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Existence of weak solutions for a non-classical sharp interface model for a two-phase flow of viscous, incompressible fluids

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Abstract

We introduce a new sharp interface model for the flow of two immiscible, viscous, incompressible fluids. In contrast to classical models for two-phase flows we prescribe an evolution law for the interfaces that takes diffusional effects into account. This leads to a coupled system of Navier–Stokes and Mullins–Sekerka type parts that coincides with the asymptotic limit of a diffuse interface model. We prove the long-time existence of weak solutions, which is an open problem for the classical two-phase model. We show that the phase interfaces have in almost all points a generalized mean curvature.

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1. Introduction

We study the flow of two incompressible, viscous and immiscible fluids like oil and water inside a bounded domain Ω or in $\Omega = \mathbb{T}^d$, $d = 2, 3$. The fluids fill domains $\Omega^+(t)$ and $\Omega^-(t)$, $t > 0$, respectively, with a common interface $\Gamma(t)$ between both fluids. The flow is described in terms of the velocity $v : \Omega \times (0, \infty) \to \mathbb{R}^d$ and the pressure $p : \Omega \times (0, \infty) \to \mathbb{R}$ in both fluids in Eulerian coordinates. We assume the fluids to be of Newtonian type, i.e., the stress tensors are of the form $T^\pm(v, p) = \nu^\pm Dv - pI$ with constant viscosities $\nu^\pm > 0$ and $2Dv = \nabla v + \nabla v^T$. Moreover, we consider the case with surface tension at the interface and assume that the densities are the same (and set to 1 for simplicity). For the evolution of the phases we take diffusional effects into account and consider a contribution to the flux that is proportional to the negative gradient of the chemical potential μ. Precise assumptions are made below.

To formulate our model we introduce some notation first. Denote by n the unit normal of $\Gamma(t)$ that points inside $\Omega^+(t)$ and by V and H the normal velocity and scalar mean curvature of $\Gamma(t)$ with respect to n. By $[\cdot]$ we denote the jump of a quantity across the interface in direction of n, i.e., $[f](x) = \lim_{h\to 0}(f(x + hn) - f(x - hn))$ for $x \in \Gamma(t)$. Then our model is described by the following equations

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$$\partial_t v + v\cdot\nabla v - \operatorname{div} T^{\pm}(v,p) = 0 \quad \text{in } \Omega^{\pm}(t),\ t>0, \tag{1.1}$$
$$\operatorname{div} v = 0 \quad \text{in } \Omega^{\pm}(t),\ t>0, \tag{1.2}$$
$$m\Delta\mu = 0 \quad \text{in } \Omega^{\pm}(t),\ t>0, \tag{1.3}$$
$$-n\cdot\big[T(v,p)\big] = \kappa H n \quad \text{on } \Gamma(t),\ t>0, \tag{1.4}$$
$$V = n\cdot v|_{\Gamma(t)} - m[n\cdot\nabla\mu] \quad \text{on } \Gamma(t),\ t>0, \tag{1.5}$$
$$\mu|_{\Gamma(t)} = \kappa H \quad \text{on } \Gamma(t),\ t>0, \tag{1.6}$$
$$v|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega,\ t>0, \tag{1.7}$$
$$n_\Omega\cdot m\nabla\mu|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega,\ t>0, \tag{1.8}$$
$$\Omega^+(0) = \Omega_0^+, \tag{1.9}$$
$$v|_{t=0} = v_0 \quad \text{in } \Omega, \tag{1.10}$$

where v_0, Ω_0^+ are given initial data satisfying $\partial\Omega_0^+\cap\partial\Omega=\emptyset$ and where $\kappa, m>0$ are a surface tension and a mobility constant, respectively. Implicitly it is assumed that v, μ do not jump across $\Gamma(t)$, i.e.,

$$[v] = [\mu] = 0 \quad \text{on } \Gamma(t),\ t>0.$$

Eqs. (1.1)–(1.2) describe the conservation of linear momentum and mass in both fluids and (1.4) is the balance of forces at the boundary. The equations for v are complemented by the non-slip condition (1.7) at the boundary of Ω. The conditions (1.3), (1.8) describe together with (1.5) a continuity equation for the mass of the phases, and (1.6) relates the chemical potential μ to the L^2-gradient of the surface area, which is given by the mean curvature of the interface. In this formulation of the model we assume (if $\Omega\neq\mathbb{T}^d$) that $\Gamma(t)$ does not touch $\partial\Omega$.

For $m=0$ the velocity field v is independent of μ. In this case, (1.5) describes the usual kinematic condition that the interface is transported by the flow of the surrounding fluids and (1.1)–(1.10) reduces to the classical model of a two-phase Navier–Stokes flow as for example studied by cf. Denisova and Solonnikov [7], where short time existence of strong solutions is shown. On the other hand, if $m>0$, Eqs. (1.3), (1.6), (1.8) with $v=0$ define the Mullins–Sekerka flow of a family of interfaces. This evolution describes the gradient flow for the surface area functional with respect to the $H^{-1}(\Omega)$ scalar product. Therefore we will also call (1.1)–(1.10) the Navier–Stokes/Mullins–Sekerka system.

The motivation to consider (1.1)–(1.10) with $m>0$ is twofold: First of all, the modified system gives a regularization of the classical model $m=0$ since we change from a parabolic–hyperbolic system to a purely parabolic system (cf. also the effect of $m>0$ in (1.13) below). Secondly, (1.1)–(1.10) appears as sharp interface limit of the following diffuse interface model, introduced by Hohenberg and Halperin [12] and rigorously derived by Gurtin et al. [10]:

$$\partial_t v + v\cdot\nabla v - \operatorname{div}\big(\nu(c)Dv\big) + \nabla p = -\varepsilon\operatorname{div}(\nabla c\otimes\nabla c) \quad \text{in } \Omega\times(0,\infty), \tag{1.11}$$
$$\operatorname{div} v = 0 \quad \text{in } \Omega\times(0,\infty), \tag{1.12}$$
$$\partial_t c + v\cdot\nabla c = m\Delta\mu \quad \text{in } \Omega\times(0,\infty), \tag{1.13}$$
$$\mu = \varepsilon^{-1}f'(c) - \varepsilon\Delta c \quad \text{in } \Omega\times(0,\infty), \tag{1.14}$$
$$v|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega\times(0,\infty), \tag{1.15}$$
$$\partial_n c|_{\partial\Omega} = \partial_n\mu|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega\times(0,\infty), \tag{1.16}$$
$$(v,c)|_{t=0} = (v_0,c_0) \quad \text{in } \Omega. \tag{1.17}$$

Here c is the concentration of one of the fluids, where we note that a partial mixing of both fluids is assumed in the model, and f is a suitable "double-well potential" e.g. $f(c)=c^2(1-c)^2$. Moreover, $\varepsilon>0$ is a small parameter related to the interface thickness, μ is the so-called chemical potential and $m>0$ the mobility. We refer to [1,5] for some analytic results for this model and to [13] for results for a non-Newtionian variant of this model. The convergence (1.11)–(1.17) to varifold solutions of (1.1)–(1.10) is discussed in Appendix A.

Sufficiently smooth solutions of (1.1)–(1.10) satisfy the following energy equality,

$$\frac{d}{dt}\frac{1}{2}\int_\Omega \big|v(t)\big|^2\,dx + \kappa\frac{d}{dt}\mathcal{H}^{d-1}\big(\Gamma(t)\big) = -\int_\Omega \nu(\mathcal{X})|Dv|^2\,dx - m\int_\Omega |\nabla\mu|^2\,dx, \tag{1.18}$$

where $\nu(0) = \nu_-$ and $\nu(1) = \nu_+$ and $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure. This identity can be verified by multiplying (1.1) and (1.3) with v, μ, resp., integrating and using the boundary and interface conditions (1.4)–(1.8). This energy equality motivates the choice of solution spaces in our weak formulation and shows that the regularization introduced for $m > 0$ yields an additional dissipation term. In particular, we expect $\mu(\cdot,t) \in H^{1,2}(\Omega)$ for almost all $t \in \mathbb{R}_+$ and formally, using Sobolev inequality and (1.7), that $H(\cdot,t) \in L^4(\Gamma(t))$ for $d \leqslant 3$. This gives some indication of extra regularity properties of the phase interfaces in the model with $m > 0$.

Our main result is the existence of weak solutions of (1.1)–(1.10) for large times. For the definitions of the function spaces we refer to Section 2 below; the concept of generalized mean curvature for non-smooth phase interfaces is taken from [20], see Definition 4.4 below.

Theorem 1.1. *Let $d = 2, 3$, $T > 0$, let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with smooth boundary or let $\Omega = \mathbb{T}^d$ and set $\Omega_T = \Omega \times (0,T)$. Moreover, let $\nu(0) := \nu_-$, $\nu(1) := \nu_+$ and $\kappa, m > 0$. Then for any $v_0 \in L^2_\sigma(\Omega)$, $\mathcal{X}_0 \in BV(\Omega;\{0,1\})$ there are $v \in L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;H^1_0(\Omega)^d)$, $\mathcal{X} \in L^\infty_{\omega*}(0,T;BV(\Omega;\{0,1\}))$, $\mu \in L^2(0,T;H^1(\Omega))$, that satisfy* (1.1)–(1.10) *in the following sense*: *For almost all $t \in (0,T)$ the phase interface $\partial^*\{\mathcal{X}(\cdot,t) = 1\}$ has a generalized mean curvature vector $H(t) \in L^s(d|\nabla\mathcal{X}^h(t)|)^d$ with $s = 4$ if $d = 3$ and $1 \leqslant s < \infty$ arbitrary if $d = 2$, such that*

$$\int_{\Omega_T} \big(-v\partial_t\varphi + v\cdot\nabla v\varphi + \nu(\mathcal{X})Dv : D\varphi\big)\,d(x,t) - \int_\Omega \varphi|_{t=0}\cdot v_0\,dx = \kappa\int_0^\infty\int_\Omega H(t)\cdot\varphi(t)\,d\big|\nabla\mathcal{X}(t)\big|\,dt, \tag{1.19}$$

holds for all $\varphi \in C^\infty([0,T];C^\infty_{0,\sigma}(\Omega))$ with $\varphi|_{t=T} = 0$,

$$\int_{\Omega_T} \mathcal{X}\big(\partial_t\psi + \operatorname{div}(\psi v)\big)\,dx\,dt + \int_\Omega \mathcal{X}_0(x)\psi(0,x)\,dx = m\int_{\Omega_T}\nabla\mu\cdot\nabla\psi\,dx\,dt \tag{1.20}$$

holds for all $\psi \in C^\infty([0,T]\times\overline{\Omega})$ with $\psi|_{t=T} = 0$ and

$$\kappa H(t,\cdot) = \mu(t,\cdot)\frac{\nabla\mathcal{X}(\cdot,t)}{|\nabla\mathcal{X}|(\cdot,t)} \quad \mathcal{H}^{d-1}\text{-almost everywhere on } \partial^*\big\{\mathcal{X}(t,\cdot) = 1\big\} \tag{1.21}$$

holds for almost all $0 < t < T$.

Remark 1.2. Eq. (1.19) is the weak formulation of (1.1), (1.4), and (1.10). It is obtained from testing (1.1) with φ in $\Omega^\pm(t)$, integrating over $\Omega^+(t) \cup \Omega^-(t)$ and using (1.5) together with Gauss' theorem. Similarly, (1.20) is a weak formulation of (1.3), (1.5), (1.8), and (1.9). The conditions (1.2), (1.7) and $[v] = [\mu] = 0$ on $\Gamma(t)$ are included in the choice of the function spaces, namely $v(t) \in H^1_0(\Omega)$, $\mu(t) \in H^1(\Omega)$ for almost every $t \geqslant 0$, and (1.6) is formulated in (1.21).

Finally, we note that, because of (1.21), (1.19) is equivalent to

$$\int_{\Omega_T} \big(-v\partial_t\varphi + v\cdot\nabla v\varphi + \nu(\mathcal{X})Dv : D\varphi\big)\,d(x,t) - \int_\Omega v_0\cdot\varphi|_{t=0}\,dx = -\int_{\Omega_T}\mathcal{X}\nabla\mu\cdot\varphi\,d(x,t) \tag{1.22}$$

for all $\varphi \in C^\infty([0,T];C^\infty_{0,\sigma}(\Omega))$ with $\varphi|_{t=T} = 0$. The latter form will be used for the construction of weak solutions.

Remark 1.3. Compared to the available long-time existence results for the classical model $m = 0$ and as a consequence of the diffusive effects that are included in our model, Theorem 1.1 yields the long-time existence of more regular solutions. In the classical case $m = 0$ Plotnikov [19] and Abels [2,3] have shown the long-time existence of generalized solutions. However, in their formulations the phase interfaces are in general not regular enough to define a mean curvature. Condition (1.4) is only satisfied by a varifold that may depend on the construction process of the solutions, that is in general not rectifiable, and lacks a $(d-1)$-dimensional character (due to concentration and oscillation effects of the interface and e.g. the formation of "infinitesimal small droplets", cf. the discussion in [3]). In contrast, in our weak formulation the phase interfaces have a generalized mean curvature that enjoys the integrability property that are expected, in the smooth case, from (1.7), the energy equality, and the Sobolev inequality for the chemical potential. A similar result for the case $m = 0$ is an open problem.

