ISOPERIMETRIC INEQUALITIES FOR PARAMETRIC VARIATIONAL PROBLEMS

INÉGALITÉS ISOPÉRIMÉTRIQUES POUR LES PROBLÈMES VARIATIONNELS PARAMÉTRIQUES

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ABSTRACT. – We prove isoperimetric inequalities for general parametric variational double integrals $\mathcal{F}$, whose Lagrangians F depend on the position vector X and on the surface normal N. As an essential tool we introduce Sauvigny's F-conformal parameters adapted to the parametric integrand and use the notion of generalized mean and Gaussian curvature. The special cases of minimal surfaces, surfaces of bounded mean curvature and $\mathcal{F}$-minimizing surfaces are also discussed.

RÉSUMÉ. – Nous démontrerons des inégalités isopérimétriques pour les intégrales doubles variationnelles paramétriques générales $\mathcal{F}$ dont la fonction de Lagrange F dépend du vecteur de position X et de la surface normal N. Comme outil essentiel, nous introduiserons des paramètres F-conformes de Sauvigny adaptés l'intégrand paramétrique et nous employerons la notion de la courbure moyenne généralisée et gaussienne généralisée. Nous discuterons également les cas particuliers des surfaces minimales, des surfaces de courbure moyenne limitée et des surfaces minimalisant $\mathcal{F}$.

1. Introduction

We consider general parametric functionals of the form

$$\mathcal{F}(X) := \int_B F(X, X_u \wedge X_v)\, du\, dv, \tag{1}$$

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where $B \subset \mathbb{R}^2$ is the open unit disk in the plane. The Lagrangian F is of class $C^0(\mathbb{R}^3 \times \mathbb{R}^3) \cap C^{3,\alpha}(\mathbb{R}^3 \times (\mathbb{R}^3 - \{0\}))$ for some $\alpha \in (0, 1)$, and satisfies the homogeneity condition

$$F(y, tz) = t F(y, z) \quad \text{for all } t > 0 \text{ and } (y, z) \in \mathbb{R}^3 \times \mathbb{R}^3. \tag{H}$$

Moreover, we assume that F is elliptic, i.e., that the mapping $F_{zz}(y, z) : z^\perp \to z^\perp$ is positive definite for all $(y, z) \in \mathbb{R}^3 \times (\mathbb{R}^3 - \{0\})$, where [1]

$$z^\perp = \{\zeta \in \mathbb{R}^3 \mid \langle \zeta, z \rangle = 0\}.$$

Since F_{zz} is homogeneous of degree -1 in its second argument by (H), this implies that there exist constants $0 < M_1 \leqslant M_2 < \infty$ such that

$$M_1 |P_{z^\perp} \xi|^2 \leqslant |z| \langle \xi, F_{zz}(y, z)\xi \rangle \leqslant M_2 |P_{z^\perp} \xi|^2 \tag{E}$$

for all $(y, z) \in \overline{B_1(0)} \times (\mathbb{R}^3 - \{0\})$, $\xi \in \mathbb{R}^3$, where $P_{z^\perp} \xi = \xi - \langle \xi, z \rangle z / |z|^2$ is the projection onto $z^\perp$ and where $B_1(0)$ denotes the open unit ball in $\mathbb{R}^3$.

In addition, we suppose that X is an immersed surface of class $C^{2,\alpha}(\overline{B}, \mathbb{R}^3)$ that maps the boundary ∂B topologically onto a given closed Jordan curve $\Gamma \subset \mathbb{R}^3$ with length $L(\Gamma)$. The aim of the present paper is to prove an isoperimetric inequality for immersions X that are $\mathcal{F}$-critical, i.e., stationary for the parametric functional (1). To be more precise we are going to estimate the area

$$\mathcal{A}(X) := \int_X dA = \int_B |X_u \wedge X_v| \, du \, dv \tag{2}$$

in terms of quantities depending only on Γ and F. The area functional $\mathcal{A}$ itself is a special parametric functional of the form (1) with the integrand $F(y, z) = A(z) := |z|$. It is a well-known fact that $\mathcal{A}$-critical surfaces, i.e., minimal surfaces of the type of the disk satisfy the classical isoperimetric inequality

$$\mathcal{A}(X) \leqslant \frac{1}{4\pi} L^2(\Gamma), \tag{3}$$

see e.g. [6, Ch. 6.3]. Note that this is true for all surfaces with nonpositive Gauß curvature K (cf. [2,1]). More generally, disk-type surfaces of prescribed bounded mean curvature H, whose Gauß curvature may have varying sign, can be obtained as critical points of the parametric functional with the integrand

$$F(y, z) = E(y, z) := |z| + \langle Q(y), z \rangle, \tag{4}$$

where Q is a weakly differentiable vector field on $\mathbb{R}^3$ with $\operatorname{div} Q = H$ on $\mathbb{R}^3$, compare with Hildebrandt [8]. Under the assumption that

$$\|X(.)\|_{\infty, \overline{B}} := \sup_{w \in \overline{B}} |X(w)| \leqslant 1 \quad \text{and} \quad h := \|H(.)\|_{\infty, \mathbb{R}^3} < 2, \tag{5}$$

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[1] Note that (H) implies $F_{zz}(y, z)z = 0$ for all $(y, z) \in \mathbb{R}^3 \times (\mathbb{R}^3 - \{0\})$.

one can show that

$$\mathcal{A}(X) \leqslant \frac{1}{8(1-(h/2))} L^2(\Gamma) \tag{6}$$

for such surfaces, which is a result of Heinz and Hildebrandt [7]. [2] In [4] we generalized (3) to critical immersions of parametric functionals with an integrand $F(y, z) = F(z)$ depending only on z, which does not cover the case of surfaces of prescribed bounded mean curvature, cf. (4). In the present work we are going to prove the following isoperimetric inequality for critical immersions of general parametric double integrals (1), which contains all the previous cases.

