Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN 26 (2009) 2457–2479

ANNALES
DE L'INSTITUT
HENRI
POINCARÉ
ANALYSE
NON LINÉAIRE

www.elsevier.com/locate/anihpc

Homogenization of p-Laplacian in perforated domain

B. Amaziane [a,*], S. Antontsev [b], L. Pankratov [a,c], A. Piatnitski [d,e]

[a] *Laboratoire de Mathématiques et leur Applications, CNRS-UMR5142, Université de Pau, Av. de l'Université, 64000 Pau, France*
[b] *CMAF, Universidade de Lisboa, Av. Prof. Gama Pinto, 2, 1649-003, Lisboa, Portugal*
[c] *Department of Mathematics, B.Verkin Institut for Low Temperature Physics and Engineering, 47, av. Lenin, 61103, Kharkov, Ukraine*
[d] *Narvik University College, Postbox 385, Narvik, 8505, Norway*
[e] *Lebedev Physical Institute RAS, leninski prospect 53, Moscow, 119991, Russia*

Received 4 February 2007; accepted 10 June 2009

Abstract

We study the homogenization of the following nonlinear Dirichlet variational problem:

$$\inf\left\{\int\limits_{\Omega^\varepsilon}\left\{\frac{1}{p_\varepsilon(x)}|\nabla u|^{p_\varepsilon(x)}+\frac{1}{p_\varepsilon(x)}|u|^{p_\varepsilon(x)}-f(x)u\right\}dx\colon\ u\in W_0^{1,p_\varepsilon(\cdot)}\left(\Omega^\varepsilon\right)\right\}$$

in a perforated domain $\Omega^\varepsilon=\Omega\setminus\mathcal{F}^\varepsilon\subset\mathbb{R}^n$, $n\geqslant 2$, where ε is a small positive parameter that characterizes the scale of the microstructure. The non-standard exponent $p_\varepsilon(x)$ is assumed to be an oscillating continuous function in $\bar{\Omega}$ such that, for any $\varepsilon>0$, $1<p_\varepsilon(x)\leqslant n$ in Ω; for any $x,y\in\Omega$, $|p_\varepsilon(x)-p_\varepsilon(y)|\leqslant\omega_\varepsilon(|x-y|)$ with $\overline{\lim}_{\tau\to 0}\,\omega_\varepsilon(\tau)\ln(1/\tau)=0$; and converges uniformly in Ω to a function p_0 which satisfies the same properties. Moreover, we assume that $p_\varepsilon(x)\geqslant p_0(x)$ in Ω. Denoting u^ε a minimizer in the above variational problem, without any periodicity assumption, for a large range of perforated domains we find, by means of the variational homogenization technique, the global behavior of u^ε as ε tends to zero. It is shown that u^ε extended by zero in $\mathcal{F}^\varepsilon$, converges weakly in $W^{1,p_0(\cdot)}(\Omega)$ to the solution of the following nonlinear variational problem:

$$\min\left\{\int\limits_{\Omega}\left\{\frac{1}{p_0(x)}|\nabla u|^{p_0(x)}+\frac{1}{p_0(x)}|u|^{p_0(x)}+c(x,u)-f(x)u\right\}dx\colon\ u\in W_0^{1,p_0(\cdot)}(\Omega)\right\},$$

where the function $c(x,u)$ is defined in terms of the local characteristic of Ω^ε. This result is then illustrated with a periodic and a non-periodic examples.

Résumé

Nous étudions l'homogénéisation du problème variationnel de Dirichlet nonlinéaire suivant :

$$\inf\left\{\int\limits_{\Omega^\varepsilon}\left\{\frac{1}{p_\varepsilon(x)}|\nabla u|^{p_\varepsilon(x)}+\frac{1}{p_\varepsilon(x)}|u|^{p_\varepsilon(x)}-f(x)u\right\}dx\colon\ u\in W_0^{1,p_\varepsilon(\cdot)}\left(\Omega^\varepsilon\right)\right\}$$

---

[*] Corresponding author.
E-mail addresses: brahim.amaziane@univ-pau.fr (B. Amaziane), antontsevsn@mail.ru (S. Antontsev), leonid.pankratov@univ-pau.fr (L. Pankratov), andrey@sci.lebedev.ru (A. Piatnitski).

0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2009.06.004

dans un domaine perforé $\Omega^\varepsilon = \Omega \setminus \mathcal{F}^\varepsilon \subset \mathbb{R}^n$, $n \geqslant 2$, où $\varepsilon > 0$ est un petit paramètre qui caractérise la taille des perforations. La fonction puissance $p_\varepsilon(x)$ est nonstandard et supposée être une fonction continue et oscillante dans $\bar{\Omega}$. Elle vérifie, pour tout $\varepsilon > 0$, $1 < p_\varepsilon(x) \leqslant n$ dans Ω, pour tout $x, y \in \Omega$, $|p_\varepsilon(x) - p_\varepsilon(y)| \leqslant \omega_\varepsilon(|x-y|)$ avec $\overline{\lim}_{\tau\to 0}\, \omega_\varepsilon(\tau)\ln(1/\tau) = 0$; et elle est uniformément convergente dans Ω vers une fonction p_0 qui vérifie les mêmes propriétés. De plus, on suppose que $p_\varepsilon(x) \geqslant p_0(x)$ dans Ω. On note u^ε une solution du problème de minimisation variationnel ci-dessus, sans hypothèse de périodicité et pour différents milieux perforés, on trouve le problème limite décrivant le comportement global de u^ε lorsque ε tend vers zéro, en utilisant la technique de l'homogénéisation variationnelle. On montre que u^ε, prolongée par zéro dans $\mathcal{F}^\varepsilon$, converge faiblement dans $W^{1,p_0(\cdot)}(\Omega)$, quand ε tend vers zéro, vers la solution u du problème variationel nonlinéaire suivant :

$$\min\left\{ \int_\Omega \left\{ \frac{1}{p_0(x)}|\nabla u|^{p_0(x)} + \frac{1}{p_0(x)}|u|^{p_0(x)} + c(x,u) - f(x)u \right\} dx\colon\ u \in W_0^{1,p_0(\cdot)}(\Omega) \right\},$$

où la fonction $c(x,u)$ est définie à partir des caractéristiques géométriques locales du domaine Ω^ε. Enfin, nous présentons deux exemples, un périodique et l'autre nonpériodique, pour illustrer les résultats obtenus.

MSC: 35B40; 35J60; 46E35; 74Q05; 76M50

Keywords: Homogenization; Nonlinear; Non-standard growth

---

1. Introduction

In this paper we study the homogenization of the following nonlinear problem:

$$-\operatorname{div}\left(\left|\nabla u^\varepsilon\right|^{p_\varepsilon(x)-2}\nabla u^\varepsilon\right) + \left|u^\varepsilon\right|^{p_\varepsilon(x)-2}u^\varepsilon = f(x) \quad \text{in } \Omega^\varepsilon,\ u^\varepsilon \in W_0^{1,p_\varepsilon(\cdot)}\left(\Omega^\varepsilon\right), \tag{1.1}$$

where ε is a small positive parameter, $\Omega^\varepsilon = \Omega \setminus \mathcal{F}^\varepsilon$ is a perforated domain in $\mathbb{R}^n$ ($n \geqslant 2$) with Ω being a bounded Lipschitz domain, and p_ε is a smooth positive oscillating function in Ω satisfying some conditions which will be specified in Section 3, and uniformly converging in Ω to a smooth function p_0. f is a given function. Equations of such type are called $p_\varepsilon(x)$-Laplacian equations with non-standard growth conditions.

In recent years, there has been an increasing interest in the study of such equations (in the case where there is no dependence on the small parameter) motivated by their applications to the mathematical modeling in continuum mechanics. These equations arise, for example, from the modeling of non-Newtonian fluids with thermo-convective effects (see, e.g., [7,9]), the modeling of electro-rheological fluids (see, e.g., [30,31]), the thermistor problem (see, e.g., [39]), the problem of image recovery (see, e.g., [24]), and the motion of a compressible fluid in a heterogeneous anisotropic porous medium obeying to the nonlinear Darcy law (see, e.g., [8,11]).

Eq. (1.1) is an idealized model for a variety of interesting physical problems; we motivate our work by describing one of them. We consider a steady flow of a compressible barotropic gaz through a porous medium. The nonlinear Darcy law with the continuity equation lead to the equation given by [10]

$$-\operatorname{div}\left(K(x)|\nabla u|^{p(x)-2}\nabla u\right) + R(x)|u|^{p(x)-2}u = f(x,t). \tag{1.2}$$

u stands for the fluid pressure, f is a source term and K, p, R are characteristic functions of the heterogeneous porous medium. For more details on the formulation of such problems see for instance [10,13]. We refer to [10,11,17,18] and the references therein for a detailed analysis of such equations.

In the present paper we deal with the Dirichlet boundary value problem for the nonlinear equation (1.1). More precisely, we consider the corresponding variational problem:

$$\inf\left\{ \int_{\Omega^\varepsilon} \left\{ \frac{1}{p_\varepsilon(x)}|\nabla u|^{p_\varepsilon(x)} + \frac{1}{p_\varepsilon(x)}|u|^{p_\varepsilon(x)} - f(x)u \right\} dx\colon\ u \in W_0^{1,p_\varepsilon(\cdot)}\left(\Omega^\varepsilon\right) \right\}. \tag{1.3}$$

The homogenization of the Dirichlet boundary value problem was studied for the first time in [25] and then it was revisited by many authors (see, e.g., [12,15,16,20,26,33], and the references therein). Note also that the homogenization of nonlinear elliptic equations is a long-standing problem and a number of methods have been developed. There is an extensive literature on this subject. We will not attempt a review of the literature here, but merely mention a few references, see for instance [2,14,16,29], and the references therein. Let us mention that the homogenization problems for the Lagrangians with variable exponents were first studied in [22,34–37] (see also the book [38]) which focus on the variational functionals with non-standard growth conditions. In particular, the homogenization and Γ-convergence problems for Lagrangians with variable rapidly oscillating exponents $p(x)$ were considered in [35,36]. Variational functionals with non-standard growth conditions have also been considered in the book [14], namely Chapter 21 of this book focuses on the Γ-convergence of such functionals in L^p spaces. The Dirichlet homogenization problem and related questions for Lagrangians of $p_\varepsilon(x)$ growth in $W^{1,p_\varepsilon(\cdot)}(\Omega^\varepsilon)$, where Ω^ε is a perforated domain, have been studied recently in [3–6].

Following the approach developed in [20], instead of a classical periodicity assumption on the structure of the perforated domain Ω^ε, we impose certain conditions on the so-called *local energy characteristics* associated with the boundary value problem (1.1). It will be shown that the asymptotic behavior, as $\varepsilon \to 0$, of the solution u^ε is described by the following variational problem:

$$\inf\left\{ \int_\Omega \left\{ \frac{1}{p_0(x)}|\nabla u|^{p_0(x)} + \frac{1}{p_0(x)}|u|^{p_0(x)} + c(x,u) - f(x)u \right\} dx\colon\ u \in W_0^{1,p_0(\cdot)}(\Omega) \right\}, \tag{1.4}$$

where the function $c(x,u)$ is calculated by the local energy characteristic of Ω^ε.

The proof of the main result is based on the variational homogenization technique which is nowadays widely used in the homogenization theory (see, e.g., [14,26,38] and the references therein). Let us also mention that another non-periodic homogenization approach was proposed recently in [28] for nonlinear monotone operators.

The paper is organized as follows. In Section 2, for the sake of completeness, we recall the definition and the main results on the Lebesgue and Sobolev spaces with variable exponents which will be used in the sequel. In Section 3 we state the problem and formulate the main result which will be proved in Section 4. Two examples of periodic and locally periodic structures are considered in Section 5.

2. Sobolev spaces with variable exponents

In this section we introduce the function spaces used throughout the paper and describe their basic properties, see for instance [19,21,27,32].

We assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ and the function $p(x)$ satisfies the following conditions:

$$1 < p^{(-)} = \inf_\Omega p(x) \leqslant p(x) \leqslant \sup_\Omega p(x) = p^{(+)} < +\infty \quad \text{with } p^{(+)} \leqslant n. \tag{2.1}$$

For all $x, y \in \Omega$,

$$\left|p(x) - p(y)\right| \leqslant \omega\left(|x-y|\right) \quad \text{with } \varlimsup_{\tau\to 0} \omega(\tau)\ln\left(\frac{1}{\tau}\right) = 0. \tag{2.2}$$

1. By $L^{p(\cdot)}(\Omega)$ we denote the space of measurable functions f in Ω such that
$$A_{p(\cdot)}(f) = \int_\Omega \left|f(x)\right|^{p(x)} dx < +\infty.$$
The space $L^{p(\cdot)}(\Omega)$ equipped with the norm
$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0\colon\ A_{p(\cdot)}\left(\frac{f}{\lambda}\right) \leqslant 1\right\} \tag{2.3}$$
becomes a Banach space.
2. The following inequalities hold
$$\begin{cases} \min\left(\|f\|^{p^{(-)}}_{L^{p(\cdot)}(\Omega)}, \|f\|^{p^{(+)}}_{L^{p(\cdot)}(\Omega)}\right) \leqslant A_{p(\cdot)}(f) \leqslant \max\left(\|f\|^{p^{(-)}}_{L^{p(\cdot)}(\Omega)}, \|f\|^{p^{(+)}}_{L^{p(\cdot)}(\Omega)}\right), \\ \min\left(A_{p(\cdot)}^{\frac{1}{p^{(-)}}}, A_{p(\cdot)}^{\frac{1}{p^{(+)}}}\right) \leqslant \|f\|_{L^{p(\cdot)}(\Omega)} \leqslant \max\left(A_{p(\cdot)}^{\frac{1}{p^{(-)}}}, A_{p(\cdot)}^{\frac{1}{p^{(+)}}}\right). \end{cases} \tag{2.4}$$

3. Let $f \in L^{p(\cdot)}(\Omega)$, $g \in L^{q(\cdot)}(\Omega)$ with
$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1, \quad 1 < p^{(-)} \leqslant p(x) \leqslant p^{(+)} < \infty,\ 1 < q^{(-)} \leqslant q(x) \leqslant q^{(+)} < +\infty.$$
Then the Hölder's inequality holds
$$\int_{\Omega} |fg|\,dx \leqslant 2\|f\|_{L^{p(\cdot)}(\Omega)}\|g\|_{L^{q(\cdot)}(\Omega)}. \tag{2.5}$$
4. According to (2.5), for every $1 \leqslant q = \text{const} < p^{(-)} \leqslant p(x) < +\infty$
$$\|f\|_{L^q(\Omega)} \leqslant C\|f\|_{L^{p(\cdot)}(\Omega)} \quad \text{with the constant } C = 2\|1\|_{L^{\frac{p(\cdot)}{p(\cdot)-q}}(\Omega)}. \tag{2.6}$$
It is straightforward to check that for domains Ω such that $\operatorname{meas}\Omega < +\infty$,
$$\|1\|_{L^{p(\cdot)}(\Omega)} \leqslant 2\max\bigl\{[\operatorname{meas}\Omega]^{2/p^{(-)}}, [\operatorname{meas}\Omega]^{1/2p^{(+)}}\bigr\}. \tag{2.7}$$
5. The space $W^{1,p(\cdot)}(\Omega)$, $p(\cdot) \in [p^{(-)}, p^{(+)}] \subset]1, +\infty[$, is defined by
$$W^{1,p(\cdot)}(\Omega) = \bigl\{f \in L^{p(\cdot)}(\Omega)\colon |\nabla f| \in L^{p(\cdot)}(\Omega)\bigr\}.$$
If condition (2.2) is satisfied, $W_0^{1,\,p(\cdot)}(\Omega)$ is the closure of the set $C_0^\infty(\Omega)$ with respect to the norm of $W^{1,\,p(\cdot)}(\Omega)$. If the boundary of Ω is Lipschitz-continuous and $p(x)$ satisfies (2.2), then $C_0^\infty(\Omega)$ is dense in $W_0^{1,\,p(\cdot)}(\Omega)$. The norm in the space $W_0^{1,p(\cdot)}$ is defined by
$$\|u\|_{W_0^{1,p(\cdot)}} = \sum_i \|D_i u\|_{L^{p(\cdot)}(\Omega)} + \|u\|_{L^{p(\cdot)}(\Omega)}.$$
If the boundary of Ω is Lipschitz and $p \in C^0(\Omega)$, then the norm $\|\cdot\|_{W_0^{1,p(\cdot)}(\Omega)}$ is equivalent to the norm
$$\widetilde{\|u\|}_{W_0^{1,p(x)}(\Omega)} = \sum_i \|D_i u\|_{L^{p(\cdot)}(\Omega)}. \tag{2.8}$$
6. If $p \in C^0(\overline{\Omega})$, then $W^{1,p(\cdot)}(\Omega)$ is separable and reflexive.
7. If $p, q \in C^0(\overline{\Omega})$,
$$p_*(x) = \begin{cases} \frac{p(x)n}{n-p(x)} & \text{if } p(x) < n, \\ +\infty & \text{if } p(x) > n, \end{cases} \quad \text{and} \quad 1 < q(x) \leqslant \sup_{\Omega} q(x) < \inf_{\Omega} p_*(x),$$
then the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and compact.
8. Friedrich's inequality is valid in the following form: if $p(x)$ satisfies conditions (2.1)–(2.2), then there exists a constant $C > 0$ such that for every $f \in W_0^{1,p(\cdot)}(\Omega)$
$$\|f\|_{L^{p(\cdot)}(\Omega)} \leqslant C\|\nabla f\|_{L^{p(\cdot)}(\Omega)}. \tag{2.9}$$