We note that a similar but different regularization was proposed by Liu and Shen [14]. In their model (1.5) is replaced by

$$V = n \cdot v|_{\Gamma(t)} + mH.$$

Local in time well-posedness for the latter system was proved by Maekawa [16]. Physically, this model has the disadvantage that the mass of the fluids, i.e., $|\Omega^{\pm}(t)|$, is not preserved in time, while this is the case for our system (1.1)–(1.10).

Finally, we mention that Hoffmann and Starovoitov [11] constructed weak solutions for a model of a two-phase flow with phase transition.

Remark 1.4. We note that our concept of weak solution does not include a formulation of a contact angle condition in the case that Ω is a bounded domain and the phase boundary $\partial^*\{\mathcal{X}(t,\cdot)=1\}$ meets the boundary of the domain $\partial\Omega$. Even for weak solutions of the Mullins–Sekerka flow as constructed in [20] the formulation of boundary conditions is an open problem.

For simplicity we will assume $\kappa = m = 1$ in the following. All statements below and the proof of Theorem 1.1 will be valid for general $m, \kappa > 0$ if modified accordingly. The structure of the article is as follows: First the basic notation and some preliminaries are summarized in Section 2. Then weak solutions of a time-discrete approximate system are constructed in Section 3. Our main theorem is proved in Section 4 by passing to the limit in the approximate system. Finally, in Appendix A we prove the convergence of the diffuse interface model (1.11)–(1.17) to (1.1)–(1.10). However, in this limit we have to work with a weaker notion of generalized solutions, compared to the notion of solutions that we use in Theorem 1.1.

2. Notation and preliminaries

For $A, B \in \mathbb{R}^{d\times d}$ we denote $A : B = \operatorname{tr}(AB)$ and $|A| = \sqrt{A:A}$. Given $a \in \mathbb{R}^d$ we define $a \otimes a \in \mathbb{R}^{d\times d}$ as the matrix with the entries $a_i a_j$, $i, j = 1, \ldots, d$. The space of all k-dimensional unoriented linear subspaces of $\mathbb{R}^d$ is denoted by G_k. If X is a Banach space, X^* denotes its dual and $\langle x^*, x\rangle \equiv \langle x^*, x\rangle_{X^*,X}$, $x^* \in X^*$, $x \in X$, the duality product. If H is a Hilbert space, then $(\cdot,\cdot)_H$ denotes its inner product. Moreover, we use the abbreviation $(\cdot,\cdot)_M = (\cdot,\cdot)_{L^2(M)}$.

For $s > 0$ we denote by $[s]$ the integer part of s and for $f : \mathbb{R} \to X$ we define the backward and forward difference quotients by

$$\partial^-_{t,h} f := \frac{f(t) - f(t-h)}{h}, \qquad \partial^+_{t,h} f := \frac{f(t+h) - f(t)}{h}.$$

2.1. Measures and BV-functions

Let X be a locally compact separable metric space and let $C_0(X;\mathbb{R}^m)$ by the closure of compactly supported continuous functions $f : X \to \mathbb{R}^m$, $m \in \mathbb{N}$, in the supremum norm. Moreover, denote by $\mathcal{M}(X;\mathbb{R}^m)$ the space of all finite $\mathbb{R}^m$-valued Radon measures, $\mathcal{M}(X) := \mathcal{M}(X;\mathbb{R})$, and $\mathcal{M}_1(X)$ denotes the space of all probability measures on X. By the Riesz representation theorem $\mathcal{M}(X;\mathbb{R}^m) = C_0(X;\mathbb{R}^m)^*$, cf. e.g. [4, Theorem 1.54]. Given $\lambda \in \mathcal{M}(X;\mathbb{R}^m)$ we denote by $|\lambda|$ the total variation measure defined by

$$|\lambda|(A) = \sup\left\{\sum_{k=0}^{\infty} \bigl|\lambda(A_k)\bigr|:\ A_k \in \mathcal{B}(X) \text{ pairwise disjoint},\ A = \bigcup_{k=0}^{\infty} A_k\right\}$$

for every $A \in \mathcal{B}(X)$, where $\mathcal{B}(X)$ denotes the σ-algebra of Borel sets of X. Moreover, $\frac{\lambda}{|\lambda|} : X \to \mathbb{R}^m$ denotes the Radon–Nikodym derivative of λ with respect to $|\lambda|$. The restriction of a measure μ to a μ-measurable set A is denoted by $(\mu \llcorner A)(B) = \mu(A \cap B)$. Finally, the s-dimensional Hausdorff measure on $\mathbb{R}^d$, $0 \leqslant s \leqslant d$, is denoted by $\mathcal{H}^s$.

Let $U \subseteq \mathbb{R}^d$ be an open set. Recall that

$$BV(U) = \bigl\{f \in L^1(U):\ \nabla f \in \mathcal{M}\bigl(U;\mathbb{R}^d\bigr)\bigr\},$$
$$\|f\|_{BV(U)} = \|f\|_{L^1(U)} + \|\nabla f\|_{\mathcal{M}(U;\mathbb{R}^d)},$$

where ∇f denotes the distributional derivative. Moreover, $BV(U;\{0,1\})$ denotes the set of all $\mathcal{X} \in BV(U)$ such that $\mathcal{X}(x) \in \{0,1\}$ for almost all $x \in U$.

A set $E \subseteq U$ is said to have finite perimeter in U if $\mathcal{X}_E \in BV(U)$. By the structure theorem of sets of finite perimeter $|\nabla \mathcal{X}_E| = \mathcal{H}^{d-1} \llcorner \partial^* E$, where $\partial^* E$ is the so-called reduced boundary of E and for all $\varphi \in C_0(U, \mathbb{R}^d)$

$$-\langle \nabla \mathcal{X}_E, \varphi\rangle = \int_E \operatorname{div}\varphi \, dx = -\int_{\partial^* E} \varphi \cdot n_E \, d\mathcal{H}^{d-1},$$

where $n_E(x) = \frac{\nabla \mathcal{X}_E}{|\nabla \mathcal{X}_E|}$, cf. e.g. [4]. Note that, if E is a domain with C^1-boundary, then $\partial^* E = \partial E$ and n_E coincides with the interior unit normal.

2.2. Function spaces

As usual the space of smooth and compactly supported functions in an open set U is denoted by $C_0^\infty(U)$. Moreover, $C^\infty(\overline{U})$ denotes the set of all smooth functions with continuous derivatives on $\overline{U}$. If X is a Banach-space, the X-valued variants are denoted by $C_0^\infty(U;X)$ and $C^\infty(\overline{U};X)$. For $0 < T \leqslant \infty$, we denote by $L^p_{\mathrm{loc}}([0,T);X)$, $1 \leqslant p \leqslant \infty$, the space of all strongly measurable $f:(0,T) \to X$ such that $f \in L^p(0,T';X)$ for all $0 < T' < T$.

Furthermore, $C^\infty_{0,\sigma}(\Omega) = \{\varphi \in C_0^\infty(\Omega)^d\colon \operatorname{div}\varphi = 0\}$ and

$$L^2_\sigma(\Omega) := \overline{C^\infty_{0,\sigma}(\Omega)}^{L^2(\Omega)},$$

cf. e.g. [24] for other characterizations and properties of $L^2_\sigma(\Omega)$.

If $Y = X'$ is a dual space and $Q \subseteq \mathbb{R}^N$ is open, then $L^\infty_{\omega*}(Q;Y)$ denotes the space of all functions $\nu : Q \to Y$ that are weakly-$*$ measurable and essentially bounded, i.e.

$$x \mapsto \langle \nu_x, F(x,\cdot)\rangle_{X',X}$$

is measurable for each $F \in L^1(Q;X)$ and

$$\|\nu\|_{L^\infty_{\omega*}(Q;Y)} := \operatorname*{ess\,sup}_{x\in Q} \|\nu_x\|_Y < \infty.$$

Moreover, we note that there is [4] a separable Banach space X such that $X' = BV(\Omega)$. As a consequence [8] we obtain that $L^\infty_{\omega*}(0,T;BV(\Omega)) = (L^1(0,T;X))^*$ and that uniformly bounded sets in $L^\infty_{\omega*}(0,T;BV(\Omega))$ are weakly*-compact.

Finally, we will use the following notation:

$$H^1_{(0)}(\Omega) := H^1(\Omega) \cap \left\{\mu\colon \int_\Omega \mu = 0\right\},$$

$$H^1_{0,\sigma}(\Omega) := H^1_0(\Omega, \mathbb{R}^d) \cap L^2_\sigma(\Omega),$$

$$BV_{(m_0)}(\Omega;\{0,1\}) := BV(\Omega;\{0,1\}) \cap \left\{\int_\Omega \mathcal{X} = m_0\right\},$$

$$H^{-1}_{(0)}(\Omega) := H^1_{(0)}(\Omega)^*.$$

Here we note that $H^1_{(0)}(\Omega)$ is equipped with the norm $\|f\|_{H^1_{(0)}(\Omega)} = \|\nabla f\|_{L^2(\Omega)}$ and $H^{-1}_{(0)}(\Omega)$ with the dual norm associated to the latter norm. In particular, this yields the useful relation

$$\|f\|_{H^{-1}_{(0)}(\Omega)} = \left\|\nabla(-\Delta_N)^{-1} f\right\|_{L^2(\Omega)} \quad \text{for all } f \in H^{-1}_{(0)}(\Omega), \tag{2.1}$$

where $-\Delta_N : H^1_{(0)}(\Omega) \to H^1_{(0)}(\Omega)^*$ is the weak Laplace operator with Neumann boundary conditions defined by

$$\langle -\Delta_N w, \varphi\rangle_{H^1_{(0)}(\Omega)^*, H^1_{(0)}(\Omega)} = \int_\Omega \nabla w \cdot \nabla \varphi \, dx \quad \text{for all } \varphi \in H^1_{(0)}(\Omega). \tag{2.2}$$

3. Time-discrete approximation

In this section we will construct weak solutions of an approximate time-discrete system. Fortunately the coupling of the Navier–Stokes to the Mullins–Sekerka system can be treated explicitely. The main result of this section is

Proposition 3.1. *Let the assumptions of Theorem* 1.1 *be valid and let* $m_0 = \int_\Omega \mathcal{X}_0\,dx$. *Then for all* $h > 0$ *sufficiently small there exist time-discrete solutions* $v^h \in L^\infty(-h, T; H^1_{0,\sigma}(\Omega))$, $\mathcal{X}^h \in L^\infty(-h, T; BV_{(m_0)}(\Omega; \{0,1\}))$, $\mu_0^h \in L^\infty(0,\infty; H^1_{(0)}(\Omega))$, *and Lagrange multipliers* $\lambda^h : [0,T] \to \mathbb{R}$ *such that for all* $t \in (0,T)$ *the following equations hold*

$$\int_\Omega \left(\frac{1}{h}\big(v^h(t) - v^h(t-h)\big)\eta + v^h(t-h)\cdot\nabla v^h(t)\eta\right)dx + \int_\Omega \nu\big(\mathcal{X}^h(t)\big)Dv^h(t) : D\eta\,dx$$

$$= -\int_\Omega \mathcal{X}^h(t)\nabla\mu_0^h(t)\cdot\eta\,dx \tag{3.1}$$

for all $\eta \in H^1_{0,\sigma}(\Omega)$,

$$v^h(t) = v_0, \qquad \mathcal{X}^h(t) = \mathcal{X}_0 \tag{3.2}$$

for $-h \leqslant t \leqslant 0$,

$$\int_\Omega \left(\frac{\mathcal{X}^h(t) - \mathcal{X}^h(t-h)}{h}\xi - \big(v^h\mathcal{X}^h\big)(t-h)\cdot\nabla\xi\right)dx = -\int_\Omega \nabla\mu_0^h(t)\cdot\nabla\xi\,dx \tag{3.3}$$

for all $\xi \in H^1(\Omega)$, *and with* $\mu^h(t) := \mu_0^h(t) + \lambda^h(t)$

$$\int_\Omega \left(\operatorname{div}\eta - \frac{\nabla\mathcal{X}^h(t)}{|\nabla\mathcal{X}^h(t)|}\cdot D\eta\frac{\nabla\mathcal{X}^h(t)}{|\nabla\mathcal{X}^h(t)|}\right)d\big|\nabla\mathcal{X}^h(t)\big| = \int_\Omega \mathcal{X}^h(t)\operatorname{div}\big(\mu^h(t)\eta\big) \tag{3.4}$$

for all $\eta \in C^1(\overline{\Omega}; \mathbb{R}^d)$ *with* $\eta\cdot n_\Omega = 0$ *on* $\partial\Omega$. *Moreover, we have the estimates*

$$\sup_{t\in(0,T)}\frac{1}{2}\big\|v^h(t)\big\|^2_{L^2(\Omega)} + \sup_{t\in(0,T)}\int_\Omega d\big|\nabla\mathcal{X}^h(t)\big| + \nu_{\min}\big\|Dv^h\big\|^2_{L^2(\Omega_T)} + \frac{1}{4}\big\|\nabla\mu_0^h\big\|^2_{L^2(\Omega_T)}$$

$$\leqslant C\left(\|v_0\|_{L^2(\Omega)}, \int_\Omega d|\nabla\mathcal{X}_0|\right) \tag{3.5}$$

and for all $t \in (0,T)$

$$\big\|\mu_0^h(t) + \lambda^h(t)\big\|_{H^1(\Omega)} \leqslant C\big(1 + \big\|\nabla\mu_0^h(t)\big\|_{L^2(\Omega)}\big) \tag{3.6}$$

holds, where C *depends only on* d, Ω, T, m_0, *the initial data, and* $\nu_{\min} = \min(\nu(0), \nu(1))$.