THEOREM 1. – *Let $X \in C^{2,\alpha}(\overline{B}, \mathbb{R}^3)$, $\alpha \in (0, 1)$, be an $\mathcal{F}$-critical immersion with surface area $\mathcal{A}(X)$, which maps ∂B topologically onto a closed Jordan curve Γ. Assume that the parametric elliptic integrand $F = F(y, z)$ is of class $C^0(\mathbb{R}^3 \times \mathbb{R}^3) \cap C^{3,\alpha}(\mathbb{R}^3 \times (\mathbb{R}^3 - \{0\}))$, and set $\|F_{yz}\| := \|F_{y^i z^i}(.,.)\|_{\infty, \overline{B_1(0)} \times S^2}$. Then there is a constant $C = C(F)$ depending solely on F, such that if*

$$\|X(.)\|_{\infty, B} \leqslant 1, \tag{7}$$

$$h_F := \left[C(F)(1 + \|F_{yz}\|^2) + (\|F_{yz}\|/M_1) \right] < 2, \tag{8}$$

then

$$\mathcal{A}(X) \leqslant \frac{2C(F)\frac{M_2}{M_1}[\int_\Gamma k\, ds - 2\pi]L(\Gamma) + \sqrt{\frac{M_2}{M_1}} L^2(\Gamma)}{8(1-(h_F/2))}. \tag{9}$$

If $F(y, z) = A(z) = |z|$, or if $F(y, z) = E(y, z) = |z| + \langle Q(y), z \rangle$ for some $Q \in W_{loc}^{1,\infty}(\mathbb{R}^3, \mathbb{R}^3)$, then $C(F) = 0$ and $M_1 = M_2 = 1$.

Remarks. – 1. For the area integrand $A(z)$ one has $h_F = 0$, and (9) reduces to (3) for minimal surfaces. [3] In the case of immersed surfaces with bounded mean curvature as critical points of the parametric functional with the integrand $E(y, z)$, the inequality (9) simplifies to the estimate (6), since $h_F = h$ in this situation.

2. For a geometric interpretation of the term $\|F_{yz}\|$ in the definition of h_F in (8) we recall the notion of the *F-mean curvature* $H_F(X, N) = H_F := -\operatorname{tr}(A_F S)$ and the *F-Gauß curvature* $K_F(X, N) = K_F := \det(A_F S)$, as introduced in [4]. Here, $S: T_w B \to T_w B$ is the shape operator defined by $DX \circ S := DN$ on the tangent space $T_w B$, and $A_F: T_w B \to T_w B$ is the symmetric endomorphism given by

$$V \mapsto A_F(V) := (DX)^{-1}(F_{zz}(X, N) DX(V)). \tag{10}$$

For the special parametric integrands $A(z) = |z|$ and $E(y, z) = |z| + \langle Q(y), z \rangle$ the curvature functions H_F and K_F reduce to the classical mean curvature H and Gauß

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[2] In contrast to [7] and [8] we adopt the convention that the mean curvature H is the sum of the principal curvatures, which accounts for the factor $1/2$ in the denominator in (6).

[3] The factor 4π instead of 8 may be obtained using Wirtinger's Inequality without any smallness condition of the form (7), see e.g. [6, Ch. 6.3].

curvature K, respectively, since one has $A_A = A_E = \mathrm{Id}_{T_w B}$. The first author proved in [3] that the Euler equation for $\mathcal{F}$ can be written as

$$H_F = \sum_{i=1}^{3} F_{y^i z^i}(X, N), \tag{11}$$

which shows that (9) may be regarded as an isoperimetric inequality for immersed $\mathcal{F}$-critical surfaces with bounded F-mean curvature. In the proof of Theorem 1 we actually work with $\|H_F\|_{\infty, B}$ instead of $\|F_{yz}\|$. Consequently, (9) *is also valid for immersions that are not necessarily $\mathcal{F}$-critical but whose F-mean curvature H_F is a prescribed bounded function on $\overline{B} \subset \mathbb{R}^2$*, we merely have to replace the term $\|F_{yz}\|$ in (8) by $\|H_F\|_{\infty, B}$. Hence, the isoperimetric inequality in [7, Theorem 1′] is a special case of Theorem 1 above in the context of immersed surfaces.

