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$p(x)$-Harmonic functions with unbounded exponent in a subdomain

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To the memory of Oded Schramm

Abstract

We study the Dirichlet problem $-\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = 0$ in Ω, with $u = f$ on $\partial\Omega$ and $p(x) = \infty$ in D, a subdomain of the reference domain Ω. The main issue is to give a proper sense to what a solution is. To this end, we consider the limit as $n \to \infty$ of the solutions u_n to the corresponding problem when $p_n(x) = p(x) \wedge n$, in particular, with $p_n = n$ in D. Under suitable assumptions on the data, we find that such a limit exists and that it can be characterized as the unique solution of a variational minimization problem which is, in addition, ∞-harmonic within D. Moreover, we examine this limit in the viscosity sense and find the boundary value problem it satisfies in the whole of Ω.

MSC: 35J20; 35J60; 35J70

Keywords: $p(x)$-Laplacian; Infinity-Laplacian; Viscosity solutions

1. Introduction

The goal of this paper is to study the elliptic problem

$$\begin{cases} -\Delta_{p(x)} u(x) = 0, & x \in \Omega \subset \mathbb{R}^N, \\ u(x) = f(x), & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_{p(x)} u(x) := \mathrm{div}(|\nabla u(x)|^{p(x)-2}\nabla u(x))$ is the $p(x)$-Laplacian operator and the variable exponent $p(x)$ verifies

$$p(x) = +\infty, \quad x \in D, \tag{1.2}$$

for some subdomain $D \subset \Omega$. We assume that Ω and D are bounded and convex domains with smooth boundaries, at least of class C^1. On the complementary domain $\Omega \setminus \overline{D}$ we assume that $p(x)$ is a continuously differentiable bounded function.

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On the variable exponent, apart from (1.2), we also require that

$$p_- := \inf_{x\in\Omega} p(x) > N, \tag{1.3}$$

so that we will always be dealing with continuous solutions for (1.1); to fix notation, we define

$$p_+ := \sup_{x\in\Omega\setminus\overline{D}} p(x).$$

The boundary data f is taken to be Lipschitz continuous.

Our strategy to solve (1.1) is to replace $p(x)$ by a sequence of bounded functions $p_n(x)$ such that $p_n(x)$ is increasing and converging to $p(x)$. For definiteness, we consider, for $n > N$,

$$p_n(x) := \min\{p(x), n\}.$$

We will use the notation $(1.1)_n$ to refer to problem (1.1) for the variable exponents $p_n(x)$.

Since $p(x)$ is bounded in $\Omega \setminus D$, we have, for large n, specifically for $n > p_+$,

$$p_n(x) = \begin{cases} p(x), & x \in \Omega \setminus D, \\ n, & x \in D. \end{cases}$$

Moreover, still for large n, the boundary of the set $\{p(x) > n\}$ coincides with the boundary of D and thus does not depend on n. This fact is important when passing to the limit.

Using a variational method, we solve $(1.1)_n$ obtaining solutions u_n; if the limit

$$\lim_{n\to\infty} u_n \tag{1.4}$$

exists, we call it u_∞. It is a natural candidate to be a solution to (1.1) with the original variable exponent $p(x)$. A crucial role in this process will be played by the set

$$S = \left\{u \in W^{1,p^-}(\Omega)\colon\ u|_{\Omega\setminus\overline{D}} \in W^{1,p(x)}(\Omega \setminus \overline{D}),\ \|\nabla u\|_{L^\infty(D)} \leqslant 1 \text{ and } u|_{\partial\Omega} = f\right\}$$

and by the infinity Laplacian

$$\Delta_\infty u := \left(D^2 u \nabla u\right) \cdot \nabla u = \sum_{i,j=1}^{N} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i x_j}.$$

Our main results are contained in the following theorem.

Theorem. *There exists a unique solution u_n to $(1.1)_n$. If $S \neq \emptyset$, then the uniform limit*

$$u_\infty := \lim_{n\to\infty} u_n$$

exists and is characterized as the unique function that is a minimizer of the integral

$$\int_{\Omega\setminus\overline{D}} \frac{|\nabla u|^{p(x)}}{p(x)}\,dx \tag{1.5}$$

in S and, in addition, verifies

$$-\Delta_\infty u_\infty = 0 \quad \text{in } D,$$

in the viscosity sense. Moreover, u_∞ is a viscosity solution of

$$\begin{cases} -\Delta_{p(x)} u(x) = 0, & x \in \Omega \setminus \overline{D}, \\ -\Delta_\infty u(x) = 0, & x \in D, \\ \operatorname{sgn}\big(|\nabla u|(x) - 1\big)\operatorname{sgn}\left(\dfrac{\partial u}{\partial \nu}(x)\right) = 0, & x \in \partial D \cap \Omega, \\ u(x) = f(x), & x \in \partial\Omega, \end{cases}$$

where ν is the exterior unit normal vector to ∂D in Ω.

Finally, if $\partial\Omega \cap \overline{D} \neq \emptyset$ and the Lipschitz constant of $f|_{\partial\Omega\cap\overline{D}}$ is strictly greater than one, then $S = \emptyset$ and we have

$$\liminf_{n\to\infty}\left(\int_D \frac{|\nabla u_n|^n}{n}\,dx\right)^{\frac{1}{n}} > 1;$$

hence, the natural energy associated to u_n is unbounded.

Remark 1.1. The boundedness of Ω is used to ensure compactness of minimizing sequences for (1.5), while the convexity of Ω and D guarantees that the Lipschitz constant of $W^{1,\infty}$ functions coincides with the L^∞ norm of their gradients, which will be instrumental in some of the proofs.

Remark 1.2. The characterization of the non-emptiness of S is an interesting open problem that strongly depends on the geometry of Ω and D, and on the boundary data f. When $\partial\Omega \cap \overline{D} = \emptyset$, S is always non-empty. When $\partial\Omega \cap \overline{D} \neq \emptyset$, the condition that the Lipschitz constant of $f|_{\partial\Omega\cap\overline{D}}$ is less than or equal to one is necessary but, in general, it is not sufficient (cf. Section 4).