In the remainder of this section we prove Proposition 3.1. The first step gives the solvability and estimates for a time-discrete and regularized Navier–Stokes equation.

Lemma 3.2. *Let* $\tilde{v} \in H^1_{0,\sigma}(\Omega)$, $\mathcal{X} \in BV_{(m_0)}(\Omega; \{0,1\})$, $\mu \in H^1(\Omega)$, *and* $h > 0$ *be given. Then there exists a solution* $v \in H^1_{0,\sigma}(\Omega)$ *of*

$$\int_\Omega \left(\frac{1}{h}(v - \tilde{v}) + \tilde{v}\cdot\nabla v\right)\varphi\,dx + \int_\Omega \nu(\mathcal{X})Dv : D\varphi\,dx = -\int_\Omega \mathcal{X}\nabla\mu\cdot\varphi\,dx \tag{3.7}$$

for all $\varphi \in C^\infty_{0,\sigma}(\Omega)$. *For each solution* $v \in H^1_{0,\sigma}(\Omega)$

$$\frac{1}{2}\|v\|^2_{L^2(\Omega)} + h\int_\Omega \nu(\mathcal{X})|Dv|^2 \leqslant \frac{1}{2}\|\tilde{v}\|^2_{L^2(\Omega)} - h\int_\Omega \mathcal{X}\nabla\mu\cdot v\,dx \tag{3.8}$$

holds.

Proof. We show the existence of a solution $v \in H^1_{0,\sigma}(\Omega)$ of (3.7) that satisfies (3.8) with the aid of the Leray–Schauder principle, cf. e.g. [24, Chapter II, Lemma 3.1.1]. To this end, we define $L : H^1_{0,\sigma}(\Omega) \to H^1_{0,\sigma}(\Omega)^*$ and $G : H^1_{0,\sigma}(\Omega) \to L^3_\sigma(\Omega)^* \cong L^{3/2}_\sigma(\Omega)$ by

$$\langle Lv,\varphi\rangle_{H^1_{0,\sigma}(\Omega)',H^1_{0,\sigma}(\Omega)} = \int_\Omega \nu(\mathcal{X})Dv : D\varphi\,dx,$$

$$\big(G(v),\psi\big)_\Omega = \int_\Omega \left(-\mathcal{X}\nabla\mu - \frac{1}{h}(v-\tilde{v}) - \tilde{v}\cdot\nabla v\right)\cdot\psi\,dx$$

for all $v,\varphi \in H^1_{0,\sigma}(\Omega)$, $\psi \in L^3_\sigma(\Omega)$. By the lemma of Lax–Milgram, $L : H^1_{0,\sigma}(\Omega) \to H^1_{0,\sigma}(\Omega)'$ is an isomorphism. Moreover, it is easy to check that $G : H^1_{0,\sigma}(\Omega) \to L^3_\sigma(\Omega)^*$ is a continuous mapping, where we note that $\tilde{v}\cdot\nabla v \in L^{3/2}(\Omega)$. Since $L^3_\sigma(\Omega)^* \hookrightarrow H^1_{0,\sigma}(\Omega)^*$ compactly, $G : H^1_{0,\sigma}(\Omega) \to H^1_{0,\sigma}(\Omega)^*$ is completely continuous. Thus $F = L^{-1}G : H^1_{0,\sigma}(\Omega) \to H^1_{0,\sigma}(\Omega)$ is completely continuous and (3.7) is equivalent to the fixed-point problem $v = F(v)$. In order to apply the Leray–Schauder principle, let $R > 0$ be such that $R^2 = M^2\|\nabla\mu\|^2_{L^2(\Omega)} + \frac{M}{h}\|\tilde{v}\|^2_{L^2(\Omega)}$, where M is a constant such that

$$\|v\|^2_{H^1(\Omega)} \leqslant M\int_\Omega \nu(\mathcal{X})|Dv|^2\,dx \quad \text{for all } v \in H^1_{0,\sigma}(\Omega).$$

It remains to show that for all $v \in H^1_{0,\sigma}(\Omega)$

$$v = \lambda F(v),\quad \lambda\in[0,1] \quad\Rightarrow\quad \|v\|_{H^1_{0,\sigma}(\Omega)} \leqslant R. \tag{3.9}$$

To this end let $v = \lambda F(v)$ for some $\lambda \in [0,1]$. Then $Lv = \lambda G(v)$ and therefore

$$\frac{\lambda}{h}\int_\Omega (v-\tilde{v})\varphi\,dx + \int_\Omega \lambda\tilde{v}\cdot\nabla v\varphi\,dx + \int_\Omega \nu(\mathcal{X})Dv : D\varphi\,dx = -\lambda\int_\Omega \mathcal{X}\nabla\mu\cdot\varphi\,dx$$

for all $\varphi \in H^1_{0,\sigma}(\Omega)$. Choosing $\varphi = v$ yields

$$\frac{\lambda}{2h}\|v\|^2_{L^2(\Omega)} + \int_\Omega \nu(\mathcal{X})|Dv|^2\,dx \leqslant -\lambda\int_\Omega \mathcal{X}\nabla\mu\cdot v\,dx + \frac{\lambda}{2h}\|\tilde{v}\|^2_{L^2(\Omega)}, \tag{3.10}$$

where we have used $(v-\tilde{v})\cdot v = \frac{1}{2}|v|^2 - \frac{1}{2}|\tilde{v}|^2 + \frac{1}{2}|v-\tilde{v}|^2 \geqslant \frac{1}{2}|v|^2 - \frac{1}{2}|\tilde{v}|^2$. Hence, using that $|\int_\Omega \mathcal{X}\nabla\mu\cdot v| \leqslant \|\nabla\mu\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}$,

$$\begin{aligned}\|v\|^2_{H^1(\Omega)} &\leqslant M\int_\Omega \nu(\mathcal{X})Dv : Dv\,dx \leqslant M\left(\|\nabla\mu\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \frac{1}{2h}\|\tilde{v}\|^2_{L^2(\Omega)}\right)\\ &\leqslant \frac{M^2}{2}\|\nabla\mu\|^2_{L^2(\Omega)} + \frac{M}{2h}\|\tilde{v}\|^2_{L^2(\Omega)} + \frac{1}{2}\|v\|^2_{H^1(\Omega)}\end{aligned}$$

and therefore $\|v\|_{H^1(\Omega)} \leqslant R$.

Because of (3.9), the Leray–Schauder principle implies the existence of a fixed point $v \in H^1_{0,\sigma}(\Omega)$ which solves (3.7). Finally, (3.8) follows from (3.10) with $\lambda = 1$. □

Next we solve the appropriate versions of the Mullins–Sekerka part (3.3), (3.4). We follow Luckhaus–Sturzenhecker [15] and use that the Mullins–Sekerka flow is the H^{-1}-gradient flow of the surface-area-functional.

Lemma 3.3. *For $\tilde{\mathcal{X}} \in BV_{(m_0)}(\Omega;\{0,1\})$ and $\tilde{v} \in H^1_{0,\sigma}(\Omega)$ there exist $\mathcal{X} \in BV_{(m_0)}(\Omega;\{0,1\})$, $\mu_0 \in H^1_{(0)}(\Omega)$, and a constant $\lambda \in \mathbb{R}$ such that*

$$\int_\Omega \left(\nabla\mu_0 \cdot \nabla\xi + \frac{1}{h}(\mathcal{X} - \tilde{\mathcal{X}})\xi - \tilde{v}\tilde{\mathcal{X}} \cdot \nabla\xi\right) dx = 0 \tag{3.11}$$

for all $\xi \in H^1(\Omega)$, such that

$$\int_\Omega \left(\operatorname{div}\eta - \frac{\nabla\mathcal{X}}{|\nabla\mathcal{X}|} \cdot D\eta \frac{\nabla\mathcal{X}}{|\nabla\mathcal{X}|}\right) d|\nabla\mathcal{X}| = \int_\Omega \mathcal{X} \operatorname{div}\big((\mu_0 + \lambda)\eta\big)\, dx \tag{3.12}$$

for all $\eta \in C^1(\Omega, \mathbb{R}^d)$ with $\eta \cdot n_\Omega = 0$ on $\partial\Omega$, and such that

$$\int_\Omega d|\nabla\mathcal{X}| + \frac{h}{2}\|\nabla\mu_0\|^2_{L^2(\Omega)} \leqslant \int_\Omega d|\nabla\tilde{\mathcal{X}}| + \frac{h}{2}\|\tilde{v}\|^2_{L^2(\Omega)}. \tag{3.13}$$

Moreover, we have

$$|\lambda| \leqslant C(m_0, n, \Omega)\left(1 + \int_\Omega d|\nabla\mathcal{X}|\right)\left(\int_\Omega d|\nabla\mathcal{X}| + \|\nabla\mu_0\|_{L^2(\Omega)}\right), \tag{3.14}$$

$$\|\mu_0 + \lambda\|_{H^1(\Omega)} \leqslant C(m_0, n, \Omega)\left(1 + \int_\Omega d|\nabla\mathcal{X}|\right)\left(\int_\Omega d|\nabla\mathcal{X}| + \|\nabla\mu_0\|_{L^2(\Omega)}\right). \tag{3.15}$$

Proof. We split the proof into several steps.

Step 1: There exist $\mu_0 \in H^1_{(0)}(\Omega)$, $\mathcal{X} \in BV_{(m_0)}(\Omega;\{0,1\})$ satisfying (3.11), (3.13), and enjoying a minimizing property from which we will deduce (3.12).

Define a functional $F^h : BV_{(m_0)}(\Omega;\{0,1\}) \to \mathbb{R}$,

$$F^h(\sigma) = \int_\Omega |\nabla\sigma| + \frac{1}{2h}\|\sigma - \tilde{\mathcal{X}} + h\tilde{v} \cdot \nabla\tilde{\mathcal{X}}\|^2_{H^{-1}_{(0)}(\Omega)} \tag{3.16}$$

for $\sigma \in BV_{(m_0)}(\Omega;\{0,1\})$. We remark that $\tilde{v} \cdot \nabla\tilde{\mathcal{X}} \in H^1_{(0)}(\Omega)^*$ is defined by

$$\langle \tilde{v} \cdot \nabla\tilde{\mathcal{X}}, \zeta\rangle = -\int_\Omega \tilde{\mathcal{X}}\tilde{v} \cdot \nabla\zeta\, dx, \tag{3.17}$$

where we note that $\operatorname{div}\tilde{v} = 0$. Because of (2.2),

$$\|\sigma - \tilde{\mathcal{X}} + h\tilde{v} \cdot \nabla\tilde{\mathcal{X}}\|^2_{H^{-1}_{(0)}(\Omega)} = \big\langle \sigma - \tilde{\mathcal{X}} + h\tilde{v} \cdot \nabla\tilde{\mathcal{X}}, (-\Delta_N)^{-1}(\sigma - \tilde{\mathcal{X}} + h\tilde{v} \cdot \nabla\tilde{\mathcal{X}})\big\rangle_{H^1_{(0)}(\Omega)^*, H^1_{(0)}(\Omega)}. \tag{3.18}$$

By the $L^1(\Omega)$-compactness of bounded sequences in $BV(\Omega)$, the lower semi-continuity of the perimeter under $L^1(\Omega)$-convergence, and the continuity of the embedding $L^2_{(0)}(\Omega) \to H^{-1}_{(0)}(\Omega)$ there exists a minimizer $\mathcal{X} \in BV_{(m_0)}(\Omega;\{0,1\})$ of F^h. Moreover,

$$\mu_0 := -(-\Delta_N)^{-1}\left(\frac{1}{h}(\mathcal{X} - \tilde{\mathcal{X}}) + \tilde{v} \cdot \nabla\tilde{\mathcal{X}}\right) \tag{3.19}$$

satisfies (3.11). We deduce now (3.13) from $F^h(\mathcal{X}) - F^h(\tilde{\mathcal{X}}) \leqslant 0$ and (2.2), (3.19). In fact,

$$\begin{aligned}\int_\Omega d|\nabla\mathcal{X}| + \frac{1}{2h}\int_\Omega |h\nabla\mu_0|^2\, dx &\leqslant \int_\Omega d|\nabla\tilde{\mathcal{X}}| + \frac{h}{2}\big\langle \tilde{v} \cdot \nabla\tilde{\mathcal{X}}, (-\Delta_N)^{-1}(\tilde{v} \cdot \nabla\tilde{\mathcal{X}})\big\rangle_{H^{-1}_{(0)}, H^1_{(0)}} \\ &= \int_\Omega d|\nabla\tilde{\mathcal{X}}| + \frac{h}{2}\int_\Omega \big|\nabla(-\Delta_N)^{-1}\tilde{v} \cdot \nabla\tilde{\mathcal{X}}\big|^2\, dx\end{aligned}$$

$$\leqslant \int_\Omega d|\nabla\tilde{\mathcal{X}}| + \frac{h}{2}\|\tilde{v}\|^2_{L^2(\Omega)},$$

where in the last step we have used that by (2.2), (3.17)

$$\begin{aligned}\big\|\nabla(-\Delta_N)^{-1}\tilde{v}\cdot\nabla\tilde{\mathcal{X}}\big\|^2_{L^2(\Omega)} &= \big\langle \tilde{v}\cdot\nabla\tilde{\mathcal{X}}, (-\Delta_N)^{-1}\tilde{v}\cdot\nabla\tilde{\mathcal{X}}\big\rangle_{H^{-1}_{(0)},H^1_{(0)}} \\ &= -\int_\Omega \tilde{\mathcal{X}}\tilde{v}\cdot\nabla(-\Delta_N)^{-1}\tilde{v}\cdot\nabla\tilde{\mathcal{X}}\,dx \\ &\leqslant \|\tilde{v}\|_{L^2(\Omega)}\big\|\nabla(-\Delta_N)^{-1}\tilde{v}\cdot\nabla\tilde{\mathcal{X}}\big\|_{L^2(\Omega)},\end{aligned}$$

hence $\|\nabla(-\Delta_N)^{-1}\tilde{v}\cdot\nabla\tilde{\mathcal{X}}\|_{L^2(\Omega)} \leqslant \|\tilde{v}\|_{L^2(\Omega)}$.