3. Statement of the problem and the main result

Let Ω be a bounded domain in $\mathbb{R}^n$ ($n \geqslant 2$) with sufficiently smooth boundary. Let $\mathcal{F}^\varepsilon$ be a closed subset in Ω. Here ε is a small parameter characterizing the scale of the microstructure. We assume that $\mathcal{F}^\varepsilon$ is distributed in an asymptotically regular way in Ω, i.e., for any ball $V(y, r)$ of radius r centered at $y \in \Omega$ and $\varepsilon > 0$ small enough ($\varepsilon \leqslant \varepsilon_0(r)$), $V(y,r) \cap \mathcal{F}^\varepsilon \neq \emptyset$ and $V(y,r) \cap (\Omega \setminus \mathcal{F}^\varepsilon) \neq \emptyset$. We set
$$\Omega^\varepsilon = \Omega \setminus \mathcal{F}^\varepsilon. \tag{3.1}$$

Let $p_\varepsilon = p_\varepsilon(x)$ be a continuous function defined in the domain $\overline{\Omega}$. We assume that, for any $\varepsilon > 0$, it satisfies the following conditions:

(i) this function is bounded in the following sense:

$$1 < \mathsf{p}^{(-)} \leqslant p_\varepsilon^{(-)} \equiv \min_{x\in\overline{\Omega}} p_\varepsilon(x) \leqslant p_\varepsilon(x) \leqslant \max_{x\in\overline{\Omega}} p_\varepsilon(x) \equiv p_\varepsilon^{(+)} \leqslant \mathsf{p}^{(+)} \leqslant n \quad \text{in } \overline{\Omega}; \tag{3.2}$$

(ii) for any $x, y \in \Omega$, we have

$$\bigl|p_\varepsilon(x) - p_\varepsilon(y)\bigr| \leqslant \omega_\varepsilon\bigl(|x-y|\bigr) \quad \text{with } \overline{\lim_{\tau\to 0}}\, \omega_\varepsilon(\tau)\ln\left(\frac{1}{\tau}\right) = 0; \tag{3.3}$$

(iii) the function p_ε converges uniformly in Ω to a function p_0, i.e.,

$$\lim_{\varepsilon\to 0} \|p_\varepsilon - p_0\|_{C^0(\overline{\Omega})} = 0, \tag{3.4}$$

where the limit function p_0 is assumed to be bounded in the sense of the condition (2.1) and satisfies (2.2);

(iv) the function p_ε is such that

$$p_\varepsilon(x) \geqslant p_0(x) \quad \text{in } \Omega. \tag{3.5}$$

We consider the following variational problem:

$$\begin{gathered} \min\bigl\{J^\varepsilon[u]:\ u \in W_0^{1,p_\varepsilon(\cdot)}\bigl(\Omega^\varepsilon\bigr)\bigr\}, \\ J^\varepsilon[u] = \int\limits_{\Omega^\varepsilon} \left\{ \frac{1}{p_\varepsilon(x)}|\nabla u|^{p_\varepsilon(x)} + \frac{1}{p_\varepsilon(x)}|u|^{p_\varepsilon(x)} - f(x)u \right\} dx, \end{gathered} \tag{3.6}$$

where $f \in C^1(\Omega)$. It is known from [1,10,11,17] that, for each $\varepsilon > 0$, there exists a unique solution (minimizer) $u^\varepsilon \in W^{1,p_\varepsilon(\cdot)}(\Omega^\varepsilon)$ of problem (3.6). Let us extend u^ε in $\mathcal{F}^\varepsilon$ by zero (keeping for it the same notation). Then we obtain the family $\{u^\varepsilon\} \subset W^{1,p_\varepsilon(\cdot)}(\Omega)$. We study the asymptotic behavior of u^ε as $\varepsilon \to 0$.

Instead of the classical periodicity assumption on the microstructure of the perforated domain Ω^ε, we impose certain conditions on the local energy characteristic of the set $\mathcal{F}^\varepsilon$. To this end we introduce K_h^z an open cube centered at $z \in \Omega$ with length equal to h $(0 < \varepsilon \ll h < 1)$ and we set

$$c^{\varepsilon,h}(z,b) = \inf_{v^\varepsilon} \int\limits_{K_h^z} \left\{ \frac{1}{p_\varepsilon(x)}\bigl|\nabla v^\varepsilon\bigr|^{p_\varepsilon(x)} + h^{-\mathsf{p}^{(+)}-\gamma}\mathfrak{S}\bigl(v^\varepsilon - b\bigr) \right\} dx, \tag{3.7}$$

where $\gamma > 0$,

$$\mathfrak{S}\bigl(v^\varepsilon - b\bigr) = \bigl|v^\varepsilon - b\bigr|^{p_\varepsilon(x)} + \bigl|v^\varepsilon - b\bigr|^{p_0(x)}, \tag{3.8}$$

and the infimum is taken over $v^\varepsilon \in W^{1,p_\varepsilon(\cdot)}(\Omega)$ that equal zero in $\mathcal{F}^\varepsilon$. We assume that:

(C.1) there exists a continuous function $c(x,b)$ such that for any $x \in \Omega$, and any $b \in \mathbb{R}$, and a certain $\gamma = \gamma_0 > 0$,

$$\lim_{h\to 0} \overline{\lim_{\varepsilon\to 0}}\, h^{-n} c^{\varepsilon,h}(z,b) = \lim_{h\to 0} \underline{\lim_{\varepsilon\to 0}}\, h^{-n} c^{\varepsilon,h}(z,b) = c(x,b);$$

(C.2) there exists a constant A independent of ε such that, for any $x \in \Omega$,

$$\lim_{h\to 0} \overline{\lim_{\varepsilon\to 0}}\, h^{-n} c^{\varepsilon,h}(z,b) \leqslant A\bigl(1 + |b|^{p_0(x)}\bigr).$$

The examples of the functions $p_\varepsilon(x)$ and the domains Ω^ε which satisfy all the above conditions, will be given in Section 5.

The main result of the paper is the following.

Theorem 3.1. *Let conditions* (i)–(iv) *on the function p_ε and conditions* (C.1)–(C.2) *on the local characteristic be satisfied. Then u^ε the solution* (*minimizer*) *of the variational problem* (3.6) (*extended by zero in $\mathcal{F}^\varepsilon$*) *converges weakly in $W^{1,p_0(\cdot)}(\Omega)$ to u the solution* (*minimizer*) *of*

$$\inf\left\{\int\limits_{\Omega}\left\{\frac{1}{p_0(x)}|\nabla u|^{p_0(x)}+\frac{1}{p_0(x)}|u|^{p_0(x)}+c(x,u)-f(x)u\right\}dx\colon\ u\in W_0^{1,p_0(\cdot)}(\Omega)\right\}. \tag{3.9}$$

Remark 1. The condition (C.1) and the definition of the local energy characteristic $c^{\varepsilon,h}(z,b)$ imply that $\operatorname{meas}[\mathcal{F}^\varepsilon\cap K_h^x]=o(h^n)$ for sufficiently small ε ($\varepsilon\leqslant\tilde{\varepsilon}(h)$), uniformly with respect to $z\in\Omega$. Therefore, $\operatorname{meas}\mathcal{F}^\varepsilon\to 0$ as $\varepsilon\to 0$.

Notation. In what follows C, C_1, C_2, etc. are generic constants independent of ε.

4. Proof of Theorem 3.1

It follows from (3.6), (2.4), and the regularity properties of the functions f, p_ε that

$$\|u^\varepsilon\|_{W^{1,p_\varepsilon(\cdot)}(\Omega^\varepsilon)}\leqslant C. \tag{4.1}$$

We extend u^ε by zero to the set $\mathcal{F}^\varepsilon$ and consider $\{u^\varepsilon\}$ as a sequence in the space $W^{1,p_\varepsilon(\cdot)}(\Omega)$. It follows from (4.1) that

$$\|u^\varepsilon\|_{W^{1,p_\varepsilon(\cdot)}(\Omega)}\leqslant C. \tag{4.2}$$

Condition (iv) and (4.2) immediately imply that

$$\|u^\varepsilon\|_{W^{1,p_0(\cdot)}(\Omega)}\leqslant C. \tag{4.3}$$

Hence, one can extract a subsequence $\{u^\varepsilon,\ \varepsilon=\varepsilon_k\to 0\}$ that converges weakly to a function $u\in W^{1,p_0(\cdot)}(\Omega)$. We will show that $u=u(x)$ is the solution of the variational problem (3.9). The proof will be done in two mains steps.

4.1. Step 1. Upper bound

Let $\{x^\alpha\}$ be a periodic grid in Ω with a period $h'=h-h^{1+\gamma/\mathsf{p}^{(+)}}$ ($\varepsilon\ll h\ll 1$, $0<\gamma<\mathsf{p}^{(+)}$). Let us cover the domain Ω by the cubes K_h^α of length $h>0$ centered at the points x^α. We associate with this covering a partition of unity $\{\varphi_\alpha\}$: $0\leqslant\varphi_\alpha(x)\leqslant 1$; $\varphi_\alpha(x)=0$ for $x\notin K_h^\alpha$; $\varphi_\alpha(x)=1$ for $x\in K_h^\alpha\setminus\bigcup_{\beta\neq\alpha}K_h^\beta$; $\sum_\alpha\varphi_\alpha(x)=1$ for $x\in\Omega$; $|\nabla\varphi_\alpha(x)|\leqslant Ch^{-1-\gamma/p^{(+)}}$.

Now let $v_\alpha^\varepsilon=v_\alpha^\varepsilon(x)$ be a function minimizing the functional (3.7)–(3.8) with $b=b_\alpha$ and $z=x^\alpha$, where b_α will be specified later. It follows from condition (C.1) that, as $h\to 0$,

$$\varlimsup_{\varepsilon\to 0}\int\limits_{K_h^\alpha\cap\Omega^\varepsilon}\frac{1}{p_\varepsilon(x)}\left|\nabla v_\alpha^\varepsilon\right|^{p_\varepsilon(x)}dx=O\left(h^n\right);\qquad \varlimsup_{\varepsilon\to 0}\int\limits_{K_h^\alpha\cap\Omega^\varepsilon}\mathfrak{S}\left(v_\alpha^\varepsilon-b_\alpha\right)dx=O\left(h^{n+\mathsf{p}^{(+)}+\gamma}\right). \tag{4.4}$$

Moreover, condition (iv) implies that

$$\varlimsup_{\varepsilon\to 0}\int\limits_{K_h^\alpha\cap\Omega^\varepsilon}\left|\nabla v_\alpha^\varepsilon\right|^{p_0(x)}dx=O\left(h^n\right)\quad\text{as } h\to 0. \tag{4.5}$$

Denote by $K_{h'}^\alpha$ and Π_h^α the cube of length h' centered at the point x^α, and the set $K_h^\alpha\setminus K_{h'}^\alpha$, respectively. It follows from condition (C.1) of Theorem 3.1 that, as $h\to 0$,

$$\varlimsup_{\varepsilon\to 0}\int\limits_{\Pi_h^\alpha\cap\Omega^\varepsilon}\frac{1}{p_\varepsilon(x)}\left|\nabla v_\alpha^\varepsilon\right|^{p_\varepsilon(x)}dx=o\left(h^n\right);\qquad \varlimsup_{\varepsilon\to 0}\int\limits_{\Pi_h^\alpha\cap\Omega^\varepsilon}\mathfrak{S}\left(v_\alpha^\varepsilon-b_\alpha\right)dx=o\left(h^{n+\mathsf{p}^{(+)}+\gamma}\right). \tag{4.6}$$

Moreover, condition (iv) implies that

$$\varlimsup_{\varepsilon\to 0}\int\limits_{\Pi_h^\alpha\cap\Omega^\varepsilon}\left|\nabla v_\alpha^\varepsilon\right|^{p_0(x)}dx=o\left(h^n\right)\quad\text{as } h\to 0. \tag{4.7}$$

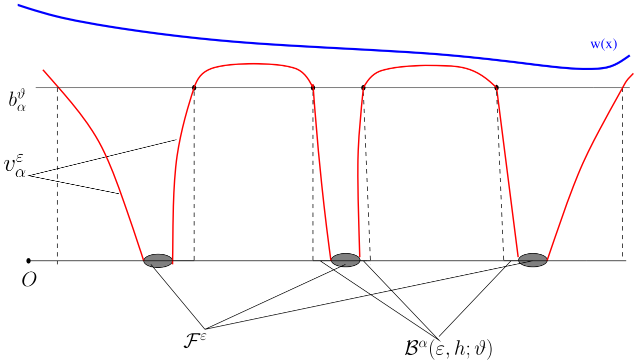

Fig. 1. The set $\mathcal{B}^{\alpha}(\varepsilon, h; \vartheta)$ and the function v^{ε}_{α}.

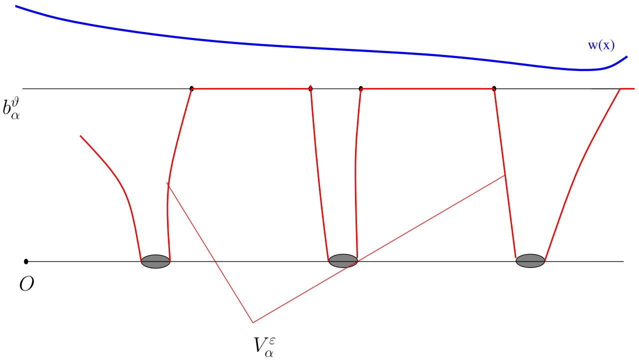

Fig. 2. The function V^{ε}_{α}.

