mapping:

$$\mathscr{T}:u(t)\to S(t)u_0-\mathrm{i}\mathscr{A}\left(\sum_{i=1}^{n}\lambda_i\partial_{x_i}u^{\kappa_i+1}\right).$$

For convenience, we denote

$$\|u\|_{Y_i}=\sum_{k\in\mathbb{Z}^n,\,|k_i|>4}\langle k_i\rangle\|\Box_k u\|_{L^\infty_{x_i}L^2_{(x_j)_{j\neq i}}L^2_t(\mathbb{R}^{1+n})}.$$

In order to estimate $\rho_1(u)$, it suffices to control $\|\cdot\|_{Y_1}$. By (2.3) and Plancherel's identity, we have

$$\begin{aligned}\big\|S(t)u_0\big\|_{Y_1}&\lesssim\sum_{k\in\mathbb{Z}^n,\,|k_1|>4}\langle k_1\rangle\big\|\Box_k D^{-1/2}_{x_1}u_0\big\|_{L^2(\mathbb{R}^n)}\\&\lesssim\sum_{k\in\mathbb{Z}^n}\langle k_1\rangle^{1/2}\|\Box_k u_0\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

By (3.3), Lemma 3.2, we have

$$\rho_i\big(S(t)u_0\big)\lesssim\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{1/2}\|\Box_k u_0\|_{L^2(\mathbb{R}^n)},\quad i=2,3.$$

Denote

$$\mathbb{S}^{(i)}_{\ell,1}:=\big\{\big(k^{(1)},\dots,k^{(\kappa_\ell+1)}\big)\in\big(\mathbb{Z}^n\big)^{k_\ell+1}:\ \big|k^{(1)}_i\big|\vee\cdots\vee\big|k^{(\kappa_\ell+1)}_i\big|>4\big\},$$
$$\mathbb{S}^{(i)}_{\ell,2}:=\big\{\big(k^{(1)},\dots,k^{(\kappa_\ell+1)}\big)\in\big(\mathbb{Z}^n\big)^{k_\ell+1}:\ \big|k^{(1)}_i\big|\vee\cdots\vee\big|k^{(\kappa_\ell+1)}_i\big|\leqslant4\big\}.$$

Using the frequency-uniform decomposition, we have

$$\begin{aligned}u^{\kappa_\ell+1}&=\sum_{k^{(1)},\dots,k^{(\kappa_\ell+1)}\in\mathbb{Z}^n}\Box_{k^{(1)}}u\dots\Box_{k^{(\kappa_\ell+1)}}u\\&=\sum_{\mathbb{S}^{(i)}_{\ell,1}}\Box_{k^{(1)}}u\dots\Box_{k^{(\kappa_\ell+1)}}u+\sum_{\mathbb{S}^{(i)}_{\ell,2}}\Box_{k^{(1)}}u\dots\Box_{k^{(\kappa_\ell+1)}}u.\end{aligned}\tag{5.1}$$

Using (3.21) and (3.24), we obtain that

$$\begin{aligned}\big\|\mathscr{A}\partial_{x_1}u^{\kappa_1+1}\big\|_{Y_1}\lesssim&\sum_{k\in\mathbb{Z}^n,\,|k_1|>4}\langle k_1\rangle\sum_{\mathbb{S}^{(1)}_{1,1}}\big\|\Box_k(\Box_{k^{(1)}}u\dots\Box_{k^{(\kappa_1+1)}}u)\big\|_{L^1_{x_1}L^2_{x_2,\dots,x_n}L^2_t(\mathbb{R}^{1+n})}\\&+\sum_{k\in\mathbb{Z}^n,\,|k_1|>4}\langle k_1\rangle^{3/2}\sum_{\mathbb{S}^{(1)}_{1,2}}\big\|\Box_k(\Box_{k^{(1)}}u\dots\Box_{k^{(\kappa_1+1)}}u)\big\|_{L^{(2+\kappa)/(1+\kappa)}_{t,x}(\mathbb{R}^{1+n})}\\:=&\,I+II.\end{aligned}\tag{5.2}$$

In view of the support property of $\widehat{\Box_k u}$, we see that

$$\Box_k(\Box_{k^{(1)}}u\dots\Box_{k^{(\kappa_1+1)}}u)=0,\quad \text{if } \left|k-k^{(1)}-\dots-k^{(\kappa_1+1)}\right|\geqslant C. \tag{5.3}$$

Hence, by Lemma 3.4,

$$I\lesssim\sum_{k\in\mathbb{Z}^n,\,|k_1|>4}\langle k_1\rangle\sum_{\mathbb{S}^{(1)}_{1,1}}\|\Box_{k^{(1)}}u\dots\Box_{k^{(\kappa_1+1)}}u\|_{L^1_{x_1}L^2_{x_2,\dots,x_n}L^2_t(\mathbb{R}^{1+n})}\chi_{|k-k^{(1)}-\dots-k^{(\kappa_1+1)}|\leqslant C}. \tag{5.4}$$

By Hölder's inequality and $\|\Box_k u\|_{L^\infty_x}\lesssim\|\Box_k u\|_{L^2_x}$ uniformly holds for all $k\in\mathbb{Z}^n$, we have

$$\begin{aligned}&\|\Box_{k^{(1)}}u\dots\Box_{k^{(\kappa_1+1)}}u\|_{L^1_{x_1}L^2_{x_2,\dots,x_n}L^2_t(\mathbb{R}^{1+n})}\\&\quad\leqslant\|\Box_{k^{(1)}}u\|_{L^\infty_{x_1}L^2_{x_2,\dots,x_n}L^2_t(\mathbb{R}^{1+n})}\prod_{i=2}^{\kappa_1+1}\|\Box_{k^{(i)}}u\|_{L^\kappa_{x_1}L^\infty_{x_2,\dots,x_n}L^\infty_t\cap L^\infty_tL^2_x(\mathbb{R}^{1+n})}.\end{aligned}$$

Since $|k-k^{(1)}-\dots-k^{(\kappa_1+1)}|\leqslant C$ implies that $|k_1-k_1^{(1)}-\dots-k_1^{(\kappa_1+1)}|\leqslant C$, we see that $|k_1|\leqslant C\max_{i=1,\dots,\kappa_1+1}|k_1^{(i)}|$. We may assume that $|k_1^{(1)}|=\max_{i=1,\dots,\kappa_1+1}|k_1^{(i)}|$ in the summation $\sum_{\mathbb{S}^{(1)}_{1,1}}$ in (5.4) above. So,

$$\begin{aligned}I&\lesssim\sum_{k^{(1)}\in\mathbb{Z}^n,\ |k_1^{(1)}|>4}\langle k_1^{(1)}\rangle\|\Box_{k^{(1)}}u\|_{L^\infty_{x_1}L^2_{x_2,\dots,x_n}L^2_t(\mathbb{R}^{1+n})}\sum_{k^{(2)},\dots,k^{(\kappa_1+1)}\in\mathbb{Z}^n}\prod_{i=2}^{\kappa_1+1}\|\Box_{k^{(i)}}u\|_{L^\kappa_{x_1}L^\infty_{x_2,\dots,x_n}L^\infty_t\cap L^\infty_tL^2_x(\mathbb{R}^{1+n})}\\&\lesssim\rho_1(u)\big(\rho_2(u)+\rho_3(u)\big)^{\kappa_1}.\end{aligned} \tag{5.5}$$

In view of (5.3) we easily see that $|k_1|\leqslant C$ in II of (5.2). Hence,

$$\begin{aligned}II&\lesssim\sum_{k\in\mathbb{Z}^n,\,|k_1|>4}\sum_{\mathbb{S}^{(1)}_{1,2}}\|\Box_{k^{(1)}}u\dots\Box_{k^{(\kappa_1+1)}}u\|_{L^{(2+\kappa)/(1+\kappa)}_{x,t}(\mathbb{R}^{1+n})}\chi_{|k-k^{(1)}-\dots-k^{(\kappa_1+1)}|\leqslant C}\\&\lesssim\sum_{\mathbb{S}^{(1)}_{1,2}}\|\Box_{k^{(1)}}u\dots\Box_{k^{(\kappa_1+1)}}u\|_{L^{(2+\kappa)/(1+\kappa)}_{x,t}(\mathbb{R}^{1+n})}\\&\lesssim\sum_{\mathbb{S}^{(1)}_{1,2}}\prod_{i=1}^{\kappa_1+1}\|\Box_{k^{(i)}}u\|_{L^{2+\kappa}_{x,t}\cap L^\infty_tL^2_x(\mathbb{R}^{1+n})}\lesssim\rho_3(u)^{1+\kappa_1}.\end{aligned} \tag{5.6}$$

Hence, we have

$$\left\|\mathscr{A}\partial_{x_1}u^{\kappa_1+1}\right\|_{Y_1}\lesssim\rho_1(u)\big(\rho_2(u)+\rho_3(u)\big)^{\kappa_1}+\rho_3(u)^{1+\kappa_1}. \tag{5.7}$$

Next, we estimate $\|\mathscr{A}\partial_{x_2}u^{\kappa_2+1}\|_{Y_1}$. Let ψ_i be as in Lemma 4.2. For convenience, we write

$$P_i=\mathscr{F}^{-1}_{\xi_1,\xi_2}\psi_i\mathscr{F}_{x_1,x_2},\quad i=1,2. \tag{5.8}$$

We have

$$\left\|\mathscr{A}\partial_{x_2}u^{\kappa_2+1}\right\|_{Y_1}\lesssim\left\|P_1\partial_{x_2}\mathscr{A}u^{\kappa_2+1}\right\|_{Y_1}+\left\|P_2\partial_{x_2}\mathscr{A}u^{\kappa_2+1}\right\|_{Y_1}:=III+IV. \tag{5.9}$$

Using the decomposition (5.1),

$$\begin{aligned}III&\leqslant\left\|P_1\partial_{x_2}\mathscr{A}\sum_{\mathbb{S}^{(1)}_{2,1}}(\Box_{k^{(1)}}u\dots\Box_{k^{(\kappa_2+1)}}u)\right\|_{Y_1}+\left\|P_1\partial_{x_2}\mathscr{A}\sum_{\mathbb{S}^{(1)}_{2,2}}(\Box_{k^{(1)}}u\dots\Box_{k^{(\kappa_2+1)}}u)\right\|_{Y_1}\\&:=III_1+III_2.\end{aligned} \tag{5.10}$$

Applying Lemma 4.2 and then following the same way as in the estimate to (5.4),

$$III_1 \lesssim \sum_{k\in\mathbb{Z}^n,\,|k_1|>4} \langle k_1\rangle \sum_{\mathbb{S}^{(1)}_{2,1}} \big\| \Box_k (\Box_{k^{(1)}} u \ldots \Box_{k^{(\kappa_2+1)}} u) \big\|_{L^1_{x_1} L^2_{x_2,\ldots,x_n} L^2_t(\mathbb{R}^{1+n})}$$
$$\lesssim \rho_1(u)\big(\rho_2(u)+\rho_3(u)\big)^{\kappa_2}. \tag{5.11}$$