Step 2: We compute the first variation of F^h in $\mathcal{X}$ with respect to volume preserving variations. With this aim we consider a smooth family $(\Phi_s)_{s\in(-\varepsilon,\varepsilon)}$ of smooth diffeomorphisms $\Phi_s : \Omega \to \Omega$ with $\Phi_0 = \mathrm{Id}$ and variation field η such that

$$\eta = \frac{\partial}{\partial s}\Big|_{s=0}\Phi_s \in C^\infty\big(\Omega,\mathbb{R}^d\big), \qquad \eta\cdot n_\Omega = 0 \quad \text{on } \partial\Omega.$$

Assume that the variations Φ_s conserve the volume of $\{\mathcal{X}=1\}$, that means that $\sigma_s := \mathcal{X}\circ\Phi_s^{-1}$ satisfy

$$\int_\Omega \sigma_s\,dx = m_0 \quad \text{for all } -\varepsilon < s < \varepsilon,$$

in particular $\sigma_s \in BV_{(m_0)}(\Omega;\{0,1\})$ and $\int_\Omega \mathcal{X}\operatorname{div}\eta = 0$. Since $\mathcal{X}$ minimizes F^h in Z we have $\frac{d}{ds}|_{s=0}F^h(\mathcal{X}_s) = 0$. The first part of F^h is given by the perimeter-functional $\mathcal{P}_\Omega$ and

$$\frac{d}{ds}\Big|_{s=0}\mathcal{P}_\Omega(\sigma_s) = \delta\mathcal{P}_\Omega(\mathcal{X})(\eta) = \int_\Omega\left(\operatorname{div}\eta - \frac{\nabla\mathcal{X}}{|\nabla\mathcal{X}|}\cdot D\eta\frac{\nabla\mathcal{X}}{|\nabla\mathcal{X}|}\right)d|\nabla\mathcal{X}|. \tag{3.20}$$

Let K^h denote the second part of F^h. Since $(-\Delta_N)$ is linear and symmetric, since $\frac{\partial}{\partial s}|_{s=0}\sigma_s = -\nabla\mathcal{X}\cdot\eta$, and by (3.19) we obtain that

$$\frac{d}{ds}\Big|_{s=0}K^h(\sigma_s) = \delta K^h(\mathcal{X})(\eta) = \langle\nabla\mathcal{X}\cdot\eta,\mu_0\rangle_{H^{-1}_{(0)},H^1_{(0)}} = -\int_\Omega \mathcal{X}\operatorname{div}(\eta\mu_0)\,dx. \tag{3.21}$$

We therefore deduce from the minimality of $\mathcal{X}$ that

$$0 = \int_\Omega\left(\operatorname{div}\eta - \frac{\nabla\mathcal{X}}{|\nabla\mathcal{X}|}\cdot D\eta\frac{\nabla\mathcal{X}}{|\nabla\mathcal{X}|}\right)d|\nabla\mathcal{X}| - \int_\Omega \mathcal{X}\operatorname{div}(\eta\mu_0)\,dx. \tag{3.22}$$

Step 3: We next prove (3.12). Fix $\xi \in C_0^\infty(\Omega,\mathbb{R}^d)$ such that

$$\int_\Omega \mathcal{X}\operatorname{div}\xi\,dx \neq 0, \qquad \xi\cdot n_\Omega = 0 \quad \text{on } \partial\Omega,$$

and choose a family $(h_r)_{r\in(-\varepsilon_1,\varepsilon_1)}$ of smooth diffeomorphisms of Ω with $h_0 = \mathrm{Id}$ and $\frac{\partial}{\partial r}|_{r=0}h_r = \xi$. Similarly, for given $\eta \in C^\infty(\Omega,\mathbb{R}^d)$ with $\eta\cdot n_\Omega = 0$ on $\partial\Omega$ we let $(g_s)_{s\in(-\varepsilon_1,\varepsilon_1)}$ be a family of smooth diffeomorphisms of Ω with $g_0 = \mathrm{Id}$ and $\frac{\partial}{\partial s}|_{s=0}g_s = \eta$. Then the function $f : (-\varepsilon_1,\varepsilon_1)^2 \to \mathbb{R}$ defined by

$$f(s,r) := \int_\Omega \mathcal{X}\circ(g_s\circ h_r)^{-1}\,dx - m_0 = \int_\Omega \mathcal{X}\big(\det\big((Dg_s\circ h_r)Dh_r\big) - 1\big)\,dx$$

for sufficiently small $\varepsilon_1 > 0$ satisfies

$$f(0,0)=0, \qquad \partial_r|_{s=r=0} f(s,r) = \int_\Omega \mathcal{X} \operatorname{div} \xi \neq 0.$$

Since f is smooth we obtain by the implicit function theorem that there exists $0 < \varepsilon \leqslant \varepsilon_1$ and a smooth function $\varrho : (-\varepsilon, \varepsilon) \to (-\varepsilon_1, \varepsilon_1)$ such that

$$f\bigl(s, \varrho(s)\bigr) = 0 \quad \text{for all } s \in (-\varepsilon, \varepsilon). \tag{3.23}$$

We therefore deduce that

$$0 = \frac{d}{ds}\bigg|_{s=0} f\bigl(s, \varrho(s)\bigr) = \int_\Omega \bigl(\operatorname{div} \eta + \varrho'(0) \operatorname{div} \xi\bigr) \mathcal{X}\, dx,$$

hence

$$\varrho'(0) = -\biggl(\int_\Omega \mathcal{X} \operatorname{div} \xi \, dx\biggr)^{-1} \int_\Omega \mathcal{X} \operatorname{div} \eta \, dx. \tag{3.24}$$

By (3.23) the family $(g_s \circ h_{r(s)})_{s\in(-\varepsilon,\varepsilon)}$ defines a variation of Ω that conserves the volume of $\mathcal{X}$. The corresponding variation field is given by

$$\tilde\eta := \frac{\partial}{\partial s}\bigg|_{s=0} (g_s \circ h_r) = \eta + \varrho'(0)\xi.$$

Step 2 therefore implies that

$$0 = \int_\Omega \biggl(\operatorname{div} \tilde\eta - \frac{\nabla \mathcal{X}}{|\nabla \mathcal{X}|} \cdot D\tilde\eta \frac{\nabla \mathcal{X}}{|\nabla \mathcal{X}|}\biggr) d|\nabla \mathcal{X}| - \int_\Omega \mathcal{X} \operatorname{div}(\tilde\eta \mu_0)\, dx$$

and yields by (3.24)

$$\begin{aligned}
&\int_\Omega \biggl(\operatorname{div} \eta - \frac{\nabla \mathcal{X}}{|\nabla \mathcal{X}|} \cdot D\eta \frac{\nabla \mathcal{X}}{|\nabla \mathcal{X}|}\biggr) d|\nabla \mathcal{X}| - \int_\Omega \mathcal{X} \operatorname{div}(\eta \mu_0)\, dx \\
&\quad = -\varrho'(0)\biggl[\int_\Omega \biggl(\operatorname{div} \xi - \frac{\nabla \mathcal{X}}{|\nabla \mathcal{X}|} \cdot D\xi \frac{\nabla \mathcal{X}}{|\nabla \mathcal{X}|}\biggr) d|\nabla \mathcal{X}| - \int_\Omega \mathcal{X} \operatorname{div}(\xi \mu_0)\, dx\biggr] \\
&\quad = \lambda \int_\Omega \mathcal{X} \operatorname{div} \eta
\end{aligned} \tag{3.25}$$

with

$$\lambda := \biggl(\int_\Omega \mathcal{X} \operatorname{div} \xi \, dx\biggr)^{-1} \biggl[\int_\Omega \biggl(\operatorname{div} \xi - \frac{\nabla \mathcal{X}}{|\nabla \mathcal{X}|} \cdot D\xi \frac{\nabla \mathcal{X}}{|\nabla \mathcal{X}|}\biggr) d|\nabla \mathcal{X}| - \int_\Omega \mathcal{X} \operatorname{div}(\xi \mu_0)\, dx\biggr]. \tag{3.26}$$

This proves (3.12).

Step 4: Finally we derive (3.15) by choosing a particular ξ in Step 3. We adapt the proof given in [6]. First we choose a Dirac sequence $(\varphi_\delta)_{\delta>0}$ with kernel $\varphi \in C_c^\infty(B_1(0))$, $0 \leqslant \varphi \leqslant 1$, and set

$$\mathcal{X}_\delta := \mathcal{X} * \varphi_\delta, \qquad \bar{\mathcal{X}}_\delta := \frac{1}{|\Omega|}\int_\Omega \mathcal{X}_\delta.$$

Let $\psi : \Omega \to \mathbb{R}$ be the solution of

$$\begin{aligned}
&\Delta\psi = \mathcal{X}_\delta - \bar{\mathcal{X}}_\delta \quad \text{in } \Omega, \\
&\nabla\psi \cdot n_\Omega = 0 \quad \text{on } \partial\Omega, \qquad \int_\Omega \psi = 0
\end{aligned}$$

and choose $\xi := \nabla\psi$ in Step 3. We observe that $|\mathcal{X}_\delta - \bar{\mathcal{X}}_\delta| \leqslant 1$ and $|\nabla\mathcal{X}_\delta| \leqslant C(\Omega)\delta^{-1}$. By standard elliptic estimates we conclude that

$$\|\psi\|_{C^2(\bar{\Omega})} \leqslant \frac{1}{\delta}C(\Omega). \tag{3.27}$$

Moreover, we compute that

$$\|\mathcal{X} - \mathcal{X}_\delta\|_{L^1(\Omega)} \leqslant C(\Omega)\delta\biggl(1 + \int_\Omega d|\nabla\mathcal{X}|\biggr), \tag{3.28}$$

and

$$\bar{\mathcal{X}}_\delta = \frac{1}{|\Omega|}\int_\Omega \mathcal{X}_\delta \leqslant \frac{1}{|\Omega|}\int_{\mathbb{R}^d} \mathcal{X}_\delta = \frac{m_0}{|\Omega|}. \tag{3.29}$$

Therefore we deduce the estimate

$$\begin{aligned}\int_\Omega \mathcal{X}\operatorname{div}\xi\,dx &= \int_\Omega \mathcal{X}(\mathcal{X}_\delta - \bar{\mathcal{X}}_\delta)\,dx = (1-\bar{\mathcal{X}}_\delta)m_0 + \int_\Omega (\mathcal{X}_\delta - \mathcal{X})\mathcal{X} \\ &\geqslant \biggl(1 - \frac{m_0}{|\Omega|}\biggr)m_0 - C(\Omega)\delta\biggl(1 + \int_\Omega d|\nabla\mathcal{X}|\biggr) \\ &\geqslant c(m_0,\Omega),\end{aligned} \tag{3.30}$$

for $\delta = \delta_0(m_0,\Omega)(1+\int_\Omega d|\nabla\mathcal{X}|)^{-1}$. Further we compute that

$$\begin{aligned}&\biggl|\int_\Omega \biggl(\operatorname{div}\xi - \frac{\nabla\mathcal{X}}{|\nabla\mathcal{X}|}\cdot D\xi\frac{\nabla\mathcal{X}}{|\nabla\mathcal{X}|}\biggr)d|\nabla\mathcal{X}| - \int_\Omega \mathcal{X}\operatorname{div}(\xi\mu_0)\,dx\biggr| \\ &\quad\leqslant \|\psi\|_{C^2(\overline{\Omega})}\int_\Omega d|\nabla\mathcal{X}| + 2\|\mu_0\|_{H^1(\Omega)}\|\psi\|_{C^2(\overline{\Omega})} \\ &\quad\leqslant \frac{C(\Omega)}{\delta}\int_\Omega d|\nabla\mathcal{X}| + C(n,\Omega)\frac{1}{\delta}\|\nabla\mu_0\|_{L^2(\Omega)} \\ &\quad\leqslant C(m_0,n,\Omega)\biggl(1+\int_\Omega d|\nabla\mathcal{X}|\biggr)\biggl(\int_\Omega d|\nabla\mathcal{X}| + \|\nabla\mu_0\|_{L^2(\Omega)}\biggr),\end{aligned} \tag{3.31}$$

where in the last two steps we have used (3.27) and Poincaré's inequality. We now obtain (3.14) from (3.26), (3.30), and (3.31). The estimate (3.15) follows again from Poincaré's inequality. □