Finally, for parametric integrands $F(y, z) = F(z)$ depending only on the z-variable, $\mathcal{F}$-critical immersions have vanishing F-mean curvature H_F. By a slight modification of the proof of Theorem 1 in Section 3 below, one obtains the isoperimetric inequality presented in [4].

3. The regularity assumption on X (which also implies $\Gamma \in C^{2,\alpha}$) is due to the use of the Gauß–Bonnet Theorem. It is not clear whether one can relax the assumptions to $X \in C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$ and to closed curves Γ that are merely rectifiable as in the case of surfaces with bounded mean curvature. In addition, it is an open question how to treat $\mathcal{F}$-critical surfaces with branch points for general parametric functionals.

We are going to prove an isoperimetric inequality slightly stronger than (9) using the radius $R_\Gamma(X)$ of the smallest ball in $\mathbb{R}^3$ containing the curve Γ, which can be expressed as

$$R_\Gamma(X) := \inf_{q \in \mathbb{R}^3} \|X(.) - q\|_{\infty, \partial B}. \tag{12}$$

THEOREM 2. – *Under the assumptions of Theorem* 1 *the following isoperimetric inequality holds true*:

$$\mathcal{A}(X) \leqslant R_\Gamma(X) \cdot \frac{2C(F)\frac{M_2}{M_1}[\int_\Gamma k\, ds - 2\pi] + \sqrt{\frac{M_2}{M_1}}L(\Gamma)}{2 - R_\Gamma(X)h_F}. \tag{13}$$

If the boundary curve Γ is long in comparison to the radius $R_\Gamma(X)$, then the estimate (13) is better than (9), since it depends at most linearly on the length $L(\Gamma)$. Theorem 2 reduces to the *linear isoperimetric inequality* for minimal surfaces presented in [6, p. 388], because $h_F = C(F) = 0$ and $M_1 = M_2 = 1$ in that case. Notice that Theorem 1 follows from Theorem 2 by (7) and a simple geometric observation that leads to

$$R_\Gamma(X) \leqslant \min\left\{1, \frac{L(\Gamma)}{4}\right\}. \tag{14}$$

If X *minimizes* the functional (1) within the class of surfaces bounded by Γ, one merely needs to assume that the continuous parametric integrand F satisfies the growth

condition

$$m_1|z| \leqslant F(y, z) \leqslant m_2|z| \tag{15}$$

for some constants $0 < m_1 \leqslant m_2 < \infty$ without further regularity assumptions on F. Moreover, the minimizer X does not have to be immersed, and it suffices to know that X is in the Sobolev class $W^{1,2}(B, \mathbb{R}^3)$, and that Γ is a rectifiable closed Jordan curve.

THEOREM 3. – *Let $X \in W^{1,2}(B, \mathbb{R}^3)$ be a minimizer for the parametric functional* (1) *within the class of mappings $Z \in W^{1,2}(B, \mathbb{R}^3)$ such that the trace $Z|_{\partial B}$ on ∂B is a continuous, weakly monotonic mapping* [4] *of ∂B onto a closed rectifiable Jordan curve Γ. Assume that the Lagrangian F of class $C^0(\mathbb{R}^3 \times \mathbb{R}^3)$ satisfies* (15). *Then*

$$\mathcal{A}(X) \leqslant \frac{m_2}{4\pi m_1} L^2(\Gamma). \tag{16}$$

Remark. – In this context no ellipticity condition is needed. In order to prove the *existence* of a minimizer for (1), however, one has to assume convexity of F in the second argument, see [9,10].

To describe the strategy for proving Theorem 1 without getting involved with the technical details which arise in the general case, let us give a short proof of the isoperimetric inequality (6) for surfaces of bounded mean curvature. Such surfaces satisfy the partial differential equations

$$\Delta X = H(X) X_u \wedge X_v, \tag{17}$$

$$|X_u|^2 = |X_v|^2, \qquad \langle X_u, X_v \rangle = 0 \tag{18}$$

on the domain B, where H is a given bounded function on $\mathbb{R}^3$. Using (18) and integrating by parts we may write

$$\mathcal{A}(X) = \mathcal{D}(X) := \frac{1}{2} \int_B |\nabla X(w)|^2 \, dw$$

$$\leqslant -\frac{1}{2} \int_B \langle \Delta X(w), X(w) - q \rangle + \frac{1}{2} \int_{\partial B} |X_r(w)| \cdot |X(1, \theta) - q| \, d\theta \tag{19}$$

for any $q \in \mathbb{R}^3$, where for $w = (u, v) = re^{i\theta} \in \overline{B}$ we have identified $X(w)$ with $X(r, \theta)$. We observe that there is a vector $q^* \in \mathbb{R}^3$ with $\|X(.) - q^*\|_{\infty, \partial B} = R_\Gamma(X) \leqslant 1$ by assumption (5). Inserting Eq. (17) into (19) we infer for $q = q^*$

$$\mathcal{A}(X) \leqslant \frac{1}{2} \int_B |H(X(w))||X(w) - q^*||X_u \wedge X_v| \, dw + \frac{1}{2} \|X(.) - q^*\|_{\infty, \partial B} L(\Gamma)$$

$$\leqslant \frac{1}{2} h R_\Gamma(X) \mathcal{A}(X) + \frac{1}{2} R_\Gamma(X) L(\Gamma),$$

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[4] See [6, Ch. 4.2] for the notion of weakly monotonic mappings on the boundary.

where we have used that $|X_\theta| = |X_r|$ on ∂B by (18), and the fact that

$$\|X(.) - q^*\|_{\infty, B} \leqslant \|X(.) - q^*\|_{\infty, \partial B} \tag{20}$$

by an application of the maximum principle for (17). In fact, one can show that the function $f(w) := |X(w) - q^*|^2$ is a subsolution for the Laplace operator under the smallness assumption (5), see [7, Lemma 1]. With (14) we arrive at (6).