Now let w be a smooth function in Ω such that $w(x) = 0$ on $\partial\Omega$ and let $\mathcal{K}_{\theta}$ denotes a subset of the cubes K^{α}_{h} covering Ω such that $|w(x)| > \theta > 0$ for any $x \in K^{\alpha}_{h}$. We set

$$b_{\alpha} = w\big(x^{\alpha}\big) \quad \text{for } K^{\alpha}_{h} \in \mathcal{K}_{\theta} \quad \text{and} \quad b_{\alpha} = 1 \quad \text{for } K^{\alpha}_{h} \notin \mathcal{K}_{\theta}.$$

For any K^{α}_{h}, we also define the set (see Fig. 1)

$$\mathcal{B}^{\alpha}(\varepsilon, h; \vartheta) = \big\{x \in K^{\alpha}_{h}\colon\ v^{\varepsilon}_{\alpha}(x) \operatorname{sign} b_{\alpha} \leqslant |b_{\alpha}| - \vartheta\big\} \tag{4.8}$$

and the function (see Fig. 2)

$$V^{\varepsilon}_{\alpha}(x) = \begin{cases} v^{\varepsilon}_{\alpha}(x) & \text{in } \mathcal{B}^{\alpha}(\varepsilon, h; \vartheta); \\ b^{\vartheta}_{\alpha} \equiv (|b_{\alpha}| - \vartheta) \operatorname{sign} b_{\alpha} & \text{in } K^{\alpha}_{h} \setminus \mathcal{B}^{\alpha}(\varepsilon, h; \vartheta), \end{cases} \tag{4.9}$$

where $0 < \vartheta \ll \theta/2 \ll 1$.

Now let us estimate meas $\mathcal{B}^{\alpha}(\varepsilon, h; \vartheta)$. For ε sufficiently small, from (4.5), we have

$$\vartheta^{\mathsf{p}^{(-)}} \operatorname{meas} \mathcal{B}^{\alpha}(\varepsilon, h; \vartheta) \leqslant \int\limits_{\mathcal{B}^{\alpha}(\varepsilon, h; \vartheta) \cap \Omega^{\varepsilon}} \big|v^{\varepsilon}_{\alpha} - b_{\alpha}\big|^{p_{\varepsilon}(x)}\, dx \leqslant \int\limits_{K^{\alpha}_{h} \cap \Omega^{\varepsilon}} \big|v^{\varepsilon}_{\alpha} - b_{\alpha}\big|^{p_{\varepsilon}(x)}\, dx \leqslant C h^{n + \mathsf{p}^{(+)} + \gamma}.$$

We set $\vartheta = h$. Then

$$\varlimsup_{\varepsilon \to 0} \operatorname{meas} \mathcal{B}^{\alpha}(\varepsilon, h; \vartheta) = O\big(h^{n + (\mathsf{p}^{(+)} - \mathsf{p}^{(-)}) + \gamma}\big) = o\big(h^{n}\big) \quad \text{as } h \to 0. \tag{4.10}$$

In the domain Ω^{ε} we introduce the function

$$w^{\varepsilon}_{h}(x) = w(x) + \sum_{\alpha} \frac{w(x)}{b^{\vartheta}_{\alpha}} \big(V^{\varepsilon}_{\alpha}(x) - b^{\vartheta}_{\alpha}\big) \varphi_{\alpha}(x). \tag{4.11}$$

From the definition of the functions $\{\varphi_{\alpha}\}$ and (4.9) we have that $w^{\varepsilon}_{h} \in W^{1, p_{\varepsilon}(\cdot)}_{0}(\Omega^{\varepsilon})$.

Since u^ε is the solution of the variational problem (3.6) then we have

$$J^\varepsilon\big[u^\varepsilon\big] \leqslant J^\varepsilon\big[w^\varepsilon_h\big]. \tag{4.12}$$

Let us estimate the right-hand side of the inequality (4.12). It is clear that

$$J^\varepsilon\big[w^\varepsilon_h\big] \leqslant \sum_\alpha \int\limits_{K^\alpha_{h'}\cap\Omega^\varepsilon} \mathsf{F}_\varepsilon\big(x, w^\varepsilon_h, \nabla w^\varepsilon_h\big)\,dx + \sum_{\alpha,\beta} \int\limits_{(K^\alpha_h\cap K^\beta_h)\cap\Omega^\varepsilon} \big|\mathsf{F}_\varepsilon\big(x, w^\varepsilon_h, \nabla w^\varepsilon_h\big)\big|\,dx, \tag{4.13}$$

where

$$\mathsf{F}_\varepsilon(x, u, \nabla u) = \frac{1}{p_\varepsilon(x)}|\nabla u|^{p_\varepsilon(x)} + \frac{1}{p_\varepsilon(x)}|u|^{p_\varepsilon(x)} - f(x)u. \tag{4.14}$$

First, we consider the second term on the right-hand side of (4.13). It follows from the definition of the partition of unity $\{\varphi_\alpha\}$ that for any intersection $K^\alpha_h \cap K^\beta_h$ the number of terms in the sum over α, β is finite and does not depend on ε. Then to estimate the second term on the right-hand side of (4.13) it is sufficient to consider the following integral:

$$\begin{aligned}\mathbf{j}^\varepsilon\big[w^\varepsilon_h\big] = &\int\limits_{(K^\alpha_h\cap K^\beta_h)\cap\Omega^\varepsilon} \left\{\frac{1}{p_\varepsilon}\left|\nabla\left(w + \frac{w}{b^\vartheta_\alpha}\big(V^\varepsilon_\alpha - b^\vartheta_\alpha\big)\varphi_\alpha\right)\right|^{p_\varepsilon} + \frac{1}{p_\varepsilon}\left|w + \frac{w}{b^\vartheta_\alpha}\big(V^\varepsilon_\alpha - b^\vartheta_\alpha\big)\varphi_\alpha\right|^{p_\varepsilon}\right.\\ &\left. - f(x)\left(w + \frac{w}{b^\vartheta_\alpha}\big(V^\varepsilon_\alpha - b^\vartheta_\alpha\big)\varphi_\alpha\right)\right\}dx\\ \equiv{}& \mathbf{j}^\varepsilon_1\big[w^\varepsilon_h\big] + \mathbf{j}^\varepsilon_2\big[w^\varepsilon_h\big] + \mathbf{j}^\varepsilon_3\big[w^\varepsilon_h\big].\end{aligned} \tag{4.15}$$

For the first term on the right-hand side of (4.15) we have

$$\begin{aligned}\mathbf{j}^\varepsilon_1\big[w^\varepsilon_h\big] = &\int\limits_{(K^\alpha_h\cap K^\beta_h)\cap\Omega^\varepsilon} \frac{1}{p_\varepsilon(x)}\left|\nabla\left(w + \frac{w}{b^\vartheta_\alpha}\big(V^\varepsilon_\alpha - b^\vartheta_\alpha\big)\varphi_\alpha\right)\right|^{p_\varepsilon(x)} dx\\ \leqslant{}& C_1 \int\limits_{(K^\alpha_h\cap K^\beta_h)\cap\Omega^\varepsilon} |\nabla w|^{p_\varepsilon(x)}\,dx + C_1 \int\limits_{(K^\alpha_h\cap K^\beta_h)\cap\Omega^\varepsilon} \left|\nabla w\frac{1}{b^\vartheta_\alpha}\big(V^\varepsilon_\alpha - b^\vartheta_\alpha\big)\varphi_\alpha\right|^{p_\varepsilon(x)} dx\\ &+ C_1 \int\limits_{(K^\alpha_h\cap K^\beta_h)\cap\Omega^\varepsilon} \frac{1}{p_\varepsilon(x)}\left|\frac{w}{b^\vartheta_\alpha}\nabla v^\varepsilon_\alpha\varphi_\alpha\right|^{p_\varepsilon(x)} dx + C_1 \int\limits_{(K^\alpha_h\cap K^\beta_h)\cap\Omega^\varepsilon} \left|\frac{w}{b^\vartheta_\alpha}\big(V^\varepsilon_\alpha - b^\vartheta_\alpha\big)\nabla\varphi_\alpha\right|^{p_\varepsilon(x)} dx.\end{aligned} \tag{4.16}$$

First, it is clear that

$$\overline{\lim_{\varepsilon\to 0}} \int\limits_{(K^\alpha_h\cap K^\beta_h)\cap\Omega^\varepsilon} |\nabla w|^{p_\varepsilon(x)}\,dx = o\big(h^n\big) \quad \text{as } h \to 0. \tag{4.17}$$

For the second term on the right-hand side of (4.16), from (4.6), we have, as $h \to 0$,

$$\overline{\lim_{\varepsilon\to 0}} \int\limits_{(K^\alpha_h\cap K^\beta_h)\cap\Omega^\varepsilon} \left|\nabla w\frac{1}{b^\vartheta_\alpha}\big(V^\varepsilon_\alpha - b^\vartheta_\alpha\big)\varphi_\alpha\right|^{p_\varepsilon} dx \leqslant C_2\, \overline{\lim_{\varepsilon\to 0}} \int\limits_{(K^\alpha_h\cap K^\beta_h)\cap\Omega^\varepsilon} \big|v^\varepsilon_\alpha - b_\alpha\big|^{p_\varepsilon}\,dx = o\big(h^n\big). \tag{4.18}$$

For the third term on the right-hand side of (4.16), from (4.6), we have, as $h \to 0$,

$$\overline{\lim_{\varepsilon\to 0}} \int\limits_{(K^\alpha_h\cap K^\beta_h)\cap\Omega^\varepsilon} \frac{1}{p_\varepsilon}\left|\frac{w}{b^\vartheta_\alpha}\nabla v^\varepsilon_\alpha\varphi_\alpha\right|^{p_\varepsilon} dx \leqslant C_3\, \overline{\lim_{\varepsilon\to 0}} \int\limits_{(K^\alpha_h\cap K^\beta_h)\cap\Omega^\varepsilon} \frac{1}{p_\varepsilon}\big|\nabla v^\varepsilon_\alpha\big|^{p_\varepsilon}\,dx = o\big(h^n\big). \tag{4.19}$$

Finally, for the fourth term on the right-hand side of (4.16), from (4.6) and the properties of φ_α, we have

$$\begin{aligned}&\overline{\lim_{\varepsilon\to0}}\int\limits_{(K_h^\alpha\cap K_h^\beta)\cap\Omega^\varepsilon}\left|\frac{w}{b_\alpha^\vartheta}\left(V_\alpha^\varepsilon-b_\alpha^\vartheta\right)\nabla\varphi_\alpha\right|^{p_\varepsilon(x)}dx\\&\quad\leqslant C_4h^{-\mathsf{p}^{(+)}-\gamma}\,\overline{\lim_{\varepsilon\to0}}\int\limits_{(K_h^\alpha\cap K_h^\beta)\cap\Omega^\varepsilon}\left|v_\alpha^\varepsilon-b_\alpha\right|^{p_\varepsilon(x)}dx=o\left(h^n\right)\quad\text{as } h\to0.\end{aligned}\tag{4.20}$$

Thus, from (4.15)–(4.20) we get

$$\lim_{h\to0}\overline{\lim_{\varepsilon\to0}}\,\mathbf{j}_1^\varepsilon\left[w_h^\varepsilon\right]=0.\tag{4.21}$$

In a similar way we can estimate the integrals $\mathbf{j}_2^\varepsilon[w_h^\varepsilon]$, $\mathbf{j}_3^\varepsilon[w_h^\varepsilon]$. Therefore, for the second term on the right-hand side of (4.13), we get

$$\lim_{h\to0}\overline{\lim_{\varepsilon\to0}}\sum_{\alpha,\beta}\int\limits_{(K_h^\alpha\cap K_h^\beta)\cap\Omega^\varepsilon}\left|\mathsf{F}_\varepsilon\left(x,w_h^\varepsilon,\nabla w_h^\varepsilon\right)\right|dx=0.\tag{4.22}$$

Consider now the first term on the right-hand side of (4.13). First, let us denote:

$$\mathcal{B}_1^\alpha(\varepsilon,h)=\left(K_{h'}^\alpha\cap\Omega^\varepsilon\right)\cap\mathcal{B}^\alpha(\varepsilon,h;\vartheta)\quad\text{and}\quad\mathcal{B}_2^\alpha(\varepsilon,h)=\left(K_{h'}^\alpha\cap\Omega^\varepsilon\right)\setminus\mathcal{B}_1^\alpha(\varepsilon,h),\tag{4.23}$$

where the set $\mathcal{B}^\alpha(\varepsilon,h;\vartheta)$ is defined in (4.8) with $\vartheta=h$. Then $w_h^\varepsilon(x)=w(x)$ in $\mathcal{B}_2^\alpha(\varepsilon,h)$ and

$$\int\limits_{\mathcal{B}_2^\alpha(\varepsilon,h)}\mathsf{F}_\varepsilon\left(x,w_h^\varepsilon,\nabla w_h^\varepsilon\right)dx=\int\limits_{\mathcal{B}_2^\alpha(\varepsilon,h)}\mathsf{F}_\varepsilon(x,w,\nabla w)\,dx=\int\limits_{\mathcal{B}_2^\alpha(\varepsilon,h)}\mathsf{F}_0(x,w,\nabla w)\,dx+I_{\mathsf{F}}^\varepsilon,\tag{4.24}$$

where

$$\mathsf{F}_0(x,w,\nabla w)=\frac{1}{p_0(x)}|\nabla w|^{p_0(x)}+\frac{1}{p_0(x)}|w|^{p_0(x)}-f(x)w\tag{4.25}$$

and

$$I_{\mathsf{F}}^\varepsilon=\int\limits_{\mathcal{B}_2^\alpha(\varepsilon,h)}\left\{\mathsf{F}_\varepsilon(x,w,\nabla w)-\mathsf{F}_0(x,w,\nabla w)\right\}dx.\tag{4.26}$$

Moreover, it follows from (3.4) that

$$\overline{\lim_{\varepsilon\to0}}\left|I_{\mathsf{F}}^\varepsilon\right|=0.\tag{4.27}$$

Therefore, from (4.24)–(4.27) and the regularity properties of the functions w, f, we have

$$\overline{\lim_{\varepsilon\to0}}\int\limits_{\mathcal{B}_2^\alpha(\varepsilon,h)}\mathsf{F}_\varepsilon\left(x,w_h^\varepsilon,\nabla w_h^\varepsilon\right)dx\leqslant\int\limits_{K_{h'}^\alpha}\mathsf{F}_0(x,w,\nabla w)\,dx+o\left(h^n\right)\quad\text{as } h\to0.\tag{4.28}$$

Let us consider now the integral over the set $\mathcal{B}_1^\alpha(\varepsilon,h)$ ($K_h^\alpha\in\mathcal{K}_\theta$). In the set $\mathcal{B}_1^\alpha(\varepsilon,h)$ the function w_h^ε has the form:

$$w_h^\varepsilon(x)=w(x)+\frac{w(x)}{b_\alpha^\vartheta}\left(v_\alpha^\varepsilon-b_\alpha^\vartheta\right)\quad\text{in } \mathcal{B}_1^\alpha(\varepsilon,h).\tag{4.29}$$

Therefore, we have

$$\int\limits_{\mathcal{B}_1^\alpha(\varepsilon,h)}\mathsf{F}_\varepsilon\left(x,w_h^\varepsilon,\nabla w_h^\varepsilon\right)dx=\int\limits_{\mathcal{B}_1^\alpha(\varepsilon,h)}\frac{1}{p_\varepsilon}\left|\nabla w_h^\varepsilon\right|^{p_\varepsilon}dx+\int\limits_{\mathcal{B}_1^\alpha(\varepsilon,h)}\left\{\frac{1}{p_\varepsilon}\left|w_h^\varepsilon\right|^{p_\varepsilon}-w_h^\varepsilon f\right\}dx.\tag{4.30}$$

Now it follows from the regularity properties of the functions w, f, the estimate for the measure of the set $\mathcal{B}^\alpha(\varepsilon,h;\vartheta)$ (see (4.10)) and the boundedness of the function v_α^ε on the set $\mathcal{B}^\alpha(\varepsilon,h;\vartheta)$ that

$$\varlimsup_{\varepsilon\to 0}\left|\int\limits_{\mathcal{B}_1^\alpha(\varepsilon,h)}\left\{\frac{1}{p_\varepsilon(x)}\left|w_h^\varepsilon\right|^{p_\varepsilon(x)}-w_h^\varepsilon(x)f(x)\right\}dx\right|=o\left(h^n\right)\quad\text{as } h\to 0. \tag{4.31}$$