Partial differential equations involving variable exponents became popular a few years ago in relation to applications to elasticity and electrorheological fluids. Meanwhile, the underlying functional analytical tools have been extensively developed and new applications, *e.g.* to image processing, have kept the subject as the focus of an intensive research activity. For general references on the $p(x)$-Laplacian we refer to [10], that includes a thorough bibliography, and [14], a seminal paper where many of the basic properties of variable exponent spaces were established. The delicate regularity properties of $p(x)$-harmonic functions have been established in [1] and [2].

In the literature, the variable exponent $p(x)$ is always assumed to be bounded, a necessary condition to define a proper norm in the corresponding Lebesgue spaces. To the best of our knowledge, this paper is the first attempt at analyzing a problem where the exponent $p(\cdot)$ becomes infinity in some part of the domain. For constant exponents, limits as $p \to \infty$ in p-Laplacian type problems have been widely studied, see for example [7], and are related to optimal transport problems (cf. [3]).

Organization of the paper. The rest of the paper is organized as follows: in Section 2 we show existence and uniqueness of solutions with $p(x) = p_n(x) = p \wedge n$ using a variational argument; moreover we find the equation that they verify in the viscosity sense and prove some useful independent of n estimates; in Section 3 we pass to the limit in the variational formulation of the problem and we deal with the limit in the viscosity sense; in Section 4 we discuss necessary and sufficient conditions related to the non-emptiness of S and present examples and counter-examples. Finally, in Section 5 we present a detailed analysis of the one-dimensional case.

2. Weak and viscosity approximate solutions

To start with, let us establish the existence and uniqueness of the approximations u_n in the weak sense.

Lemma 2.1. *There exists a unique weak solution u_n to $(1.1)_n$, which is the unique minimizer of the functional*

$$F_n(u) = \int_\Omega \frac{|\nabla u|^{p_n(x)}}{p_n(x)}\,dx = \int_D \frac{|\nabla u|^n}{n}\,dx + \int_{\Omega\setminus\overline{D}} \frac{|\nabla u|^{p(x)}}{p(x)}\,dx \tag{2.1}$$

in

$$S_n = \left\{u \in W^{1,p_n(x)}(\Omega):\ u|_{\partial\Omega} = f\right\}. \tag{2.2}$$

Proof. Although the exponent $p_n(\cdot)$ might be discontinuous, functions in the variable exponent Sobolev space $W^{1,p_n(\cdot)}(\Omega)$ are continuous thanks to assumption (1.3). Indeed, for n sufficiently large, we have $p_n(\cdot) \geqslant (p_n)_- \geqslant p_- > N$ and the continuous embedding in

$$W^{1,p_n(\cdot)}(\Omega) \hookrightarrow W^{1,p_-}(\Omega) \subset C(\overline{\Omega}) \tag{2.3}$$

follows from [14, Theorem 2.8 and (3.2)]. That the boundedness away from the dimension is not superfluous when the exponent is not continuous is shown by a counter-example in [11, Example 3.3].

We can then take the boundary condition $u|_{\partial\Omega} = f$ in the classical sense (recall that f is assumed to be Lipschitz) and the results of [12] apply since the jump condition (cf. [12, (4.1)–(4.2)]) is trivially satisfied by the variable exponent because $p_n(\cdot) \geqslant N$. This is a sufficient condition for a $p_n(\cdot)$-Poincaré inequality to hold in $W_0^{1,p_n(\cdot)}(\Omega)$ which, in turn, is instrumental in obtaining the coercivity of the functional. The lower semi-continuity is standard as is the strict convexity, that also gives the uniqueness.

It is also standard that the minimizer of F_n in S_n is the unique weak solution of $(1.1)_n$, i.e., $u_n = f$ on $\partial\Omega$ and it satisfies the weak form of the equation, namely,

$$\int_\Omega |\nabla u_n|^{p_n(x)-2}\nabla u_n \cdot \nabla\varphi\,dx = 0, \quad \forall\varphi \in C_0^\infty(\Omega). \qquad \square \tag{2.4}$$

Lemma 2.2. *Problem* $(1.1)_n$ *can be rewritten as*

$$\begin{cases} -\Delta_{p(x)} u_n(x) = 0, & x \in \Omega \setminus \overline{D}, \\ -\Delta_n u_n(x) = 0, & x \in D, \\ \left|\nabla u_n(x)\right|^{n-2} \dfrac{\partial u_n}{\partial\nu}(x) = \left|\nabla u_n(x)\right|^{p(x)-2} \dfrac{\partial u_n}{\partial\nu}(x), & x \in \partial D \cap \Omega, \\ u_n(x) = f(x), & x \in \partial\Omega, \end{cases} \tag{2.5}$$

where ν *is the exterior unit normal to* ∂D *in* Ω.

Proof. Just notice that the weak form of this problem is exactly the same as the one that holds for $(1.1)_n$. This follows since after multiplying by a test function and integrating by parts one arrives at (2.4) for both problems. $\square$

Next, we investigate the problem satisfied by u_n from the point of view of viscosity solutions.

Let us recall the definition of viscosity solution (see [9] and [6]) for a problem like (2.5), which involves a transmission condition across the boundary $\partial D \cap \Omega$. Assume we are given a family of continuous functions

$$F_i : \overline{\Omega} \times \mathbb{R}^N \times \mathbb{S}^{N\times N} \to \mathbb{R}.$$

The associated equations

$$F_i\big(x, \nabla u, D^2 u\big) = 0$$

are called (degenerate) elliptic if

$$F_i(x, \xi, X) \leqslant F_i(x, \xi, Y) \quad \text{whenever } X \geqslant Y.$$