For the estimate of III_2, noticing the fact that $\operatorname{supp}\psi_1 \subset \{\xi\colon |\xi_2| \leqslant 4|\xi_1|\}$ and using the multiplier estimate, then applying (4.2), we have

$$III_2 \lesssim \sum_{k\in\mathbb{Z}^n,\,|k_1|>4,\,|k_2|\lesssim|k_1|} \langle k_1\rangle^{3/2} \sum_{\mathbb{S}^{(1)}_{2,2}} \big\| \Box_k (\Box_{k^{(1)}} u \ldots \Box_{k^{(\kappa_2+1)}} u) \big\|_{L^{(2+\kappa)/(1+\kappa)}_{t,x}(\mathbb{R}^{1+n})}$$
$$\lesssim \rho_3(u)^{1+\kappa_2}. \tag{5.12}$$

We need to further control IV. Using the decomposition (5.1),

$$IV \leqslant \Big\| P_2 \partial_{x_2} \mathscr{A} \sum_{\mathbb{S}^{(2)}_{2,1}} (\Box_{k^{(1)}} u \ldots \Box_{k^{(\kappa_2+1)}} u) \Big\|_{Y_1} + \Big\| P_2 \partial_{x_2} \mathscr{A} \sum_{\mathbb{S}^{(2)}_{2,2}} (\Box_{k^{(1)}} u \ldots \Box_{k^{(\kappa_2+1)}} u) \Big\|_{Y_1}$$
$$:= IV_1 + IV_2. \tag{5.13}$$

By Lemma 4.2,

$$IV_1 \lesssim \sum_{k\in\mathbb{Z}^n,\,|k_2|>4} \langle k_2\rangle \sum_{\mathbb{S}^{(2)}_{2,1}} \big\| \Box_k (\Box_{k^{(1)}} u \ldots \Box_{k^{(\kappa_2+1)}} u) \big\|_{L^1_{x_1} L^2_{x_2,\ldots,x_n} L^2_t(\mathbb{R}^{1+n})}. \tag{5.14}$$

By symmetry of $k^{(1)},\ldots,k^{(\kappa_2+1)}$, we can assume that $|k^{(1)}_2| = \max_{1\leqslant i\leqslant \kappa_2+1} |k^{(i)}_2|$ in $\mathbb{S}^{(2)}_{2,1}$. Using the same way as in the estimate of I, we have

$$IV_1 \lesssim \sum_{\mathbb{S}^{(2)}_{2,1},\,|k^{(1)}_2|>4} \big\langle k^{(1)}_2 \big\rangle \|\Box_{k^{(1)}} u \ldots \Box_{k^{(\kappa_2+1)}} u\|_{L^1_{x_1} L^2_{x_2,\ldots,x_n} L^2_t(\mathbb{R}^{1+n})}. \tag{5.15}$$

By Hölder's inequality,

$$\|\Box_{k^{(1)}} u \ldots \Box_{k^{(\kappa_2+1)}} u\|_{L^1_{x_1} L^2_{x_2,\ldots,x_n} L^2_t(\mathbb{R}^{1+n})}$$
$$\lesssim \big\| \Box_{k^{(1)}} u |\Box_{k^{(2)}} u \ldots \Box_{k^{(\kappa_2+1)}} u|^{1/2} \big\|_{L^2_{x,t}(\mathbb{R}^{1+n})} \big\| |\Box_{k^{(2)}} u \ldots \Box_{k^{(\kappa_2+1)}} u|^{1/2} \big\|_{L^2_{x_1} L^\infty_{x_2,\ldots,x_n} L^\infty_t(\mathbb{R}^{1+n})}$$
$$\lesssim \|\Box_{k^{(1)}} u\|_{L^\infty_{x_2} L^2_{x_1,x_3,\ldots,x_n} L^2_t(\mathbb{R}^{1+n})} \prod_{i=2}^{\kappa_2+1} \|\Box_{k^{(i)}} u\|_{L^{\kappa_2}_{x_2} L^\infty_{x_1,x_3,\ldots,x_n} L^\infty_t(\mathbb{R}^{1+n})} \prod_{i=2}^{\kappa_2+1} \|\Box_{k^{(i)}} u\|_{L^{\kappa_2}_{x_1} L^\infty_{x_2,\ldots,x_n} L^\infty_t(\mathbb{R}^{1+n})}. \tag{5.16}$$

In view of the inclusion $L^\kappa_{x_1} L^\infty_{x_2,\ldots,x_n} L^\infty_t \cap L^\infty_{x,t} \subset L^{\kappa_2}_{x_1} L^\infty_{x_2,\ldots,x_n} L^\infty_t$, we immediately have

$$IV_1 \lesssim \rho_1(u)\big(\rho_2(u)+\rho_3(u)\big)^{\kappa_2}. \tag{5.17}$$

Noticing the fact that $\operatorname{supp}\psi_2 \subset \{\xi\colon |\xi_2| \geqslant 2|\xi_1|\}$ and applying (4.2), we have

$$IV_2 \lesssim \sum_{k\in\mathbb{Z}^n,\,|k_2|>4} \langle k_2\rangle^{3/2} \sum_{\mathbb{S}^{(2)}_{2,2}} \big\| \Box_k (\Box_{k^{(1)}} u \ldots \Box_{k^{(\kappa_2+1)}} u) \big\|_{L^{(2+\kappa)/(1+\kappa)}_{t,x}(\mathbb{R}^{1+n})}$$
$$\lesssim \sum_{k\in\mathbb{Z}^n,\,|k_2|>4} \sum_{\mathbb{S}^{(2)}_{2,2}} \big\| \Box_k (\Box_{k^{(1)}} u \ldots \Box_{k^{(\kappa_2+1)}} u) \big\|_{L^{(2+\kappa)/(1+\kappa)}_{t,x}(\mathbb{R}^{1+n})}$$
$$\lesssim \rho_3(u)^{1+\kappa_2}. \tag{5.18}$$

The other terms in $\rho_1(\cdot)$ can be bounded in a similar way. So, we have shown that

$$\rho_1\bigg(\mathscr{A}\bigg(\sum_{i=1}^n \lambda_i \partial_{x_i} u^{\kappa_i+1} \bigg)\bigg) \lesssim \sum_{i=1}^n \big(\rho_1(u)\big(\rho_2(u)+\rho_3(u)\big)^{\kappa_i} + \rho_3(u)^{1+\kappa_i} \big). \tag{5.19}$$

We estimate $\rho_2(\cdot)$. Denote

$$\|u\|_{Z_i} = \sum_{k\in\mathbb{Z}^n} \langle k\rangle^{1/2-1/\kappa} \|\Box_k u\|_{L^\kappa_{x_i} L^\infty_{(x_j)_{j\neq i}} L^\infty_t(\mathbb{R}^{1+n})}. \tag{5.20}$$

We have

$$\rho_2\left(\mathscr{A}\left(\sum_{j=1}^n \lambda_i \partial_{x_i} u^{\kappa_i+1}\right)\right) \lesssim \sum_{i=1}^n \left\| \mathscr{A}\left(\sum_{i=1}^n \lambda_i \partial_{x_i} u^{\kappa_i+1}\right)\right\|_{Z_j}. \tag{5.21}$$

Due to the symmetry of $Z_1,\dots,Z_n$, it suffices to consider the estimate of $\|\cdot\|_{Z_1}$. Recall that $k_{\max} = |k_1|\vee\cdots\vee|k_n|$. We have

$$\begin{aligned}\|v\|_{Z_1} &\leqslant \left(\sum_{k\in\mathbb{Z}^n,\, k_{\max}>4} + \sum_{k\in\mathbb{Z}^n,\, k_{\max}\leqslant 4}\right)\langle k\rangle^{1/2-1/\kappa}\|\Box_k v\|_{L^\kappa_{x_1} L^\infty_{x_2,\dots,x_n} L^\infty_t(\mathbb{R}^{1+n})} \\ &:= \Gamma_1(v) + \Gamma_2(v).\end{aligned} \tag{5.22}$$

In view of Lemma 4.3 and Hölder's inequality,

$$\begin{aligned}\Gamma_2\left(\mathscr{A}\left(\sum_{i=1}^n \lambda_i \partial_{x_i} u^{\kappa_i+1}\right)\right) &\leqslant \sum_{k\in\mathbb{Z}^n,\, k_{\max}\leqslant 4}\left\|\Box_k \mathscr{A}\left(\sum_{i=1}^n \lambda_i \partial_{x_i} u^{\kappa_i+1}\right)\right\|_{L^\kappa_{x_1} L^\infty_{x_2,\dots,x_n} L^\infty_t(\mathbb{R}^{1+n})} \\ &\lesssim \sum_{i=1}^n \sum_{k^{(1)},\dots,k^{(\kappa_i+1)}\in\mathbb{Z}^n} \|\Box_{k^{(1)}} u \dots \Box_{k^{(\kappa_i+1)}} u\|_{L^{\frac{2+\kappa_i}{1+\kappa_i}}_{t,x}(\mathbb{R}^{1+n})} \\ &\lesssim \sum_{i=1}^n \sum_{k^{(1)},\dots,k^{(\kappa_i+1)}\in\mathbb{Z}^n} \|\Box_{k^{(1)}} u\|_{L^{2+\kappa_i}_{t,x}(\mathbb{R}^{1+n})} \dots \|\Box_{k^{(\kappa_i+1)}} u\|_{L^{2+\kappa_i}_{t,x}(\mathbb{R}^{1+n})} \\ &\lesssim \sum_{i=1}^n \rho_3(u)^{\kappa_i+1}.\end{aligned} \tag{5.23}$$

It is easy to see that

$$\begin{aligned}\Gamma_1(v) &\leqslant \left(\sum_{k\in\mathbb{Z}^n,\, |k_1|=k_{\max}>4} + \cdots + \sum_{k\in\mathbb{Z}^n,\, |k_n|=k_{\max}>4}\right)\langle k\rangle^{1/2-1/\kappa}\|\Box_k v\|_{L^\kappa_{x_1} L^\infty_{x_2,\dots,x_n} L^\infty_t(\mathbb{R}^{1+n})} \\ &:= \Gamma_1^1(v) + \cdots + \Gamma_1^n(v).\end{aligned} \tag{5.24}$$

Using (5.1), Lemmas 4.3 and 4.4, we have

$$\begin{aligned}\Gamma_1^1\left(\mathscr{A}\left(\sum_{i=1}^n \lambda_i \partial_{x_i} u^{\kappa_i+1}\right)\right) &\lesssim \sum_{i=1}^n \sum_{k\in\mathbb{Z}^n,\, |k_1|>4} \langle k_1\rangle \sum_{\mathbb{S}^{(1)}_{i,1}} \left\|\Box_k(\Box_{k^{(1)}} u \dots \Box_{k^{(\kappa_i+1)}} u)\right\|_{L^1_{x_1} L^2_{x_2,\dots,x_n} L^2_t(\mathbb{R}^{1+n})} \\ &\quad + \sum_{i=1}^n \sum_{k\in\mathbb{Z}^n,\, |k_1|>4} \langle k_1\rangle^{3/2} \sum_{\mathbb{S}^{(1)}_{i,2}} \left\|\Box_k(\Box_{k^{(1)}} u \dots \Box_{k^{(\kappa_i+1)}} u)\right\|_{L^{(2+\kappa_i)/(1+\kappa_i)}_{t,x}(\mathbb{R}^{1+n})}.\end{aligned} \tag{5.25}$$