Proof of Proposition 3.1. We construct iteratively time-discrete solutions v^h, $\mathcal{X}^h$. First set

$$v^h(t) := v_0, \qquad \mathcal{X}^h(t) := \mathcal{X}_0 \quad \text{for } -h < t \leqslant 0.$$

Given functions $v^h(t-h)$, $\mathcal{X}^h(t-h)$ for $t \in (kh,(k+1)h] \subset [0,T]$, $k \in \mathbb{N}_0$, Lemma 3.3 yields a solution $\mathcal{X}^h(t) \in BV_{(m_0)}(\Omega;\{0,1\})$, $\mu_0^h(t) \in H^1_{(0)}(\Omega)$, $\lambda^h(t) \in \mathbb{R}$ that satisfy (3.3), (3.4) and

$$\int_\Omega d\bigl|\nabla\mathcal{X}^h(t)\bigr| + \frac{1}{2h}\bigl\|\nabla\mu_0^h(t)\bigr\|^2_{L^2(\Omega)} \leqslant \int_\Omega d\bigl|\nabla\mathcal{X}^h(t-h)\bigr| + \frac{h}{2}\bigl\|v^h(t-h)\bigr\|^2_{L^2(\Omega)}, \tag{3.32}$$

$$\bigl\|\mu_0^h(t)+\lambda^h(t)\bigr\|_{H^1(\Omega)} \leqslant C(m_0,n,\Omega)\biggl(1+\int_\Omega d\bigl|\nabla\mathcal{X}^h(t)\bigr|\biggr)\biggl(\int_\Omega d\bigl|\nabla\mathcal{X}^h(t)\bigr| + \bigl\|\nabla\mu^h(t)\bigr\|_{L^2(\Omega)}\biggr). \tag{3.33}$$

Then we deduce from Lemma 3.2 the existence of $v^h(t) \in H^1_{0,\sigma}(\Omega)$ that satisfies (3.1) and the estimate

$$\frac{1}{2}\|v^h(t)\|^2_{L^2(\Omega)} + h\int_\Omega \nu(\mathcal{X})|Dv^h(t)|^2 \leqslant \frac{1}{2}\|v^h(t-h)\|^2_{L^2(\Omega)} + h\|\nabla\mu_0^h(t)\|_{L^2(\Omega)}\|v^h(t)\|_{L^2(\Omega)}$$
$$\leqslant \frac{1}{2}\|v^h(t-h)\|^2_{L^2(\Omega)} + \frac{h}{4}\|\nabla\mu_0^h(t)\|^2_{L^2(\Omega)} + h\|v^h(t)\|^2_{L^2(\Omega)}. \tag{3.34}$$

By construction v^h, $\mathcal{X}^h$ are constant on each subinterval $(kh,(k+1)h] \subset [0,T]$, $k \in \mathbb{N}$. Summing (3.32), (3.34) for $t_k = kh$, $k = 1,\dots,[t/h]$ we obtain that

$$\frac{1}{2}\|v^h(t)\|^2_{L^2(\Omega)} + \int_\Omega d|\nabla\mathcal{X}^h(t)| + \int_0^{h[t/h]}\int_\Omega \left(\nu_{\min}|Dv^h(\tau)|^2 + \frac{1}{2}|\nabla\mu_0^h(\tau)|^2\right)dx\,d\tau$$
$$\leqslant \frac{1}{2}\|v_0\|^2_{L^2(\Omega)} + \int_\Omega d|\nabla\mathcal{X}_0| + 2\int_{-h}^{h[t/h]}\int_\Omega |v^h(\tau)|^2\,dx\,d\tau + \frac{1}{4}\int_0^{h[t/h]}\int_\Omega |\nabla\mu_0^h(t)|^2\,dx\,d\tau. \tag{3.35}$$

Using Gronwall's lemma we deduce (3.5). Finally, (3.6) follows from (3.5) and (3.33). □

4. Passing time-discrete approximations to a limit

We first show strong compactness of v^h, $\mathcal{X}^h$. To this end, we will apply the following theorem by Simon [22, Theorem 6].

Theorem 4.1. *Let $X \subset B \subset Y$ be Banach spaces with compact embedding $X \hookrightarrow B$. Let $T > 0$ and let $\mathcal{F}$ be a bounded subset of $L^q(0,T;X)$, $1 < q \leqslant \infty$. Assume that for every $0 < t_1 < t_2 < T$*

$$\sup_{f\in\mathcal{F}} \|\tau_s f - f\|_{L^1(t_1,t_2;Y)} \to 0 \quad \text{as } s \to 0, \tag{4.1}$$

where $\tau_s f(t) := f(t+s)$ for every $t \in (0, T-s)$. Then $\mathcal{F}$ is relatively compact in $L^p(0,T;B)$ for every $p \in [1,q)$.

First of all, because of (3.1), $v^h(\cdot-h)\cdot\nabla v^h = \operatorname{div}(v^h(\cdot-h)\otimes v^h)$, and since $v^h(\cdot-h)\otimes v^h \in L^{\frac{4}{3}}(0,T;L^2(\Omega))$ is bounded, $\partial^-_{t,h} v^h \in L^{\frac{4}{3}}(0,T;H^{-1}(\Omega))$, $h \in (0,1)$ is bounded. This implies that

$$\|\tau_{kh}v^h - v^h\|_{L^{\frac{4}{3}}(0,T-kh;H^{-1})} \leqslant kh\|\partial^-_{t,h}v^h\|_{L^{\frac{4}{3}}(0,T;H^{-1})} \leqslant Ckh \tag{4.2}$$

for all $k \in \mathbb{N}$ such that $kh < T$. Therefore (4.1) holds for $\mathcal{F} = \{v^h\colon h \in (0,1)\}$ and $Y = H^{-1}$ since v^h is piecewise constant. Moreover, since $v^h \in L^4(0,T;H^{\frac{1}{2}}(\Omega))$ is bounded, Theorem 4.1 implies that $(v^h)_{0<h<1}$ is relatively compact in $L^2(\Omega_T)$.

Similarly, (3.3) implies that $\partial^-_{t,h}\mathcal{X}^h \in L^2(0,T;H^{-1}(\Omega))$, $h \in (0,1)$ is bounded. Moreover, since $\mathcal{X}^h \in L^\infty(0,T;BV(\Omega))$ is bounded and $BV(\Omega) \hookrightarrow L^1(\Omega)$ compactly, we obtain that $\mathcal{X}^h \in L^1(\Omega_T)$, $0 < h < 1$, is relatively compact. Since $\|\mathcal{X}^h\|_{L^\infty(\Omega_T)} = 1$, $(\mathcal{X}^h)_{0<h<1}$, is relatively compact in $L^p(\Omega_T)$ for every $1 \leqslant p < \infty$.

As a corollary we obtain the compactness of time-discrete approximations.

Proposition 4.2. *Choose a sequence $h \to 0$ and let $(v^h, \mathcal{X}^h, \mu^h, \lambda^h)$ denote the time-discrete solutions constructed in Proposition 3.1. Then there exists a subsequence $h_k \to 0$ $(k \to \infty)$ and $v \in L^2(0,T;H^1(\Omega)^d) \cap L^\infty(0,T;L^2_\sigma(\Omega))$, $\mu \in L^2(0,T;H^1(\Omega))$, $\mathcal{X} \in L^\infty_{\omega*}(0,T;BV_{(m_0)}(\Omega;\{0,1\}))$, and $\lambda \in L^2(0,T)$ such that*

$$v_k \underset{k\to\infty}{\longrightarrow} v \quad \text{in } L^2(\Omega_T), \tag{4.3}$$

$$v_k \underset{k\to\infty}{\rightharpoonup} v \quad \text{in } L^2(0,T;H^1(\Omega)), \tag{4.4}$$

$$\mathcal{X}_k \underset{k\to\infty}{\longrightarrow} \mathcal{X} \quad \text{in } L^p(\Omega_T) \text{ for all } 1 \leqslant p < \infty, \tag{4.5}$$

$$\mu_k \underset{k\to\infty}{\rightharpoonup} \mu \quad \text{in } L^2(0,T;H^1(\Omega)), \tag{4.6}$$

$$\lambda_k \underset{k\to\infty}{\longrightarrow} \lambda \quad \text{in } L^2(0,T), \tag{4.7}$$

where $(v_k, \mathcal{X}_k, \mu_k, \lambda_k) := (v^{h_k}, \mathcal{X}^{h_k}, \mu^{h_k}, \lambda^{h_k})$, $k \in \mathbb{N}$.

4.1. Convergence in (3.1) and (3.3)

We first verify the equations in the bulk.

Proposition 4.3. *Let* $v, \mu, \mathcal{X}$ *be the limits obtained in Proposition* 4.2. *Then*

$$\int_{\Omega_T} \big(-v\partial_t\varphi + v\cdot\nabla v\varphi + \nu(\mathcal{X})Dv : D\varphi\big)\,d(x,t) - \int_\Omega \varphi(0,x)\cdot v_0(x)\,dx = -\int_{\Omega_T} \mathcal{X}\nabla\mu\cdot\varphi\,d(x,t), \tag{4.8}$$

holds for all $\varphi \in C^\infty([0,T]; C^\infty_{0,\sigma}(\Omega))$ *with* $\varphi|_{t=T} = 0$ *and*

$$\int_{\Omega_T} \nabla\mu\cdot\nabla\zeta\,d(x,t) = \int_{\Omega_T} \partial_t\zeta\mathcal{X} + \operatorname{div}(\zeta v)\mathcal{X}\,d(x,t) + \int_\Omega \zeta(0,x)\mathcal{X}_0(x)\,dx \tag{4.9}$$

holds for all $\zeta \in C^\infty([0,T]\times\overline{\Omega})$ *with* $\zeta|_{t=T} = 0$.

Proof. If we test in (3.1) with $\varphi(\cdot,t)$, where $\varphi \in C^\infty([0,T]; C^\infty_{0,\sigma}(\Omega))$ with $\varphi|_{t=T} = 0$, and integrate over $t \in (0,T)$, we deduce that

$$\int_{\Omega_T} \big(-v^h\partial^+_{t,h}\varphi + v^h(\cdot - h)\cdot\nabla v^h\varphi + \nu\big(\mathcal{X}^h\big)Dv^h : D\varphi\big)\,d(x,t) - \frac{1}{h}\int_0^h\int_\Omega \varphi(x,t)v_0(x)\,dx\,dt$$
$$= -\int_{\Omega_T} \mathcal{X}^h\nabla\mu^h\cdot\varphi\,d(x,t)$$

for all sufficiently small $h > 0$, where we set $\varphi(t) = 0$ for $t > T$. By (4.3), (4.4), (4.6) we can pass to the limit $h \to 0$ in this equality and obtain

$$\int_{\Omega_T} \big(-v^h\partial_t\varphi + v^h\cdot\nabla v^h + \nu(\mathcal{X})Dv : D\varphi\,d(x,t)\varphi\big)\,d(x,t) - \int_\Omega \varphi(0,x)v_0(x)\,dx$$
$$= -\int_{\Omega_T} \mathcal{X}\nabla\mu\cdot\varphi\,d(x,t). \tag{4.10}$$

Similarly we obtain from (3.3) that for all $\zeta \in C^\infty([0,T]\times\overline{\Omega})$ with $\zeta|_{t=T} = 0$

$$\int_{\Omega_T} \nabla\mu^h\cdot\nabla\zeta\,d(x,t) = \int_{\Omega_T} \big(\mathcal{X}^h\partial^+_{t,h}\zeta + \big(v^h\mathcal{X}^h\big)(\cdot - h)\cdot\nabla\zeta\big)\,d(x,t) + \int_{-h}^0\int_\Omega \mathcal{X}_0(x)\zeta(x,t)\,dx\,dt$$

holds and again we can pass to the limit in this equality and obtain (4.9). □

4.2. Convergence in the Gibbs–Thomson law

The main difficulty in passing the approximate solutions to a limit is the convergence in the Gibbs–Thomson condition (3.4). In particular, we cannot exclude that parts of the phase boundary $\partial^*\{\mathcal{X}^h(\cdot,t) = 1\}$ cancel in the limit $h \to 0$. To overcome such difficulties we consider the limit of the phase boundaries in the sense of measures and use varifold theory. For the definition of varifolds and weak mean curvature for varifolds we refer to [23].

Let $\vartheta_t^h := d|\nabla\mathcal{X}^h(\cdot,t)|$ denote the surface measure of the phase interface $\partial^*\{\mathcal{X}^h(\cdot,t)=1\}$,

$$\vartheta_t^h(\eta) := \int_\Omega \eta\, d\big|\nabla\mathcal{X}^h(\cdot,t)\big| \quad \text{for } \eta \in C_0(\Omega) \tag{4.11}$$

and let $n^h(t)$ denote the inner normal of $\partial^*\{\mathcal{X}^h(\cdot,t)=1\}$, i.e.,

$$n^h(x,t) = \frac{\nabla\mathcal{X}^h(\cdot,t)}{|\nabla\mathcal{X}^h(\cdot,t)|}(x),$$

which is well-defined for $\mathcal{H}^{d-1}$-almost all $x \in \partial^*\{\mathcal{X}^h(\cdot,t)=1\}$. By (3.12) the first variation of ϑ_t^h is given as

$$\begin{aligned}\delta\vartheta_t^h(\eta) &= \int_\Omega \big(\operatorname{div}\eta - n^h(\cdot,t)\cdot D\eta n^h(\cdot,t)\big)\, d\big|\nabla\mathcal{X}_t^h\big| \\ &= \int_\Omega \mathcal{X}^h(t)\operatorname{div}\big(\big(\mu_0^h(\cdot,t)+\lambda^h(t)\big)\eta\big)\, dx\end{aligned}$$

for all $\eta \in C_0^1(\Omega,\mathbb{R}^d)$.