Definition 2.3. Consider the problem

$$\begin{aligned} &F_1\big(x, \nabla u, D^2 u\big) = 0, \quad \text{in } \Omega \setminus \overline{D}, \\ &F_2\big(x, \nabla u, D^2 u\big) = 0, \quad \text{in } D, \end{aligned} \tag{2.6}$$

with a transmission condition

$$B(x, u, \nabla u) = 0, \quad \text{on } \partial D \cap \Omega, \tag{2.7}$$

and a boundary condition

$$u = f, \quad \text{on } \partial\Omega. \tag{2.8}$$

A lower semi-continuous function u is a viscosity supersolution of (2.6)–(2.8) if $u \geqslant f$ on $\partial\Omega$ and for every $\phi \in C^2(\overline{\Omega})$ such that $u - \phi$ has a strict minimum at the point $x_0 \in \Omega$, with $u(x_0) = \phi(x_0)$, we have

$$F_1\big(x_0, \nabla\phi(x_0), D^2\phi(x_0)\big) \geqslant 0 \quad \text{if } x_0 \in \Omega \setminus \overline{D},$$
$$F_2\big(x_0, \nabla\phi(x_0), D^2\phi(x_0)\big) \geqslant 0 \quad \text{if } x_0 \in D,$$
$$\max\left\{\begin{array}{l} F_1(x_0, \nabla\phi(x_0), D^2\phi(x_0)) \\ F_2(x_0, \nabla\phi(x_0), D^2\phi(x_0)) \\ B(x_0, \phi(x_0), \nabla\phi(x_0)) \end{array}\right\} \geqslant 0 \quad \text{if } x_0 \in \partial D \cap \Omega.$$

An upper semi-continuous function u is a viscosity subsolution of (2.6)–(2.8) if $u \leqslant f$ on $\partial\Omega$ and for every $\psi \in C^2(\overline{\Omega})$ such that $u - \psi$ has a strict maximum at the point $x_0 \in \Omega$, with $u(x_0) = \psi(x_0)$, we have

$$F_1\big(x_0, \nabla\psi(x_0), D^2\psi(x_0)\big) \leqslant 0 \quad \text{if } x_0 \in \Omega \setminus \overline{D},$$
$$F_2\big(x_0, \nabla\psi(x_0), D^2\psi(x_0)\big) \leqslant 0 \quad \text{if } x_0 \in D,$$
$$\min\left\{\begin{array}{l} F_1(x_0, \nabla\psi(x_0), D^2\psi(x_0)) \\ F_2(x_0, \nabla\psi(x_0), D^2\psi(x_0)) \\ B(x_0, \psi(x_0), \nabla\psi(x_0)) \end{array}\right\} \leqslant 0 \quad \text{if } x_0 \in \partial D \cap \Omega.$$

Finally, u is a viscosity solution if it is both a viscosity supersolution and a viscosity subsolution.

In the sequel, we will use the notation as in the definition: ϕ will always stand for a test function touching the graph of u from below and ψ for a test function touching the graph of u from above.

Proposition 2.4. *Let u_n be a continuous weak solution of* $(1.1)_n$. *Then u_n is a viscosity solution of* (2.5) *in the sense of Definition* 2.3.

Proof. To simplify, we omit in the proof the subscript n. Let $x_0 \in \Omega \setminus \overline{D}$ and a let ϕ be a test function such that $u(x_0) = \phi(x_0)$ and $u - \phi$ has a strict minimum at x_0. We want to show that

$$\begin{aligned} -\Delta_{p(x_0)}\phi(x_0) &= -\big|\nabla\phi(x_0)\big|^{p(x_0)-2}\Delta\phi(x_0) - \big(p(x_0)-2\big)\big|\nabla\phi(x_0)\big|^{p(x_0)-4}\Delta_\infty\phi(x_0) \\ &\quad - \big|\nabla\phi(x_0)\big|^{p(x_0)-2}\ln\big(|\nabla\phi|\big)(x_0)\big\langle\nabla\phi(x_0), \nabla p(x_0)\big\rangle \\ &\geqslant 0. \end{aligned}$$

Assume, *ad contrarium*, that this is not the case; then there exists a radius $r > 0$ such that $B(x_0, r) \subset \Omega \setminus \overline{D}$ and

$$\begin{aligned} -\Delta_{p(x)}\phi(x) &= -\big|\nabla\phi(x)\big|^{p(x)-2}\Delta\phi(x) - \big(p(x)-2\big)\big|\nabla\phi(x)\big|^{p(x)-4}\Delta_\infty\phi(x) \\ &\quad - \big|\nabla\phi(x)\big|^{p(x)-2}\ln\big(|\nabla\phi|\big)(x)\big\langle\nabla\phi(x), \nabla p(x)\big\rangle \\ &< 0, \end{aligned}$$

for every $x \in B(x_0, r)$. Set $m = \inf_{|x-x_0|=r}(u - \phi)(x)$ and let $\Phi(x) = \phi(x) + m/2$. This function Φ verifies $\Phi(x_0) > u(x_0)$ and

$$-\Delta_{p(x)}\Phi = -\operatorname{div}\big(|\nabla\Phi|^{p(x)-2}\nabla\Phi\big) < 0 \quad \text{in } B(x_0, r). \tag{2.9}$$

Multiplying (2.9) by $(\Phi - u)^+$, which vanishes on the boundary of $B(x_0, r)$, we get

$$\int\limits_{B(x_0,r)\cap\{\Phi>u\}} |\nabla\Phi|^{p(x)-2}\nabla\Phi \cdot \nabla(\Phi - u)\,dx < 0.$$

On the other hand, taking $(\Phi - u)^+$, extended by zero outside $B(x_0, r)$, as test function in the weak formulation of $(1.1)_n$, we obtain

$$\int\limits_{B(x_0,r)\cap\{\Phi>u\}} |\nabla u|^{p(x)-2}\nabla u \cdot \nabla(\Phi - u)\,dx = 0,$$

since $p_n(x) = p(x)$ in $\Omega \setminus \overline{D}$. Upon subtraction and using a well-know inequality, see for example [15], we conclude

$$0 > \int_{B(x_0,r)\cap\{\Phi>u\}} \left(|\nabla\Phi|^{p(x)-2}\nabla\Phi - |\nabla u|^{p(x)-2}\nabla u\right)\cdot\nabla(\Phi-u)\,dx$$
$$\geqslant c \int_{B(x_0,r)\cap\{\Phi>u\}} |\nabla\Phi - \nabla u|^{p(x)}\,dx,$$

a contradiction. Here c is a constant that depends on N, p^- and $\sup_{x\in B(x_0,r)} p(x)$.

If $x_0 \in D$ the proof is entirely analogous, albeit simpler due to the absence of the logarithmic term, and we obtain

$$-\Delta_n\phi(x_0) = -\left|\nabla\phi(x_0)\right|^{n-2}\Delta\phi(x_0) - (n-2)\left|\nabla\phi(x_0)\right|^{n-4}\Delta_\infty\phi(x_0) \geqslant 0.$$

The constant c in this case depends on N and n.

