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STABILITY FOR SEMILINEAR PARABOLIC EQUATIONS WITH DECAYING POTENTIALS IN $\mathbb{R}^n$ AND DYNAMICAL APPROACH TO THE EXISTENCE OF GROUND STATES

STABILITÉ POUR LES ÉQUATIONS PARABOLIQUES AVEC POTENTIEL TENDANT VERS 0 À L'INFINI ET APPROCHE DYNAMIQUE POUR L'EXISTENCE D'UN ÉTAT FONDAMENTAL

Philippe SOUPLET [a,b], **Qi S. ZHANG** [c,1]
[a] *Département de Mathématiques, INSSET, Université de Picardie, 02109 St-Quentin, France*
[b] *Laboratoire de Mathématiques Appliquées, UMR CNRS 7641, Université de Versailles, 78035 Versailles, France*
[c] *Department of Mathematics, University of Memphis, Memphis, TN 38152, USA*

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ABSTRACT. – Consider the elliptic problem

$$\Delta u - V(x)u + u^p = 0 \quad \text{in } \mathbb{R}^n, \tag{1}$$

with $1 < p < \frac{n+2}{n-2}$, $n \geqslant 2$, and $0 \leqslant V(x) \in L^\infty$, which may decay to 0 at infinity. We prove that if V is radial and satisfies

$$\frac{a_1}{1+|x|^b} \leqslant V(x) \leqslant a_2 \quad \text{and} \quad 0 \leqslant b < \frac{2(n-1)(p-1)}{p+3},$$

then (1) admits a (ground state) positive solution. We do not use traditional variational methods and the result relies on the study of global solutions of the parabolic problem

$$\Delta u - V(x)u + u^p - \partial_t u = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty), \quad u(x,0) = u_0(x). \tag{2}$$

Indeed, we will show that, under suitable conditions on V (not necessarily radial), (2) admits global positive solutions and that when V and u_0 are radial some global solutions have ω-limit sets containing a positive equilibrium. The method also covers nonlinearities more general

E-mail addresses: souplet@math.uvsq.fr (P. Souplet), qizhang@math.ucr.edu (Q.S. Zhang).
[1] Current address: Mathematics Department, University of California, Riverside, CA 92521, USA.

than u^p, in which case the standard variational method may be hard to apply.

RÉSUMÉ. – Nous considérons le problème elliptique

$$\Delta u - V(x)u + u^p = 0 \quad \text{dans } \mathbb{R}^n, \tag{1}$$

avec $1 < p < \frac{n+2}{n-2}$, $n \geqslant 2$, et $0 \leqslant V(x) \in L^\infty$, $V(x)$ pouvant tendre vers 0 à l'infini. Nous prouvons que si V est radial et satisfait

$$\frac{a_1}{1+|x|^b} \leqslant V(x) \leqslant a_2 \quad \text{et} \quad 0 \leqslant b < \frac{2(n-1)(p-1)}{p+3},$$

alors (1) admet une solution positive (état fondamental). Nous n'utilisons pas les méthodes variationnelles traditionnelles : le résultat repose sur l'étude des solutions globales du problème parabolique

$$\Delta u - V(x)u + u^p - \partial_t u = 0 \quad \text{dans } \mathbb{R}^n \times (0,\infty), \quad u(x,0) = u_0(x). \tag{2}$$

En effet, nous montrons que, sous des conditions appropriées sur V (non nécessairement radial), (2) admet des solutions globales positives et que, lorsque V et u_0 sont radiales, l'ensemble ω-limite de certaines solutions globales contient un état d'équilibre positif. La méthode s'applique également à des non-linéarités plus générales que u^p, pour lesquelles la méthode variationnelle classique pourrait être difficilement applicable.

1. Introduction

We are concerned with the existence of positive solutions to the elliptic equation

$$\Delta u - V(x)u + u^p = 0 \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

and with the asymptotic behavior of global positive solutions to the corresponding parabolic equation

$$\begin{cases} \Delta u - V(x)u + u^p - \partial_t u = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x), \end{cases} \tag{1.2}$$

where n is a positive integer and $1 < p < p_S = \frac{n+2}{n-2}$ ($p_S = \infty$ if $n \leqslant 2$). In what follows, unless otherwise stated, we will assume that $V = V(x)$ is a locally Hölder continuous, nonnegative and bounded function.

Problems such as (1.1) and (1.2) arise from diverse fields such as mathematical physics, differential geometry and biology, etc. Consequently these problems have played a central role in nonlinear analysis over the past few decades.