Therefore, from (4.30), (4.31) we obtain

$$\varlimsup_{\varepsilon\to 0}\int\limits_{\mathcal{B}_1^\alpha(\varepsilon,h)}\mathsf{F}_\varepsilon\left(x,w_h^\varepsilon,\nabla w_h^\varepsilon\right)dx=\varlimsup_{\varepsilon\to 0}\int\limits_{\mathcal{B}_1^\alpha(\varepsilon,h)}\frac{1}{p_\varepsilon(x)}\left|\nabla w_h^\varepsilon\right|^{p_\varepsilon(x)}dx+o\left(h^n\right)\quad\text{as } h\to 0. \tag{4.32}$$

Consider now the integral on the right-hand side of (4.32). We have

$$\begin{aligned}
&\int\limits_{\mathcal{B}_1^\alpha(\varepsilon,h)}\frac{1}{p_\varepsilon(x)}\left|\nabla w_h^\varepsilon\right|^{p_\varepsilon(x)}dx\\
&\quad=\int\limits_{\mathcal{B}_1^\alpha(\varepsilon,h)}\frac{1}{p_\varepsilon(x)}\left|\nabla\left(\frac{w}{b_\alpha^\vartheta}\left(v_\alpha^\varepsilon-b_\alpha^\vartheta\right)\right)\right|^{p_\varepsilon(x)}dx\\
&\qquad+\int\limits_{\mathcal{B}_1^\alpha(\varepsilon,h)}\frac{1}{p_\varepsilon(x)}\left\{\left|\nabla\left(w+\frac{w}{b_\alpha^\vartheta}\left(v_\alpha^\varepsilon-b_\alpha^\vartheta\right)\right)\right|^{p_\varepsilon(x)}-\left|\nabla\left(\frac{w}{b_\alpha^\vartheta}\left(v_\alpha^\varepsilon-b_\alpha^\vartheta\right)\right)\right|^{p_\varepsilon(x)}\right\}dx.
\end{aligned} \tag{4.33}$$

To estimate the second term on the right-hand side of (4.33) we make use of the following inequality:

$$\left|(\xi+\eta)^{p_\varepsilon(\cdot)}-\xi^{p_\varepsilon(\cdot)}\right|\leqslant A\eta\left(1+\xi^{p_\varepsilon(\cdot)-1}+\eta^{p_\varepsilon(\cdot)-1}\right), \tag{4.34}$$

where $\xi,\eta\geqslant 0$ and $A=A(\mathsf{p}^{(-)},\mathsf{p}^{(+)})$ is a constant. We have

$$\begin{aligned}
&\left|\int\limits_{\mathcal{B}_1^\alpha(\varepsilon,h)}\frac{1}{p_\varepsilon(x)}\left\{\left|\nabla\left(w+\frac{w}{b_\alpha^\vartheta}\left(v_\alpha^\varepsilon-b_\alpha^\vartheta\right)\right)\right|^{p_\varepsilon(x)}-\left|\nabla\left(\frac{w}{b_\alpha^\vartheta}\left(v_\alpha^\varepsilon-b_\alpha^\vartheta\right)\right)\right|^{p_\varepsilon(x)}\right\}dx\right|\\
&\quad\leqslant C_5\left\{\operatorname{meas}\mathcal{B}_1^\alpha(\varepsilon,h)+\int\limits_{\mathcal{B}_1^\alpha(\varepsilon,h)}\left|v_\alpha^\varepsilon-b_\alpha\right|^{p_\varepsilon(x)-1}dx+\int\limits_{\mathcal{B}_1^\alpha(\varepsilon,h)}\left|\nabla v_\alpha^\varepsilon\right|^{p_\varepsilon(x)-1}dx\right\}.
\end{aligned} \tag{4.35}$$

Consider the second term on the right-hand side of (4.35). Since v_α^ε is bounded in $\mathcal{B}_1^\alpha(\varepsilon,h)$, then

$$\varlimsup_{\varepsilon\to 0}\int\limits_{\mathcal{B}_1^\alpha(\varepsilon,h)}\left|v_\alpha^\varepsilon-b_\alpha\right|^{p_\varepsilon(x)-1}dx=o\left(h^n\right)\quad\text{as } h\to 0. \tag{4.36}$$

Finally, we consider the third term on the right-hand side of (4.35). To this end we define the following sets

$$\mathcal{B}_{1,<}^\alpha(\varepsilon,h)=\left\{x\in\mathcal{B}_1^\alpha(\varepsilon,h)\colon\ \left|\nabla v_\alpha^\varepsilon\right|\leqslant\mu_{\varepsilon,h}\right\};\qquad \mathcal{B}_{1,>}^\alpha(\varepsilon,h)=\left\{x\in\mathcal{B}_1^\alpha(\varepsilon,h)\colon\ \left|\nabla v_\alpha^\varepsilon\right|>\mu_{\varepsilon,h}\right\},$$

where

$$\mu_{\varepsilon,h}=\mu_{\varepsilon,h}(x)=h^{-\frac{(\mathsf{p}^{(+)}-\mathsf{p}^{(-)})}{p_\varepsilon(x)-1}}. \tag{4.37}$$

Then

$$\int\limits_{\mathcal{B}_{1,<}^\alpha(\varepsilon,h)}\left|\nabla v_\alpha^\varepsilon\right|^{p_\varepsilon(x)-1}dx\leqslant\operatorname{meas}\mathcal{B}_1^\alpha(\varepsilon,h)h^{-(\mathsf{p}^{(+)}-\mathsf{p}^{(-)})}$$

and it follows from the definition of the set $\mathcal{B}_1^\alpha(\varepsilon,h)$, (4.23), and (4.10) that

$$\varlimsup_{\varepsilon\to 0}\int\limits_{\mathcal{B}_{1,<}^\alpha(\varepsilon,h)}\left|\nabla v_\alpha^\varepsilon\right|^{p_\varepsilon(x)-1}dx=o\left(h^n\right)\quad\text{as } h\to 0. \tag{4.38}$$

Furthermore, in the set $\mathcal{B}^\alpha_{1,>}(\varepsilon,h)$ we have that

$$\mu_{\varepsilon,h}\left|\nabla v^\varepsilon_\alpha\right|^{p_\varepsilon(x)-1}<\left|\nabla v^\varepsilon_\alpha\right|^{p_\varepsilon(x)}$$

therefore

$$\int\limits_{\mathcal{B}^\alpha_{1,>}(\varepsilon,h)}\left|\nabla v^\varepsilon_\alpha\right|^{p_\varepsilon(x)-1}dx\leqslant\int\limits_{\mathcal{B}^\alpha_{1,>}(\varepsilon,h)}\frac{1}{\mu_{\varepsilon,h}}\left|\nabla v^\varepsilon_\alpha\right|^{p_\varepsilon(x)}dx. \tag{4.39}$$

Now it follows from (4.39), (4.4), and (4.37) that

$$\overline{\lim_{\varepsilon\to0}}\int\limits_{\mathcal{B}^\alpha_{1,>}(\varepsilon,h)}\left|\nabla v^\varepsilon_\alpha\right|^{p_\varepsilon(x)-1}dx=o\big(h^n\big)\quad\text{as } h\to0. \tag{4.40}$$

Finally, from (4.38), (4.40) we conclude that the third term on the right-hand side of (4.35) satisfies the estimate:

$$\overline{\lim_{\varepsilon\to0}}\int\limits_{\mathcal{B}^\alpha_{1}(\varepsilon,h)}\left|\nabla v^\varepsilon_\alpha\right|^{p_\varepsilon(x)-1}dx=o\big(h^n\big)\quad\text{as } h\to0. \tag{4.41}$$

With (4.36) the inequality (4.41) means that the second term on the right-hand side of (4.33) is of order $o(h^n)$ as $h\to0$. Thus

$$\int\limits_{\mathcal{B}^\alpha_{1}(\varepsilon,h)}\frac{1}{p_\varepsilon(x)}\left|\nabla w^\varepsilon_h\right|^{p_\varepsilon(x)}dx=\int\limits_{\mathcal{B}^\alpha_{1}(\varepsilon,h)}\frac{1}{p_\varepsilon(x)}\left|\nabla\left(\frac{w}{b^\vartheta_\alpha}\big(v^\varepsilon_\alpha-b^\vartheta_\alpha\big)\right)\right|^{p_\varepsilon(x)}dx+o\big(h^n\big)\quad\text{as } h\to0. \tag{4.42}$$

Then we can conclude that, as $h\to0$,

$$\overline{\lim_{\varepsilon\to0}}\int\limits_{\mathcal{B}^\alpha_{1}(\varepsilon,h)}\frac{1}{p_\varepsilon(x)}\left|\nabla\left(\frac{w}{b^\vartheta_\alpha}\big(v^\varepsilon_\alpha-b^\vartheta_\alpha\big)\right)\right|^{p_\varepsilon(x)}dx=\overline{\lim_{\varepsilon\to0}}\int\limits_{\mathcal{B}^\alpha_{1}(\varepsilon,h)}\frac{1}{p_\varepsilon(x)}\left|\nabla v^\varepsilon_\alpha\right|^{p_\varepsilon(x)}dx+o\big(h^n\big). \tag{4.43}$$

Now from (4.32), (4.33), (4.42), and (4.43) we have

$$\overline{\lim_{\varepsilon\to0}}\int\limits_{\mathcal{B}^\alpha_{1}(\varepsilon,h)}\mathsf{F}_\varepsilon\big(x,w^\varepsilon_h,\nabla w^\varepsilon_h\big)dx=\overline{\lim_{\varepsilon\to0}}\int\limits_{\mathcal{B}^\alpha_{1}(\varepsilon,h)}\frac{1}{p_\varepsilon(x)}\left|\nabla v^\varepsilon_\alpha\right|^{p_\varepsilon(x)}dx+o\big(h^n\big)\quad\text{as } h\to0. \tag{4.44}$$

Finally, from (4.28), (4.44), and (3.7)–(3.8), for any $K^\alpha_h\in\mathcal{K}_\theta$, we get

$$\overline{\lim_{\varepsilon\to0}}\int\limits_{K^\alpha_{h'}\cap\Omega^\varepsilon}\mathsf{F}_\varepsilon\big(x,w^\varepsilon_h,\nabla w^\varepsilon_h\big)dx\leqslant\int\limits_{K^\alpha_{h'}\cap\Omega}\mathsf{F}_0(x,w,\nabla w)\,dx+\overline{\lim_{\varepsilon\to0}}\,c^{\varepsilon,h}\big(x^\alpha,w\big(x^\alpha\big)\big)+o\big(h^n\big) \tag{4.45}$$

as $h\to0$.

In a similar way, for any $K^\alpha_h\notin\mathcal{K}_\theta$, we can obtain the following inequality:

$$\overline{\lim_{\varepsilon\to0}}\int\limits_{K^\alpha_{h'}\cap\Omega^\varepsilon}\mathsf{F}_\varepsilon\big(x,w^\varepsilon_h,\nabla w^\varepsilon_h\big)dx\leqslant\int\limits_{K^\alpha_{h'}\cap\Omega}\mathsf{F}_0(x,w,\nabla w)\,dx+j(\theta)O\big(h^n\big)+o\big(h^n\big)\quad\text{as } h\to0, \tag{4.46}$$

where $j(\theta)\to0$ as $\theta\to0$.

Now we take the union in (4.45) and (4.46) over the corresponding cubes and pass to the limit first as $\varepsilon\to0$, then as $h\to0$, and, finally, as $\theta\to0$. The relations (4.13), (4.22), and condition (C.1) of Theorem 3.1 imply that

$$\lim_{h\to0}\overline{\lim_{\varepsilon\to0}}\,J^\varepsilon\big[w^\varepsilon_h\big]\leqslant J_{hom}[w]\equiv\int\limits_{\Omega}\big\{\mathsf{F}_0(x,w,\nabla w)+c(x,w)\big\}\,dx. \tag{4.47}$$

Therefore, we have

$$\overline{\lim_{\varepsilon\to 0}}\, J^{\varepsilon}\big[u^{\varepsilon}\big] \leqslant J_{hom}[w]. \tag{4.48}$$

This inequality was obtained under the assumption that $w \in C_0^{\infty}(\Omega)$. It remains true for any $w \in W_0^{1,p_0(\cdot)}(\Omega)$ due to the density of $C_0^{\infty}(\Omega)$ in $W_0^{1,p_0(\cdot)}(\Omega)$ (see Section 2) and the following lemma.

Lemma 4.1. *The functional J_{hom} is continuous in the space $W^{1,p_0(\cdot)}(\Omega)$.*

Proof. It is similar to the proof of Lemma 3.2 of [5]. □

4.2. Step 2. Lower bound

Let $u \in W^{1,p_0(\cdot)}(\Omega)$ be a weak limit in $W^{1,p_0(\cdot)}(\Omega)$ of the sequence $\{u^{\varepsilon}\} \subset W_0^{1,p_{\varepsilon}(\cdot)}(\Omega^{\varepsilon}) \cap W_0^{1,p_0(\cdot)}(\Omega^{\varepsilon}) = W_0^{1,p_{\varepsilon}(\cdot)}(\Omega^{\varepsilon})$ (extended by zero in $\mathcal{F}^{\varepsilon}$) by a subsequence $\varepsilon = \varepsilon_k$. Let us show that

$$\underline{\lim}_{\varepsilon=\varepsilon_k\to 0}\, J^{\varepsilon}\big[u^{\varepsilon}\big] \geqslant J_{hom}[u], \tag{4.49}$$

where the functional J_{hom} is defined in (4.47).

First we will obtain some auxiliary results. In what follows we will use the notation:

$$W\big(\Omega, \mathcal{F}^{\varepsilon}\big) = \big\{u \in W^{1,p_{\varepsilon}(\cdot)}\big(\Omega^{\varepsilon}\big) \;\big|\; u = 0 \text{ in } \mathcal{F}^{\varepsilon}\big\}.$$

The following result holds.

Lemma 4.2. *Let w be an arbitrary function from the space $W_0^{1,p_0(\cdot)}(\Omega)$ such that*

$$\|w\|_{W^{1,p_0(\cdot)}(\Omega)} < 1 \tag{4.50}$$

and let the conditions of Theorem 3.1 *be fulfilled. Then there exists a sequence of functions $\{W^{\varepsilon}\} \subset W(\Omega, \mathcal{F}^{\varepsilon})$ which converges weakly to the function w in $W^{1,p_0(\cdot)}(\Omega)$ and satisfies the following estimate*

$$\big\|W^{\varepsilon}\big\|_{W^{1,p_{\varepsilon}(\cdot)}(\Omega)} \leqslant C\big(\|w\|_{W^{1,p_0(\cdot)}(\Omega)}\big)^{1/\mathsf{p}^{(-)}}. \tag{4.51}$$

Proof. Since $C_0^1(\Omega)$ is dense in the space $W_0^{1,p_0(\cdot)}(\Omega)$, then it is sufficient to prove the lemma for $w \in C_0^1(\Omega)$.

Let w_h^{ε} be the function defined in (4.11). Due to (4.4)–(4.7) $w_h^{\varepsilon} \in W_0^{1,p_{\varepsilon}(\cdot)}(\Omega^{\varepsilon})$. Repeating the proof of the inequality (4.48) one can show that

$$\int_{\Omega} \frac{1}{p_{\varepsilon}(x)}\big|\nabla w_h^{\varepsilon}\big|^{p_{\varepsilon}(x)}\,dx \leqslant 2\int_{\Omega}\left\{\frac{1}{p_0(x)}|\nabla w|^{p_0(x)} + c(x,w)\right\}dx \tag{4.52}$$

for sufficiently small θ, h, and ε ($\theta < \tilde{\theta}$, $h \leqslant \tilde{h}(w,\theta)$, $\varepsilon \leqslant \tilde{\varepsilon}(h)$).