If $x_0 \in \partial D\cap\Omega$ we want to prove that

$$\max\begin{Bmatrix} -\Delta_{p(x_0)}\phi(x_0) \\ -\Delta_n\phi(x_0) \\ |\nabla\phi(x_0)|^{n-2}\frac{\partial\phi}{\partial\nu}(x_0) - |\nabla\phi(x_0)|^{p(x_0)-2}\frac{\partial\phi}{\partial\nu}(x_0) \end{Bmatrix} \geqslant 0.$$

If this is not the case, there exists a radius $r > 0$ such that

$$-\Delta_{p(x)}\phi(x) < 0 \quad\text{and}\quad -\Delta_n\phi(x) < 0,$$

for every $x \in B(x_0, r)$. Set $m = \inf_{|x-x_0|=r}(u-\phi)(x)$ and let $\Phi(x) = \phi(x) + m/2$. This function Φ verifies $\Phi(x_0) > u(x_0)$,

$$-\Delta_{p(x)}\Phi < 0 \quad \text{in } B(x_0,r)\cap(\Omega\setminus\overline{D}) \tag{2.10}$$

and

$$-\Delta_n\Phi < 0 \quad \text{in } B(x_0,r)\cap D. \tag{2.11}$$

Moreover, we can assume (taking r smaller if necessary) that

$$\left|\nabla\Phi(x)\right|^{n-2}\frac{\partial\Phi}{\partial\nu}(x) - \left|\nabla\Phi(x)\right|^{p(x)-2}\frac{\partial\Phi}{\partial\nu}(x) < 0 \quad \text{in } B(x_0,r)\cap\partial D. \tag{2.12}$$

Multiplying both (2.10) and (2.11) by $(\Phi-u)^+$, integrating by parts and adding, we obtain

$$\int_{B(x_0,r)\cap\Omega\setminus\overline{D}} |\nabla\Phi|^{p(x)-2}\nabla\Phi\cdot\nabla(\Phi-u)^+\,dx + \int_{B(x_0,r)\cap D} |\nabla\Phi|^{n-2}\nabla\Phi\cdot\nabla(\Phi-u)^+\,dx$$
$$< \int_{B(x_0,r)\cap\partial D} \left(|\nabla\Phi|^{n-2}\frac{\partial\Phi}{\partial\nu} - |\nabla\Phi|^{p(x)-2}\frac{\partial\Phi}{\partial\nu}\right)(\Phi-u)^+\,dS,$$

taking also into account that the test function vanishes on the boundary of $B(x_0, r)$. Using (2.12), we finally get

$$\int_{B(x_0,r)\cap(\Omega\setminus\overline{D})\cap\{\Phi>u\}} |\nabla\Phi|^{p(x)-2}\nabla\Phi\cdot\nabla(\Phi-u)\,dx + \int_{B(x_0,r)\cap D\cap\{\Phi>u\}} |\nabla\Phi|^{n-2}\nabla\Phi\cdot\nabla(\Phi-u)\,dx < 0.$$

On the other hand, taking $(\Phi-u)^+$, extended by zero outside $B(x_0, r)$, as test function in the weak formulation of $(1.1)_n$, we reach a contradiction as in the previous cases. This proves that u is a viscosity supersolution.

The proof that u is a viscosity subsolution runs as above and we omit the details. □

We next obtain uniform estimates (independent of n) for the sequence of approximations $(u_n)_n$.

Proposition 2.5. *Assume the set*

$$S = \left\{u \in W^{1,p^-}(\Omega)\colon\ u|_{\Omega \setminus \overline{D}} \in W^{1,p(x)}(\Omega \setminus \overline{D}),\ \|\nabla u\|_{L^\infty(D)} \leqslant 1 \text{ and } u|_{\partial\Omega} = f\right\}$$

is non-empty. Then u_n, the minimizer of F_n in S_n, satisfies

$$F_n(u_n) = \int_\Omega \frac{|\nabla u_n|^{p_n(x)}}{p_n(x)}\,dx \leqslant \int_D \frac{|\nabla v|^n}{n}\,dx + \int_{\Omega\setminus\overline{D}} \frac{|\nabla v|^{p(x)}}{p(x)}\,dx,$$

for every $v \in S$. Hence, the sequence $(F_n(u_n))_n$ is uniformly bounded and the sequence $(u_n)_n$ is uniformly bounded in $W^{1,p_-}(\Omega)$ and equicontinuous.

Proof. Recalling (2.2), the definition of S_n, observe that $S \subset S_n$, for every n. Since u_n is a minimizer, we have

$$F_n(u_n) \leqslant F_n(v), \quad \forall v \in S.$$

Hence, picking an element $v \in S \neq \emptyset$,

$$\begin{aligned} F_n(u_n) = \int_\Omega \frac{|\nabla u_n|^{p_n(x)}}{p_n(x)}\,dx &\leqslant \int_\Omega \frac{|\nabla v|^{p_n(x)}}{p_n(x)}\,dx \\ &= \int_D \frac{|\nabla v|^n}{n}\,dx + \int_{\Omega\setminus\overline{D}} \frac{|\nabla v|^{p(x)}}{p(x)}\,dx \\ &\leqslant |D| + \int_{\Omega\setminus\overline{D}} \frac{|\nabla v|^{p(x)}}{p(x)}\,dx \equiv C_*. \end{aligned}$$

In order to estimate the Sobolev norm, we first use Poincaré inequality and the boundary data, to obtain

$$\begin{aligned} \|u_n\|_{W^{1,p_-}(\Omega)} &\leqslant \|u_n - f\|_{W_0^{1,p_-}(\Omega)} + \|f\|_{W^{1,p_-}(\Omega)} \\ &\leqslant C\big\|\nabla(u_n - f)\big\|_{L^{p_-}(\Omega)} + \|f\|_{W^{1,\infty}(\Omega)} \\ &\leqslant C\|\nabla u_n\|_{L^{p_-}(\Omega)} + (C+1)\|f\|_{W^{1,\infty}(\Omega)}. \end{aligned}$$