In Section 2 we introduce the tools necessary to take up the approach outlined above. Proposition 2.2 gives the inclusion principle generalizing (20), the conformality relations are suitably adapted to the general parametric integrand F in (29), (30). The partial differential equation (17) will be replaced by a differential inequality in Proposition 2.4 containing first derivatives of X and of its normal N on the right-hand side. Therefore it is necessary to estimate the Dirichlet energy $\mathcal{D}(N)$ of the normal of an $\mathcal{F}$-critical immersion, which will be done using the Gauß–Bonnet Theorem, see Lemma 2.5. Section 3 contains the proof for Theorem 2 along the lines of the arguments described above, as well as a short proof of Theorem 3.

2. Inclusion theorem and F-conformal parameters

Let $X : M \hookrightarrow \mathbb{R}^3$ be an immersion of an orientable smooth manifold M of dimension 2 into $\mathbb{R}^3$, where $X \in C^2(M, \mathbb{R}^3)$. We are going to work with the induced metric

$$g(V, W) := \langle DX(V), DX(W) \rangle \quad \text{for } V, W \in T_p M$$

and the globally well-defined normal mapping $N : M \to S^2$. Consider the parametric variational integral

$$\mathcal{F}(X) := \int_M F(X, N) \, dA \tag{21}$$

with an elliptic parametric Lagrangian $F = F(y, z) \in C^0(\mathbb{R}^3 \times \mathbb{R}^3) \cap C^{3,\alpha}(\mathbb{R}^3 \times (\mathbb{R}^3 - \{0\}))$ satisfying the homogeneity condition (H).

As a starting point we give a generalization of the well-known identity $\Delta_M X = HN$, where $\Delta_M = \operatorname{div}_M \operatorname{grad}_M$ is the Laplace–Beltrami operator associated to M. To this end we introduce the differential operator

$$\theta_F[\varphi] := \operatorname{div}_M(A_F \operatorname{grad}_M \varphi) - (\operatorname{div}_M A_F)[\varphi] \quad \text{for } \varphi \in C^2(M),$$

where A_F is defined in (10) of the introduction. In [3] the following result is proved:

THEOREM 2.1. – *Let $X \in C^2(M, \mathbb{R}^3)$ be an immersion with normal N and F-mean curvature H_F. Then we have*

$$\theta_F X = H_F N. \tag{22}$$

Let us point out that according to the ellipticity condition (E) the eigenvalues of $F_{zz}(y, z) : z^\perp \to z^\perp$ are bounded by M_1 and M_2 for $|y| \leqslant 1$ in the case of elliptic integrands as defined in (E). Obviously, θ_F is an elliptic operator, if F is elliptic.

This allows us to prove the following inclusion principle for $\mathcal{F}$-critical immersions of a smooth manifold M with boundary ∂M, where we set $\overline{M} := M \cup \partial M$.

PROPOSITION 2.2. – *Let F be an elliptic integrand and X be an immersion of class* $C^2(M, \mathbb{R}^3) \cap C^0(\overline{M}, \mathbb{R}^3)$ *such that*

$$X(\overline{M}) \subset \overline{B_1(0)} \subset \mathbb{R}^3, \quad and \quad \|H_F\|_{\infty, M} < 2M_1. \tag{23}$$

Then $X(\partial M) \subset \overline{B_r(p)}$ *for some* $p \in \mathbb{R}^3$ *and some radius* $0 < r \leqslant 1$, *implies* $X(\overline{M}) \subset \overline{B_r(p)}$.

Proof. – We apply a continuity method as in [7]. For $\sigma \in [0, 1]$ we consider the family of surfaces $X_\sigma(w) := X(w) - \sigma p$ and note that by our assumptions (23) one has

$$\|X_\sigma(.)\|_{\infty, \partial M} = \|\sigma(X(.) - p) + (1 - \sigma)X(.)\|_{\infty, \partial M} \leqslant 1. \tag{24}$$