Using the same way as in (5.5) and (5.6), one easily sees that

$$\Gamma_1^1\left(\mathscr{A}\left(\sum_{i=1}^n \lambda_i \partial_{x_i} u^{\kappa_i+1}\right)\right) \lesssim \sum_{i=1}^n \left(\rho_1(u)\big(\rho_2(u)+\rho_3(u)\big)^{\kappa_i} + \rho_3(u)^{1+\kappa_i}\right). \tag{5.26}$$

We estimate $\Gamma_1^2(\cdot)$. By Lemmas 4.3 and 4.4,

$$\begin{aligned}\Gamma_1^2\left(\mathscr{A}\left(\sum_{i=1}^n \lambda_i \partial_{x_i} u^{\kappa_i+1}\right)\right) &\lesssim \sum_{i=1}^n \sum_{k\in\mathbb{Z}^n,\,|k_2|=k_{\max}>4} \langle k\rangle^{1/2-1/\kappa} \left\|\Box_k(\mathscr{A}\partial_{x_i} u^{\kappa_i+1})\right\|_{L^\kappa_{x_1} L^\infty_{x_2,\dots,x_n} L^\infty_t(\mathbb{R}^{1+n})} \\ &\lesssim \sum_{i=1}^n \sum_{k\in\mathbb{Z}^n,\,|k_2|=k_{\max}>4} \langle k_2\rangle \sum_{\mathbb{S}^{(2)}_{i,1}} \left\|\Box_k(\Box_{k^{(1)}} u \dots \Box_{k^{(\kappa_i+1)}} u)\right\|_{L^1_{x_2} L^2_{x_1,x_3,\dots,x_n} L^2_t(\mathbb{R}^{1+n})} \\ &\quad + \sum_{i=1}^n \sum_{k\in\mathbb{Z}^n,\,|k_2|>4} \langle k_2\rangle^{3/2} \sum_{\mathbb{S}^{(2)}_{i,2}} \left\|\Box_k(\Box_{k^{(1)}} u \dots \Box_{k^{(\kappa_i+1)}} u)\right\|_{L^{(2+\kappa_i)/(1+\kappa_i)}_{t,x}(\mathbb{R}^{1+n})}. \end{aligned} \tag{5.27}$$

This reduces the same estimate as $\Gamma_1^1(\cdot)$. We easily see that $\Gamma_1^i(\cdot)$ for $3 \leqslant i \leqslant n$ can be controlled in a similar way as $\Gamma_1^2(\cdot)$. Hence, we have shown that

$$\left\|\mathscr{A}\left(\sum_{i=1}^n \lambda_i \partial_{x_i} u^{\kappa_i+1}\right)\right\|_{Z_1} \lesssim \sum_{i=1}^n \left(\rho_1(u)\left(\rho_2(u)+\rho_3(u)\right)^{\kappa_i} + \rho_3(u)^{1+\kappa_i}\right). \tag{5.28}$$

For the estimates of $\rho_3(\mathscr{A}\partial_{x_i} u^{\kappa_i+1})$, we have from (3.13) and Lemma 3.4 that

$$\begin{aligned}\|\Box_k \mathscr{A}\partial_{x_i} f\|_{L^\infty_t L^2_x \cap L^{2+\kappa}_{t,x}(\mathbb{R}^{1+n})} &\lesssim \|\Box_k \partial_{x_i} f\|_{L^{(2+\kappa)/(1+\kappa)}_{t,x}(\mathbb{R}^{1+n})} \\ &\lesssim \langle k_i\rangle \|\Box_k f\|_{L^{(2+\kappa)/(1+\kappa)}_{t,x}(\mathbb{R}^{1+n})}. \end{aligned} \tag{5.29}$$

Hence, using (5.1), (3.26) and (3.23), we obtain that can be controlled by the right-hand side of (5.25).

$$\begin{aligned}\rho_3\left(\mathscr{A}\partial_{x_1} u^{\kappa_1+1}\right) &\lesssim \sum_{k\in\mathbb{Z}^n,\,|k_1|\leqslant 4} \langle k_1\rangle^{3/2} \sum_{k^{(1)},\dots,k^{(\kappa_1+1)}\in\mathbb{Z}^n} \left\|\Box_k(\Box_{k^{(1)}} u \dots \Box_{k^{(\kappa_1+1)}} u)\right\|_{L^{(2+\kappa)/(1+\kappa)}_{t,x}(\mathbb{R}^{1+n})} \\ &\quad + \sum_{k\in\mathbb{Z}^n,\,|k_1|>4} \langle k_1\rangle \sum_{\mathbb{S}^{(1)}_{1,1}} \left\|\Box_k(\Box_{k^{(1)}} u \dots \Box_{k^{(\kappa_1+1)}} u)\right\|_{L^1_{x_1} L^2_{x_2,\dots,x_n} L^2_t(\mathbb{R}^{1+n})} \\ &\quad + \sum_{k\in\mathbb{Z}^n,\,|k_1|>4} \langle k_1\rangle^{3/2} \sum_{\mathbb{S}^{(1)}_{1,2}} \left\|\Box_k(\Box_{k^{(1)}} u \dots \Box_{k^{(\kappa_1+1)}} u)\right\|_{L^{(2+\kappa)/(1+\kappa)}_{t,x}(\mathbb{R}^{1+n})}. \end{aligned} \tag{5.30}$$

By (5.5) and (5.6), we have

$$\rho_3\left(\mathscr{A}\partial_{x_1} u^{\kappa_1+1}\right) \lesssim \sum_{i=1}^n \left(\rho_1(u)\left(\rho_2(u)+\rho_3(u)\right)^{\kappa_i} + \rho_3(u)^{1+\kappa_i}\right). \tag{5.31}$$

Hence, we have shown that

$$\|\mathscr{T} u\|_X \lesssim \|u_0\|_{M^{1/2}_{2,1}} + \sum_{i=1}^n \|u\|_X^{1+\kappa_i}. \tag{5.32}$$

Using a standard contraction mapping argument, we can finish the proof of Theorem 1.3. □

6. Proof of Theorem 1.1

Roughly speaking, we will prove our Theorem 1.1 by following some ideas as in the proof of Theorem 1.3. However, due to the nonlinearity contains $u^{\kappa+1}$, and $(\nabla u)^\nu$ and $u^\kappa(\nabla u)^\nu$ as special cases, the proof of Theorem 1.3 can not be directly applied. We construct the space X as follows. Denote

$$\begin{aligned}\varrho_1^{(i)}(u) &= \sum_{k\in\mathbb{Z}^n,|k_i|>4} \langle k_i\rangle \|\Box_k u\|_{L^\infty_{x_i} L^2_{(x_j)_{j\neq i}} L^2_t(\mathbb{R}^{1+n})}, \\ \varrho_2^{(i)}(u) &= \sum_{k\in\mathbb{Z}^n} \langle k\rangle^{1/2-1/m} \|\Box_k u\|_{L^m_{x_i} L^\infty_{(x_j)_{j\neq i}} L^\infty_t(\mathbb{R}^{1+n})}, \\ \varrho_3^{(i)}(u) &= \sum_{k\in\mathbb{Z}^n} \langle k_i\rangle^{1/2} \|\Box_k u\|_{L^{2+m}_{x,t} \cap L^\infty_t L^2_x(\mathbb{R}^{1+n})}. \end{aligned}$$

Put

$$X := \left\{ u \in \mathscr{S}'(\mathbb{R}^{1+n}):\ \|u\|_X := \sum_{\ell=1}^{3} \sum_{\alpha=0,1} \sum_{i,j=1}^{n} \varrho_\ell^{(i)}\big(\partial_{x_j}^\alpha u\big) \leqslant \delta \right\}.$$

Considering the following mapping:

$$\mathscr{T} : u(t) \to S(t)u_0 - \mathrm{i}\mathscr{A} F(u, \bar{u}, \nabla u, \nabla \bar{u}),$$

we will show that $\mathscr{T} : X \to X$ is a contraction mapping.

Since $\|u\|_X = \|\bar{u}\|_X$, we may assume, without loss of generality that

$$F(u, \bar{u}, \nabla u, \nabla \bar{u}) = F(u, \nabla u) := \sum_{m+1 \leqslant \kappa + |\nu| < \infty} c_{\kappa\nu} u^\kappa (\nabla u)^\nu,$$

where $(\nabla u)^\nu = u_{x_1}^{\nu_1} \dots u_{x_n}^{\nu_n}$. For the sake of convenience, we denote

$$v_1 = \dots = v_\kappa = u, \quad v_{\kappa+1} = \dots = v_{\kappa+\nu_1} = u_{x_1}, \quad \dots, \quad v_{\kappa+|\nu|-\nu_n+1} = \dots = v_{\kappa+|\nu|} = u_{x_n}.$$

By (2.3), for $\alpha = 0, 1$,

$$\varrho_1^{(i)}\big(\partial_{x_j}^\alpha S(t)u_0\big) \lesssim \sum_{k \in \mathbb{Z}^n,\ |k_i|>4} \langle k_i\rangle^{1/2} \langle k_j\rangle \|\Box_k u_0\|_{L^2(\mathbb{R}^n)} \leqslant \|u_0\|_{M_{2,1}^{3/2}}.$$

By (3.19), (3.20), we have for $\alpha = 0, 1$,

$$\varrho_2^{(i)}\big(\partial_{x_j}^\alpha S(t)u_0\big) + \varrho_3^{(i)}\big(\partial_{x_j}^\alpha S(t)u_0\big) \lesssim \|u_0\|_{M_{2,1}^{3/2}}.$$

Hence,

$$\big\|S(t)u_0\big\|_X \lesssim \|u_0\|_{M_{2,1}^{3/2}}.$$

In order to estimate $\varrho_1^{(i)}(\mathscr{A}\partial_{x_j}^\alpha(v_1 \dots v_{\kappa+|\nu|}))$, $i, j = 1, \dots, n$, it suffices to estimate $\varrho_1^{(1)}(\mathscr{A}\partial_{x_1}^\alpha(v_1 \dots v_{\kappa+|\nu|}))$ and $\varrho_1^{(1)}(\mathscr{A}\partial_{x_2}^\alpha(v_1 \dots v_{\kappa+|\nu|}))$. Similarly as in (5.1), we will use the decomposition

$$\Box_k(v_1 \dots v_{\kappa+|\nu|}) = \sum_{\mathbb{S}_1^{(i)}} \Box_k(\Box_{k^{(1)}} v_1 \dots \Box_{k^{(\kappa+|\nu|)}} v_{\kappa+|\nu|}) + \sum_{\mathbb{S}_2^{(i)}} \Box_k(\Box_{k^{(1)}} v_1 \dots \Box_{k^{(\kappa+|\nu|)}} v_{\kappa+|\nu|}), \tag{6.1}$$

where

$$\mathbb{S}_1^{(i)} := \big\{\big(k^{(1)}, \dots, k^{(\kappa+|\nu|)}\big):\ \big|k_i^{(1)}\big| \vee \dots \vee \big|k_i^{(\kappa+|\nu|)}\big| > 4\big\},$$
$$\mathbb{S}_2^{(i)} := \big\{\big(k^{(1)}, \dots, k^{(\kappa+|\nu|)}\big):\ \big|k_i^{(1)}\big| \vee \dots \vee \big|k_i^{(\kappa+|\nu|)}\big| \leqslant 4\big\}.$$