We proceed, using Hölder inequality and elementary computations, to get

$$\begin{aligned} \|\nabla u_n\|_{L^{p_-}(\Omega)} &= \left(\int_\Omega |\nabla u_n|^{p_-}\,dx\right)^{\frac{1}{p_-}} \\ &\leqslant \left(\int_D |\nabla u_n|^{p_-}\,dx\right)^{\frac{1}{p_-}} + \left(\int_{\Omega\setminus\overline{D}} |\nabla u_n|^{p_-}\,dx\right)^{\frac{1}{p_-}} \\ &= \left(\int_D |\nabla u_n|^{p_-}\,dx\right)^{\frac{1}{p_-}} + \left(\int_{(\Omega\setminus\overline{D})\cap\{|\nabla u_n|\leqslant 1\}} |\nabla u_n|^{p_-}\,dx\right)^{\frac{1}{p_-}} \\ &\quad + \left(\int_{(\Omega\setminus\overline{D})\cap\{|\nabla u_n|> 1\}} |\nabla u_n|^{p_-}\,dx\right)^{\frac{1}{p_-}} \\ &\leqslant |D|^{\frac{1}{p_-}-\frac{1}{n}}\left(\int_D |\nabla u_n|^n\right)^{\frac{1}{n}} + |\Omega| + \left(\int_{\Omega\setminus\overline{D}} |\nabla u_n|^{p(x)}\,dx\right)^{\frac{1}{p_-}}. \end{aligned}$$

Since we have the bounds

$$\left(\int_D |\nabla u_n|^n\right)^{\frac{1}{n}} = n^{\frac{1}{n}}\left(\int_D \frac{|\nabla u_n|^n}{n}\,dx\right)^{\frac{1}{n}} \leqslant n^{\frac{1}{n}}\big(F_n(u_n)\big)^{\frac{1}{n}} \leqslant 2C_*$$

and

$$\int_{\Omega\setminus\overline{D}} |\nabla u_n|^{p(x)}\,dx \leqslant p_+ \int_{\Omega\setminus\overline{D}} \frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx \leqslant p_+ F_n(u_n) \leqslant p_+ C_*,$$

we conclude that the sequence $(u_n)_n$ is uniformly bounded in $W^{1,p_-}(\Omega)$ and, recalling the embedding in (2.3), that it is equicontinuous. □

3. Variational and viscosity limit

We first analyze the case in which $\partial\Omega \cap \overline{D} \neq \emptyset$ and the Lipschitz constant of $f|_{\partial\Omega\cap\overline{D}}$ is greater than one. Note that, in this case, $S = \emptyset$ since any Lipschitz extension u of this datum to D verifies $\|\nabla u\|_{L^\infty(D)} > 1$.

Theorem 3.1. *Assume that $\partial\Omega \cap \overline{D} \neq \emptyset$ and the Lipschitz constant of $f|_{\partial\Omega\cap\overline{D}}$ is strictly greater than one. Then, we have*

$$\liminf_{n\to\infty}\big(F_n(u_n)\big)^{\frac{1}{n}} > 1;$$

hence, $F_n(u_n) \to \infty$ and the natural energy associated to u_n is unbounded.

Proof. Consider the absolutely minimizing Lipschitz extension (AMLE) of $f|_{\partial\Omega\cap\overline{D}}$ to D, which is well defined even if the datum $f|_{\partial\Omega\cap\overline{D}}$ is not given in the whole ∂D. In this case, the AMLE is characterized, as proved in [16] and [8], as the unique solution of the problem

$$\begin{cases} -\Delta_\infty u(x) = 0, & x \in D,\\ u(x) = f(x), & x \in \partial\Omega \cap \overline{D},\\ \dfrac{\partial u}{\partial \nu}(x) = 0, & x \in \partial D \setminus \partial\Omega. \end{cases}$$

Let $\lambda > 1$ be the Lipschitz constant in D of this AMLE. Suppose that

$$\liminf_{n\to\infty}\big(F_n(u_n)\big)^{\frac{1}{n}} = \beta < \lambda$$

and consequently that

$$\liminf_{n\to\infty}\left(\int_D \frac{|\nabla u_n|^n}{n}\,dx\right)^{\frac{1}{n}} \leqslant \beta.$$

Fix $m \geqslant p_-$ and take $n > m$. By Hölder's inequality,

$$\left(\int_D |\nabla u_n|^m\right)^{\frac{1}{m}} \leqslant |D|^{\frac{1}{m}-\frac{1}{n}}\left(\int_D |\nabla u_n|^n\right)^{\frac{1}{n}} \leqslant |D|^{\frac{1}{m}-\frac{1}{n}} n^{\frac{1}{n}}\left(\int_D \frac{|\nabla u_n|^n}{n}\,dx\right)^{\frac{1}{n}}.$$

Taking the limit in n, we conclude

$$\liminf_{n\to\infty}\left(\int_D |\nabla u_n|^m\right)^{\frac{1}{m}} \leqslant |D|^{\frac{1}{m}}\beta,$$

so, for a subsequence, there exists a weak limit in $W^{1,m}(D)$, that we denote by u_∞. This weak limit has to verify the inequality

$$\left(\int_D |\nabla u_\infty|^m\right)^{\frac{1}{m}} \leqslant |D|^{\frac{1}{m}}\beta$$

for every m. Thus, taking the limit $m\to\infty$, we get that $u_\infty \in W^{1,\infty}(D)$ and, moreover,

$$|\nabla u_\infty| \leqslant \beta, \quad \text{a.e. } x\in D.$$

But this is a contradiction since λ is the Lipschitz constant in D of the AMLE of $f|_{\partial\Omega\cap\overline{D}}$ to D. We conclude that

$$\liminf_{n\to\infty}\bigl(F_n(u_n)\bigr)^{\frac{1}{n}} \geqslant \lambda$$

and the result follows. □

Remark 3.2. The AMLE problem has been extensively studied in the literature: see [4,13], the survey [5], and the recent approach using tug-of-war games of [8,16] and [17].

Remark 3.3. If $\partial\Omega\cap\overline{D}=\emptyset$, then $S\neq\emptyset$; indeed, we can consider a function that is constant in D and coincides with f on $\partial\Omega$, and extend it as a Lipschitz function to the whole of Ω, thus obtaining an element of S.