We will prove that for almost all $t \in (0,T)$ the phase boundary $\partial^*\{\mathcal{X}(\cdot,t)=1\}$ has a *generalized mean curvature* in the following sense.

Definition 4.4. Let $E \subset \Omega$ and $\mathcal{X}_E \in BV(\Omega)$. If there exists an integral $(d-1)$-varifold ϑ on Ω such that

$$\partial^*E \subset \operatorname{supp}(\vartheta), \quad \vartheta \text{ has weak mean curvature vector } H_\vartheta,\ H_\vartheta \in L^s_{\mathrm{loc}}(\vartheta),\ s > d-1,\ s \geqslant 2$$

then we call

$$H := H_\vartheta|_{\partial^*E}$$

the generalized mean curvature vector of ∂^*E.

This definition was justified in [20], where it is shown that under the above conditions H is a property of E and independent of the choice of ϑ. Moreover, for any C^2-hypersurface $M \subset \mathbb{R}^d$ the mean curvature H of ∂^*E coincides $\mathcal{H}^{d-1}$-almost everywhere on $M \cap \partial^*E$ with the mean curvature of M.

Lemma 4.5. *Let $s=4$ if $d=3$ and $1 \leqslant s < \infty$ arbitrary if $d=2$. Then for almost all $t \in (0,T)$ the phase boundary $\partial^*\{\mathcal{X}(\cdot,t)=1\}$ has a generalized mean curvature $H(t) \in L^s(d|\nabla\mathcal{X}^h(\cdot,t),\mathbb{R}^d|)$,*

$$\int_\Omega \big|H(\cdot,t)\big|^s\, d\big|\nabla\mathcal{X}(\cdot,t)\big| \leqslant C \liminf_{h\to 0} \big\|\mu^h(\cdot,t)\big\|_{H^1(\Omega)}. \tag{4.12}$$

Further, $H(\cdot,t)$ determines the limit of the first variations $\delta\vartheta_t^h$: For any subsequence $h_i \to 0$ $(i\to\infty)$ of $h \to 0$ such that

$$\limsup_{i\in\mathbb{N}} \big\|\mu^{h_i}(\cdot,t)\big\|_{H^1(\Omega)} < \infty \tag{4.13}$$

and for all $\eta \in C_0^1(\Omega,\mathbb{R}^d)$ we obtain

$$\delta\vartheta_t^{h_i}(\eta) \to \int_\Omega -H(t)\cdot\xi\, d\big|\nabla\mathcal{X}(\cdot,t)\big| \quad \textit{as } i\to\infty. \tag{4.14}$$

Proof. By Fatou's lemma and (3.5) we deduce that $t \mapsto \liminf_{h\to 0}\|\mu^h(\cdot,t)\|_{H^1(\Omega)}$ belongs to $L^2(0,T)$ and that the right-hand side of (4.12) is finite for almost all $t \in (0,T)$. In the following let $t \in (0,T)$ be such that $\liminf_{h\to 0}\|\mu^h(\cdot,t)\|_{H^1(\Omega)}$ is finite.

Since $\int_\Omega d|\nabla\mathcal{X}^h(\cdot,t)|$ is uniformly bounded by (3.5) and recalling (4.13) we can extract a subsequence (not relabeled) $h_i \to 0$ ($i\to\infty$) such that

$$\vartheta_t^{h_i} \to \vartheta_t \quad \text{as Radon measures,} \tag{4.15}$$
$$\mu^{h_i}(\cdot,t) \rightharpoonup w_t \quad \text{in } H^1(\Omega), \tag{4.16}$$

for a Radon measure ϑ_t on Ω and $w_t \in H^1(\Omega)$. We then deduce from [21, Theorem 1.1] that

ϑ_t is an integral varifold,

$\vartheta_t^{h_i} \to \vartheta_t$ as varifolds, for a subsequence $h_i \to 0$ ($i\to\infty$),

ϑ_t has weak mean curvature $H_{\vartheta_t} \in L^s(\vartheta_t)$,

$\left|\nabla\mathcal{X}(\cdot,t)\right| \leqslant \vartheta_t$,

and that

$$H_{\vartheta_t} = w_t n(t)$$

holds ϑ_t-almost everywhere, with

$$n(\cdot,t) = \begin{cases} \frac{\nabla\mathcal{X}(\cdot,t)}{|\nabla\mathcal{X}(\cdot,t)|} & \text{on } \partial^*\{\mathcal{X}(\cdot,t)=1\},\\ 0 & \text{elsewhere.}\end{cases}$$

According to [20] $H(\cdot,t) := H_{\vartheta_t}|_{\partial^*\{\mathcal{X}(\cdot,t)=1\}}$ is a property of $\mathcal{X}(\cdot,t)$ and independent of the choice of subsequence in (4.15), (4.16). Moreover, due to [21, Theorem 1.2], we have

$$H(\cdot,t) = 0 \quad \vartheta_t\text{-almost everywhere in } \left\{\theta^{d-1}(\vartheta_t,\cdot) \neq 1\right\},$$

where $\theta^{d-1}(\vartheta_t,\cdot)$ denotes the $(d-1)$-dimensional density of ϑ_t, cf. [23]. Since the first variation is continuous under varifold-convergence, we then obtain that

$$\lim_{i\to\infty} \delta\vartheta_t^{h_i}(\eta) = \delta\vartheta_t(\eta) = \int_\Omega -H_{\vartheta_t}\cdot\eta\, d\vartheta_t = \int_\Omega -H(\cdot,t)\cdot\eta\, d\left|\nabla\mathcal{X}(\cdot,t)\right|. \qquad \square$$

We still have to relate the generalized mean curvature $H(\cdot,t)$ that we obtained for almost all $t\in(0,T)$ with the weak limit μ of μ^h in $L^2(0,T;H^1(\Omega))$.

Lemma 4.6. *For all* $\xi \in L^2(0,T;C_0^1(\Omega;\mathbb{R}^d))$

$$\int_0^T \int -H(\cdot,t)\cdot\xi(\cdot,t)\, d\left|\nabla\mathcal{X}(\cdot,t)\right| dt = \int_{\Omega_T} \mathcal{X}(x,t)\operatorname{div}\bigl(\mu(x,t)\xi(x,t)\bigr)\, d(x,t). \tag{4.17}$$

In particular, for almost all $t\in(0,T)$,

$$H(\cdot,t) = \mu(\cdot,t)n(\cdot,t) \tag{4.18}$$

holds $\mathcal{H}^{d-1}$*-almost everywhere on* $\partial^*\{\mathcal{X}(\cdot,t)=1\}$.

Proof. Define for all $t\in(0,T)$ such that $H(\cdot,t)\in L^s(|\nabla\mathcal{X}(\cdot,t)|)$ exists

$$\bigl\langle T(t),\psi\bigr\rangle := \int_\Omega -H(x,t)\cdot\psi\, d\left|\nabla\mathcal{X}(x,t)\right| \tag{4.19}$$

for all $\psi \in C_0^1(\Omega,\mathbb{R}^d)$. Then

$$\left|\bigl\langle T(t),\psi\bigr\rangle\right| \leqslant C\|\psi\|_{C_0(\Omega)}\left\|H(\cdot,t)\right\|_{L^s(d|\nabla\mathcal{X}_t^h|)} \tag{4.20}$$

and we deduce from (4.12) that $T\in L^2(0,T;C_0(\Omega)^*)$.

Similarly we define for $\psi \in C_0^1(\Omega)$

$$\langle T^h(t), \psi\rangle := \delta\vartheta_t^h(\psi).$$

From (3.4) we deduce that

$$\begin{aligned} \left|\langle T^h(t), \xi(\cdot,t)\rangle\right| &= \left|\int_\Omega \mathcal{X}^h \nabla\cdot\left(\xi\mu^h\right) dx\right| \\ &\leqslant C\left\|\mu^h(t)\right\|_{H^1(\Omega)}\left\|\xi(t)\right\|_{C_0^1(\Omega)}, \end{aligned} \tag{4.21}$$

and

$$\left\|T^h\right\|_{L^2(0,T;C_0^1(\Omega)^*)} \leqslant C\left\|\mu^h\right\|_{L^2(0,T;H^1(\Omega))}\|\xi\|_{L^2(0,T;C_0^1(\Omega))}. \tag{4.22}$$

Moreover, by (4.6), (4.7) there exists a subsequence $h \to 0$ such that

$$\lim_{h\to 0}\int_0^T \langle T^h(t), \xi(\cdot,t)\rangle\, dt = \int_0^T\int_\Omega \mathcal{X}\nabla\cdot(\xi\mu)\, dx\, dt. \tag{4.23}$$

For $\alpha > 0$ we now define functions $T_\alpha^h : (0,T) \to C_0^1(\Omega;\mathbb{R}^d)^*$:

$$\langle T_\alpha^h(t), \psi\rangle := \begin{cases} \langle T_t^h, \psi\rangle & \text{if } \|\mu^h(\cdot,t)\|_{H^1(\Omega)} \leqslant \alpha, \\ \langle T(t), \psi\rangle & \text{else.} \end{cases} \tag{4.24}$$

Fix an arbitrary $\xi \in L^2(0,T;C_0^1(\Omega;\mathbb{R}^d))$. Then we deduce from Lemma 4.5 that for almost all $t \in (0,T)$

$$\langle T_\alpha^h(t), \xi(\cdot,t)\rangle \to \int_\Omega -H\cdot\xi(\cdot,t)\, d\left|\nabla\mathcal{X}(\cdot,t)\right|. \tag{4.25}$$

We also see from (4.21) that

$$\left|\langle T_\alpha^h(t), \xi(\cdot,t)\rangle\right| \leqslant \left\|\xi(\cdot,t)\right\|_{C_0^1(\Omega;\mathbb{R}^d)} C\left(\alpha + \left\|T(t)\right\|_{C_0(\Omega)^*}\right), \tag{4.26}$$

which gives by (4.20) an $L^1(0,T)$-dominator for the left-hand side. By (4.25), (4.26) and Lebesgues Dominated Convergence theorem we deduce that for all $\alpha > 0$

$$\int_0^T \langle T_\alpha^h(t), \xi(\cdot,t)\rangle\, dt \to \int_0^T \langle T(t)\xi(\cdot,t)\rangle\, dt \quad \text{as } h \to 0. \tag{4.27}$$

Next, consider the sets $A^h := \{t \in (0,T)\colon \|\mu^h(\cdot,t)\|_{H^1(\Omega)} > \alpha\}$ and observe

$$\begin{aligned} \left|\int_0^T \langle T^h(t) - T_\alpha^h(t), \xi(\cdot,t)\rangle\, dt\right| &\leqslant \int_{A^h} \left|\langle T^h(t) - T(t), \xi(\cdot,t)\rangle\right| dt \\ &\leqslant \left(\int_{A^h} \|\xi(\cdot,t)\|_{C_0^1(\Omega;\mathbb{R}^d)}^2\, dt\right)^{\frac{1}{2}} \left(\left\|T^h\right\|_{L^2(0,T;C_0^1(\Omega;\mathbb{R}^d)^*)} + \|T\|_{L^2(0,T;C_0(\Omega;\mathbb{R}^d)^*)}\right). \end{aligned}$$

By (4.20) and (4.22) $\|T^h\|_{L^2(0,T;C_0^1(\Omega;\mathbb{R}^d)^*)}$ and $\|T\|_{L^2(0,T;C_0^1(\Omega;\mathbb{R}^d)^*)}$ are bounded uniformly in $h > 0$. Since

$$\left|A^h\right| \leqslant \frac{1}{\alpha^2}\left\|\mu^h\right\|_{L^2(0,T;H^1(\Omega))}^2 \leqslant \frac{1}{\alpha^2}C,$$

we end up with

$$\int_0^T \langle T^h(t) - T^h_\alpha(t), \xi(\cdot,t)\rangle\, dt \to 0 \quad \text{as } \alpha \to \infty \tag{4.28}$$

uniformly in $h > 0$. Thus we obtain from (4.25), (4.28) that

$$\int_{\Omega_T} \mathcal{X} \operatorname{div}(\mu\xi)\, d(x,t) = \lim_{h\to 0} \int_{\Omega_T} \mathcal{X}^h \operatorname{div}(\mu^h \xi)\, d(x,t) \tag{4.29}$$

$$= \lim_{h\to 0} \int_0^T \langle \xi(\cdot,t), \delta\vartheta^h_t\rangle\, dt = \int_0^T \langle \xi(\cdot,t), T(t)\rangle\, dt, \tag{4.30}$$

which proves (4.17). Since no time derivative is involved, we deduce that for almost all $t \in (0,T)$ and all $\xi \in C^1_0(\Omega;\mathbb{R}^d)$,

$$\int_\Omega \mathcal{X}(\cdot,t) \operatorname{div}(\mu(\cdot,t)\xi)\, dx = \langle T(t), \xi\rangle$$

holds. The Gauss–Green theorem [9, Theorem 5.8.1] and (4.19) yield

$$\int_\Omega \mu(\cdot,t) n(\cdot,t)\cdot\xi\, d|\nabla\mathcal{X}(\cdot,t)| = \int_\Omega H(\cdot,t)\cdot\xi\, d|\nabla\mathcal{X}(\cdot,t)|,$$

with $n(\cdot,t) = \frac{\nabla\mathcal{X}(\cdot,t)}{|\nabla\mathcal{X}(\cdot,t)|}$ on $\partial^*\{\mathcal{X}(\cdot,t) = 1\}$. This finally proves (4.18). □