Let us estimate the right-hand side of (4.52). Using conditions (C.1), (C.2) of Theorem 3.1 and the properties of the function $p_0 = p_0(x)$ we have

$$\int_{\Omega}\left\{\frac{1}{p_0(x)}|\nabla w|^{p_0(x)} + c(x,w)\right\}dx \leqslant C_1\int_{\Omega}\big\{|\nabla w|^{p_0(x)} + |w|^{p_0(x)} + |w|\big\}\,dx, \tag{4.53}$$

where C_1 is a constant independent of w. Now it follows from (4.50) and (2.4) that

$$\int_{\Omega}\big\{|\nabla w|^{p_0(x)} + |w|^{p_0(x)}\big\}\,dx \leqslant \|w\|_{W^{1,p_0(\cdot)}(\Omega)}^{p_0^{(-)}}, \tag{4.54}$$

where

$$p_0^{(-)} = \min_{x\in\overline{\Omega}} p_0(x). \tag{4.55}$$

From Hölder's inequality (2.5), we get

$$\int\limits_{\Omega} |w|\,dx \leqslant C_2\|w\|_{L^{p_0(\cdot)}(\Omega)}, \tag{4.56}$$

where

$$C_2 = 2\|1\|_{L^{p_0'(\cdot)}(\Omega)} \quad \text{with } p_0'(\cdot) = \frac{p_0(\cdot)}{p_0(\cdot)-1}.$$

Now it follows from (4.50), (4.54), and (4.56) that

$$\int\limits_{\Omega} \left\{ \frac{1}{p_0(x)}|\nabla w|^{p_0(x)} + c(x,w) \right\} dx \leqslant C_3\|w\|_{W^{1,p_0(\cdot)}(\Omega)}. \tag{4.57}$$

Consider now the left-hand side of (4.52). From (2.4), (2.9), and (3.2) we have

$$\int\limits_{\Omega} \frac{1}{p_\varepsilon(x)}\left|\nabla w_h^\varepsilon\right|^{p_\varepsilon(x)} dx \geqslant C_4 \int\limits_{\Omega} \left|\nabla w_h^\varepsilon\right|^{p_\varepsilon(x)} dx \geqslant C_5 \min\left\{ \left\|w_h^\varepsilon\right\|^{p_\varepsilon^{(-)}}_{W^{1,p_\varepsilon(\cdot)}(\Omega)}, \left\|w_h^\varepsilon\right\|^{p_\varepsilon^{(+)}}_{W^{1,p_\varepsilon(\cdot)}(\Omega)} \right\}. \tag{4.58}$$

Then it follows from (4.52), (4.57), and (4.58) that

$$\min\left\{ \left\|w_h^\varepsilon\right\|^{p_\varepsilon^{(-)}}_{W^{1,p_\varepsilon(\cdot)}(\Omega)}, \left\|w_h^\varepsilon\right\|^{p_\varepsilon^{(+)}}_{W^{1,p_\varepsilon(\cdot)}(\Omega)} \right\} \leqslant C_6\|w\|_{W^{1,p_0(\cdot)}(\Omega)}. \tag{4.59}$$

To obtain the estimate for $\|w_h^\varepsilon\|_{W^{1,p_\varepsilon(\cdot)}(\Omega)}$ we consider two different cases. First we suppose that

$$\min\left\{ \left\|w_h^\varepsilon\right\|^{p_\varepsilon^{(-)}}_{W^{1,p_\varepsilon(\cdot)}(\Omega)}, \left\|w_h^\varepsilon\right\|^{p_\varepsilon^{(+)}}_{W^{1,p_\varepsilon(\cdot)}(\Omega)} \right\} = \left\|w_h^\varepsilon\right\|^{p_\varepsilon^{(-)}}_{W^{1,p_\varepsilon(\cdot)}(\Omega)}.$$

Then

$$\left\|w_h^\varepsilon\right\|_{W^{1,p_\varepsilon(\cdot)}(\Omega)} \leqslant C_7\left(\|w\|_{W^{1,p_0(\cdot)}(\Omega)}\right)^{1/p_\varepsilon^{(-)}}. \tag{4.60}$$

Now if

$$\min\left\{ \left\|w_h^\varepsilon\right\|^{p_\varepsilon^{(-)}}_{W^{1,p_\varepsilon(\cdot)}(\Omega)}, \left\|w_h^\varepsilon\right\|^{p_\varepsilon^{(+)}}_{W^{1,p_\varepsilon(\cdot)}(\Omega)} \right\} = \left\|w_h^\varepsilon\right\|^{p_\varepsilon^{(+)}}_{W^{1,p_\varepsilon(\cdot)}(\Omega)}$$

then

$$\left\|w_h^\varepsilon\right\|_{W^{1,p_\varepsilon(\cdot)}(\Omega)} \leqslant C_8\left(\|w\|_{W^{1,p_0(\cdot)}(\Omega)}\right)^{1/p_\varepsilon^{(+)}}. \tag{4.61}$$

Therefore, from (4.50), (4.60), and (4.61), for sufficiently small θ, h, and ε ($\theta < \tilde{\theta}$, $h \leqslant \tilde{h}(w,\theta)$, $\varepsilon \leqslant \tilde{\varepsilon}(h)$), we obtain that

$$\left\|w_h^\varepsilon\right\|_{W^{1,p_\varepsilon(\cdot)}(\Omega)} \leqslant C_9\left(\|w\|_{W^{1,p(\cdot)}(\Omega)}\right)^{1/\mathsf{p}^{(-)}}. \tag{4.62}$$

We set $W^\varepsilon(x) = w_h^\varepsilon(x)$, where $h = h(\varepsilon) = 1/m$ for $\tilde{\varepsilon}(1/(m+1)) < \varepsilon \leqslant \tilde{\varepsilon}(1/m)$, $m = 1, 2, \ldots$. It is clear that $h(\varepsilon) \to 0$ as $\varepsilon \to 0$, and W^ε satisfies (4.62). Thus the inequality (4.51) is proved. Finally, using the explicit form of the function w_h^ε, given by (4.11), it is easy to check that the sequence $\{w^\varepsilon\}$ converges weakly in $W^{1,p_0(\cdot)}(\Omega)$ to the function w. This completes the proof of Lemma 4.2. □

Now let us prove (4.49). Let $u \in W_0^{1,p_0(\cdot)}(\Omega)$ be a weak limit in $W^{1,p_0(\cdot)}(\Omega)$ of the sequence $\{u^\varepsilon\} \subset W_0^{1,p_\varepsilon(\cdot)}(\Omega^\varepsilon) \cap W_0^{1,p_0(\cdot)}(\Omega^\varepsilon)$ (extended by zero in $\mathcal{F}^\varepsilon$) by a subsequence $\varepsilon = \varepsilon_k$. For any $\delta > 0$, we introduce a function $u_\delta \in C_0^1(\Omega)$ such that

$$\|u - u_\delta\|_{W^{1,p_0(\cdot)}(\Omega)} < \delta. \tag{4.63}$$

It follows from Lemma 4.2 that there exists a sequence $\{w_\delta^\varepsilon\} \subset W(\Omega, \mathcal{F}^\varepsilon)$ which converges weakly in $W^{1,p_0(\cdot)}(\Omega)$ to the function $(u - u_\delta)$. We set

$$u_\delta^\varepsilon = u^\varepsilon + w_\delta^\varepsilon. \tag{4.64}$$

Therefore, by (4.63) and Lemma 4.2 we have

$$\lim_{\delta\to 0}\overline{\lim_{\varepsilon=\varepsilon_k\to 0}}\left\|u_\delta^\varepsilon - u^\varepsilon\right\|_{W^{1,p_\varepsilon(\cdot)}(\Omega^\varepsilon)} = 0. \tag{4.65}$$

Using this inequality we can easily show that

$$\lim_{\delta\to 0}\overline{\lim_{\varepsilon=\varepsilon_k\to 0}}\left|J^\varepsilon\left[u_\delta^\varepsilon\right] - J^\varepsilon\left[u^\varepsilon\right]\right| = 0. \tag{4.66}$$

Moreover, it follows from Lemma 4.1 and (4.63) that

$$\lim_{\delta\to 0} J_{hom}[u_\delta] = J_{hom}[u]. \tag{4.67}$$

Thus, we can easily see that to obtain (4.49), it is sufficient to prove the inequality:

$$\underline{\lim}_{\varepsilon=\varepsilon_k\to 0} J^\varepsilon\left[u_\delta^\varepsilon\right] \geqslant J_{hom}[u_\delta]. \tag{4.68}$$

Let us prove (4.68). To this end let us cover the space $\mathbb{R}^n$ by cubes K_h^α centered at the points x^α forming a periodic, with the period h, grid in $\mathbb{R}^n$ and with nonintersecting interiors. Let us introduce the following notation:

$$\Omega_\theta^\pm = \{x \in \Omega \mid \pm u_\delta > \theta > 0\}; \qquad \Omega_{\theta,h}^\pm = \left\{\bigcup_\alpha K_h^\alpha \,\middle|\, K_h^\alpha \subset \Omega_\theta^\pm\right\};$$

$$\Omega_\theta = \Omega_\theta^+ \cup \Omega_\theta^-; \qquad \Omega_{\theta,h} = \Omega_{\theta,h}^+ \cup \Omega_{\theta,h}^-; \qquad \mathcal{O}_\theta = \Omega \setminus \Omega_\theta;$$

$$\Omega_\theta^\varepsilon = \Omega_\theta \cap \Omega^\varepsilon; \qquad \Omega_{\theta,h}^\varepsilon = \Omega_{\theta,h} \cap \Omega^\varepsilon; \qquad \mathcal{O}_\theta^\varepsilon = \mathcal{O}_\theta \cap \Omega^\varepsilon.$$

Since u_δ is a smooth function in Ω, then

$$\lim_{h\to 0} \operatorname{meas}[\Omega_\theta \setminus \Omega_{\theta,h}] = 0. \tag{4.69}$$

Let us rewrite $J^\varepsilon[u_\delta^\varepsilon]$ in the following way:

$$\begin{aligned} J^\varepsilon\left[u_\delta^\varepsilon\right] = \int_{\Omega^\varepsilon} \mathsf{F}_\varepsilon\left(x, u_\delta^\varepsilon, \nabla u_\delta^\varepsilon\right) dx &= \int_{\Omega_{\theta,h}^\varepsilon} \mathsf{F}_\varepsilon\left(x, u_\delta^\varepsilon, \nabla u_\delta^\varepsilon\right) dx + \int_{\Omega_\theta^\varepsilon\setminus\Omega_{\theta,h}^\varepsilon} \mathsf{F}_\varepsilon\left(x, u_\delta^\varepsilon, \nabla u_\delta^\varepsilon\right) dx \\ &\quad + \int_{\mathcal{O}_\theta^\varepsilon} \mathsf{F}_\varepsilon\left(x, u_\delta^\varepsilon, \nabla u_\delta^\varepsilon\right) dx. \end{aligned} \tag{4.70}$$

To estimate the right-hand side of (4.70) from below we use the inequality (see [23, Chapter 5]):

$$\mathsf{f}_\varepsilon(x, \nabla u) - \mathsf{f}_\varepsilon(x, \nabla v) - \sum_{i=1}^n \frac{\partial \mathsf{f}_\varepsilon}{\partial u_{x_i}}(x, \nabla v)\left(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i}\right) \geqslant 0, \tag{4.71}$$

where

$$\mathsf{f}_\varepsilon(x, \nabla u) = \frac{1}{p_\varepsilon(x)}|\nabla u|^{p_\varepsilon(x)}.$$

Consider now the second term on the right-hand side of (4.70). We have

$$\begin{aligned} \int_{\Omega_\theta^\varepsilon\setminus\Omega_{\theta,h}^\varepsilon} \mathsf{F}_\varepsilon\left(x, u_\delta^\varepsilon, \nabla u_\delta^\varepsilon\right) dx &= \int_{\Omega_\theta^\varepsilon\setminus\Omega_{\theta,h}^\varepsilon} \mathsf{f}_\varepsilon\left(x, \nabla u_\delta^\varepsilon\right) dx + \int_{\Omega_\theta^\varepsilon\setminus\Omega_{\theta,h}^\varepsilon} \frac{1}{p_\varepsilon(x)}\left|u_\delta^\varepsilon\right|^{p_\varepsilon(x)} dx - \int_{\Omega_\theta^\varepsilon\setminus\Omega_{\theta,h}^\varepsilon} f(x) u_\delta^\varepsilon\, dx \\ &\equiv \mathbf{i}_1^\varepsilon(\delta, h, \theta) + \mathbf{i}_2^\varepsilon(\delta, h, \theta) + \mathbf{i}_3^\varepsilon(\delta, h, \theta). \end{aligned} \tag{4.72}$$

According to the inequality (4.71), for the first term in the right-hand side of (4.72) we have

$$
\begin{aligned}
\mathbf{i}_1^\varepsilon(\delta,h,\theta) \geqslant{}& \int\limits_{\Omega_\theta^\varepsilon\setminus\Omega_{\theta,h}^\varepsilon} \mathsf{f}_\varepsilon(x,\nabla u_\delta)\,dx + \sum_{i=1}^n \int\limits_{\Omega_\theta^\varepsilon\setminus\Omega_{\theta,h}^\varepsilon} \frac{\partial \mathsf{f}_\varepsilon}{\partial u_{x_i}}(x,\nabla u_\delta)\left(\frac{\partial u_\delta^\varepsilon}{\partial x_i}-\frac{\partial u_\delta}{\partial x_i}\right)dx \\
={}& \int\limits_{\Omega_\theta^\varepsilon\setminus\Omega_{\theta,h}^\varepsilon} \frac{1}{p_0(x)}|\nabla u_\delta|^{p_0(x)}\,dx + \int\limits_{\Omega_\theta^\varepsilon\setminus\Omega_{\theta,h}^\varepsilon} \left\{\frac{1}{p_\varepsilon(x)}|\nabla u_\delta|^{p_\varepsilon(x)} - \frac{1}{p_0(x)}|\nabla u_\delta|^{p_0(x)}\right\}dx \\
&+ \sum_{i=1}^n \int\limits_{\Omega_\theta^\varepsilon\setminus\Omega_{\theta,h}^\varepsilon} \frac{\partial u_\delta}{\partial x_i}|\nabla u_\delta|^{p_0(x)-2}\left(\frac{\partial u_\delta^\varepsilon}{\partial x_i}-\frac{\partial u_\delta}{\partial x_i}\right)dx \\
&+ \sum_{i=1}^n \int\limits_{\Omega_\theta^\varepsilon\setminus\Omega_{\theta,h}^\varepsilon} \frac{\partial u_\delta}{\partial x_i}\left\{|\nabla u_\delta|^{p_\varepsilon(x)-2} - |\nabla u_\delta|^{p_0(x)-2}\right\}\left(\frac{\partial u_\delta^\varepsilon}{\partial x_i}-\frac{\partial u_\delta}{\partial x_i}\right)dx. \qquad (4.73)
\end{aligned}
$$

Considering the facts that p_ε converges uniformly to p_0, the function u_δ^ε converges weakly in $W^{1,p_0(\cdot)}(\Omega)$ (and strongly in $L^{p_0(\cdot)}(\Omega)$) to the function u_δ (which is a smooth function in Ω), and the measure of the set $\Omega_\theta^\varepsilon \setminus \Omega_{\theta,h}^\varepsilon$ satisfies (4.69), from (4.73) we get

$$
\lim_{h\to 0}\ \varliminf_{\varepsilon=\varepsilon_k\to 0} \mathbf{i}_1^\varepsilon(\delta,h,\theta)\,dx \geqslant 0. \qquad (4.74)
$$