We now focus on the main case $S\neq\emptyset$. Recall that solutions to $(1.1)_n$ are minima of the functional

$$F_n(u)=\int_\Omega \frac{|\nabla u|^{p_n(x)}}{p_n(x)}\,dx$$

in

$$S_n=\bigl\{u\in W^{1,p_n(x)}(\Omega)\colon\ u|_{\partial\Omega}=f\bigr\}.$$

The limit of these variational problems is given by minimizing

$$F(u)=\int_{\Omega\setminus\overline{D}} \frac{|\nabla u|^{p(x)}}{p(x)}\,dx \tag{3.1}$$

in

$$S=\bigl\{u\in W^{1,p^-}(\Omega)\colon\ u|_{\Omega\setminus\overline{D}}\in W^{1,p(x)}(\Omega\setminus\overline{D}),\ \|\nabla u\|_{L^\infty(D)}\leqslant 1 \text{ and } u|_{\partial\Omega}=f\bigr\}.$$

Theorem 3.4. *Assume that $S\neq\emptyset$ and let u_n be minimizers of F_n in S_n. Then, along subsequences, $(u_n)_n$ converges uniformly in $\overline{\Omega}$, weakly in $W^{1,m}(D)$, for every $m\geqslant p_-$, and weakly in $W^{1,p(x)}(\Omega\setminus\overline{D})$ to u_∞, a minimizer of F in S. Moreover, the limit u_∞ is ∞-harmonic in D, i.e.,*

$$-\Delta_\infty u_\infty=0 \quad \text{in } D,$$

in the viscosity sense. Finally, the limit u_∞ is unique, in the sense that any other minimizer of F in S that is ∞-harmonic in D coincides with u_∞.

Proof. We use the estimates obtained in the previous section. Since the sequence $(u_n)_n$ is equicontinuous and uniformly bounded, by Arzelà–Ascoli theorem it converges (along subsequences) uniformly in $\overline{\Omega}$; the weak convergence in the space $W^{1,m}(D)$, for every $m\geqslant p_-$, is obtained as in the proof of Theorem 3.1 and the weak convergence in $W^{1,p(x)}(\Omega\setminus\overline{D})$ follows from the estimates in Proposition 2.5.

Also as before, we get that $u_\infty \in W^{1,\infty}(D)$, with $|\nabla u_\infty| \leqslant 1$, a.e. $x \in D$, thus concluding that $u_\infty \in S$. On the other hand, also from Proposition 2.5, we get

$$\int_{\Omega\setminus\overline{D}} \frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx \leqslant F_n(u_n) \leqslant F_n(v) \to \int_{\Omega\setminus\overline{D}} \frac{|\nabla v|^{p(x)}}{p(x)}\,dx$$

and we conclude that

$$F(u_\infty) = \int_{\Omega\setminus\overline{D}} \frac{|\nabla u_\infty|^{p(x)}}{p(x)}\,dx \leqslant \int_{\Omega\setminus\overline{D}} \frac{|\nabla v|^{p(x)}}{p(x)}\,dx = F(v), \quad \forall v \in S$$

so that u_∞ is a minimizer for F in S.

That a uniform limit of n-harmonic functions is ∞-harmonic is a well-known fact (cf., for example, [7] or [13]).

To prove the uniqueness, suppose we have two minimizers in S, u_1 and u_2. Then, considering

$$v = \frac{u_1 + u_2}{2} \in S,$$

we obtain that they coincide in $\Omega \setminus \overline{D}$ since F is a strictly convex functional in S. Using the uniqueness of solutions of the Dirichlet problem for the ∞-Laplacian in D (note that u_1 coincides with u_2 on the whole of ∂D), we conclude that $u_1 = u_2$ also in D. We conclude, in particular, that the whole sequence u_n converges uniformly in $\overline{\Omega}$. $\square$

Our next task is to pass to the limit in (2.5), the problem satisfied by u_n in the viscosity sense, to identify the problem solved by u_∞. We are under the assumption $S \neq \emptyset$ and we recall that

$$u_n \to u_\infty$$

uniformly in $\overline{\Omega}$.

Theorem 3.5. *Every uniform limit of a sequence $\{u_n\}$ of solutions of $(1.1)_n$ is a viscosity solution of*

$$\begin{cases} -\Delta_{p(x)} u(x) = 0, & x \in \Omega \setminus \overline{D}, \\ -\Delta_\infty u(x) = 0, & x \in D, \\ \operatorname{sgn}\big(|\nabla u|(x) - 1\big)\operatorname{sgn}\left(\dfrac{\partial u}{\partial \nu}(x)\right) = 0, & x \in \partial D \cap \Omega, \\ u(x) = f(x), & x \in \partial\Omega. \end{cases} \tag{3.2}$$

Proof. Since $u_n(x) = f(x)$, for $x \in \partial\Omega$, it is clear that $u(x) = f(x)$, for $x \in \partial\Omega$.

Let u_∞ be a uniform limit of $\{u_n\}$ and let ϕ be a test function such that $u_\infty(x_0) = \phi(x_0)$ and $u_\infty - \phi$ has a strict minimum at $x_0 \in \Omega$. Depending on the location of the point x_0 we have different situations.

If $x_0 \in D$, we encounter the standard fact the uniform limit of n-harmonic functions is ∞-harmonic.

If $x_0 \in \Omega \setminus \overline{D}$, consider a sequence of points x_n such that $x_n \to x_0$ and $u_n - \phi$ has a minimum at x_n, with $x_n \in \Omega \setminus \overline{D}$ for n large. Using the fact that u_n is a viscosity solution of (2.5), we obtain

$$-\Delta_{p_n(x_n)}\phi(x_n) \geqslant 0.$$

Now we observe that $p_n(x) = p(x)$ in a neighborhood of x_0 and hence, taking the limit as $n \to \infty$, we get

$$-\Delta_{p(x_0)}\phi(x_0) \geqslant 0.$$

That is, u_∞ is a viscosity supersolution of $-\Delta_{p(x)} u_\infty = 0$ in $\Omega \setminus \overline{D}$.

If $x_0 \in \partial D \cap \Omega$, we have to show that

$$\max \begin{Bmatrix} -\Delta_{p(x_0)}\phi(x_0) \\ -\Delta_\infty \phi(x_0) \\ \operatorname{sgn}(|\nabla\phi|(x_0) - 1)\operatorname{sgn}(\frac{\partial\phi}{\partial\nu}(x_0)) \end{Bmatrix} \geqslant 0.$$

Again, since u_n converges to u uniformly, there exists a sequence of points x_n converging to x_0 such that $u_n - \phi$ has a minimum at x_n. We distinguish several cases.