In view of (3.8) and (3.15),

$$\begin{aligned}\varrho_1^{(1)}\big(\mathscr{A}\partial_{x_1}^\alpha(v_1 \dots v_{\kappa+|\nu|})\big) \lesssim{}& \sum_{k \in \mathbb{Z}^n,\ |k_1|>4} \langle k_1\rangle \sum_{\mathbb{S}_1^{(1)}} \big\|\Box_k(\Box_{k^{(1)}} v_1 \dots \Box_{k^{(\kappa+|\nu|)}} v_{\kappa+|\nu|})\big\|_{L_{x_1}^1 L_{x_2,\dots,x_n}^2 L_t^2(\mathbb{R}^{1+n})} \\ &+ \sum_{k \in \mathbb{Z}^n,\ |k_1|>4} \langle k_1\rangle^{3/2} \sum_{\mathbb{S}_2^{(1)}} \big\|\Box_k(\Box_{k^{(1)}} v_1 \dots \Box_{k^{(\kappa+|\nu|)}} v_{\kappa+|\nu|})\big\|_{L_{t,x}^{\frac{\kappa+|\nu|+1}{\kappa+|\nu|}}(\mathbb{R}^{1+n})} \\ :={}& I + II.\end{aligned} \tag{6.2}$$

Similar to (5.5),

$$\begin{aligned} I \lesssim{}& \sum_{k^{(1)} \in \mathbb{Z}^n,\ |k_1^{(1)}|>2} \big\langle k_1^{(1)}\big\rangle \|\Box_{k^{(1)}} v_1\|_{L_{x_1}^\infty L_{x_2,\dots,x_n}^2 L_t^2(\mathbb{R}^{1+n})} \\ &\times \sum_{k^{(2)},\dots,k^{(\kappa+|\nu|)} \in \mathbb{Z}^n} \prod_{i=2}^{\kappa+|\nu|} \|\Box_{k^{(i)}} v_i\|_{L_{x_1}^{\kappa+|\nu|-1} L_{x_2,\dots,x_n}^\infty L_t^\infty(\mathbb{R}^{1+n})}. \end{aligned} \tag{6.3}$$

By Hölder's inequality and Lemma 3.3,

$$\begin{aligned}\|\Box_{k^{(i)}} v_i\|_{L^{\kappa+|\nu|-1}_{x_1} L^\infty_{x_2,\ldots,x_n} L^\infty_t(\mathbb{R}^{1+n})} &\leqslant \|\Box_{k^{(i)}} v_i\|^{\frac{m}{\kappa+|\nu|-1}}_{L^m_{x_1} L^\infty_{x_2,\ldots,x_n} L^\infty_t(\mathbb{R}^{1+n})} \|\Box_{k^{(i)}} v_i\|^{1-\frac{m}{\kappa+|\nu|-1}}_{L^\infty_{x,t}(\mathbb{R}^{1+n})} \\ &\lesssim \|\Box_{k^{(i)}} v_i\|^{\frac{m}{\kappa+|\nu|-1}}_{L^m_{x_1} L^\infty_{x_2,\ldots,x_n} L^\infty_t(\mathbb{R}^{1+n})} \|\Box_{k^{(i)}} v_i\|^{1-\frac{m}{\kappa+|\nu|-1}}_{L^\infty_t L^2_x(\mathbb{R}^{1+n})}. \end{aligned} \tag{6.4}$$

Hence, noticing that $v_i = u$ or $v_i = u_{x_j}$, we have from (6.3) and (6.4),

$$I \lesssim \|u\|_X^{\kappa+|\nu|}. \tag{6.5}$$

Similar to (5.6), we see that $|k_1| \leqslant C$ in the summation of II. Again, in view of Hölder's inequality and Lemma (3.3),

$$\begin{aligned}\|\Box_{k^{(1)}} v_1 \ldots \Box_{k^{(\kappa+|\nu|)}} v_{\kappa+|\nu|}\|_{L^{\frac{\kappa+|\nu|+1}{\kappa+|\nu|}}_{x,t}(\mathbb{R}^{1+n})} &\leqslant \prod_{i=1}^{\kappa+|\nu|} \|\Box_{k^{(i)}} v_i\|_{L^{\kappa+|\nu|+1}_{x,t}(\mathbb{R}^{1+n})} \\ &\lesssim \prod_{i=1}^{\kappa+|\nu|} \|\Box_{k^{(i)}} v_i\|_{L^{2+m}_{x,t} \cap L^\infty_t L^2_x(\mathbb{R}^{1+n})}. \end{aligned} \tag{6.6}$$

Hence, using a similar way as in (5.6),

$$II \lesssim \|u\|_X^{\kappa+|\nu|}. \tag{6.7}$$

We now give the estimate of $\varrho_1^{(1)}(\mathscr{A}\partial^\alpha_{x_2}(v_1 \ldots v_{\kappa+|\nu|}))$. Since we have obtained the estimate in the case $\alpha = 0$, it suffices to consider the case $\alpha = 1$. Let ψ_i $(i = 1, 2)$ be as in Lemma 4.2 and $P_i = \mathscr{F}^{-1}\psi_i\mathscr{F}$. We have

$$\begin{aligned}\varrho_1^{(1)}\big(\mathscr{A}\partial_{x_2}(v_1 \ldots v_{\kappa+|\nu|})\big) &\leqslant \sum_{k\in\mathbb{Z}^n,\,|k_1|>4} \langle k_1\rangle \big\|P_1\Box_k\big(\mathscr{A}\partial_{x_2}(v_1 \ldots v_{\kappa+|\nu|})\big)\big\|_{L^\infty_{x_1} L^2_{x_2,\ldots,x_n} L^2_t} \\ &\quad + \sum_{k\in\mathbb{Z}^n,\,|k_1|>4} \langle k_1\rangle \big\|P_2\Box_k\big(\mathscr{A}\partial_{x_2}(v_1 \ldots v_{\kappa+|\nu|})\big)\big\|_{L^\infty_{x_1} L^2_{x_2,\ldots,x_n} L^2_t} \\ &:= III + IV. \end{aligned} \tag{6.8}$$

Using the decomposition (6.1),

$$\begin{aligned} III &\leqslant \sum_{k\in\mathbb{Z}^n,\,|k_1|>4} \langle k_1\rangle \sum_{\mathbb{S}_1^{(1)}} \big\|P_1\Box_k\big(\mathscr{A}\partial_{x_2}\big(\Box_{k^{(1)}} v_1 \ldots \Box_{k^{(\kappa+|\nu|)}} v_{\kappa+|\nu|}\big)\big)\big\|_{L^\infty_{x_1} L^2_{x_2,\ldots,x_n} L^2_t} \\ &\quad + \sum_{k\in\mathbb{Z}^n,\,|k_1|>4} \langle k_1\rangle \sum_{\mathbb{S}_2^{(1)}} \big\|P_1\Box_k\big(\mathscr{A}\partial_{x_2}\big(\Box_{k^{(1)}} v_1 \ldots \Box_{k^{(\kappa+|\nu|)}} v_{\kappa+|\nu|}\big)\big)\big\|_{L^\infty_{x_1} L^2_{x_2,\ldots,x_n} L^2_t} \\ &:= III_1 + III_2. \end{aligned} \tag{6.9}$$

By Lemma 4.2,

$$III_1 \lesssim \sum_{\mathbb{S}_1^{(1)}} \sum_{k\in\mathbb{Z}^n,\,|k_1|>4} \langle k_1\rangle \big\|\Box_k(\Box_{k^{(1)}} v_1 \ldots \Box_{k^{(\kappa+|\nu|)}} v_{\kappa+|\nu|})\big\|_{L^1_{x_1} L^2_{x_2,\ldots,x_n} L^2_t}. \tag{6.10}$$

By symmetry, we may assume $|k_1^{(1)}| = \max(|k_1^{(1)}|, \ldots, |k_1^{(\kappa+|\nu|)}|)$ in $\mathbb{S}_1^{(1)}$. Hence,

$$\begin{aligned} III_1 &\lesssim \sum_{\mathbb{S}_1^{(1)},\ |k_1^{(1)}|>4} \big\langle k_1^{(1)}\big\rangle \|\Box_{k^{(1)}} v_1\|_{L^\infty_{x_1} L^2_{x_2,\ldots,x_n} L^2_t} \prod_{i=2}^{\kappa+|\nu|} \|\Box_{k^{(i)}} v_i\|_{L^{\kappa+|\nu|-1}_{x_1} L^\infty_{x_2,\ldots,x_n} L^\infty_t} \\ &\lesssim \varrho_1^{(1)}(v_1) \prod_{i=2}^{\kappa+|\nu|} \big(\varrho_2^{(1)}(v_i) + \varrho_3^{(1)}(v_i)\big) \lesssim \|u\|_X^{\kappa+|\nu|}. \end{aligned} \tag{6.11}$$

Applying (4.2) and using a similar way as in (5.12),

$$III_2 \lesssim \sum_{k\in\mathbb{Z}^n,\,|k_1|>4,\,|k_2|\lesssim|k_1|} \langle k_1\rangle^{3/2} \sum_{\mathbb{S}_2^{(1)}} \big\| \Box_k(\Box_{k^{(1)}} v_1 \dots \Box_{k^{(\kappa+|\nu|)}} v_{\kappa+|\nu|})\big\|_{L^{(2+m)/(1+m)}_{t,x}(\mathbb{R}^{1+n})}$$
$$\lesssim \prod_{i=1}^{\kappa+|\nu|} \varrho_3^{(1)}(v_i) \leqslant \|u\|_X^{\kappa+|\nu|}. \tag{6.12}$$

So, we have shown that

$$III \lesssim \|u\|_X^{\kappa+|\nu|}. \tag{6.13}$$

Now we estimate IV. Using the decomposition (6.1),

$$IV \leqslant \sum_{k\in\mathbb{Z}^n,\,|k_1|>4} \langle k_1\rangle \sum_{\mathbb{S}_1^{(2)}} \big\| P_2\Box_k\big(\mathscr{A}\partial_{x_2}(\Box_{k^{(1)}} v_1 \dots \Box_{k^{(\kappa+|\nu|)}} v_{\kappa+|\nu|})\big)\big\|_{L^\infty_{x_1} L^2_{x_2,\dots,x_n} L^2_t}$$
$$+ \sum_{k\in\mathbb{Z}^n,\,|k_1|>4} \langle k_1\rangle \sum_{\mathbb{S}_2^{(2)}} \big\| P_2\Box_k\big(\mathscr{A}\partial_{x_2}(\Box_{k^{(1)}} v_1 \dots \Box_{k^{(\kappa+|\nu|)}} v_{\kappa+|\nu|})\big)\big\|_{L^\infty_{x_1} L^2_{x_2,\dots,x_n} L^2_t}$$
$$:= IV_1 + IV_2. \tag{6.14}$$

By Lemma 4.2,

$$IV_1 \lesssim \sum_{\mathbb{S}_1^{(2)}} \sum_{k\in\mathbb{Z}^n,\,|k_2|>4} \langle k_2\rangle \big\|\Box_k(\Box_{k^{(1)}} v_1 \dots \Box_{k^{(\kappa+|\nu|)}} v_{\kappa+|\nu|})\big\|_{L^1_{x_1} L^2_{x_2,\dots,x_n} L^2_t}. \tag{6.15}$$