In a similar way we prove that

$$
\lim_{h\to 0}\ \varliminf_{\varepsilon=\varepsilon_k\to 0} \mathbf{i}_2^\varepsilon(\delta,h,\theta)\,dx \geqslant 0 \quad \text{and} \quad \lim_{h\to 0}\ \varliminf_{\varepsilon=\varepsilon_k\to 0} \mathbf{i}_3^\varepsilon(\delta,h,\theta)\,dx \geqslant 0. \qquad (4.75)
$$

Thus, it follows from (4.74)–(4.75) that

$$
\lim_{h\to 0}\ \varliminf_{\varepsilon=\varepsilon_k\to 0} \int\limits_{\Omega_\theta^\varepsilon\setminus\Omega_{\theta,h}^\varepsilon} \mathsf{F}_\varepsilon\big(x,u_\delta^\varepsilon,\nabla u_\delta^\varepsilon\big)\,dx \geqslant 0. \qquad (4.76)
$$

In a similar way, for the third term on the right-hand side of (4.70), we have

$$
\lim_{h\to 0}\ \varliminf_{\varepsilon=\varepsilon_k\to 0} \int\limits_{\mathcal{O}_\theta^\varepsilon} \mathsf{F}_\varepsilon\big(x,u_\delta^\varepsilon,\nabla u_\delta^\varepsilon\big)\,dx \geqslant \int\limits_{\mathcal{O}_\theta} \mathsf{F}_0(x,u_\delta,\nabla u_\delta)\,dx. \qquad (4.77)
$$

Consider the first term on the right-hand side of (4.70). Let K_h^α be an arbitrary cube from $\Omega_{\theta,h}^+$. We get

$$
b_\alpha^{min} = \min_{K_h^\alpha} u_\delta(x) - h, \qquad b_\alpha = b_\alpha^{min} - h.
$$

Let us represent the set $K_h^\alpha \cap \Omega^\varepsilon$ as the union of three nonintersecting sets

$$
K_h^\alpha \cap \Omega^\varepsilon = \omega_{1,\alpha}^\varepsilon \cup \omega_{2,\alpha}^\varepsilon \cup \omega_{3,\alpha}^\varepsilon,
$$

where

$$
\begin{aligned}
&\omega_{1,\alpha}^\varepsilon = \big\{x \in K_h^\alpha \cap \Omega^\varepsilon \ \big|\ u_\delta^\varepsilon < h\big\}; \qquad \omega_{2,\alpha}^\varepsilon = \big\{x \in K_h^\alpha \cap \Omega^\varepsilon \ \big|\ h \leqslant u_\delta^\varepsilon \leqslant b_\alpha^{min}\big\}; \\
&\omega_{3,\alpha}^\varepsilon = \big\{x \in K_h^\alpha \cap \Omega^\varepsilon \ \big|\ u_\delta^\varepsilon > b_\alpha^{min}\big\}.
\end{aligned}
$$

Since $u_\delta^\varepsilon \in W_0^{1,p_\varepsilon(\cdot)}(\Omega^\varepsilon) \cap W_0^{1,p_0(\cdot)}(\Omega^\varepsilon)$ converges weakly in $W^{1,p_0(\cdot)}(\Omega)$ to the function u_δ, then one can show that for sufficiently small ε ($\varepsilon \leqslant \tilde{\varepsilon}(h)$),

$$
\int\limits_{K_h^\alpha} \mathfrak{S}\big(u_\delta^\varepsilon - u_\delta\big)\,dx = O\big(h^{n+2\mathsf{p}^{(+)}+2\gamma}\big). \qquad (4.78)
$$

Therefore,

$$h^{\mathsf{p}^{(+)}}\operatorname{meas}\left[\omega^{\varepsilon}_{1,\alpha}\cup\omega^{\varepsilon}_{2,\alpha}\right]\leqslant\int\limits_{\omega^{\varepsilon}_{1,\alpha}\cup\omega^{\varepsilon}_{2,\alpha}}\mathfrak{S}\left(u^{\varepsilon}_{\delta}-u_{\delta}\right)dx=O\left(h^{n+2\mathsf{p}^{(+)}+2\gamma}\right)\tag{4.79}$$

and

$$\operatorname{meas}\left[\omega^{\varepsilon}_{1,\alpha}\cup\omega^{\varepsilon}_{2,\alpha}\right]=O\left(h^{n+\mathsf{p}^{(+)}+2\gamma}\right).\tag{4.80}$$

Using the same arguments which were used to obtain (4.76), (4.77) we get

$$\underline{\lim}_{\varepsilon=\varepsilon_k\to 0}\int\limits_{\omega^{\varepsilon}_{1,\alpha}\cup\omega^{\varepsilon}_{3,\alpha}}\mathsf{F}_{\varepsilon}\left(x,u^{\varepsilon}_{\delta},\nabla u^{\varepsilon}_{\delta}\right)dx\geqslant\int\limits_{K^{\alpha}_{h}}\mathsf{F}_{0}(x,u_{\delta},\nabla u_{\delta})\,dx+o\left(h^{n}\right)\quad\text{as } h\to 0.\tag{4.81}$$

To estimate the integral over the set $\omega^{\varepsilon}_{2,\alpha}$ we introduce the function:

$$v^{\varepsilon}_{\alpha}(x)=\begin{cases}0 & \text{in } \omega^{\varepsilon}_{1,\alpha}\cup(\mathcal{F}^{\varepsilon}\cap K^{\alpha}_{h});\\ (u^{\varepsilon}_{\delta}-h) & \text{in } \omega^{\varepsilon}_{2,\alpha};\\ b_{\alpha} & \text{in } \omega^{\varepsilon}_{3,\alpha}.\end{cases}\tag{4.82}$$

Since u^{ε}_{δ} is bounded in $\omega^{\varepsilon}_{2,\alpha}$, then using (4.80), for sufficiently small ε ($\varepsilon\leqslant\tilde{\varepsilon}(h)$), we have

$$\int\limits_{\omega^{\varepsilon}_{2,\alpha}}\mathsf{F}_{\varepsilon}\left(x,u^{\varepsilon}_{\delta},\nabla u^{\varepsilon}_{\delta}\right)dx=\int\limits_{K^{z}_{h}}\left\{\frac{1}{p_{\varepsilon}}\left|\nabla v^{\varepsilon}_{\alpha}\right|^{p_{\varepsilon}(x)}dx+h^{-\mathsf{p}^{(+)}-\gamma}\mathfrak{S}\left(v^{\varepsilon}_{\alpha}-b_{\alpha}\right)\right\}dx+o\left(h^{n}\right)\quad\text{as } h\to 0.$$

Therefore, from the definition of $c^{\varepsilon,h}(x^{\alpha},b_{\alpha})$,

$$\int\limits_{\omega^{\varepsilon}_{2,\alpha}}\mathsf{F}_{\varepsilon}\left(x,u^{\varepsilon}_{\delta},\nabla u^{\varepsilon}_{\delta}\right)dx\geqslant c^{\varepsilon,h}\left(x^{\alpha},b_{\alpha}\right)+o\left(h^{n}\right)\quad\text{as } h\to 0.\tag{4.83}$$

Thus, it follows from (4.81) and (4.83) that, for any $K^{\alpha}_{h}\subset\Omega^{+}_{\theta}$, as $h\to 0$,

$$\underline{\lim}_{\varepsilon=\varepsilon_k\to 0}\int\limits_{K^{\alpha}_{h}\cap\Omega^{\varepsilon}}\mathsf{F}_{\varepsilon}\left(x,u^{\varepsilon}_{\delta},\nabla u^{\varepsilon}_{\delta}\right)dx\geqslant\int\limits_{K^{\alpha}_{h}}\mathsf{F}_{0}(x,u_{\delta},\nabla u_{\delta})\,dx+\underline{\lim}_{\varepsilon=\varepsilon_k\to 0}c^{\varepsilon,h}\left(x^{\alpha},b_{\alpha}\right)+o\left(h^{n}\right).\tag{4.84}$$

We can easily obtain the same inequality for any $K^{\alpha}_{h}\subset\Omega^{-}_{\theta}$.

Summing up these inequalities over all cubes $K^{\alpha}_{h}\subset\Omega_{\theta,h}$ and passing to the limit, as $h\to 0$, by (4.76), (4.77) and condition (C.1) of Theorem 3.1 we get

$$\underline{\lim}_{\varepsilon=\varepsilon_k\to 0}\int\limits_{\Omega^{\varepsilon}}\mathsf{F}_{\varepsilon}\left(x,u^{\varepsilon}_{\delta},\nabla u^{\varepsilon}_{\delta}\right)dx\geqslant\int\limits_{\Omega}\mathsf{F}_{0}(x,u_{\delta},\nabla u_{\delta})\,dx+\int\limits_{\Omega_{\theta}}c(x,u)\,dx.\tag{4.85}$$

It is clear that $\bigcup_{\theta>0}\Omega_{\theta}=\{x\in\Omega\mid |u_{\delta}|>0\}$. It is also easy to see from the conditions of Theorem 3.1 that $c(x,0)=0$. Now we pass to the limit as $\theta\to 0$ in (4.85) and immediately obtain (4.68) and, therefore, (4.49).

Finally it follows from (4.48) and (4.49) that

$$J_{hom}[u]\leqslant J_{hom}[w]$$

for any $w\in W^{1,p_0(\cdot)}(\Omega)$ such that $w=0$ on $\partial\Omega$. This means that any weak limit of the solution of problem (3.6) extended to the set $\mathcal{F}^{\varepsilon}$ by zero, is the solution of the homogenized problem (3.9). This completes the proof of Theorem 3.1.

5. Periodic and non-periodic examples

As an application of the previous general result, we now give two examples of perforated media, where the distribution of the perforated domain and the growth function are specified.

Theorem 3.1 of Section 3 provides sufficient conditions for the existence of the homogenized problem (3.9). The goals of this section are to prove that, for appropriate examples, all the conditions of Theorem 3.1 are satisfied and to compute the function $c(x,u)$ in the homogenized problem (3.9) explicitly.

5.1. A periodic example

Let Ω be a bounded domain in $\mathbb{R}^3$ with sufficiently smooth boundary. Let $\mathcal{F}^\varepsilon$ be a union of balls $\mathcal{F}_i^\varepsilon$ ($i = 1, 2, \dots, N_\varepsilon$) periodically distributed in the domain Ω with a period ε. We assume that the ball $\mathcal{F}_i^\varepsilon$ is centered at the point $x^{i,\varepsilon}$ and its radius r_ε is defined by

$$r_\varepsilon = \mathfrak{r}\varepsilon^3, \tag{5.1}$$

where $\mathfrak{r} > 0$. It is clear that $\operatorname{meas} \mathcal{F}^\varepsilon \to 0$ as $\varepsilon \to 0$.

We will study the following variational problem:

$$\begin{gathered}\inf\bigl\{J^\varepsilon[u]\colon\ u \in W_0^{1,p_\varepsilon(\cdot)}\bigl(\Omega^\varepsilon\bigr)\bigr\},\\ J^\varepsilon[u] = \int\limits_{\Omega^\varepsilon} \left\{\frac{1}{p_\varepsilon(x)}|\nabla u|^{p_\varepsilon(x)} + \frac{1}{p_\varepsilon(x)}|u|^{p_\varepsilon(x)} - f(x)u\right\} dx,\end{gathered} \tag{5.2}$$

where $f \in C^1(\Omega)$, and the function $p_\varepsilon \in C^1(\Omega)$ is defined as follows.

Definition 5.1. Let $\mathcal{B}_{\varepsilon/8}^i$ and $\mathcal{B}_{\varepsilon/4}^i$ be the balls centered at the point $x^{i,\varepsilon}$ and of radii $\varepsilon/8$ and $\varepsilon/4$, respectively. The function p_ε is a smooth ε-periodic function in Ω such that

$$p_\varepsilon(x) = \begin{cases} 2 & \text{in } \mathcal{B}_{\varepsilon/8}^i\ (i = 1, 2, \dots, N_\varepsilon);\\ 2 + \pi_\varepsilon(|x - x^{i,\varepsilon}|) & \text{in } \mathcal{B}_{\varepsilon/4}^i \setminus \mathcal{B}_{\varepsilon/8}^i\ (i = 1, 2, \dots, N_\varepsilon);\\ 2 + \varepsilon & \text{in } \Omega \setminus \bigcup_i \mathcal{B}_{\varepsilon/4}^i, \end{cases} \tag{5.3}$$

where π_ε is a smooth ε-periodic function in Ω such that $0 \leqslant \pi_\varepsilon(x) \leqslant \varepsilon$.

It is clear that the function p_ε satisfies the conditions (i)–(iv), $p_\varepsilon^{(+)} = 2 + \varepsilon$, $p_\varepsilon^{(-)} = \mathsf{p}^{(-)} = 2$, and it converges uniformly in Ω to the function $p_0(x) = 2$.

The following result holds.

Theorem 5.2. *Let u^ε be the solution* (*minimizer*) *of the variational problem* (5.2). *Then u^ε converges weakly in $H_0^1(\Omega)$ to u the minimizer of the following variational problem*:

$$\begin{gathered}\inf\bigl\{J_{hom}[u]\colon\ u \in H_0^1(\Omega)\bigr\},\\ J_{hom}[u] = \int\limits_{\Omega} \left\{\frac{1}{2}|\nabla u|^2 + \left(\frac{1}{2} + 4\pi\mathfrak{r}\right)|u|^2 - f(x)u\right\} dx.\end{gathered} \tag{5.4}$$

5.1.1. Proof of Theorem 5.2

We have to verify the conditions of Theorem 3.1 and to calculate the function $c(x, b)$ in the condition (C.1) explicitly.

First we notice that the local energy characteristic (3.7) in this case has the form:

$$c^{\varepsilon,h}(z, b) = \inf_{v^\varepsilon} \int\limits_{K_h^z} \left\{\frac{1}{p_\varepsilon(x)}\bigl|\nabla v^\varepsilon\bigr|^{p_\varepsilon(x)} + h^{-\mathsf{p}^{(+)}-\gamma}\mathfrak{S}\bigl(v^\varepsilon - b\bigr)\right\} dx, \tag{5.5}$$

where $0 < \gamma < \mathsf{p}^{(+)}$,

$$\mathfrak{S}\bigl(v^\varepsilon - b\bigr) = \bigl|v^\varepsilon - b\bigr|^{p_\varepsilon(x)} + \bigl|v^\varepsilon - b\bigr|^2, \tag{5.6}$$

and the infimum is taken over $v^\varepsilon \in W^{1,p_\varepsilon(\cdot)}(\Omega)$ that equal zero in $\mathcal{F}^\varepsilon$.

Condition (C.2). We set

$$\alpha_\varepsilon = a\varepsilon^{1+\kappa}, \tag{5.7}$$

where $a > 0$ and $0 < \kappa < 1$. We denote by v_b^ε the solution of the following boundary value problem:

$$\begin{cases} \dfrac{1}{\rho^2}\dfrac{\partial}{\partial\rho}\left(\rho^2\dfrac{\partial \mathsf{v}_b^\varepsilon}{\partial\rho}\right) = 0 \quad \text{for } r_\varepsilon < \rho < \alpha_\varepsilon; \\ \mathsf{v}_b^\varepsilon(r_\varepsilon) = b; \\ \mathsf{v}_b^\varepsilon(\alpha_\varepsilon) = 0. \end{cases} \tag{5.8}$$

It is clear that

$$\mathsf{v}_b^\varepsilon(\rho) = b\frac{\frac{1}{\rho} - \frac{1}{\alpha_\varepsilon}}{\frac{1}{r_\varepsilon} - \frac{1}{\alpha_\varepsilon}}. \tag{5.9}$$

Let us introduce the following function:

$$W^\varepsilon(x) = \begin{cases} 0 & \text{in } K_h^z \cap \mathcal{F}^\varepsilon; \\ b - \sum_i \mathsf{v}_b^\varepsilon(|x - x^{i,\varepsilon}|)\varphi(\frac{|x-x^{i,\varepsilon}|}{\alpha_\varepsilon}) & \text{in } K_h^z \setminus \mathcal{F}^\varepsilon, \end{cases} \tag{5.10}$$

where $\varphi(t)$ is a smooth positive function defined by: $\varphi \in C^2(\mathbb{R}_+)$ with $\varphi(t) = 1$ for $t \leqslant 1/2$; $\varphi(t) = 0$ for $t \geqslant 1$. It is clear that $W^\varepsilon \in W^{1,p_\varepsilon(\cdot)}(\Omega)$ and it equals zero in $\mathcal{F}^\varepsilon$.