Appendix A. Sharp interface limit

Here we discuss the relation between (1.1)–(1.10) and its diffuse interface analog (1.11)–(1.17). First we consider the corresponding energy identities. For the Navier–Stokes/Mullins–Sekerka system we recall that by (1.18) every sufficiently smooth solution of (1.1)–(1.10) satisfies

$$\frac{d}{dt}\frac{1}{2}\int_\Omega |v(t)|^2\, dx + \kappa\frac{d}{dt}\mathcal{H}^{d-1}(\Gamma(t)) = -\int_\Omega \nu(\mathcal{X})|Dv|^2\, dx - m\int_\Omega |\nabla\mu|^2\, dx. \tag{A.1}$$

On the other hand, every sufficiently smooth solution of (1.11)–(1.17) satisfies

$$\frac{d}{dt}\frac{1}{2}\int_\Omega |v(t)|^2\, dx + \frac{d}{dt}E_\varepsilon(c(t)) = -\int_\Omega \bar\nu(c)|Dv|^2\, dx - m\int_\Omega |\nabla\mu|^2\, dx, \tag{A.2}$$

where

$$E_\varepsilon(c) = \frac{\varepsilon}{2}\int_\Omega |\nabla c|^2\, dx + \frac{1}{\varepsilon}\int_\Omega f(c)\, dx.$$

Moreover, by Modica and Mortola [18] or Modica [17], we have

$$E_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \mathcal{P} \quad \text{w.r.t. } L^1\text{-}\Gamma\text{-convergence},$$

where

$$\mathcal{P}(u) = \begin{cases} \kappa\mathcal{H}^{d-1}(\partial^* E) & \text{if } u = 2\mathcal{X}_E - 1 \text{ and } E \text{ has finite perimeter,} \\ +\infty & \text{else.} \end{cases}$$

Here $\kappa = \int_0^1 \sqrt{2f(s)}\, ds$, $f : \mathbb{R} \to [0,\infty)$ is a suitable function such that $f(s) = 0$ if and only if $s = 0, 1$, and $\partial^* E$ denotes the reduced boundary. Note that $\partial^* E = \partial E$ if E is a sufficiently regular domain. Therefore we see that for constant $m > 0$ the energy identity (A.1) is formally identical to the sharp interface limit of the energy identity (A.2) of the diffuse interface model (1.11)–(1.17). In contrast, if we would choose $m = m_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0$ in (1.11)–(1.17), we formally obtain in the sharp interface limit the energy identity of the classical two-phase flow (1.1)–(1.10).

Now we will adapt the arguments of Chen [6] to show that as $\varepsilon \to 0$ and if $m = m(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} m_0 > 0$ solutions of the diffuse interface model (1.11)–(1.17) converge to *varifold solutions* of the system (1.1)–(1.10), which are defined as follows:

Definition A.1. Let $v_0 \in L^2_\sigma(\Omega)$, $E_0 \subset \Omega$ be a set of finite perimeter, and let $Q = \Omega \times (0,\infty)$. Then (v, E, μ, V) is called *varifold solution* of (1.1)–(1.10) with initial values (v_0, E_0) if the following conditions are satisfied:

(1) $v \in L^\infty(0,\infty; L^2_\sigma(\Omega)) \cap L^2(0,\infty; H^1_0(\Omega)^d)$, $\mu \in L^2_{\mathrm{loc}}([0,\infty); H^1(\Omega))$ and $\nabla\mu \in L^2(Q)$.

(2) $E = \bigcup_{t\geqslant 0} E_t \times \{t\}$ is a measurable subset of $\Omega \times [0,\infty)$ such that $\mathcal{X}_E \in C([0,\infty); L^1(\Omega)) \cap L^\infty_{\omega *}(0,\infty; BV(\Omega))$, $\int_\Omega \mathcal{X}_E(t)\,dx = m_0$ for all $t \geqslant 0$, and $\mathcal{X}_E|_{t=0} = \mathcal{X}_{E_0}$ in $L^1(\Omega)$.

(3) V is a Radon measure on $\Omega \times G_{d-1} \times (0,\infty)$ such that $V = V^t\,dt$ where V^t is for almost all $t \in (0,\infty)$ a general varifold on Ω, i.e., a Radon measure on $\Omega \times G_{d-1}$. Moreover, for almost all $t \in (0,\infty)$ V^t has the representation

$$\int\limits_{\Omega\times G_{d-1}} \psi(x,p)\,dV^t(x,p) = \sum_{i=1}^d \int\limits_\Omega b_i^t(x)\psi\big(x, p_i^t(x)\big)\,d\lambda^t(x)$$

for all $\psi \in C_0(\Omega \times G_{d-1})$, where λ^t is a Radon measure on $\overline{\Omega}$, $b_1^t, \dots, b_N^t$ and $p_1^t, \dots, p_N^t$ are measurable $\mathbb{R}$- and $\mathbb{R}^d$-valued functions, respectively, such that

$$0 \leqslant b_i^t \leqslant 1, \qquad \sum_{i=1}^d b_i^t \geqslant 1, \qquad \sum_{i=1}^d p_i^t \otimes p_i^t = I \quad \mu^t\text{-almost everywhere},$$

$$\frac{|\nabla \mathcal{X}_{E_t}|}{\lambda^t} \leqslant \frac{1}{\kappa} \quad \lambda^t\text{-a.e. with } \kappa = \int\limits_0^1 \sqrt{2f(s)}\,ds.$$

(4) Moreover,

$$\int\limits_Q \big(-v\partial_t\varphi + v\cdot\nabla v\varphi + \nu(\mathcal{X}_E)Dv : D\varphi\big)\,d(x,t) - \int\limits_\Omega \varphi|_{t=0}\cdot v_0\,dx = -\kappa\int\limits_Q \mathcal{X}_E\nabla\mu\cdot\varphi\,d(x,t) \tag{A.3}$$

holds for all $\varphi \in C^\infty([0,\infty); C^\infty_{0,\sigma}(\Omega))$ with $\operatorname{supp}\varphi \subset \Omega \times [0,T]$ for some $T > 0$,

$$m\int\limits_Q \nabla\mu\cdot\nabla\psi\,d(x,t) = \int\limits_Q \mathcal{X}\big(\partial_t\psi + \operatorname{div}(\psi v)\big)\,d(x,t) + \int\limits_\Omega \psi|_{t=0}\mathcal{X}_{E_0}\,dx \tag{A.4}$$

holds for all $\psi \in C^\infty([0,\infty) \times \overline{\Omega})$ with $\operatorname{supp}\psi \subset \overline{\Omega} \times [0,T]$ for some $T > 0$, as well as

$$\big(\mathcal{X}_{E_t}, \operatorname{div}(\mu\eta)\big)_\Omega = \big\langle \delta V^t, \eta\big\rangle := \int\limits_{\Omega\times G_{d-1}} (I - p\otimes p) : \nabla\eta(x)\,dV^t(x,p)$$

for all $\eta \in C^1_0(\Omega \times G_{d-1})$.

(5) Finally, for almost all $0 \leqslant s \leqslant t < \infty$

$$\frac{1}{2}\big\|v(t)\big\|^2_{L^2(\Omega)} + \lambda^t(\overline{\Omega}) + \int\limits_s^t\int\limits_\Omega \big(\nu(\mathcal{X}_{E_\tau})|Dv|^2 + m|\nabla\mu|^2\big)\,dx\,d\tau \leqslant \frac{1}{2}\big\|v(s)\big\|^2_{L^2(\Omega)} + \lambda^s(\overline{\Omega}). \tag{A.5}$$

Here and in the following f shall satisfy $f \in C^3(\mathbb{R})$, $f(s) \geqslant 0$ and $f(s) = 0$ if and only if $s = 0, 1$ as well as $f''(s) \geqslant c_0(1+|s|)^{p-2}$ if $s \geqslant 1 - c_0$ and if $s \leqslant c_0$ for some $c_0 > 0$, $p > 2$. Then $F(s) := f(\frac{s+1}{2})$ we will satisfy the assumption in [6]. We will even assume that $p \geqslant 3$, which we will need in the following to estimate $v_\varepsilon \cdot \nabla c_\varepsilon = \operatorname{div}(v_\varepsilon c_\varepsilon)$ uniformly in $L^2(0,T;H^{-1}(\Omega))$. One can choose e.g. $f(s) = s^2(1-s)^2$.

For the following we denote

$$e_\varepsilon(c) = \frac{\varepsilon}{2}|\nabla c|^2 + \varepsilon^{-1} f(c).$$

Theorem A.2. *Let $\Omega \subset \mathbb{R}^d$, $d = 2,3$, be a smooth bounded domain and let $\nu \in C^0(\mathbb{R})$ with $\nu(s) \geqslant \nu_0 > 0$ for all $s \in \mathbb{R}$. Moreover, let initial data $(v_{0,\varepsilon}, c_{0,\varepsilon}) \in L^2_\sigma(\Omega) \times H^1(\Omega)$ with $\frac{1}{|\Omega|}\int_\Omega c_{0,\varepsilon}\,dx = \bar{c} \in (-1,1)$ be given that satisfy*

$$\frac{1}{2}\int_\Omega |v_{0,\varepsilon}|^2\,dx + E_\varepsilon(c_{0,\varepsilon}) \leqslant R \tag{A.6}$$

uniformly in $\varepsilon \in (0,1]$ for some $R > 0$. Finally, let $(m_\varepsilon)_{\varepsilon\in(0,1]}$ with $m_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} m > 0$. Consider now (1.11)–(1.17) *with m replaced by m_ε and let for every $\varepsilon \in (0,1]$ $(v_\varepsilon, c_\varepsilon, \mu_\varepsilon)$ be weak solutions in the sense of* [1, *Definition* 1]. *Then there is a subsequence $(\varepsilon_k)_{k\in\mathbb{N}}$ such that $\varepsilon_k \searrow 0$ as $k \to \infty$ such that*:

(1) *There are measurable sets $E \subset \Omega \times [0,\infty)$ and $E_0 \subset \Omega$ such that $c_{\varepsilon_k} \underset{k\to\infty}{\longrightarrow} \mathcal{X}_E$ almost everywhere in $\Omega \times [0,\infty)$ and in $C^{\frac{1}{9}}([0,T];L^2(\Omega))$ for all $T > 0$ as well as $c_{0,\varepsilon_k} \underset{k\to\infty}{\longrightarrow} -1 + 2\mathcal{X}_{E_0}$ almost everywhere in Ω and $\mathcal{X}_{E_0} = \mathcal{X}_E|_{t=0}$ in $L^2(\Omega)$.*

(2) *There are $\mu \in L^2_{\mathrm{loc}}([0,\infty);H^1(\Omega))$, $v \in L^2(0,\infty;H^1_0(\Omega))$, $v_0 \in L^2_\sigma(\Omega)$ such that*

$$\mu_{\varepsilon_k} \underset{k\to\infty}{\rightharpoonup} \mu \quad \text{in } L^2\big(0,T;H^1(\Omega)\big) \text{ for all } T > 0, \tag{A.7}$$

$$v_{\varepsilon_k} \underset{k\to\infty}{\rightharpoonup} v \quad \text{in } L^2\big(0,\infty;H^1(\Omega)\big), \tag{A.8}$$

$$v_{0,\varepsilon_k} \underset{k\to\infty}{\rightharpoonup} v_0 \quad \text{in } L^2_\sigma(\Omega). \tag{A.9}$$

(3) *There exist Radon measures λ and λ_{ij}, $i,j = 1,\dots,n$ on $\overline{\Omega} \times [0,\infty)$ such that for every $T > 0$, $i,j = 1,\dots,n$*

$$e_{\varepsilon_k}(c_{\varepsilon_k})\,dx\,dt \underset{k\to\infty}{\rightharpoonup^*} \lambda \quad \text{in } \mathcal{M}\big(\overline{\Omega} \times [0,T]\big), \tag{A.10}$$

$$\varepsilon_k \partial_{x_i} c_{\varepsilon_k} \partial_{x_j} c_{\varepsilon_k}\,dx\,dt \underset{k\to\infty}{\rightharpoonup^*} \lambda_{ij} \quad \text{in } \mathcal{M}\big(\overline{\Omega} \times [0,T]\big). \tag{A.11}$$

(4) *There exists a Radon $V = V^t\,dt$ on $\Omega \times G_{d-1} \times (0,\infty)$ such that (v, E, μ, V) is a varifold solution of* (1.1)–(1.10) *with initial values (v_0, E_0) in the sense of Definition* A.1*, where*

$$\int_0^T \langle \delta V^t, \eta\rangle\,dt = \int_0^T \int_\Omega \nabla\eta : \big(d\lambda I - (d\lambda_{ij})_{i,j=1}^d\big)\,dt \tag{A.12}$$

for all $\eta \in C^1_0(\Omega \times [0,T];\mathbb{R}^d)$.