In view of the symmetry, one can bound IV_1 by using the same way as that of III_1 and as in (5.14)–(5.17):

$$IV_1 \lesssim \|u\|_X^{\kappa+|\nu|}. \tag{6.16}$$

For the estimate of IV_2, we apply (4.2),

$$IV_2 \lesssim \sum_{k\in\mathbb{Z}^n,\,|k_1|>4} \langle k_1\rangle^{1/2}\langle k_2\rangle \sum_{\mathbb{S}_2^{(2)}} \big\| P_2\Box_k(\Box_{k^{(1)}} v_1 \dots \Box_{k^{(\kappa+|\nu|)}} v_{\kappa+|\nu|})\big\|_{L^{(2+m)/(1+m)}_{t,x}(\mathbb{R}^{1+n})}$$
$$\lesssim \sum_{\mathbb{S}_2^{(2)}} \big\|\Box_{k^{(1)}} v_1 \dots \Box_{k^{(\kappa+|\nu|)}} v_{\kappa+|\nu|}\big\|_{L^{(2+m)/(1+m)}_{t,x}(\mathbb{R}^{1+n})} \lesssim \|u\|_X^{\kappa+|\nu|}. \tag{6.17}$$

Hence, in view of (6.16) and (6.17), we have

$$IV \lesssim \|u\|_X^{\kappa+|\nu|}. \tag{6.18}$$

Collecting (6.5), (6.7), (6.13), (6.18), we have shown that

$$\sum_{\alpha=0,1} \sum_{i,j=1}^{n} \varrho_1^{(i)}\big(\mathscr{A}\partial^\alpha_{x_j}\big(u^\kappa(\nabla u)^\nu\big)\big) \lesssim \|u\|_X^{\kappa+|\nu|}. \tag{6.19}$$

Lemma 6.1. *Let $s \geqslant 0$, $1 \leqslant p, p_i, \gamma, \gamma_i \leqslant \infty$ satisfy*

$$\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_N}, \qquad \frac{1}{\gamma} = \frac{1}{\gamma_1} + \dots + \frac{1}{\gamma_N}. \tag{6.20}$$

Then

$$\sum_{k\in\mathbb{Z}^n} \langle k_1\rangle^s \big\|\Box_k(u_1 \dots u_N)\big\|_{L^\gamma_t L^p_x(\mathbb{R}^{1+n})} \lesssim \prod_{i=1}^{N} \bigg(\sum_{k\in\mathbb{Z}^n} \langle k_1\rangle^s \|\Box_k u_i\|_{L^{\gamma_i}_t L^{p_i}_x(\mathbb{R}^{1+n})} \bigg). \tag{6.21}$$

Proof. See [31], Lemma 7.1. □

Next, we consider the estimates of $\varrho_2^{(1)}(\mathscr{A}(u^\kappa(\nabla u)^\nu))$ and $\varrho_3^{(1)}(\mathscr{A}(u^\kappa(\nabla u)^\nu))$. In view of (3.26) and (3.16),

$$\sum_{j=2,3}\varrho_j^{(1)}\big(\mathscr{A}\big(u^\kappa(\nabla u)^\nu\big)\big)\lesssim\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{1/2}\big\|\Box_k\big(u^\kappa(\nabla u)^\nu\big)\big\|_{L_{t,x}^{\frac{2+m}{1+m}}(\mathbb{R}^{1+n})}. \tag{6.22}$$

We use Lemma 6.1 to control the right-hand side of (6.22):

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{1/2}\big\|\Box_k(v_1\dots v_{\kappa+|\nu|})\big\|_{L_{t,x}^{\frac{2+m}{1+m}}(\mathbb{R}^{1+n})}\\
&\lesssim\prod_{i=1}^{m+1}\Big(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{1/2}\|\Box_k v_i\|_{L_{t,x}^{2+m}(\mathbb{R}^{1+n})}\Big)\prod_{i=m+2}^{\kappa+|\nu|}\Big(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{1/2}\|\Box_k v_i\|_{L_{t,x}^{\infty}(\mathbb{R}^{1+n})}\Big)\\
&\lesssim\prod_{i=1}^{m+1}\Big(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{1/2}\|\Box_k v_i\|_{L_{t,x}^{2+m}(\mathbb{R}^{1+n})}\Big)\prod_{i=m+2}^{\kappa+|\nu|}\Big(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{1/2}\|\Box_k v_i\|_{L_t^\infty L_x^2(\mathbb{R}^{1+n})}\Big)\\
&\lesssim\prod_{i=1}^{\kappa+|\nu|}\left(\sum_{j=1}^n\varrho_3^{(j)}(v_i)\right)\leqslant\|u\|_X^{\kappa+|\nu|}.\end{aligned} \tag{6.23}$$

We estimate $\varrho_2^{(1)}(\mathscr{A}\partial_{x_1}(u^\kappa(\nabla u)^\nu))$. Recall that $k_{\max}=|k_1|\vee\dots\vee|k_n|$.

$$\begin{aligned}\varrho_2^{(1)}\big(\mathscr{A}\partial_{x_1}(v_1\dots v_{\kappa+|\nu|})\big)&\lesssim\sum_{k\in\mathbb{Z}^n,\,k_{\max}>4}\langle k\rangle^{1/2-1/m}\big\|\Box_k\mathscr{A}\partial_{x_1}(v_1\dots v_{\kappa+|\nu|})\big\|_{L_{x_1}^m L_{x_2,\dots,x_n}^\infty L_t^\infty(\mathbb{R}^{1+n})}\\
&\quad+\sum_{k\in\mathbb{Z}^n,\,k_{\max}\leqslant 4}\langle k\rangle^{1/2-1/m}\big\|\Box_k\mathscr{A}\partial_{x_1}(v_1\dots v_{\kappa+|\nu|})\big\|_{L_{x_1}^m L_{x_2,\dots,x_n}^\infty L_t^\infty(\mathbb{R}^{1+n})}\\
&:=V+VI.\end{aligned} \tag{6.24}$$

By (3.16) and Lemma 6.1, we have

$$VI\lesssim\sum_{k\in\mathbb{Z}^n}\big\|\Box_k(v_1\dots v_{\kappa+|\nu|})\big\|_{L_{t,x}^{\frac{2+m}{1+m}}(\mathbb{R}^{1+n})}\lesssim\|u\|_X^{\kappa+|\nu|}. \tag{6.25}$$

It is easy to see that

$$\begin{aligned}V&\lesssim\bigg(\sum_{k\in\mathbb{Z}^n,\,|k_1|=k_{\max}>4}+\dots+\sum_{k\in\mathbb{Z}^n,\,|k_n|=k_{\max}>4}\bigg)\langle k\rangle^{1/2-1/m}\big\|\Box_k\mathscr{A}\partial_{x_1}(v_1\dots v_{\kappa+|\nu|})\big\|_{L_{x_1}^m L_{x_2,\dots,x_n}^\infty L_t^\infty(\mathbb{R}^{1+n})}\\
&:=\Upsilon_1(u)+\dots+\Upsilon_n(u).\end{aligned} \tag{6.26}$$

Applying the decomposition (6.1) and Lemmas 4.3 and 4.4, we obtain that

$$\begin{aligned}\Upsilon_1(u)\lesssim&\sum_{k\in\mathbb{Z}^n,\,|k_1|>4}\langle k_1\rangle\sum_{\mathbb{S}_1^{(1)}}\big\|\Box_k(\Box_{k^{(1)}}v_1\dots\Box_{k^{(\kappa+|\nu|)}}v_{\kappa+|\nu|})\big\|_{L_{x_1}^1 L_{x_2,\dots,x_n}^2 L_t^2(\mathbb{R}^{1+n})}\\
&+\sum_{k\in\mathbb{Z}^n,\,|k_1|>4}\langle k_1\rangle^{3/2}\sum_{\mathbb{S}_2^{(1)}}\big\|\Box_k(\Box_{k^{(1)}}v_1\dots\Box_{k^{(\kappa+|\nu|)}}v_{\kappa+|\nu|})\big\|_{L_{t,x}^{\frac{\kappa+|\nu|+1}{\kappa+|\nu|}}(\mathbb{R}^{1+n})},\end{aligned} \tag{6.27}$$

which reduces to the case $\alpha=1$ in (6.2). So,

$$\Upsilon_1(u)\lesssim\|u\|_X^{\kappa+|\nu|}. \tag{6.28}$$

Again, in view of Lemmas 4.3 and 4.4,

$$\Upsilon_2(u) \lesssim \sum_{k\in\mathbb{Z}^n,\,|k_2|>4} \langle k_2\rangle \sum_{\mathbb{S}_1^{(2)}} \big\| \Box_k(\Box_{k^{(1)}} v_1 \dots \Box_{k^{(\kappa+|\nu|)}} v_{\kappa+|\nu|})\big\|_{L^1_{x_2} L^2_{x_1,x_3,\dots,x_n} L^2_t(\mathbb{R}^{1+n})}$$
$$+ \sum_{k\in\mathbb{Z}^n,\,|k_2|>4} \langle k_2\rangle^{3/2} \sum_{\mathbb{S}_2^{(2)}} \big\| \Box_k(\Box_{k^{(1)}} v_1 \dots \Box_{k^{(\kappa+|\nu|)}} v_{\kappa+|\nu|})\big\|_{L_{t,x}^{\frac{\kappa+|\nu|+1}{\kappa+|\nu|}}(\mathbb{R}^{1+n})}, \tag{6.29}$$

which reduces to the same estimate as $\Upsilon_1(u)$. Using the same way as $\Upsilon_2(u)$, we can get the estimates of $\Upsilon_3(u),\dots,\Upsilon_n(u)$. So,

$$\varrho_2^{(1)}\big(\mathscr{A}\partial_{x_1}(v_1\dots v_{\kappa+|\nu|})\big) \lesssim \|u\|_X^{\kappa+|\nu|}. \tag{6.30}$$

We need to further bound $\varrho_2^{(1)}(\mathscr{A}\partial_{x_i}(v_1\dots v_{\kappa+|\nu|}))$, $i=2,\dots,n$, which is essentially the same as $\varrho_2^{(1)}(\mathscr{A}\partial_{x_1}(v_1\dots v_{\kappa+|\nu|}))$. Indeed, it is easy to see that (6.24) holds if we substitute ∂_{x_1} with ∂_{x_i}. Moreover, using Lemmas 6.1, 4.3 and 4.4, we easily get that

$$\varrho_2^{(1)}\big(\mathscr{A}\big(\partial_{x_i}(v_1\dots v_{\kappa+|\nu|})\big)\big) \lesssim \|u\|_X^{\kappa+|\nu|}. \tag{6.31}$$

By Lemma 3.4, (3.13), we see that

$$\|\Box_k \mathscr{A}\partial_{x_1} f\|_{L_t^\infty L^2 \cap L_{t,x}^{2+m}(\mathbb{R}^{1+n})} \lesssim \langle k_1\rangle \|\Box_k f\|_{L_{t,x}^{\frac{2+m}{1+m}}(\mathbb{R}^{1+n})}. \tag{6.32}$$

Hence, in view of (3.24) and (3.14), repeating the procedure as in the estimates of $\rho_3(u)$ in Theorem 1.3, $\varrho_3^{(1)}(\mathscr{A}\partial_{x_1}(v_1\dots v_{\kappa+|\nu|}))$ can be controlled by the right-hand side of (6.27) and (6.25). Summarizing the estimates as in the above, we have shown that[3]

$$\|\mathscr{T}u\|_X \leqslant C\|u_0\|_{M^{3/2}} + \sum_{m+1\leqslant \ell<\infty} \ell^{2n+2} C^\ell \|u\|_X^\ell. \tag{6.33}$$

Applying a standard contraction mapping argument, we can prove our result.