Now it follows from the definition of the functional $c^{\varepsilon,h}(z,b)$, (5.5)–(5.6), that

$$c^{\varepsilon,h}(z,b) \leqslant \int\limits_{K_h^z} \left\{\frac{1}{p_\varepsilon(x)}\left|\nabla W^\varepsilon\right|^{p_\varepsilon(x)} + h^{-\mathsf{p}^{(+)}-\gamma}\mathfrak{S}\left(W^\varepsilon - b\right)\right\} dx \equiv \mathbf{A}^{\varepsilon,h}(z). \tag{5.11}$$

Consider the first integral on the right-hand side of (5.11). According to the definition of the function p_ε and the parameter α_ε, we have

$$\int\limits_{K_h^z} \left|\nabla W^\varepsilon\right|^{p_\varepsilon(x)} dx = 4\pi \sum_{\mathcal{F}_i^\varepsilon \subset K_h^z} \left(\int\limits_{r_\varepsilon}^{\alpha_\varepsilon/2} \left|\frac{\partial W^\varepsilon}{\partial\rho}\right|^2 \rho^2 d\rho + \int\limits_{\alpha_\varepsilon/2}^{\alpha_\varepsilon} \left|\frac{\partial W^\varepsilon}{\partial\rho}\right|^2 \rho^2\, d\rho \right). \tag{5.12}$$

Here

$$\int\limits_{r_\varepsilon}^{\alpha_\varepsilon/2} \left|\frac{\partial W^\varepsilon}{\partial\rho}\right|^2 \rho^2 d\rho = b^2 r_\varepsilon\left(1 + o(1)\right) \quad \text{as } \varepsilon \to 0 \tag{5.13}$$

and

$$\int\limits_{\alpha_\varepsilon/2}^{\alpha_\varepsilon} \left|\frac{\partial W^\varepsilon}{\partial\rho}\right|^2 \rho^2 d\rho \leqslant C_2\frac{r_\varepsilon^2}{\alpha_\varepsilon} = C_3\varepsilon^{5-\kappa}. \tag{5.14}$$

Now it follows from (5.12)–(5.14) and the definition of r_ε, (5.1), that

$$\int\limits_{K_h^z} \left|\nabla W^\varepsilon\right|^{p_\varepsilon(x)} dx = 4\pi\mathfrak{r}b^2h^3\left(1 + o(1)\right) \quad \text{as } \varepsilon \to 0. \tag{5.15}$$

Consider the second term on the right-hand side of (5.11). We have that

$$h^{-\mathsf{p}^{(+)}-\gamma}\int\limits_{K_h^z} \mathfrak{S}\left(W^\varepsilon - b\right) dx \leqslant C_4 h^{-\mathsf{p}^{(+)}+3-\gamma}\frac{\alpha_\varepsilon^3}{\varepsilon^3}. \tag{5.16}$$

Therefore, from (5.15)–(5.16) we get

$$\lim_{h\to 0}\varlimsup_{\varepsilon\to 0} h^{-n}c^{\varepsilon,h}(z,b)\leqslant 4\pi\mathfrak{t}b^2 \tag{5.17}$$

and condition (C.2) is satisfied.

Condition (C.1). Now let $v^{\varepsilon}_{min}=v^{\varepsilon}_{min}(x)$ be the function that minimizes the functional (5.5). Let us represent this function in the form:

$$v^{\varepsilon}_{min}(x)=W^{\varepsilon}(x)+\zeta^{\varepsilon}(x), \tag{5.18}$$

where the function W^{ε} is defined in (5.10). Then

$$c^{\varepsilon,h}(z,b)=\int\limits_{K^z_h}\left\{\frac{1}{p_\varepsilon(x)}\left|\nabla v^{\varepsilon}_{min}\right|^{p_\varepsilon(x)}+h^{-\mathsf{p}^{(+)}-\gamma}\left|v^{\varepsilon}_{min}-b\right|^2\right\}dx. \tag{5.19}$$

We will prove that the function ζ^{ε} gives a vanishing contribution (as $\varepsilon\to 0$ and $h\to 0$) in (5.19) and, therefore, the functional (3.7) may be computed by the function W^{ε}.

It follows from (5.15)–(5.16) that

$$c^{\varepsilon,h}(z,b)\leqslant \mathbf{A}^{\varepsilon,h}(z)=4\pi\mathfrak{t}b^2h^3+\beta^{\varepsilon,h}(z)\quad\text{with }\lim_{\varepsilon\to 0}\beta^{\varepsilon,h}(z)=0, \tag{5.20}$$

where $\mathbf{A}^{\varepsilon,h}(z)$ is defined in (5.11).

Now let $\mathcal{B}^i_{\alpha_\varepsilon}$ be the ball centered at $x^{i,\varepsilon}$ and radius α_ε and let $\mathcal{B}_{\alpha_\varepsilon}=\bigcup_i\mathcal{B}^i_{\alpha_\varepsilon}$. By the definition of the functions W^{ε} and p_ε we have

$$\begin{aligned}c^{\varepsilon,h}(z,b)={}&\int\limits_{K^z_h\cap\mathcal{B}_{\alpha_\varepsilon}}\left\{\frac{1}{p_\varepsilon(x)}\left|\nabla\left(W^{\varepsilon}+\zeta^{\varepsilon}\right)\right|^{p_\varepsilon(x)}+2h^{-\mathsf{p}^{(+)}-\gamma}\left|\left(W^{\varepsilon}+\zeta^{\varepsilon}\right)-b\right|^2\right\}dx\\&+\int\limits_{K^z_h\setminus\mathcal{B}_{\alpha_\varepsilon}}\left\{\frac{1}{p_\varepsilon(x)}\left|\nabla\zeta^{\varepsilon}\right|^{p_\varepsilon(x)}+h^{-\mathsf{p}^{(+)}-\gamma}\mathfrak{S}\left(\zeta^{\varepsilon}\right)\right\}dx.\end{aligned} \tag{5.21}$$

For the first term on the right-hand side of (5.21), from the definition of the function p_ε, we obtain

$$\begin{aligned}&\int\limits_{K^z_h\cap\mathcal{B}_{\alpha_\varepsilon}}\left\{\frac{1}{p_\varepsilon(x)}\left|\nabla\left(W^{\varepsilon}+\zeta^{\varepsilon}\right)\right|^{p_\varepsilon(x)}+2h^{-\mathsf{p}^{(+)}-\gamma}\left|\left(W^{\varepsilon}+\zeta^{\varepsilon}\right)-b\right|^2\right\}dx\\&\quad=\int\limits_{K^z_h\cap\mathcal{B}_{\alpha_\varepsilon}}\left\{\frac{1}{2}\left|\nabla W^{\varepsilon}\right|^2+\frac{1}{2}\left|\nabla\zeta^{\varepsilon}\right|^2+\left(\nabla W^{\varepsilon},\nabla\zeta^{\varepsilon}\right)\right\}dx\\&\qquad+2h^{-\mathsf{p}^{(+)}-\gamma}\int\limits_{K^z_h\cap\mathcal{B}_{\alpha_\varepsilon}}\left\{\left|W^{\varepsilon}-b\right|^2+2\left(W^{\varepsilon}-b\right)\zeta^{\varepsilon}+\left|\zeta^{\varepsilon}\right|^2\right\}dx.\end{aligned} \tag{5.22}$$

Now it follows from (5.21)–(5.22) that

$$c^{\varepsilon,h}(z,b)=\mathbf{A}^{\varepsilon,h}(z)+\mathbf{J}^{\varepsilon,h}(z)+\int\limits_{K^z_h\cap\mathcal{B}_{\alpha_\varepsilon}}\left\{\left(\nabla W^{\varepsilon},\nabla\zeta^{\varepsilon}\right)+4h^{-\mathsf{p}^{(+)}-\gamma}\left(W^{\varepsilon}-b\right)\zeta^{\varepsilon}\right\}dx, \tag{5.23}$$

where

$$\mathbf{J}^{\varepsilon,h}(z)=\int\limits_{K^z_h}\left\{\frac{1}{p_\varepsilon(x)}\left|\nabla\zeta^{\varepsilon}\right|^{p_\varepsilon(x)}+h^{-\mathsf{p}^{(+)}-\gamma}\mathfrak{S}\left(\zeta^{\varepsilon}\right)\right\}dx. \tag{5.24}$$

Therefore, integrating by parts in the third term of the right-hand side of (5.23) and taking into account (5.20), we get

$$\mathbf{J}^{\varepsilon,h}(z) \leqslant 2 \int\limits_{K_h^z \cap \mathcal{B}_{\alpha_\varepsilon}} \bigl\{ \left|\Delta W^\varepsilon\right| + 4h^{-\mathsf{p}^{(+)}-\gamma} \left|\left(W^\varepsilon - b\right)\right| \bigr\} \left|\zeta^\varepsilon\right| dx. \tag{5.25}$$

Let $\eta^\varepsilon(x) = \Delta W^\varepsilon$. Then this function equals zero everywhere in the cube K_h^z except the set $\mathcal{D}_{\alpha_\varepsilon} = \bigcup_i \mathcal{D}_{\alpha_\varepsilon}^i$, where $\mathcal{D}_{\alpha_\varepsilon}^i = \{x \in K_h^z \mid \alpha_\varepsilon/2 < |x - x^{i,\varepsilon}| < \alpha_\varepsilon\}$ and

$$\eta^\varepsilon = \Delta W^\varepsilon = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left(\rho^2 \frac{\partial W^\varepsilon}{\partial \rho}(\rho) \right) \quad \text{in } \mathcal{D}_{\alpha_\varepsilon}^i.$$

Moreover, the following estimate holds

$$\left|\eta^\varepsilon\right| \leqslant C \frac{r_\varepsilon}{\alpha_\varepsilon^3} \quad \text{in } \mathcal{D}_{\alpha_\varepsilon}^i. \tag{5.26}$$

Now from (5.25), (5.26) and the Cauchy inequality we get

$$\mathbf{J}^{\varepsilon,h}(z) \leqslant C \left\{ \frac{r_\varepsilon}{\alpha_\varepsilon^3} + h^{-\mathsf{p}^{(+)}-\gamma} \right\} \left[\alpha_\varepsilon^3 \frac{h^3}{\varepsilon^3} \right]^{1/2} \Biggl(\sum_i \int\limits_{K_h^z \cap \mathcal{B}_{\alpha_\varepsilon}^i} \left|\zeta^\varepsilon\right|^2 dx \Biggr)^{1/2}. \tag{5.27}$$

To estimate the integral on the right-hand side of (5.27) we make use of the following lemma.

Lemma 5.1. *Let $\mathcal{K}^\varepsilon$ be a cube centered at the point zero and of length ε and $\mathcal{B}_{\alpha_\varepsilon}^0$ be a ball centered at zero and of radius α_ε. Then for any $v \in W^{1,p_\varepsilon(\cdot)}(\mathcal{K}^\varepsilon)$ we have*

$$\int\limits_{\mathcal{B}_{\alpha_\varepsilon}^0} |v|^2 dx \leqslant C \Biggl\{ \frac{\alpha_\varepsilon^3}{\varepsilon^3} \int\limits_{\mathcal{K}^\varepsilon} \mathfrak{S}(v)\, dx + \varepsilon \alpha_\varepsilon \int\limits_{\mathcal{K}^\varepsilon} |\nabla v|^{p_\varepsilon(x)} dx + \delta(\varepsilon) \Biggr\}, \tag{5.28}$$

where

$$\delta(\varepsilon) = \varepsilon^4 \alpha_\varepsilon. \tag{5.29}$$

Proof. We make use of the following inequality (see Section 7.5 in [26]):

$$\int\limits_{\mathcal{B}_{\alpha_\varepsilon}^0} |v|^2 dx \leqslant C \Biggl\{ \frac{\alpha_\varepsilon^3}{\varepsilon^3} \int\limits_{\mathcal{K}^\varepsilon} |v|^2 dx + \varepsilon \alpha_\varepsilon \int\limits_{\mathcal{K}^\varepsilon} |\nabla v|^2 dx \Biggr\}. \tag{5.30}$$

This inequality was proved for any $v \in H^1(\mathcal{K}^\varepsilon)$, therefore, it is valid for any $v \in W^{1,p_\varepsilon(\cdot)}(\mathcal{K}^\varepsilon)$, where the function p_ε is given by Definition 5.1.

Consider the right-hand side of (5.30), we have

$$\frac{\alpha_\varepsilon^3}{\varepsilon^3} \int\limits_{\mathcal{K}^\varepsilon} |v|^2 dx + \varepsilon \alpha_\varepsilon \int\limits_{\mathcal{K}^\varepsilon} |\nabla v|^2 dx \leqslant \frac{\alpha_\varepsilon^3}{\varepsilon^3} \int\limits_{\mathcal{K}^\varepsilon} \mathfrak{S}(v)\, dx + \varepsilon \alpha_\varepsilon \int\limits_{\mathcal{K}_>^\varepsilon} |\nabla v|^2 dx + \varepsilon \alpha_\varepsilon \int\limits_{\mathcal{K}_<^\varepsilon} |\nabla v|^2 dx, \tag{5.31}$$

where $\mathcal{K}_>^\varepsilon = \{x \in \mathcal{K}^\varepsilon \colon |\nabla v| > 1\}$ and $\mathcal{K}_<^\varepsilon = \{x \in \mathcal{K}^\varepsilon \colon |\nabla v| \leqslant 1\}$. Then it is clear that

$$\varepsilon \alpha_\varepsilon \int\limits_{\mathcal{K}_>^\varepsilon} |\nabla v|^2 dx + \varepsilon \alpha_\varepsilon \int\limits_{\mathcal{K}_<^\varepsilon} |\nabla v|^2 dx \leqslant \varepsilon \alpha_\varepsilon \int\limits_{\mathcal{K}_>^\varepsilon} |\nabla v|^{p_\varepsilon(x)} dx + \varepsilon^4 \alpha_\varepsilon \leqslant \int\limits_{\mathcal{K}^\varepsilon} |\nabla v|^{p_\varepsilon(x)} dx + \varepsilon^4 \alpha_\varepsilon.$$

This inequality together with (5.30), (5.31) proves the lemma. □

Now it follows from (5.27)–(5.29) that

$$\mathbf{J}^{\varepsilon,h}(z) \leqslant Ch^{3/2}\Bigg(\bigg[\frac{r_\varepsilon^2}{\varepsilon^6} + h^{-2\mathsf{p}^{(+)}-2\gamma}\frac{\alpha_\varepsilon^6}{\varepsilon^6}\bigg]\int\limits_{\mathcal{K}^\varepsilon} \mathfrak{S}\big(\zeta^\varepsilon\big)\,dx + \bigg[\frac{r_\varepsilon^2}{\alpha_\varepsilon^6}\frac{\alpha_\varepsilon^4}{\varepsilon^2} + h^{-\mathsf{p}^{(+)}-\gamma}\frac{\alpha_\varepsilon^4}{\varepsilon^2}\bigg]\int\limits_{\mathcal{K}^\varepsilon} \big|\nabla\zeta^\varepsilon\big|^{p_\varepsilon(x)}\,dx$$
$$+\bigg[h^3\frac{r_\varepsilon^2}{\alpha_\varepsilon^6}\frac{\alpha_\varepsilon^3}{\varepsilon^6}\delta(\varepsilon) + h^{3-\mathsf{p}^{(+)}-\gamma}\frac{\alpha_\varepsilon^3}{\varepsilon^6}\delta(\varepsilon)\bigg]\Bigg)^{\frac{1}{2}}. \tag{5.32}$$

Since $r_\varepsilon = \mathfrak{r}\varepsilon^3$, $\alpha_\varepsilon = a\varepsilon^{1+\kappa}$, and $\delta(\varepsilon)$ is given by (5.29), then, for ε sufficiently small, we have that

$$\mathbf{J}^{\varepsilon,h}(z) \leqslant Ch^{3/2+\mathsf{p}^{(+)}/2+\gamma/2}\big(\mathbf{J}^{\varepsilon,h}(z) + o(1)\big)^{1/2} \quad \text{as } \varepsilon \to 0, \tag{5.33}$$

and, therefore,

$$\varlimsup_{\varepsilon\to 0} \mathbf{J}^{\varepsilon,h}(z) = o\big(h^3\big). \tag{5.34}$$

This means that we can calculate the function $c(x,b)$ from the test function W^ε. We obtain that $c(x,b) = 4\pi\mathfrak{r}b^2$. This completes the proof of Theorem 5.2.