Case 1. There exists infinitely many n such that $x_n \in D$.

Then we have, by Proposition 2.4,

$$-\Delta_n \phi(x_n) = -\left|\nabla\phi(x_n)\right|^{n-2}\Delta\phi(x_n) - (n-2)\left|\nabla\phi(x_n)\right|^{n-4}\Delta_\infty\phi(x_n) \geqslant 0.$$

If $\nabla\phi(x_0) = 0$, we get $-\Delta_\infty\phi(x_0) = 0$. If this is not the case, we have that $\nabla\phi(x_n) \neq 0$, for large n, and then

$$-\Delta_\infty\phi(x_n) \geqslant \frac{1}{n-2}\left|\nabla\phi(x_n)\right|^2\Delta\phi(x_n) \to 0, \quad \text{as } n \to \infty.$$

We conclude that

$$-\Delta_\infty\phi(x_0) \geqslant 0.$$

Case 2. There exists infinitely many n such that $x_n \in \Omega \setminus \overline{D}$.

Then we have, by Proposition 2.4,

$$-\Delta_{p_n(x_n)}\phi(x_n) \geqslant 0.$$

Proceeding as before, we get

$$-\Delta_{p(x_0)}\phi(x_0) \geqslant 0.$$

Case 3. There exists infinitely many n such that $x_n \in \partial D \cap \Omega$.

In this case, we have

$$\left|\nabla\phi(x_n)\right|^{n-2}\frac{\partial\phi}{\partial\nu}(x_n) - \left|\nabla\phi(x_n)\right|^{p(x_n)-2}\frac{\partial\phi}{\partial\nu}(x_n) \geqslant 0.$$

Hence, we get

$$\frac{\partial\phi}{\partial\nu}(x_n) \leqslant \left|\nabla\phi(x_n)\right|^{n-p(x_n)}\frac{\partial\phi}{\partial\nu}(x_n).$$

Taking $n \to \infty$, we deduce that

$$|\nabla\phi|(x_0) > 1 \quad \Rightarrow \quad \frac{\partial\phi}{\partial\nu}(x_0) \geqslant 0,$$

and

$$|\nabla\phi|(x_0) < 1 \quad \Rightarrow \quad \frac{\partial\phi}{\partial\nu}(x_0) \leqslant 0.$$

That is

$$\operatorname{sgn}\big(|\nabla\phi|(x_0) - 1\big)\operatorname{sgn}\left(\frac{\partial\phi}{\partial\nu}(x_0)\right) \geqslant 0.$$

This concludes the proof that u_∞ is a viscosity supersolution.

The proof that u is a viscosity subsolution runs as above and we omit the details. □

4. More on the set S

We have already observed the following two facts concerning the non-emptiness of the set S:

(1) If $\partial\Omega \cap \overline{D} = \emptyset$, then $S \neq \emptyset$.
(2) If $\partial\Omega \cap \overline{D} \neq \emptyset$ and the Lipschitz constant of $f|_{\partial\Omega\cap\overline{D}}$ is greater than one, then any Lipschitz extension u of this datum to D verifies $\|\nabla u\|_{L^\infty(D)} > 1$ and, consequently, $S = \emptyset$.

The question naturally arises of whether the condition that the Lipschitz constant of $f|_{\partial\Omega\cap\overline{D}}$ is less than or equal to one is, not only necessary, but also sufficient to guarantee that $S \neq \emptyset$.

Suppose we are given a Lipschitz boundary data f such that the Lipschitz constant of $f|_{\partial\Omega\cap\overline{D}}$ is less than or equal to one. A natural attempt to construct a function in S would be the following:

- consider the unique AMLE of $f|_{\partial\Omega\cap\overline{D}}$ to D, which is such that the L^∞-norm of its gradient is less than or equal to one;
- extend it to the whole of Ω using any function in $W^{1,p(x)}(\Omega \setminus \overline{D})$ that coincides with it on ∂D and with f on $\partial\Omega$.

The boundary datum on $\partial(\Omega \setminus D)$ that one has to extend is given by f on $\partial\Omega \setminus \overline{D}$ and by the restriction of the AMLE to $\partial D \cap \Omega$. The problem is that the extension to Ω may not always be possible. However, if this boundary data on $\partial(\Omega \setminus D)$ is Lipschitz, then we could indeed consider a Lipschitz extension to Ω.

We first give an example of a particular geometric configuration for which this is the case. Therefore the condition that the Lipschitz constant of $f|_{\partial\Omega\cap\overline{D}}$ is less than or equal to one does indeed suffice to guarantee that $S \neq \emptyset$. Let $\Omega = B(0,1)$ in $\mathbb{R}^2$ and let

$$D = B(0,1) \cap \{x > 0\}$$

be the right half-ball (here, (x, y) denote coordinates in $\mathbb{R}^2$). We still denote by f the obtained boundary datum on the boundary of the half disc $B(0,1) \setminus D$, which is Lipschitz on $\partial B(0,1) \cap \{x \leqslant 0\}$ and on $\partial D = \{(x,y)\colon\ x = 0, -1 \leqslant y \leqslant 1\}$, and continuous on the whole boundary. Let $(0, y) \in \partial D \cap B(0,1)$ and $(z, w) \in \partial B(0,1)$ with $w \geqslant 0$ and $y \geqslant 0$ (the other possible cases would have to be considered separately). Adding and subtracting $f((0,1))$ in the numerator we obtain

$$\frac{|f((0,y)) - f((z,w))|}{\|(0,y) - (z,w)\|} \leqslant \frac{|f((0,y)) - f((0,1))|}{\|(0,y) - (0,1)\|} + 2\frac{|f((z,w)) - f((0,1))|}{\|(0,1) - (z,w)\|} \leqslant C,$$

since $\|(0,y) - (0,1)\| \leqslant \|(0,y) - (z,w)\|$, $\|(0,1) - (z,w)\| \leqslant 2\|(0,y) - (z,w)\|$ and f is Lipschitz on ∂D and $\partial\Omega$. This shows that f is Lipschitz on the whole boundary of $B(0,1) \setminus D$.