7. Proofs of Theorems 1.4 and 1.2

Proof of Theorem 1.4. For convenience, we denote

$$\rho_1(u) = \sum_{i=1}^n \sum_{k\in\mathbb{Z}^n,\,|k_i|>4} \langle k_i\rangle^2 \|\Box_k u\|_{L^\infty_{x_i} L^2_{(x_j)_{j\neq i}} L^2_t(\mathbb{R}^{1+n})},$$
$$\rho_2(u) = \sum_{i=1}^n \sum_{k\in\mathbb{Z}^n} \|\Box_k u\|_{L^\kappa_{x_i} L^\infty_{(x_j)_{j\neq i}} L^\infty_t(\mathbb{R}^{1+n})},$$
$$\rho_3(u) = \sum_{k\in\mathbb{Z}^n} \langle k\rangle^{3/2} \|\Box_k u\|_{L_t^\infty L_x^2 \cap L_t^3 L_x^6(\mathbb{R}^{1+n})}.$$

Comparing the definitions of $\rho_i(u)$ with those of Section 5, we see that here we drop the regularity $\langle k_i\rangle^{1/2-1/\kappa}$ in $\rho_2(u)$ and we add 1-order regularity in $\rho_1(u)$ and $\rho_3(u)$. The estimates for $\rho_1(\mathscr{T}u)$ and $\rho_3(\mathscr{T}u)$ can be shown by following the same way as in Section 5. (It is worth to notice that in Section 5, when we estimate $\rho_1(\mathscr{T}u)$ and $\rho_3(\mathscr{T}u)$, we can replace $\rho_2(u)$ defined here to substitute that in Section 5.) We also need to point out that for $n\geqslant 2$, $2/3\leqslant n(1/2-1/6)$ and so, $\|\cdot\|_{L_t^3 L_x^6(\mathbb{R}^{1+n})}$ is a modulation Strichartz norm. Moreover,

$$\|\Box_k u\|_{L_{x,t}^{2+p}} \lesssim \|\Box_k u\|_{L_t^\infty L_x^2 \cap L_t^3 L_x^6(\mathbb{R}^{1+n})}$$

uniformly holds for all $k\in\mathbb{Z}^n$ and $2\leqslant p\leqslant\infty$.

[3] Notice that $|c_\beta|\leqslant C^{|\beta|}$.

Noticing that in the proof of Theorem 1.3, we do not know if the following two inequalities hold for $m=2$,

$$\|\Box_k \mathscr{A}\partial_{x_1} f\|_{L^m_{x_1}L^\infty_{x_2,\dots,x_n}L^\infty_t(\mathbb{R}^{1+n})} \lesssim \langle k_1\rangle^{1/2+1/m}\|\Box_k f\|_{L^1_{x_1}L^2_{x_2,\dots,x_n}L^2_t(\mathbb{R}^{1+n})}, \tag{7.1}$$

$$\|\Box_k \partial_{x_i}\mathscr{A} f\|_{L^m_{x_1}L^\infty_{x_2,\dots,x_n}L^\infty_t(\mathbb{R}^{1+n})} \lesssim \langle k_i\rangle^{1/2}\langle k_1\rangle^{1/m}\|\Box_k f\|_{L^1_{x_i}L^2_{(x_j)_{j\neq i}}L^2_t(\mathbb{R}^{1+n})}. \tag{7.2}$$

So, in the case $m=2$, we need to find another way to estimate $\rho_2(\mathscr{T}u)$. Our solution is to apply the following estimate as in (3.16):

$$\|\Box_k \mathscr{A} f\|_{L^2_{x_1}L^\infty_{x_2,\dots,x_n}L^\infty_t(\mathbb{R}^{1+n})} \lesssim \langle k_1\rangle^{1/2}\|\Box_k f\|_{L^1_t L^2_x(\mathbb{R}^{1+n})}. \tag{7.3}$$

It follows that for any $\kappa \geqslant 2$,

$$\sum_{k\in\mathbb{Z}^n}\big\|\Box_k \mathscr{A}\partial_{x_i} u^{\kappa+1}\big\|_{L^2_{x_1}L^\infty_{x_2,\dots,x_n}L^\infty_t(\mathbb{R}^{1+n})} \lesssim \sum_{k\in\mathbb{Z}^n}\langle k\rangle^{3/2}\big\|\Box_k u^{\kappa+1}\big\|_{L^1_t L^2_x(\mathbb{R}^{1+n})}. \tag{7.4}$$

Using Lemma 6.1, one has that

$$\begin{aligned}\sum_{k\in\mathbb{Z}^n}\big\|\Box_k \mathscr{A}\partial_{x_i} u^{\kappa+1}\big\|_{L^2_{x_1}L^\infty_{x_2,\dots,x_n}L^\infty_t(\mathbb{R}^{1+n})} &\lesssim \Big(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{3/2}\|\Box_k u\|_{L^{\kappa+1}_t L^{2(\kappa+1)}_x(\mathbb{R}^{1+n})}\Big)^{\kappa+1}\\ &\lesssim \Big(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{3/2}\|\Box_k u\|_{L^3_t L^6_x\cap L^\infty_{x,t}(\mathbb{R}^{1+n})}\Big)^{\kappa+1}\\ &\lesssim \rho_3(u)^{1+\kappa}.\end{aligned} \tag{7.5}$$

Using (7.5), the estimates of $\rho_2(\mathscr{T}u)$ is also obtained. □

Proof of Theorem 1.2. We can follow the proof of Theorems 1.4 and 1.1 to get the proof and we omit the details of the proof. □

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Appendix A

Proof of Propositions 2.1 and 2.2. First, we show the 1/2-order smooth effect for the Schrödinger semi-group. By Plancherel's equality and Minkowski's inequality,

$$\big\|S(t)u_0\big\|_{L^\infty_{x_1}L^2_{x_2,\dots,x_n}L^2_t(\mathbb{R}^{1+n})} \leqslant \big\|\mathscr{F}^{-1}_{\xi_1}e^{\mathrm{i}t\varepsilon_1\xi_1^2}\mathscr{F}_{x_1}(\mathscr{F}_{x_2,\dots,x_n}u_0)\big\|_{L^2_{\xi_2,\dots,\xi_n}L^\infty_{x_1}L^2_t(\mathbb{R}^{1+n})}. \tag{A.1}$$

Recall the half-order smooth effect of $S(t)$ in one spatial dimension (cf. [13]),

$$\big\|\mathscr{F}^{-1}_{\xi}e^{\mathrm{i}t\xi^2}\mathscr{F}_x u_0\big\|_{L^\infty_x L^2_t(\mathbb{R}^{1+1})} \lesssim \big\|D_x^{-1/2}u_0\big\|_{L^2(\mathbb{R})}. \tag{A.2}$$

Hence, in view of (A.1) and (A.2), using Plancherel's equality, we immediately have the result, as desired. Proposition 2.2 is the dual version of the half-order smooth effect of $S(t)$.

Denote

$$u = c\mathscr{F}^{-1}_{t,x}\frac{\xi_1}{|\xi|^2_\pm-\tau}\mathscr{F}_{t,x}f. \tag{A.3}$$

We can assume, without loss of generality that $|\xi|^2_\pm = \xi_1^2+\varepsilon_2\xi_2^2+\cdots+\varepsilon_n\xi_n^n := \xi_1^2+|\bar\xi|^2_\pm$. By Plancherel's identity,

$$\|u\|_{L^\infty_{x_1}L^2_{x_2,\dots,x_n}L^2_t(\mathbb{R}^{1+n})} \leqslant \left\| \mathscr{F}^{-1}_{\xi_1} \frac{\xi_1}{\xi_1^2+|\bar\xi|^2_\pm - \tau} \mathscr{F}_{t,x} f \right\|_{L^2_{\xi_2,\dots,\xi_n}L^\infty_{x_1}L^2_\tau(\mathbb{R}^{1+n})}. \tag{A.4}$$

By changing the variable $\tau \to \mu + |\bar\xi|^2_\pm$, the right-hand side of (A.4) becomes

$$\left\| \mathscr{F}^{-1}_{\xi_1} \frac{\xi_1}{\xi_1^2 - \mu} \mathscr{F}_{t,x_1}\left(e^{-\mathrm{i}t|\bar\xi|^2_\pm} \mathscr{F}_{x_2,\dots,x_n} f\right) \right\|_{L^2_{\xi_2,\dots,\xi_n}L^\infty_{x_1}L^2_\mu(\mathbb{R}^{1+n})}. \tag{A.5}$$

Recalling the smooth effect estimate in one spatial dimension (cf. [13])

$$\left\| \mathscr{F}^{-1}_{\tau,\xi} \frac{\xi}{\xi^2 - \tau} \mathscr{F}_{t,x} f \right\|_{L^\infty_x L^2_t(\mathbb{R}^{1+1})} \lesssim \|f\|_{L^1_x L^2_t(\mathbb{R}^{1+1})}, \tag{A.6}$$

we have from (A.4), (A.5) and (A.6) that

$$\|u\|_{L^\infty_{x_1}L^2_{x_2,\dots,x_n}L^2_t(\mathbb{R}^{1+n})} \lesssim \left\|e^{-\mathrm{i}t|\bar\xi|^2_\pm}\mathscr{F}_{x_2,\dots,x_n} f\right\|_{L^2_{\xi_2,\dots,\xi_n}L^1_{x_1}L^2_t(\mathbb{R}^{1+n})}. \tag{A.7}$$

Using Minkowski's inequality and Plancherel's equality, we immediately have

$$\|u\|_{L^\infty_{x_1}L^2_{x_2,\dots,x_n}L^2_t(\mathbb{R}^{1+n})} \lesssim \|f\|_{L^1_{x_1}L^2_{x_2,\dots,x_n}L^2_t(\mathbb{R}^{1+n})}. \tag{A.8}$$

Noticing that $\partial_{x_1}\mathscr{A} f = u - \partial_{x_1} S(t)\int_{-\infty}^{\infty} S(s)\,\mathrm{sgn}(s) f(s)\,ds$, we see that (A.8) also holds if one replaces u by $\partial_{x_1}\mathscr{A} f$. □

Appendix B

In this section, we generalize the Christ–Kiselev Lemma [6] to anisotropic Lebesgue spaces. Our idea follows Molinet and Ribaud [21], and Smith and Sogge [25]. Denote

$$Tf(t) = \int_{-\infty}^{\infty} K(t,t') f(t')\,dt', \qquad T_{re} f(t) = \int_0^t K(t,t') f(t')\,dt'. \tag{B.1}$$

If $T: Y_1 \to X_1$ implies that $T_{re}: Y_1 \to X_1$, then $T: Y_1 \to X_1$ is said to be a well restriction operator.