First of all, by the definition of weak solutions in [1], we have

$$\int_Q \big(-v_\varepsilon \partial_t \psi + v_\varepsilon \cdot \nabla v_\varepsilon \psi + \nu(c_\varepsilon) Dv_\varepsilon : D\psi\big)\,d(x,t) - \int_\Omega v_0 \psi|_{t=0}\,dx = -\int_Q c_\varepsilon \nabla\mu_\varepsilon \cdot \psi\,d(x,t) \tag{A.13}$$

for all $\psi \in C^\infty([0,\infty) \times \Omega)^d$ with $\operatorname{div}\psi = 0$ and $\psi(t) = 0$ for $t \geqslant T$ for some $T > 0$, as well as

$$m_\varepsilon \int_Q \nabla\mu_\varepsilon \cdot \nabla\varphi\,d(x,t) = \int_Q c_\varepsilon\big(\partial_t\varphi + \operatorname{div}(\varphi v_\varepsilon)\big)\,d(x,t) + \int_\Omega c_{0,\varepsilon}\varphi|_{t=0}\,dx, \tag{A.14}$$

$$\int_Q \mu_\varepsilon \varphi\,d(x,t) = \int_Q f(c_\varepsilon)\varphi\,d(x,t) + \int_Q \nabla c_\varepsilon \cdot \nabla\varphi\,d(x,t) \tag{A.15}$$

for all $\varphi \in C^\infty([0,\infty)\times\overline{\Omega})$ with $\operatorname{supp}\varphi \subset \overline{\Omega}\times[0,T]$ for some $T>0$. Moreover, from the energy inequality in [1, Definition 1] we obtain

$$\frac{1}{2}\|v_\varepsilon(t)\|^2_{L^2(\Omega)} + E_\varepsilon\big(c_\varepsilon(t)\big) + \int_0^t\int_\Omega \big(\nu(c_\varepsilon)|Dv_\varepsilon|^2 + m_\varepsilon|\nabla\mu_\varepsilon|^2\big)\,dx\,dt \leqslant R$$

for all $t \in [0,\infty)$. Therefore we have

$$\|\mu_\varepsilon(\cdot,t)\|_{H^1(\Omega)} \leqslant C\big(E_\varepsilon\big(c_\varepsilon(t)\big) + \|\nabla\mu_\varepsilon(\cdot,t)\|_{L^2(\Omega)}\big) \tag{A.16}$$

for all $t>0$ and $0<\varepsilon\leqslant\varepsilon_0$ for some $C,\varepsilon_0>0$ due to [6, Lemma 3.4].

Hence there exists a subsequence $\varepsilon_k \searrow 0$ as $k\to\infty$ such that (A.7)–(A.9) holds. Moreover, using (1.11) and the lemma by Aubin–Lions, one easily derives that $v_{\varepsilon_k} \underset{k\to\infty}{\longrightarrow} v$ strongly in $L^2(\Omega\times(0,T))$ for all $T>0$ and $v_{\varepsilon_k}(t) \underset{k\to\infty}{\longrightarrow} v(t)$ strongly in $L^2(\Omega)$ for almost every $t\geqslant 0$.

Using the assumptions on f we further deduce that

$$\int_\Omega |c_\varepsilon(t)|^p\,dx \leqslant C(1+R), \qquad \int_\Omega \operatorname{dist}\big(c,\{0,1\}\big)^2\,dx \leqslant C\varepsilon R$$

uniformly in $\varepsilon\in(0,1]$, for almost every $0<t<\infty$ and $t=0$. Now we define as in [6]

$$w_\varepsilon(x,t) = W\big(c_\varepsilon(x,t)\big) \quad \text{where } W(c) = \int_0^c \sqrt{2\tilde f(s)}\,ds,\ \ \tilde f(s) = \min\big(f(s), 1+|s|^2\big).$$

Then $(\nabla w_\varepsilon)_{\varepsilon\in(0,1]}$ is uniformly bounded in $L^\infty(0,\infty;L^1(\Omega))$ since

$$\int_\Omega |\nabla w_\varepsilon(x,t)|\,dx = \int_\Omega \sqrt{2\tilde f(s)}\,|\nabla c_\varepsilon(x,t)|\,dx \leqslant \int_\Omega e_\varepsilon\big(c_\varepsilon(t)\big)\,dx \leqslant R. \tag{A.17}$$

Moreover, we have for all $u_1,u_2\in\mathbb{R}$

$$c_1|u_1-u_2|^2 \leqslant |W(u_1)-W(u_2)| \leqslant c_2|u_1-u_2|^2\big(1+|u_1|+|u_2|\big) \tag{A.18}$$

which again follows easily from the assumptions on f. Now we obtain:

Lemma A.3. *There is some C independent of $\varepsilon\in(0,1]$ such that*

$$\|w_\varepsilon\|_{C^{\frac{1}{8}}([0,\infty);L^1(\Omega))} + \|c_\varepsilon\|_{C^{\frac{1}{8}}([0,\infty);L^2(\Omega))} \leqslant C.$$

Proof. The proof is a modification of [6, Lemma 3.2]. Therefore we only give a brief presentation, describing the differences.

For sufficiently small $\eta>0$ let

$$c_\varepsilon^\eta(x,t) = \int_{B_1} \rho(y)c_\varepsilon(x-\eta y,t)\,dy, \quad x\in\Omega,\ t\geqslant 0,$$

where ρ is a standard mollifying kernel and c_ε is extended in an η_0-neighborhood of Ω as in [6, Proof of Lemma 3.2]. Then one obtains

$$\|\nabla c_\varepsilon^\eta(\cdot,t)\|_{L^2(\Omega)} \leqslant C\eta^{-1}\|c_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \leqslant C'\eta^{-1},$$
$$\|c_\varepsilon^\eta(\cdot,t) - c_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)} \leqslant C\eta\|\nabla w_\varepsilon(\cdot,t)\|_{L^1(\Omega)} \leqslant C'\eta$$

for all sufficiently small $\eta>0$, cf. [6, Proof of Lemma 3.2]. Next we use that

$$\big(c_\varepsilon(\cdot,t)-c_\varepsilon(\cdot,\tau),\varphi\big)_\Omega=-\int_\tau^t\big((\nabla\mu-v_\varepsilon c_\varepsilon)(s),\nabla\varphi\big)_\Omega\,ds$$

for all $\varphi\in C^1(\overline{\Omega})$ because of (1.13) in its weak form. Here

$$\|\nabla\mu-v_\varepsilon c_\varepsilon\|_{L^2(\Omega\times(\tau,t))}\leqslant C(R)\quad\text{for all }0\leqslant\tau\leqslant t<\infty,\ |t-\tau|\leqslant 1$$

since $v_\varepsilon\in L^2(0,\infty;L^6(\Omega))$ and $c_\varepsilon\in L^\infty(0,\infty;L^3(\Omega))$ are bounded due to $d\leqslant 3$ and $p\geqslant 3$. Hence

$$\begin{aligned}\int_\Omega\big(c_\varepsilon^\eta(x,t)-c_\varepsilon^\eta(x,\tau)\big)\big(c_\varepsilon(x,t)-c_\varepsilon(x,\tau)\big)\,dx&=-\int_\tau^t\big((\nabla\mu-v_\varepsilon c_\varepsilon)(s),\nabla c_\varepsilon^\eta(x,t)-\nabla c_\varepsilon^\eta(x,\tau)\big)_\Omega\,ds\\&\leqslant C(R)(t-\tau)^{\frac12}\sup_{s\in[0,\infty)}\big\|\nabla c_\varepsilon^\eta(\cdot,s)\big\|_{L^2(\Omega)}\\&\leqslant C(T,R)\eta^{-1}(t-\tau)^{\frac12}\end{aligned}$$

for all $0\leqslant\tau\leqslant t<\infty$, $|t-\tau|\leqslant 1$. Now, choosing $\eta=\min(\eta_0,(t-\tau)^{\frac14})$, we conclude

$$\big\|c_\varepsilon(\cdot,t)-c_\varepsilon(\cdot,\tau)\big\|^2_{L^2(\Omega)}\leqslant C|t-\tau|^{\frac14}\quad\text{for all }|t-\tau|\leqslant 1$$

from the previous estimates. Thus $c_\varepsilon\in C^{\frac18}([0,\infty);L^2(\Omega))$. Using (A.18), one derives $w_\varepsilon\in C^{\frac18}([0,\infty);L^1(\Omega))$ as in [6]. □

Note that the previous lemma and (A.17) implies that $(w_\varepsilon)_{0<\varepsilon\leqslant 1}$ is bounded in $L^\infty(0,\infty;BV(\Omega))$.

Lemma A.4. *There is a subsequence $\varepsilon_k\searrow 0$ as $k\to\infty$, a measurable function $\mathcal{E}(t)$, $t\in(0,\infty)$ and a measurable set $E\subset\Omega\times(0,\infty)$ such that*

(1) *$E_{\varepsilon_k}(c_{\varepsilon_k}(t))\to\mathcal{E}(t)$ for almost all $t\geqslant 0$.*
(2) *$w_{\varepsilon_k}\to\kappa\mathcal{X}_E$ almost everywhere in $\Omega\times(0,\infty)$ and in $C^{\frac19}([0,T];L^1(\Omega))$ for all $T>0$.*
(3) *$c_{\varepsilon_k}\to\mathcal{X}_E$ almost everywhere in $\Omega\times(0,\infty)$ and in $C^{\frac19}([0,T];L^2(\Omega))$ for all $T>0$.*

In addition, $\mathcal{X}_E\in L^\infty_{\omega*}(0,\infty;BV(\Omega))\cap C^{\frac14}([0,\infty);L^1(\Omega))$ and $E_t:=\{x\in\Omega\colon (x,t)\in E\}$ satisfies $|E_t|=|E_0|=\frac{1+\bar c}{2}|\Omega|$ for almost all $t\geqslant 0$ and

$$|\nabla\mathcal{X}_{E_t}|(\Omega)\leqslant\frac1\kappa\mathcal{E}(t)\leqslant\frac1\kappa\mathcal{E}_0.$$

Proof. The main difference to the proof of [6, Lemma 3.3] is the proof of convergence for $E_{\varepsilon_k}(c_{\varepsilon_k}(t))$. To this end one uses that

$$F_\varepsilon(t):=\frac12\big\|v_\varepsilon(t)\big\|^2_{L^2(\Omega)}+E_\varepsilon\big(c_\varepsilon(t)\big),\quad t\geqslant 0,$$

is a sequence of bounded, monotone decreasing functions and $v_{\varepsilon_k}(t)\underset{k\to\infty}{\longrightarrow}v(t)$ for almost all $t\geqslant 0$ in $L^2(\Omega)$. The rest of the proof is identical with the proof of [6, Lemma 3.3]. □

Using the previous statements one can now easily finish the proof of Theorem A.2 by the arguments of [6, Section 3.5]. In particular, (A.3) and (A.4) easily follow from (A.13) and (A.14). It mainly remains to show (A.12) and (A.5). Let $\lambda,\lambda_{i,j}$ be as in (A.10)–(A.11). To show the energy estimate (A.5) one uses that $d\lambda=d\lambda^t\,dt$ for some Radon measures λ^t on $\overline{\Omega}$ and that for almost every $0\leqslant t\leqslant s<\infty$

$$
\begin{aligned}
\lambda^t(\overline{\Omega}) &= \lim_{k\to\infty} E_{\varepsilon_k}\big(c_{\varepsilon_k}(t)\big) \\
&\leqslant \lim_{k\to\infty}\left(E_{\varepsilon_k}\big(c_{\varepsilon_k}(s)\big) - \int_s^t\int_\Omega \big(\nu(c_{\varepsilon_k})|Dv_{\varepsilon_k}|^2 + m_{\varepsilon_k}|\nabla\mu_{\varepsilon_k}|^2\big)\,dx\,d\tau\right) \\
&\quad + \lim_{k\to\infty}\left(\frac{1}{2}\big\|v_{\varepsilon_k}(s)\big\|^2_{L^2(\Omega)} - \frac{1}{2}\big\|v_{\varepsilon_k}(t)\big\|^2_{L^2(\Omega)}\right) \\
&\leqslant \lambda^t(\overline{\Omega}) - \int_s^t\int_\Omega \big(\nu(\mathcal{X}_E)|Dv|^2 + m|\nabla\mu|^2\big)\,dx\,d\tau + \frac{1}{2}\big\|v(s)\big\|^2_{L^2(\Omega)} - \frac{1}{2}\big\|v(t)\big\|^2_{L^2(\Omega)},
\end{aligned}
$$

where we have used the weak convergence of Dv_{ε_k}, $\nabla\mu_{\varepsilon_k}$ in $L^2(Q)$ and the strong convergence of $v_{\varepsilon_k}(t)$ in $L^2(\Omega)$ for almost all $t \geqslant 0$. Finally, (A.12) follows in precisely the same way as in [6, Section 3.5], where we note that the main point in the argumentation is that the discrepancy measure

$$\xi^\varepsilon(c_\varepsilon) := \frac{\varepsilon}{2}|\nabla c_\varepsilon|^2 - \frac{1}{\varepsilon}f(c_\varepsilon)$$

converges to a non-positive measure. The latter fact follows from [6, Section 3.5] since we have the same bounds on μ_ε in $H^1(\Omega)$.

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