Using the chain rule we calculate on M

$$\theta_F X_\sigma^2 = 2\sum_{i=1}^{3} X_\sigma^i \operatorname{div}_M\left(A_F \operatorname{grad}_M(X^i)\right)$$

$$+ 2\sum_{i=1}^{3} g\left(\operatorname{grad}_M(X_\sigma^i), A_F \operatorname{grad}_M(X^i)\right) - 2\sum_{i=1}^{3} X_\sigma^i(\operatorname{div}_M A_F)(X^i)$$

$$= 2\sum_{i=1}^{3} X_\sigma^i \theta_F X^i + 2\sum_{i=1}^{3} g\left(\operatorname{grad}_M(X_\sigma^i), A_F \operatorname{grad}_M(X^i)\right)$$

$$\underset{(22)}{=} 2H_F \sum_{i=1}^{3} X_\sigma^i N^i + 2\sum_{i=1}^{3} g\left(\operatorname{grad}_M(X^i), A_F \operatorname{grad}_M(X^i)\right)$$

$$\geqslant -2\|H_F\|_{\infty, M}\|X_\sigma(.)\|_{\infty, M} + 4M_1.$$

If the condition

$$\|H_F\|_{\infty, M}\|X_\sigma(.)\|_{\infty, M} \leqslant 2M_1 \tag{25}$$

holds true for every $\sigma \in [0, 1]$, then X_σ^2 and in particular $X_1^2(w) = (X(w) - p)^2$ is a subsolution for the elliptic operator θ_F, which implies by the maximum principle the statement of the proposition. The condition (25) is certainly valid for each $\sigma \in [0, 1]$ if $\|H_F\|_{\infty, M} = 0$. Hence by virtue of (23) we may assume that $0 < \|H_F\|_{\infty, M} < 2M_1$. Then one has either

$$\|X_\sigma(.)\|_{\infty, M} > \frac{2M_1}{\|H_F\|_{\infty, M}} =: c_1 \underset{(23)}{>} 1, \tag{26}$$

or (25) is true, which implies according to (24) and the maximum principle

$$\|X_\sigma(.)\|_{\infty, M} \leqslant 1. \tag{27}$$

Note that the function $\sigma \mapsto \|X_\sigma(.)\|_{\infty, M}$ is continuous on $[0, 1]$, and that $\|X_0(.)\|_{\infty, M} \leqslant 1$. If there were some $\sigma_1 \in (0, 1]$ with (26), i.e., with $\|X_{\sigma_1}(.)\|_{\infty, M} > c_1 > 1$, then

by the intermediate value theorem there would be some parameter $\sigma_2 \in (0, \sigma_1)$ with $\|X_{\sigma_2}(.)\|_{\infty, M} = c_1$. But this would contradict our observation that for any $\sigma \in [0, 1]$ either (26) or (27) must hold. Consequently, (26) is not possible for any $\sigma \in [0, 1]$, which proves (25). $\square$

From now on we focus on immersions $X : B \to \mathbb{R}^3$ of $B \subset \mathbb{R}^2$ into $\mathbb{R}^3$ of class $C^2(B, \mathbb{R}^3)$, where B is the open unit disk in $\mathbb{R}^2$. In this case a special parametrization simplifies the situation. We introduce so-called *F-conformal parameters* defined via the metric

$$g_F(V, W) := g\big(A_F^{-1} V, W\big) \quad \text{for } V, W \in T_x B \tag{28}$$

as follows. An immersion $X(u, v) : B \to \mathbb{R}^3$ is given in *F-conformal parameters* $w = (u, v)$, if g_F is diagonalized, i.e.,

$$g_F(w)\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = g_F(w)\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) = W_F, \tag{29}$$

$$g_F(w)\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = 0 \quad \text{for all } w \in B. \tag{30}$$

The following simple result from planar linear algebra proven in [4] turns out to be useful for computing the conformal factor W_F and for other calculations in F-conformal parameters:

LEMMA 2.3. – *Let V be a two-dimensional vector space with an inner product. If $\psi : V \to V$ is a positive definite symmetric endomorphism and $D^{90} : V \to V$ a rotation about an angle of $90°$, then $D^{90} \circ \psi = (\det \psi) \psi^{-1} \circ D^{90}$.*

Remark. – The lemma will be applied to the rotation $D(w) : N(w)^\perp \to N(w)^\perp$ defined by $D(w)Z := N(w) \wedge Z$ for $Z \in N(w)^\perp$.

Using the notation $\Phi(w) := F_{zz}(X(w), N(w)) : N(w)^\perp \to N(w)^\perp$ the F-conformality relations may be rewritten as

$$0 < W_F = \big\langle X_u, \Phi^{-1} X_u \big\rangle = \big\langle X_v, \Phi^{-1} X_v \big\rangle,$$
$$0 = \big\langle X_v, \Phi^{-1} X_u \big\rangle.$$

Thus we can write with Lemma 2.3

$$X_u = \mu_2\big((\Phi^{-1} X_v) \wedge N\big) = \mu_2 \Phi (\det \Phi)^{-1}(X_v \wedge N), \tag{31}$$

$$X_v = \mu_1\big(N \wedge (\Phi^{-1} X_u)\big) = \mu_1 \Phi (\det \Phi)^{-1}(N \wedge X_u) \tag{32}$$

for some numbers $\mu_1, \mu_2 \in \mathbb{R}$. Therefore one obtains for the conformal factor W_F

$$0 < W_F = \big\langle \Phi^{-1} X_v, X_v \big\rangle = \mu_2 (\det \Phi)^{-1} \langle N \wedge X_u, X_v \rangle = \mu_2 (\det \Phi)^{-1} W,$$

where $W := |X_u \wedge X_v|$. Furthermore we have $W N = X_u \wedge X_v = \mu_1 W_F N = \mu_2 W_F N$. Consequently, we get $\mu_1 = \mu_2 = W / W_F$, and then

$$W_F = W / \sqrt{\det \Phi}, \qquad \mu_1 = \mu_2 = \sqrt{\det \Phi}. \tag{33}$$