5.2. A non-periodic example

In this example, we consider a locally periodic perforated domain. More precisely, let Ω be a bounded domain in $\mathbb{R}^3$ with sufficiently smooth boundary and $\{x^{i,\varepsilon}\}$ be a periodic grid in Ω with a period ε. We define the sets $\mathcal{F}^\varepsilon$ and Ω^ε in the following way:

$$\mathcal{F}^\varepsilon = \bigcup_i \mathcal{F}_{r_\varepsilon^{(i)}} \quad \text{and} \quad \Omega^\varepsilon = \Omega \setminus \mathcal{F}^\varepsilon, \tag{5.35}$$

where $\mathcal{F}_{r_\varepsilon^{(i)}}$ $(i = 1, 2, \ldots, N_\varepsilon)$ is the closed ball centered at the point $x^{i,\varepsilon}$ and of radius $r_\varepsilon^{(i)}$ defined by

$$r_\varepsilon^{(i)} = R\big(x^i\big)\varepsilon^3. \tag{5.36}$$

Here $R = R(x)$ is a strictly positive smooth function in Ω. As in the periodic case, it is clear that $\operatorname{meas}\mathcal{F}^\varepsilon \to 0$ as $\varepsilon \to 0$.

Consider the variational problem (5.2), where $f \in C^1(\Omega)$ and the function p_ε is given by Definition 5.1. Following the lines of the proof of Theorem 5.2 (with corresponding modifications) we can obtain the following result.

Theorem 5.3. *Let u^ε be the solution of the variational problem* (5.2) *considered in the domain Ω^ε defined in* (5.35). *Then u^ε converges weakly in $H_0^1(\Omega)$ to u the solution of the following variational problem*:

$$\inf\big\{J_{hom}[u]\colon\ u \in H_0^1(\Omega)\big\},$$
$$J_{hom}[u] = \int\limits_\Omega \bigg\{\frac{1}{2}|\nabla u|^2 + \bigg(\frac{1}{2} + 4\pi R(x)\bigg)|u|^2 - f(x)u\bigg\}\,dx. \tag{5.37}$$

5.3. Some generalizations

In Sections 5.1, 5.2 the proof of (5.25) and some other inequalities relies on the fact that, in the case under consideration, p_ε equals 2 in the neighbourhood of the inclusions $\mathcal{F}_i^\varepsilon$. In more general situation, for example, if we assume that in the said neighbourhood p_ε is equal to a constant $p > 2$, the proof of similar inequalities relies on the following statement.

Lemma 5.2. *Let $p_\varepsilon = p_\varepsilon(x)$ be a continuous function satisfying the bound*

$$2 \leqslant \mathsf{p}^{(-)} \leqslant p_\varepsilon^{(-)} \equiv \min_{x\in\overline{\Omega}} p_\varepsilon(x) \leqslant p_\varepsilon(x) \leqslant \max_{x\in\overline{\Omega}} p_\varepsilon(x) \equiv p_\varepsilon^{(+)} \leqslant \mathsf{p}^{(+)} \leqslant n \quad \textit{in } \overline{\Omega}. \tag{5.38}$$

Then, for any vectors $\xi_1, \xi_2 \in \mathbb{R}^d$ $(d = 1, 2, \ldots)$*, there exists* $\delta \in (0, 1)$*, which does not depend on* ε*, such that*

$$|\xi_1 + \xi_2|^{p_\varepsilon(\cdot)} \geqslant |\xi_1|^{p_\varepsilon(\cdot)} + \delta|\xi_2|^{p_\varepsilon(\cdot)} + p_\varepsilon(\cdot)|\xi_1|^{p_\varepsilon(\cdot)-2}(\xi_1, \xi_2)_d, \tag{5.39}$$

where $(\cdot,\cdot)_d$ *is the scalar product in the space* $\mathbb{R}^d$.

Proof. Without loss of generality we may assume that $\xi_1 = \vec{e}_1$, where $\vec{e}_1$ is the first coordinate vector in $\mathbb{R}^d$. Then the inequality (5.39) is equivalent to the following inequality:

$$|\vec{e}_1 + \xi|^{p_\varepsilon(\cdot)} \geqslant 1 + \delta|\xi|^{p_\varepsilon(\cdot)} + p_\varepsilon(\cdot)\xi^1, \tag{5.40}$$

where $\xi \equiv \xi_2$ and ξ^1 is the first component of the vector ξ. We denote

$$\mathsf{G}_{p_\varepsilon(\cdot)}(\xi) = |\vec{e}_1 + \xi|^{p_\varepsilon(\cdot)} - 1 - p_\varepsilon(\cdot)\xi^1.$$

It is clear that $\mathsf{G}_{p_\varepsilon(\cdot)}(0) = 0$ and $\nabla_\xi \mathsf{G}_{p_\varepsilon(\cdot)}(\xi) = 0$ for $\xi = 0$.

It is easy to verify that there is $\varkappa > 0$ such that

$$\mathsf{G}_{p_\varepsilon(\cdot)}(\xi) \geqslant \frac{1}{2}|\xi|^{p_\varepsilon(\cdot)}$$

for all $|\xi| \geqslant \varkappa$ and all the functions p_ε satisfying condition (5.38). Therefore, it suffices to prove that

$$\mathsf{G}_{p_\varepsilon(\cdot)}(\xi) \geqslant \delta|\xi|^{p_\varepsilon(\cdot)}$$

for all $|\xi| \leqslant \varkappa$. Computing the second order derivatives of the function $\mathsf{G}_{p_\varepsilon(\cdot)}$, we conclude that

$$\frac{\partial^2 \mathsf{G}_{p_\varepsilon(\cdot)}}{\partial^2 \xi} \geqslant 0$$

and

$$\frac{\partial^2 \mathsf{G}_{p_\varepsilon(\cdot)}}{\partial^2 \xi} \geqslant \delta_1(n) I$$

for all ξ such that $|\xi| \leqslant 1/2$ with $\delta_1(n)$ independent of p_ε. Here I is the unit matrix. For $\xi \in \mathcal{B}_{1/2}$, where

$$\mathcal{B}_{1/2} = \left\{\xi\colon\ |\xi| \leqslant \frac{1}{2}\right\},$$

we have

$$\mathsf{G}_{p_\varepsilon(\cdot)}(\xi) = \mathsf{G}_{p_\varepsilon(\cdot)}(\xi) - \mathsf{G}_{p_\varepsilon(\cdot)}(0) = \mathsf{G}''_{p_\varepsilon(\cdot)}(\tilde{\xi})|\xi|^2$$

with $\tilde{\xi} \in \mathcal{B}_{1/2}$. Thus, $\mathsf{G}_{p_\varepsilon(\cdot)}(\xi) \geqslant \delta_1(n)|\xi|^2$. By convexity, for any ξ, $\frac{1}{2} \leqslant |\xi| \leqslant \varkappa$, we have

$$\mathsf{G}_{p_\varepsilon(\cdot)}(\xi) \geqslant \mathsf{G}_{p_\varepsilon(\cdot)}\left(\frac{1}{2}\frac{\xi}{|\xi|}\right) \geqslant \left(\frac{1}{2}\right)^2 \frac{|\xi|^2}{|\xi|^2}\delta_1(n) \geqslant \left(\frac{1}{2}\right)^2 \delta_1(n)\frac{1}{\varkappa^2}|\xi|^2.$$

Lemma 5.2 is proved. □

Acknowledgements

The work of S. Antontsev was partially supported by the Portuguese research project POCI/ MAT/ 61576/ 2004/ FCT/ MCES. L. Pankratov would like to thanks Departamento de Matemática, Universidade da Beira Interior (Covilhã, Portugal) and Narvik University College (Narvik, Norway) for the invitation. This work was completed when S. Antontsev and L. Pankratov were visiting the Applied Mathematics Laboratory of the University of Pau. They are grateful for the invitations and the hospitality.

References

[1] E. Acerbi, G. Mingione, Regularity results for a class of functionals with non-standard growth, Arch. Ration. Mech. Anal. 156 (2001) 121–140.

[2] B. Amaziane, L. Pankratov, A. Piatnitski, Homogenization of a class of quasilinear elliptic equations in high-contrast fissured media, Proc. Roy. Soc. Edinburgh Sect. A 136 (2006) 1131–1155.

[3] B. Amaziane, A. Antontsev, L. Pankratov, Homogenization of a class of nonlinear elliptic equations with nonstandard growth, C. R. Mécanique 335 (2007) 138–143.

[4] B. Amaziane, S. Antontsev, L. Pankratov, A. Piatnitski, Γ-convergence and homogenization of functionals on Sobolev spaces with variable exponents, J. Math. Anal. Appl. 342 (2008) 1192–1202.

[5] B. Amaziane, L. Pankratov, V.V. Pritula, Nonlocal effects in homogenization of $p_\varepsilon(x)$-Laplacian in perforated domains, Nonlinear Anal. (2009), doi:10.1016/j.na.2009.02.097.

[6] B. Amaziane, L. Pankratov, V.V. Pritula, Homogenization of $p_\varepsilon(x)$-Laplacian in perforated domains with a nonlocal transmission condition, C. R. Mécanique 337 (2009) 173–178.

[7] S.N. Antontsev, J.I. Díaz, H.B. de Oliveira, Stopping a viscous fluid by a feedback dissipative field: Thermal effects without phase changing, in: Proceedings of Trends in Partial Differential Equations of Mathematical Physics, in: Progr. Nonlinear Differential Equations Appl., vol. 61, Birkhäuser, Basel, 2005, pp. 1–14.

[8] S.N. Antontsev, S.I. Shmarev, On localization of solutions of elliptic equations with nonhomogeneous anisotropic degeneracy, Siberian Math. J. 46 (2005) 765–782.

[9] S.N. Antontsev, J.F. Rodrigues, On stationary thermo-rheological viscous flows, Ann. Univ. Ferrara Sez. Sci. Mat. 52 (2006) 19–36.

[10] S.N. Antontsev, S.I. Shmarev, Elliptic equations with anisotropic nonlinearity and nonstandard growth conditions, in: Handbook of Differential Equations, Stationary Partial Differential Equations, vol. 3, Elsevier, 2006, pp. 1–100, Chapter 1.

[11] S.N. Antontsev, S.I. Shmarev, Elliptic equations and systems with nonstandard growth conditions: Existence, uniqueness and localization properties of solutions, J. Nonlinear Anal. 65 (2006) 722–755.

[12] M. Balzano, A. Corbo Esposito, G. Paderni, Nonlinear Dirichlet problems in randomly perforated domains, Rend. Mat. Appl. (7) 17 (1997) 163–186.

[13] G. Barenblatt, V. Entov, V. Ryzhik, The Motion of Fluids and Gases in Natural Strata, Nedra Publishing House, Moscow, 1984.

[14] A. Braides, A. Defranceschi, Homogenization of Multiple Integrals, Oxford Lecture Ser. Math. Appl., vol. 12, Clarendon Press, Oxford, 1998.

[15] D. Cioranescu, F. Murat, Un terme étrange venu d'ailleurs I and II, Nonlinear Partial Differential Equations and Their Applications II (1983) 98–138.

[16] G. Dal Maso, F. Murat, Asymptotic behaviour and correctors for Dirichlet problems in perforated domains with homogeneous monotone operators, Ann. Sc. Norm. Super. Pisa 7 (1997) 765–803.

[17] X. Fan, Q. Zhang, Existence of solutions for $p(x)$-Laplacian Dirichlet problem, J. Nonlinear Anal. 52 (2003) 1843–1852.

[18] P. Harjulehto, P. Hästö, M. Koskenoja, S. Varonen, The Dirichlet energy integral and variable exponents Sobolev spaces with zero boundary values, Potential Anal. 25 (2006) 205–222.

[19] H. Hudzik, On generalized Orlicz–Sobolev space, Funct. Approx. Comment. Math. 4 (1976) 37–51.

[20] E.Ya. Khruslov, L. Pankratov, Homogenization of the Dirichlet variational problems in Orlicz–Sobolev spaces, Fields Inst. Commun. 25 (2000) 345–366.

[21] O. Kováčik, J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, Czechoslovak Math. J. 41 (1991) 592–618.

[22] S.M. Kozlov, Geometric aspects of averaging, Russian Math. Surveys 44 (1989) 91–144.

[23] O.A. Ladyzhenskaya, N.N. Ural'tseva, Linear and Quasilinear Elliptic Equations, Academic Press, New York, 1973.

[24] Y. Chen, S. Levine, M. Rao, Variable exponent, linear growth functionals in image restoration, SIAM J. Appl. Math. 66 (4) (2006) 1383–1406.

[25] V.A. Marchenko, E.Ya. Khruslov, Boundary value problems with fine-grained boundary, Mat. Sb. 65 (3) (1964) 458–472.

[26] V.A. Marchenko, E.Ya. Khruslov, Homogenization of Partial Differential Equations, Birkhäuser, Boston, 2006.

[27] J. Musielak, Orlicz Spaces and Modular Spaces, Lecture Notes in Math., vol. 1034, Springer-Verlag, Berlin, 1983.

[28] G. Nguetseng, H. Nnang, Homogenization of nonlinear monotone operators beyond the periodic setting, Electron. J. Differential Equations 36 (2003) 1–24.

[29] A. Pankov, G-Convergence and Homogenization of Nonlinear Partial Differential Operators, Kluwer Academic, Dordrecht, 1997.

[30] K. Rajagopal, M. Růžička, Mathematical modelling of electro-rheological fluids, Contin. Mech. Thermodyn. 13 (2001) 59–78.

[31] M. Růžička, Electrorheological Fluids: Modeling and Mathematical Theory, Lecture Notes in Math., vol. 1748, Springer, Berlin, 2000.

[32] I.I. Šarapudinov, The topology of the space $\mathcal{L}^{p(t)}([0,1])$, Mat. Zametki 26 (1979) 613–632.

[33] I.V. Skrypnik, Methods of Investigation of Nonlinear Elliptic Boundary Value Problems, Nauka, Moscow, 1990.

[34] V.V. Zhikov, Lavrentiev effect and the averaging of nonlinear variational problems, Differ. Equ. 27 (1991) 32–39.

[35] V.V. Zhikov, On the passage to the limit in nonlinear variational problems, Math. Sb. 183 (1992) 47–84.

[36] V.V. Zhikov, Lavrentiev phenomenon and homogenization for some variational problems, C. R. Acad. Sci. Paris Sér. I 316 (1993) 435–439.

[37] V.V. Zhikov, On Lavrentiev's phenomenon, Russ. J. Math. Phys. 3 (1995) 249–269.

[38] V.V. Zhikov, S.M. Kozlov, O.A. Oleinik, Homogenization of Differential Operators and Integral Functionals, Springer-Verlag, Berlin/Heidelberg/New York, 1994.

[39] V.V. Zhikov, On some variational problems, Russ. J. Math. Phys. 5 (1997) 105–116.