Proposition B.1. *Let T be as in* (B.1). *We have the following results.*

(1) *If $\bigwedge_{i=1}^3 p_i > (\bigvee_{i=1}^3 q_i) \vee (q_1q_3/q_2)$, then $T: L^{q_1}_{x_1}L^{q_2}_{x_2}L^{q_3}_t(\mathbb{R}^3) \to L^{p_1}_{x_1}L^{p_2}_{x_2}L^{p_3}_t(\mathbb{R}^3)$ is a well restriction operator.*
(2) *If $q_1 < \bigwedge_{i=1}^3 p_i$, then $T: L^{q_1}_t L^{q_2}_{x_1}L^{q_3}_{x_2}(\mathbb{R}^3) \to L^{p_1}_{x_1}L^{p_2}_{x_2}L^{p_3}_t(\mathbb{R}^3)$ is a well restriction operator.*
(3) *If $p_1 > (\bigvee_{i=1}^3 q_i) \vee (q_1q_3/q_2)$, then $T: L^{q_1}_{x_1}L^{q_2}_{x_2}L^{q_3}_t(\mathbb{R}^3) \to L^{p_1}_t L^{p_2}_{x_1}L^{p_3}_{x_2}(\mathbb{R}^3)$ is a well restriction operator.*
(4) *If $\bigwedge_{i=1}^3 p_i > (\bigvee_{i=1}^3 q_i) \vee (q_1q_3/q_2)$, then $T: L^{q_1}_{x_1}L^{q_2}_{x_2}L^{q_3}_t(\mathbb{R}^3) \to L^{p_2}_{x_2}L^{p_1}_{x_1}L^{p_3}_t(\mathbb{R}^3)$ is a well restriction operator.*

Let $f \in L^{q_1}_{x_1}L^{q_2}_{x_2}L^{q_3}_t(\mathbb{R}^3)$ so that $\|f\|_{L^{q_1}_{x_1}L^{q_2}_{x_2}L^{q_3}_t(\mathbb{R}^3)} = 1$. Define $F: \mathbb{R} \to [0,1]$ by

$$F(t) := \left\| \left(\int_{-\infty}^t |f(s,x)|^{q_3}\,ds \right)^{1/q_3} \right\|^{q_1}_{L^{q_1}_{x_1}L^{q_2}_{x_2}}. \tag{B.2}$$

Lemma B.2. *Let $I \subset [0,1]$ is an interval, then it holds*:

$$\|\chi_{F^{-1}(I)} f\|_{L^{q_1}_{x_1}L^{q_2}_{x_2}L^{q_3}_t(\mathbb{R}\times\mathbb{R}^2)} \leqslant |I|^{\frac{q_2}{q_1q_3}\wedge\frac{1}{q_1}\wedge\frac{1}{q_2}\wedge\frac{1}{q_3}} \tag{B.3}$$

Proof. For any $I = (A, B) \subset [0, 1]$, there exist $t_1, t_2 \in \mathbb{R}$ satisfying

$$A = \left\| \left(\int_{-\infty}^{t_1} \left| f(s,x) \right|^{q_3} ds \right)^{1/q_3} \right\|_{L^{q_1}_{x_1} L^{q_2}_{x_2}}, \qquad B = \left\| \left(\int_{-\infty}^{t_2} \left| f(s,x) \right|^{q_3} ds \right)^{1/q_3} \right\|_{L^{q_1}_{x_1} L^{q_2}_{x_2}}$$

and $F^{-1}(I) = (t_1, t_2)$. For $x = (x_1, x_2)$, we define $J(t,x)$ and $E(t,x_1)$ by:

$$J(t,x) = \left(\int_{-\infty}^{t} \left| f(s,x) \right|^{q_3} ds \right)^{1/q_3}, \qquad E(t,x_1) = \left(\int J(t,x)^{q_2}\, dx_2 \right)^{1/q_2}. \tag{B.4}$$

It is well known that for $a \geqslant b > 0$,

$$r^a - s^a \leqslant C\big(r^b - s^b\big)\big(r^{a-b} + s^{a-b}\big), \quad 0 \leqslant s \leqslant r, \tag{B.5}$$

and for $0 < a \leqslant b$,

$$r^a - s^a \leqslant \big(r^b - s^b\big)^{a/b}, \quad 0 \leqslant s \leqslant r. \tag{B.6}$$

We divide the proof into the following four cases.

Case 1. $q_3 \geqslant q_2 \geqslant q_1$. From (B.5) we have

$$\left\| \chi_{F^{-1}(I)} f(\cdot,x) \right\|^{q_3}_{L^{q_3}_t} \lesssim \big(J(t_2,x)^{q_2} - J(t_1,x)^{q_2}\big) J(\infty,x)^{q_3 - q_2}. \tag{B.7}$$

Recalling the assumption $\|f\|_{L^{q_1}_{x_1} L^{q_2}_{x_2} L^{q_3}_t(\mathbb{R}\times\mathbb{R}^2)} = 1$, by (B.7), (B.4), (B.5) and Hölder inequality, we have

$$\begin{aligned}
&\int \left(\int \left\| \chi_{F^{-1}(I)} f(\cdot,x) \right\|^{q_2}_{L^{q_3}_t} dx_2 \right)^{\frac{q_1}{q_2}} dx_1 \\
&\quad \lesssim \int \left(\int \big(J(t_2,x)^{q_2} - J(t_1,x)^{q_2}\big)^{\frac{q_2}{q_3}} J(\infty,x)^{(q_3-q_2)\frac{q_2}{q_3}}\, dx_2 \right)^{\frac{q_1}{q_2}} dx_1 \\
&\quad \leqslant \int \big(E(t_2,x_1)^{q_2} - E(t_1,x_1)^{q_2}\big)^{\frac{q_1}{q_3}} \big(E(\infty,x_1)\big)^{\frac{(q_3-q_2)q_1}{q_3}}\, dx_1 && \text{(B.8)} \\
&\quad \lesssim \int \big(E(t_2,x_1)^{q_1} - E(t_1,x_1)^{q_1}\big)^{\frac{q_1}{q_3}} \big(E(\infty,x_1)\big)^{\frac{(q_2-q_1)q_1}{q_3}} \big(E(\infty,x_1)\big)^{\frac{(q_3-q_2)q_1}{q_3}}\, dx_1 \\
&\quad \leqslant \left\| \big(E(t_2,x_1)^{q_1} - E(t_1,x_1)^{q_1}\big)^{\frac{q_1}{q_3}} \right\|_{L^{q_3/q_1}_{x_1}} \left\| E(\infty,x_1)^{\frac{(q_3-q_1)q_1}{q_3}} \right\|_{L^{1/(1-q_1/q_3)}_{x_1}} && \text{(B.9)} \\
&\quad \leqslant \big(F(t_2) - F(t_1)\big)^{\frac{q_1}{q_3}} F(\infty)^{1-q_1/q_3} \leqslant |I|^{\frac{q_1}{q_3}}. && \text{(B.10)}
\end{aligned}$$

Case 2. $q_3 \geqslant q_2$, $q_2 < q_1$. From (B.8) and (B.6), we have

$$\begin{aligned}
&\int \left(\int \left\| \chi_{F^{-1}(I)} f(\cdot,x) \right\|^{q_2}_{L^{q_3}_t} dx_2 \right)^{\frac{q_1}{q_2}} dx_1 \\
&\quad \lesssim \int \big(E(t_2,x_1)^{q_2} - E(t_1,x_1)^{q_2}\big)^{\frac{q_1}{q_3}} \big(E(\infty,x_1)\big)^{\frac{(q_3-q_2)q_1}{q_3}}\, dx_1 \\
&\quad \leqslant \int \big(E(t_2,x_1)^{q_1} - E(t_1,x_1)^{q_1}\big)^{\frac{q_2}{q_3}} \big(E(\infty,x_1)\big)^{\frac{(q_3-q_2)q_1}{q_3}}\, dx_1 \\
&\quad \leqslant \left\| \big(E(t_2,x_1)^{q_1} - E(t_1,x_1)^{q_1}\big)^{\frac{q_2}{q_3}} \right\|_{L^{q_3/q_2}_{x_1}} \left\| E(\infty,x_1)^{\frac{(q_3-q_2)q_1}{q_3}} \right\|_{L^{1/(1-q_2/q_3)}_{x_1}} \\
&\quad \leqslant \big(F(t_2) - F(t_1)\big)^{\frac{q_2}{q_3}} F(\infty)^{1-q_2/q_3} \leqslant |I|^{\frac{q_2}{q_3}}. && \text{(B.11)}
\end{aligned}$$

Case 3. $q_3 < q_2 \leqslant q_1$. From (B.6), we have

$$\left\| \chi_{F^{-1}(I)} f(\cdot,x) \right\|^{q_3}_{L^{q_3}_t} \leqslant \big(J(t_2,x)^{q_2} - J(t_1,x)^{q_2}\big)^{q_3/q_2}. \tag{B.12}$$

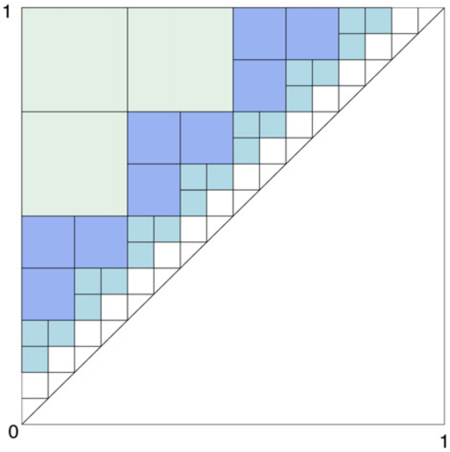

Fig. 1. Whitney's decomposition in the triangle.

Using (B.6) again, we have

$$\int\left(\int\left\|\chi_{F^{-1}(I)}f(\cdot,x)\right\|^{q_2}_{L^{q_3}_t}dx_2\right)^{\frac{q_1}{q_2}}dx_1\leqslant\int\left(E(t_2,x_1)^{q_2}-E(t_1,x_1)^{q_2}\right)^{\frac{q_1}{q_2}}dx_1 \tag{B.13}$$

$$\leqslant\int\left(E(t_2,x_1)^{q_1}-E(t_1,x_1)^{q_1}\right)dx_1$$

$$=F(t_2)-F(t_1)=|I|. \tag{B.14}$$

Case 4. $q_3<q_2$, $q_2>q_1$. From (B.13), (B.5) and Hölder inequality we have

$$\int\left(\int\left\|\chi_{F^{-1}(I)}f(\cdot,x)\right\|^{q_2}_{L^{q_3}_t}dx_2\right)^{\frac{q_1}{q_2}}dx_1\leqslant\int\left(E(t_2,x_1)^{q_2}-E(t_1,x_1)^{q_2}\right)^{\frac{q_1}{q_2}}dx_1$$

$$\leqslant\int\left(E(t_2,x_1)^{q_1}-E(t_1,x_1)^{q_1}\right)^{\frac{q_1}{q_2}}E(\infty,x_1)^{\frac{q_1(q_2-q_1)}{q_2}}dx_1$$

$$\leqslant\left(F(t_2)-F(t_1)\right)^{\frac{q_1}{q_2}}=|I|^{\frac{q_1}{q_2}}. \tag{B.15}$$

From (B.10), (B.11), (B.14) and (B.15) we get

$$\left\|\chi_{F^{-1}(I)}f\right\|_{L^{q_1}_{x_1}L^{q_2}_{x_2}L^{q_3}_t(\mathbb{R}\times\mathbb{R}^2)}\leqslant C|I|^{\frac{1}{q_1}\wedge\frac{1}{q_2}\wedge\frac{1}{q_3}\wedge\frac{q_2}{q_1q_3}}, \tag{B.16}$$

which yields (B.3), as desired. □

We will use Whitney's decomposition to the triangle $\{(x,y)\in[0,1]^2 : x<y\}$ (see Fig. 1). First, we divide $[0,1]^2$ into four congruent squares, consider the square with side-length $1/2$ in the triangle region and decompose it into four dyadic squares with side-length $1/4$, then remove the left-upper three ones in the triangle region. Secondly, considering the remaining region, we can find three squares with side-length $1/4$ in the triangle. We decompose each square into four dyadic squares in the same way as in the first step. Repeating the procedure above to the end. So, we have decomposed the triangle region into infinite squares with dyadic border. Let I and J be the dyadic subintervals of $[0,1]$ in the horizontal and perpendicular axes, respectively. We say that $I\sim J$ if they can consist the horizontal border and perpendicular border of a square described above, respectively. From the decomposition above we see that

(i) $|I|=|J|$ and $\mathrm{dist}(I,J)\geqslant|I|$ for $I\sim J$.
(ii) The squares in $\{(x,y)\in[0,1]^2\colon x<y\}$ are pairwise disjoint.
(iii) For any dyadic subinterval J, there are at most two I with $I\sim J$.