The following calculation will lead to an analytic interpretation of the F-mean curvature of an F-conformally parametrized surface $X : B \to \mathbb{R}^3$. By the definition of the F-mean curvature we have

$$H_F = -\operatorname{tr}(A_F S) = -g^{\alpha\beta} g\left(\frac{\partial}{\partial u^\alpha}, A_F S \frac{\partial}{\partial u^\beta}\right) = -g^{\alpha\beta} \langle X_{u^\alpha}, F_{zz}(X, N) N_{u^\beta}\rangle.$$

The two identities $N \wedge X_u = W g^{2\alpha} X_{u^\alpha}$, $N \wedge X_v = -W g^{1\alpha} X_{u^\alpha}$ are used to obtain

$$
\begin{aligned}
H_F &= \bigl(\langle N \wedge X_v, F_{zz}(X, N) N_u\rangle - \langle N \wedge X_u, F_{zz}(X, N) N_v\rangle\bigr)/W \\
&= \bigl(\langle \Phi(N \wedge X_v), N_u\rangle - \langle \Phi(N \wedge X_u), N_v\rangle\bigr)/W \\
&\underset{(32)(31)}{=} -\sqrt{\det \Phi}\,\bigl(\langle X_u, N_u\rangle + \langle X_v, N_v\rangle\bigr)/W \\
&\underset{(33)}{=} \langle \Delta X, N\rangle / W_F.
\end{aligned}
\tag{34}
$$

Thus we have found that the normal component of ΔX is given by $H_F W_F N$, if X is an immersion in F-conformal parameters.

For technical reasons we introduce the linear mapping

$$l(y, z) := \frac{F_{zz}(y, z)}{\sqrt{\det F_{zz}(y, z)|_{z^\perp}}} + \frac{z}{|z|} \otimes \frac{z}{|z|}, \tag{35}$$

where $y \in \mathbb{R}^3$, $z \in \mathbb{R}^3 - \{0\}$. Note that we can rewrite (31), (32) as

$$
\begin{aligned}
X_u &= l(X, N)(X_v \wedge N), \\
X_v &= l(X, N)(N \wedge X_u),
\end{aligned}
$$

since the wedge products on the right-hand side are tangential vectors in $N^\perp$. Differentiating these equations we arrive at

$$
\begin{aligned}
\Delta X = {}& \bigl(l(X, N)\bigr)_u (X_v \wedge N) + \bigl(l(X, N)\bigr)_v (N \wedge X_u) \\
& + l(X, N)(X_v \wedge N_u + N_v \wedge X_u).
\end{aligned}
\tag{36}
$$

The tangential part of ΔX may be estimated by

$$\bigl|(\Delta X)^{\mathrm{tan}}\bigr| \leqslant \bigl|\bigl(l(X, N)\bigr)_u (X_v \wedge N) + \bigl(l(X, N)\bigr)_v (N \wedge X_u)\bigr|,$$

whereas the normal part, according to (34), is given by

$$\bigl|(\Delta X)^{\mathrm{nor}}\bigr| = |\langle \Delta X, N\rangle| = |H_F| W_F. \tag{37}$$

A simple application of the chain rule leads to an a priori estimate of ΔX:

$$|\Delta X| \leqslant 2\|l_y\| |X_u| |X_v| + \|l_z\|(|X_u| |N_v| + |X_v| |N_u|) + |H_F| W_F. \tag{38}$$

Remark. – Immersed surfaces of bounded mean curvature H may be considered as critical points of the parametric functional with the integrand $E(y, z)$ defined in (4) in

the introduction. Note that the F-conformality relations (29), (30) reduce to the classical conformality relations (18) mentioned in the introduction, since $A_F = \mathrm{Id}_{T_w B}$ in that case. Moreover, $(l(X, N))_u = (l(X, N))_v = 0$ and (36) together with (34) reduce to the well-known differential equation (17) given in the introduction.

Let us summarize (34) and (38) in

PROPOSITION 2.4. – *Let $X : B \to \mathbb{R}^3$ be an F-conformally parametrized immersion of class $C^2(B, \mathbb{R}^3)$. Then*
   (i) $\langle \Delta X, N \rangle = H_F W_F$.
   (ii) $|\Delta X| \leqslant C_y(F)|X_u||X_v| + C_z(F)(|X_u||N_v| + |X_v||N_u|) + |H_F|W_F$, *where C_y, C_z are constants depending only on the integrand F.*

Note that C_y, C_z can be estimated from above in terms of $\|F\|_{C^3(\overline{B_1(0)} \times S^2)}$ and M_1, if $X(\overline{B}) \subset \overline{B_1(0)}$.