This construction does not always work since it may happen that the obtained boundary data is not a Lipschitz function. Here is a counter-example: let $\Omega = B((0,0),1)$ and $D = B((1/2,0),1/2)$ in $\mathbb{R}^2$. These two balls are tangent at the point $(1,0)$. Now let f be given in polar coordinates by

$$f(\theta) = \begin{cases} |\theta|, & 0 \leqslant |\theta| \leqslant \pi/2, \\ \pi - |\theta|, & \pi/2 < |\theta| < \pi. \end{cases}$$

This function is Lipschitz on $\partial\Omega$. The unique AMLE of $f|(1,0)$ to D is given by $u \equiv 0$. Now, we have the function

$$\tilde{f}(x) = \begin{cases} f(x), & x \in \partial B((0,0),1), \\ 0, & x \in \partial B((1/2,0),1/2), \end{cases}$$

defined on $\partial(\Omega \setminus D)$. Observe that there are points on ∂D of the form $r(\theta) = \cos(\theta)$, with $\theta \sim 0$. For $\theta > 0$,

$$\lim_{\theta\to 0+} \frac{\tilde{f}(\cos(\theta), \sin(\theta)) - \tilde{f}(\cos(\theta)(\cos(\theta), \sin(\theta)))}{\|(\cos(\theta), \sin(\theta)) - (\cos(\theta)(\cos(\theta), \sin(\theta)))\|} = \lim_{\theta\to 0+} \frac{\theta - 0}{1 - \cos(\theta)} = +\infty,$$

hence the function is not Lipschitz.

5. The one-dimensional case

In this section, we analyze with some detail the one-dimensional case, which is easier since the equation reduces to an ODE.

Let $\Omega = (0,1)$ and assume $p(x) \equiv \infty$ for $x \in (0,\xi)$. Then the problem at level n reads

$$\begin{cases} \left(\left|u_n'\right|^{p_n(x)-2} u_n'\right)'(x) = 0, \\ u_n(0) = f(0), \\ u_n(1) = f(1). \end{cases}$$

To simplify, we assume that $f(0) = 0$ and $f(1) > 0$. Then, integrating the equation, we get

$$\left|u_n'\right|^{p_n(x)-2} u_n'(x) = C_1.$$

Assuming that $u_n' \geqslant 0$, we get

$$u_n'(x) = (C_1)^{\frac{1}{p_n(x)-1}}.$$

Thus

$$u_n(x) = \int_0^x (C_1)^{\frac{1}{p_n(s)-1}}\, ds$$

and the constant C_1 (that must be positive and depends on n) verifies

$$f(1) = \int_0^1 (C_1)^{\frac{1}{p_n(s)-1}}\, ds.$$

Since $f(1)$ is finite, we conclude that C_1 must be bounded; if not,

$$\lim_{n\to\infty} u_n(x) = u_\infty(x) = +\infty$$

in the whole interval $(\xi, 1]$ and this contradicts $u_n(1) = f(1)$. Therefore, we can assume (taking a subsequence if necessary) that

$$\lim_{n\to\infty} C_1(n) = C_\infty.$$

Case 1. When $C_\infty > 0$, we conclude that the limit of u_n is given by

$$u_\infty(x) = \lim_{n\to\infty} u_n(x) = \begin{cases} x, & x \in [0,\xi], \\ \xi + \int_\xi^x (C_\infty)^{\frac{1}{p(s)-1}}\, ds, & x \in [\xi, 1]. \end{cases}$$

As $u_n(1) = f(1)$, we realize that the constant C_∞ is determined by

$$\xi + \int_\xi^1 (C_\infty)^{\frac{1}{p(s)-1}}\, ds = f(1).$$

This case, $C_\infty > 0$, actually happens when $f(1) > \xi$. Since C_∞ is uniquely determined, we obtain the convergence of the whole sequence u_n.

Note that in this case we can verify that u_∞ is a minimizer of the functional F given by (3.1). Indeed, since $|u_\infty'|(x) \leqslant 1$, for $x \in [0,\xi]$, we have that $u_\infty \in S$ and since u_∞ is a solution of

$$\left(|u'|^{p(x)-2} u'\right)'(x) = 0, \qquad u(\xi) = \xi, \qquad u(1) = f(1),$$

we have that it minimizes the functional F, which in this case is given by

$$F(u_\infty)=\int_\xi^1 \frac{(C_\infty)^{\frac{p(s)}{p(s)-1}}}{p(s)}\,ds,$$

among functions that verify $u(\xi)=\xi$ and $u(1)=f(1)$.

Now, for any function $w\in S$, we have $|w'|(x)\leqslant 1$, for $x\in[0,\xi]$, and we get $w(\xi)\leqslant\xi$. Let z be the solution of

$$\left(|z'|^{p(x)-2}z'\right)'(x)=0,\qquad z(\xi)=w(\xi)\leqslant\xi,\qquad z(1)=f(1).$$

Then we have

$$F(w)\geqslant F(z)\geqslant F(u_\infty).$$

To see that the last inequality is true just use the monotonicity of the function

$$C\mapsto\int_\xi^1 \frac{(C)^{\frac{p(s)}{p(s)-1}}}{p(s)}\,ds$$

with respect to C.

Case 2. When $C_\infty=0$, we have that

$$\lim_{n\to\infty}u_n(x)=\begin{cases}Kx, & x\in[0,\xi],\\ K\xi, & x\in[\xi,1].\end{cases}$$

Here $K\leqslant 1$ is given by

$$K=\lim_{n\to\infty}\left(C_1(n)\right)^{\frac{1}{n}}$$

(recall that we are taking $p_n(x)=p(x)\wedge n$).

As $u_n(1)=f(1)$ we get that the constant K is given by

$$K\xi=f(1).$$

This case actually happens when $f(1)\leqslant\xi$. Since K is uniquely determined, we obtain the convergence of the whole sequence u_n.

Note that in this case the limit u_∞ is not differentiable, but it is Lipschitz. Also note that it is easy to verify that u_∞ is a minimizer of the functional F given by (3.1). Indeed, $F(u_\infty)=0$ and $F(w)\geqslant 0$, for every $w\in S$.

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