Proof of Proposition B.1. First, we show the result of (1). We have

$$T_{re}f(t,x) := \int_{-\infty}^{t} K(t,t')f(t')\,dt' = \sum_{\{I,J:\ I\sim J\}} \chi_{F^{-1}(J)} T(\chi_{F^{-1}(I)} f). \tag{B.17}$$

It follows that

$$\|T_{re}f\|_{L^{p_1}_{x_1} L^{p_2}_{x_2} L^{p_3}_t(\mathbb{R}^3)} \leqslant \sum_{j=1}^{\infty} \Big\| \sum_{\{I,J:\ I\sim J,\ |I|=2^{-j}\}} \chi_{F^{-1}(J)} T(\chi_{F^{-1}(I)} f) \Big\|_{L^{p_1}_{x_1} L^{p_2}_{x_2} L^{p_3}_t(\mathbb{R}^3)}. \tag{B.18}$$

For any $p \geqslant 1$, we easily see the following fact:

$$\Big\| \sum_{\{I,J:\ I\sim J,\ |I|=2^{-j}\}} \chi_{F^{-1}(J)} T(\chi_{F^{-1}(I)} f) \Big\|^p_{L^p_t(\mathbb{R})} \leqslant 2 \sum_{J_1:\ |J_1|=2^{-j}} \int_{\mathbb{R}} \chi_{F^{-1}(J_1)} \big| T(\chi_{F^{-1}(J_1)} f)\big|^p\,dt. \tag{B.19}$$

Hence, in view of (B.18) and (B.19) we have

$$\|T_{re}f\|_{L^{p_1}_{x_1} L^{p_2}_{x_2} L^{p_3}_t(\mathbb{R}^3)} \leqslant \sum_{j=1}^{\infty} \Big\| \Big(\sum_{\{I:\ |I|=2^{-j}\}} \big\| T(\chi_{F^{-1}(I)} f)\big\|^{p_3}_{L^{p_3}_t(\mathbb{R})} \Big)^{1/p_3} \Big\|_{L^{p_1}_{x_1} L^{p_2}_{x_2}(\mathbb{R}^2)}. \tag{B.20}$$

If $p \leqslant q$, by Minkowski's inequality, we have

$$\Big\| \Big(\sum_j \|a_j(x,y)\|^p_{L^p_x} \Big)^{1/p} \Big\|_{L^q_y} \leqslant \Big(\sum_j \|a_j(x,y)\|^p_{L^q_y L^p_x} \Big)^{1/p}. \tag{B.21}$$

If $p > q$, in view of $(a+b)^\theta \leqslant a^\theta + b^\theta$ for any $0 \leqslant \theta \leqslant 1$, $a, b > 0$, we have

$$\Big\| \Big(\sum_j \|a_j(x,y)\|^p_{L^p_x} \Big)^{1/p} \Big\|_{L^q_y} \leqslant \Big(\sum_j \|a_j(x,y)\|^q_{L^q_y L^p_x} \Big)^{1/q}. \tag{B.22}$$

We divide our discussion into the following three cases.

Case 1. $p_1, p_2 \geqslant p_3$. By (B.20), using (B.21) twice, we have

$$\|T_{re}f\|_{L^{p_1}_{x_1} L^{p_2}_{x_2} L^{p_3}_t(\mathbb{R}^3)} \leqslant \sum_{j=1}^{\infty} \Big(\sum_{\{I:\ |I|=2^{-j}\}} \big\| T(\chi_{F^{-1}(I)} f)\big\|^{p_3}_{L^{p_1}_{x_1} L^{p_2}_{x_2} L^{p_3}_t(\mathbb{R}^3)} \Big)^{1/p_3}. \tag{B.23}$$

Case 2. $p_1 \leqslant p_2 \leqslant p_3$. By (B.20), using (B.22) twice, we have

$$\|T_{re}f\|_{L^{p_1}_{x_1} L^{p_2}_{x_2} L^{p_3}_t(\mathbb{R}^3)} \leqslant \sum_{j=1}^{\infty} \Big(\sum_{\{I:\ |I|=2^{-j}\}} \big\| T(\chi_{F^{-1}(I)} f)\big\|^{p_1}_{L^{p_1}_{x_1} L^{p_2}_{x_2} L^{p_3}_t(\mathbb{R}^3)} \Big)^{1/p_1}. \tag{B.24}$$

Case 3. $p_2 \leqslant p_1 \leqslant p_3$. By (B.20) and (B.22), then applying (B.21), we have

$$\|T_{re}f\|_{L^{p_1}_{x_1} L^{p_2}_{x_2} L^{p_3}_t(\mathbb{R}^3)} \leqslant \sum_{j=1}^{\infty} \Big(\sum_{\{I:\ |I|=2^{-j}\}} \big\| T(\chi_{F^{-1}(I)} f)\big\|^{p_2}_{L^{p_1}_{x_1} L^{p_2}_{x_2} L^{p_3}_t(\mathbb{R}^3)} \Big)^{1/p_2}. \tag{B.25}$$

Denote $p_{\min} = \min(p_1, p_2, p_3)$. It follows from (B.23)–(B.25) that

$$\begin{aligned} \|T_{re}f\|_{L^{p_1}_{x_1} L^{\infty}_{x_2} L^{p_2}_t} &\lesssim \sum_{j=1}^{\infty} \Big(\sum_{\{I:\ |I|=2^{-j}\}} |I|^{\frac{p_{\min} q_2}{q_1 q_3} \wedge \frac{p_{\min}}{q_3} \wedge \frac{p_{\min}}{q_2} \wedge \frac{p_{\min}}{q_1}} \Big)^{\frac{1}{p_{\min}}} \\ &\lesssim \sum_{j=1}^{\infty} 2^{-j((\frac{q_2}{q_1 q_3} \wedge \frac{1}{q_3} \wedge \frac{1}{q_2} \wedge \frac{1}{q_1}) - \frac{1}{p_{\min}})} < \infty. \end{aligned} \tag{B.26}$$

The proof of (4) is almost the same as that of (1) and we omit the details of the proof.

Next, we prove (2). We have

$$\|T_{re}f\|_{L_t^{p_1}L_{x_1}^{p_2}L_{x_2}^{p_3}(\mathbb{R}\times\mathbb{R}^2)} \leqslant \sum_{j=1}^{\infty}\Big\| \sum_{\{I,J:\ I\sim J,\ |I|=2^{-j}\}} \chi_{F^{-1}(J)}T(\chi_{F^{-1}(I)}f)\Big\|_{L_t^{p_1}L_{x_1}^{p_2}L_{x_2}^{p_3}(\mathbb{R}^3)}$$
$$\leqslant 2\sum_{j=1}^{\infty}\Big\| \sum_{\{I:\ |I|=2^{-j}\}} \chi_{F^{-1}(J)}\big\|T(\chi_{F^{-1}(I)}f)\big\|_{L_{x_1}^{p_2}L_{x_2}^{p_3}(\mathbb{R}^2)}\Big\|_{L_t^{p_1}(\mathbb{R})}.$$

Using the same way as in (B.19),

$$\|T_{re}f\|_{L_t^{p_1}L_{x_1}^{p_2}L_{x_2}^{p_3}(\mathbb{R}^3)} \lesssim \sum_{j=1}^{\infty}\bigg(\sum_{\{I:\ |I|=2^{-j}\}} \|\chi_{F^{-1}(I)}f\|^{p_1}_{L_{x_1}^{q_1}L_{x_2}^{q_2}L_t^{q_3}(\mathbb{R}^3)}\bigg)^{1/p_1},$$

So, we can control $\|T_{re}f\|_{L_t^{p_1}L_{x_1}^{p_2}L_{x_2}^{p_3}(\mathbb{R}^3)}$ by the right-hand side of (B.26) in the case $p_{\min}=p_1$.

Finally, we prove (3). We define $F_1(t)$ as follows.

$$F_1(t) := \int_{-\infty}^{t} \big\|f(s,x_1,x_2)\big\|^{q_1}_{L_{x_1}^{q_2}L_{x_2}^{q_3}}\,ds. \tag{B.27}$$

From the definition of $F_1(t)$, it is easy to see that

$$\big\|\chi_{F_1^{-1}(I)}(s)f(s)\big\|_{L_t^{q_1}L_{x_1}^{q_2}L_{x_2}^{q_3}(\mathbb{R}\times\mathbb{R}^2)} = |I|^{1/q_1}. \tag{B.28}$$

Hence, replacing (B.3) with (B.28), we can use the same way as in the proof of (1) to get the result, as desired. □

We can generalized this result to n dimensional spaces:

Lemma B.3. *Let T be as in* (B.1). *We have the following results.*

(1) *If $\min(p_1,p_2,p_3) > \max(q_1,q_2,q_3,q_1q_3/q_2)$, then $T: L_{x_1}^{q_1}L_{x_2,\dots,x_n}^{q_2}L_t^{q_3}(\mathbb{R}^{n+1}) \to L_{x_1}^{p_1}L_{x_2,\dots,x_n}^{p_2}L_t^{p_3}(\mathbb{R}^{n+1})$ is a well restriction operator.*
(2) *If $p_0 > (\bigvee_{i=1}^{3} q_i)\vee(q_1q_3/q_2)$, then $T: L_{x_1}^{q_1}L_{x_2,\dots,x_n}^{q_2}L_t^{q_3}(\mathbb{R}^{n+1}) \to L_t^{p_0}L_{x_1}^{p_1}\dots L_{x_n}^{p_n}(\mathbb{R}^{n+1})$ is a well restriction operator.*
(3) *If $q_0 < \min(p_1,p_2,p_3)$, then $T: L_t^{q_0}L_{x_1}^{q_1}\dots L_{x_n}^{q_n}(\mathbb{R}^{n+1}) \to L_{x_1}^{p_1}L_{x_2,\dots,x_n}^{p_2}L_t^{p_3}(\mathbb{R}^{n+1})$ is a well restriction operator.*
(4) *If $\min(p_1,p_2,p_3) > \max(q_1,q_2,q_3,\ q_1q_3/q_2)$, then $T: L_{x_2}^{q_1}L_{x_1,x_3,\dots,x_n}^{q_2}L_t^{q_3}(\mathbb{R}^{n+1}) \to L_{x_1}^{p_1}L_{x_2,\dots,x_n}^{p_2}L_t^{p_3}(\mathbb{R}^{n+1})$ is a well restriction operator.*

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