The theorem of Cayley–Hamilton applied to the endomorphism $A_F S : T_w B \to T_w B$ leads to the algebraic relation

$$SA_F S + K_F A_F^{-1} + H_F S = 0. \tag{39}$$

This gives us a gradient estimate for the normal N:

$$
\begin{aligned}
|\nabla N|^2 &\leqslant \frac{1}{M_1}\big[\langle \Phi N_u, N_u \rangle + \langle \Phi N_v, N_v \rangle\big] \\
&= \frac{1}{M_1}\left[g\left(SA_F S\left(\frac{\partial}{\partial u}\right), \frac{\partial}{\partial u}\right) + g\left(SA_F S\left(\frac{\partial}{\partial v}\right), \frac{\partial}{\partial v}\right)\right] \\
&\underset{(39)}{=} \frac{-1}{M_1}\Bigg[H_F\left\{\left\langle DX \circ S\left(\frac{\partial}{\partial u}\right), DX\left(\frac{\partial}{\partial u}\right)\right\rangle + \left\langle DX \circ S\left(\frac{\partial}{\partial v}\right), DX\left(\frac{\partial}{\partial v}\right)\right\rangle\right\} \\
&\quad + K_F\left\{ g\left(A_F^{-1}\left(\frac{\partial}{\partial u}\right), \frac{\partial}{\partial u}\right) + g\left(A_F^{-1}\left(\frac{\partial}{\partial v}\right), \frac{\partial}{\partial v}\right)\right\}\Bigg] \\
&= -\frac{1}{M_1}\left[H_F\{\langle N_u, X_u \rangle + \langle N_v, X_v \rangle\} + K_F\left\{ g_F\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) + g_F\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right)\right\}\right] \\
&\underset{(29)}{=} \frac{W_F}{M_1}[H_F^2 - 2K_F], \tag{40}
\end{aligned}
$$

where we have used (i) of Proposition 2.4. Now we are in the position to deduce a geometric estimate for the Dirichlet energy

$$\mathcal{D}(N) := \frac{1}{2}\int_B |\nabla N(w)|^2 \, dw$$

of the normal N of an F-conformal immersion X.

LEMMA 2.5. – *The normal N of an F-conformal immersion $X \in C^2(\overline{B}, \mathbb{R}^3)$ with $X(\overline{B}) \subset \overline{B_1(0)}$ satisfies*

$$\mathcal{D}(N) \leqslant \frac{M_2}{M_1}\left[\int_\Gamma \kappa_g \, ds - 2\pi\right] + \frac{M_2}{2M_1^3}\int_X H_F^2 \, dA. \tag{41}$$

Proof. – We may write

$$K_F = \det(A_F S) = \det \Phi \det S = K \det \Phi. \tag{42}$$

From (33), (40), and the ellipticity condition (E) we infer

$$\frac{M_1}{M_2} |\nabla N|^2 \underset{(40)}{\leqslant} \frac{W_F}{(\det \Phi)^{1/2}} \big[H_F^2 - 2K_F \big]$$

$$\underset{(33),(42)}{=} \frac{W}{\det \Phi} H_F^2 - 2KW$$

$$\leqslant \frac{W}{M_1^2} H_F^2 - 2KW.$$

The estimate (41) follows now by integrating this inequality over the domain B and applying the Gauß–Bonnet Theorem. □

3. Proofs of the main results

Proof of Theorem 2. – Since (9) is a purely geometric estimate, we may assume that X is given in F-conformal parameters. If this is not the case one may apply the uniformization theorem in a version proved by Sauvigny [11, Thm. 2] to the $C^{1,\alpha}$-metric (28), or apply our argument in [4, p. 94] to find a positively oriented diffeomorphism $w: \overline{B} \to \overline{B}$ of class $C^{2,\alpha}(\overline{B}, \mathbb{R}^2)$, such that $X \circ w^{-1} \in C^{2,\alpha}(\overline{B}, \mathbb{R}^3)$ is F-conformal. Integrating by parts we may estimate the Dirichlet energy of an F-conformal immersion for any $q \in \mathbb{R}^3$ as

$$\mathcal{D}(X) = \frac{1}{2} \int_B \langle DX(w), D(X(w) - q) \rangle \, dw$$

$$\leqslant \frac{1}{2} \int_B |\Delta X(w)| |X(w) - q| \, dw + \frac{1}{2} \int_{\partial B} |X_r(1,\theta)| |X(1,\theta) - q| \, d\theta,$$

where for $w = (u,v) = r e^{i\theta} \in \overline{B}$ we have identified $X(w)$ and $X(r,\theta)$. Using Proposition 2.4(ii) and Lemma 2.5 we obtain

$$\mathcal{D}(X) \leqslant \frac{1}{4} C_y(F) \int_B |\nabla X(w)|^2 |X(w) - q| \, dw$$

$$+ \frac{1}{2} C_z(F) \int_B \big[|X_u(w)| |N_v(w)| + |X_v(w)| |N_u(w)| \big] \cdot |X(w) - q| \, dw$$

$$+ \frac{1}{2} \int_B \big| H_F(X(w), N(w)) \big| |X(w) - q| W_F(w) \, dw$$

$$+ \frac{1}{2} \int_{\partial B} |X_r(1,\theta)| |X(1,\theta) - q| \, d\theta$$

$$
\underset{(33)}{\leqslant} \left[\frac{C_y(F)}{2} + \frac{C_z(F)}{2} + \frac{\|H_F\|_{\infty,B}}{2M_1} \right] \|X - q\|_{\infty,B} \mathcal{D}(X)
$$

$$
+ \frac{C_z(F)}{2} \|X - q\|_{\infty,B} \mathcal{D}(N) + \frac{1}{2} \int_{\partial B} |X_r(1,\theta)||X(1,\theta) - q|\, d\theta
$$

$$
\underset{(41)}{\leqslant} \left[\frac{C_y(F)}{2} + \frac{C_z(F)}{2} + \frac{\|H_F\|_{\infty,B}}{2M_1} + C_z(F)\frac{M_2}{4M_1^3}\|H_F\|_{\infty,B}^2 \right] \|X - q\|_{\infty,B} \mathcal{D}(X)
$$

$$
+ \frac{C_z(F)}{2} \|X - q\|_{\infty,B} \frac{M_2}{M_1} \left[\int_\Gamma \kappa_g\, ds - 2\pi \right]
$$

$$
+ \frac{1}{2} \int_{\partial B} |X_r(1,\theta)||X(1,\theta) - q|\, d\theta.
$$

Defining

$$
C(F) := C_y(F) + C_z(F)\left(1 + \frac{M_2}{2M_1^3} \right),
$$

we insert the Euler equation (11) to arrive at

$$
\mathcal{D}(X) \leqslant \frac{1}{2}\left[C(F)\big(1 + \|F_{yz}\|^2\big) + \frac{\|F_{yz}\|}{M_1} \right] \|X - q\|_{\infty,B} \mathcal{D}(X)
$$

$$
+ \|X - q\|_{\infty,B}\, C(F)\frac{M_2}{M_1}\left[\int_\Gamma \kappa_g\, ds - 2\pi \right]
$$

$$
+ \frac{1}{2}\|X - q\|_{\infty,\partial B}\sqrt{\frac{M_2}{M_1}}\, L(\Gamma), \tag{43}
$$

for any $q \in \mathbb{R}^3$, where we have used the ellipticity condition (E) and the conformality relations (29) and (30) in polar coordinates for the boundary integral, i.e. for $|w| = r = 1$:

$$
|X_r|^2 \leqslant M_2\langle X_r, \Phi^{-1}X_r \rangle = M_2\langle X_\theta, \Phi^{-1}X_\theta \rangle \leqslant \frac{M_2}{M_1}|X_\theta|^2.
$$

Note that $C(F)$ can be estimated from above in terms of $\|F\|_{C^3(\overline{B_1(0)} \times S^2)}$ and M_1. We notice that by assumption (7) there is a vector $q^* \in \mathbb{R}^3$, such that $\|X(.) - q^*\|_{\infty,\partial B} = R_\Gamma(X) \leqslant 1$, where $R_\Gamma(X)$ is defined in (12) in the introduction. By (7) and (8) we may apply the inclusion principle Proposition 2.2 to conclude $\|X(.) - q^*\|_{\infty,\overline{B}} \leqslant R_\Gamma(X)$. These observations together with (43) for $q = q^*$ lead to the desired result, since then we may write

$$
\mathcal{D}(X) \leqslant R_\Gamma(X) \cdot \frac{2C(F)\frac{M_2}{M_1}[\int_\Gamma \kappa_g\, ds - 2\pi] + \sqrt{\frac{M_2}{M_1}}L(\Gamma)}{2 - R_\Gamma(X)[C(F)(1 + \|F_{yz}\|^2) + \frac{\|F_{yz}\|}{M_1}]}. \qquad \Box
$$

Proof of Theorem 3. – Let Y be a disk-type minimal surface bounded by the curve Γ. Then the classical isoperimetric inequality (3) and the growth condition (15) imply that

for the minimizer X we can conclude $m_1 \mathcal{A}(X) \leqslant \mathcal{F}(X) \leqslant \mathcal{F}(Y) \leqslant m_2 \mathcal{A}(Y) \leqslant \frac{m_2}{4\pi} L^2(\Gamma)$, which proves the result.   □

Remark. – Using an isoperimetric inequality for harmonic vector functions due to Morse and Tompkins, see e.g. [5, pp. 135–138], one may use a similar comparison argument to prove a local version of (16) for minimizers:

$$\mathcal{A}_\Omega(X) := \int\limits_\Omega |X_u \wedge X_v| \, du \, dv \leqslant \frac{m_2}{4m_1} L^2\big(X(\partial\Omega)\big)$$

for all simply connected subdomains $\Omega \subset B$ whose boundary is piecewise smooth. In fact, one uses the comparison surface

$$Z := \begin{cases} Y & \text{on } \Omega, \\ X & \text{on } \overline{B} - \Omega, \end{cases}$$

where $Y \in C^0(\overline{\Omega}, \mathbb{R}^3) \cap C^2(\Omega, \mathbb{R}^3)$ satisfies $\Delta Y = 0$ on Ω, and $X - Y \in W_0^{1,2}(\Omega, \mathbb{R}^3)$.

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