Eq. (1.1) with $1 < p < p_S$ exhibits a rich history. When $V \equiv 1$, it is well known that (1.1) has a so-called ground state solution, meaning a positive solution decaying exponentially to zero near infinity. In [20] and [3,4], P.L. Lions and H. Berestycki and P.L. Lions obtained existence of nontrivial solutions to (1.1) when V is a suitable perturbation of a positive constant near infinity. Their approach is the by now famous concentration-compactness principle, which is variational in nature. Related results are also obtained in [8] and [24]. Subsequently many authors have taken up the study of the problem and produced numerous interesting results.

Despite the intensity of research, one natural question concerning (1.1) has not been addressed so far, even when V is a radial function, i.e.

When does (1.1) *have a ground state solution when the mass term* $V = V(x)$ *decays to zero near infinity*?

This question is interesting and important for a number reasons. It is well known that the concentration-compactness principle requires that $V = V(x)$ converges to a positive constant near infinity. Also, the role played by the mass term V is very delicate. In a recent paper [33], one of us shows that if V decays faster than the negative square of the distance, then (1.2) and hence (1.1) does not have any global positive solution when $1 < p < 1 + \frac{2}{n}$, thus producing a similar situation as in the case $V \equiv 0$ (see [11,18,7]). So it is natural to investigate the case when V decays slower.

The main result of this paper (Theorem 1.1) implies that for $1 < p < p_S$, $n \geqslant 2$, V radial, and under suitable conditions on V, (1.1) does admit a ground state. Indeed, we will show that, under suitable conditions on V (not necessarily radial), the parabolic problem (1.2) admits global positive solutions, even if $1 < p < 1 + \frac{2}{n}$, and that when V and u_0 are radial some global solutions have ω-limit sets containing a positive solution of (1.1).

Such an approach was used in [23] (see [23, Theorem A(iii)], and also [5]) in case when the spatial domain is bounded. However the generalization to the case of unbounded spatial domain is not obvious. First, since natural candidates are solutions lying on the boundary of the domain of attraction of 0, one must show that $u \equiv 0$ is stable in a suitable topology. This is not straightforward because, due to the decay of V, the linear term Vu is not strongly coercive. Next, due to the lack of compactness, it is not clear if the ω-limit set (in the same topology) is non-empty. To overcome these difficulties, we need to work in adapted weighted L^∞ spaces and to use suitable comparison arguments and energy estimates, together with a priori bounds on global solutions to (1.2).

To state our main results, it will be convenient to introduce the following.

Notation. – The norm in $L^q := L^q(\mathbb{R}^n)$, $1 \leqslant q \leqslant \infty$, is denoted by $\|.\|_q$, and the norm in $H^1 := H^1(\mathbb{R}^n) = W^{1,2}(\mathbb{R}^n)$ by $\|.\|_{H^1}$. For each $k \geqslant 0$, we define the weighted space

$$L^\infty_k = \Big\{ v \in L^\infty;\ \sup_{x\in\mathbb{R}^n} |x|^k |v(x)| < \infty \Big\}.$$

L^∞_k is a Banach space for the norm $\|v\|_{\infty,k} = \sup_{x\in\mathbb{R}^n} (1 + |x|^k)|v(x)|$. We also define the closed subspace

$$L^\infty_{k,0} = \Big\{ v \in L^\infty_k;\ \lim_{|x|\to\infty} |x|^k v(x) = 0 \Big\}.$$

Recall that, by standard theory, for all $u_0 \in L^\infty$, the Cauchy problem (1.2) has a unique, maximal in time, classical solution $u = u(x,t)$. The solution at time t belongs to

L^∞ and will be denoted by $u(t;u_0)$ or $S(t)u_0$, or simply $u(t)$ if no confusion arises. If u is of mixed sign, then u^p is understood as $|u|^{p-1}u$. Moreover, $u_0 \geqslant 0$ implies $u \geqslant 0$. We denote by $T = T(u_0) \in (0,\infty]$ the maximal existence time of $u(.;u_0)$. If $T(u_0) < \infty$, then $\lim_{t\to T(u_0)} \|u(t)\|_\infty = \infty$.

If $T(u_0) = \infty$, for each $k \geqslant 0$, we denote by $\omega_k(u_0)$ the ω-limit set of u with respect to the L^∞_k topology, i.e.

$$\omega_k(u_0) = \{v \in L^\infty_k;\ \exists t_j \to \infty,\ u(t_j) \to v \text{ in } L^\infty_k\}.$$

We have the following result.

THEOREM 1.1. – *Assume $p > 1$ and $V = V(x)$ is locally Hölder continuous, nonnegative and bounded.*

(a) ($L^\infty_{k,0}$-stability of $u \equiv 0$.) *Suppose $V(x) \geqslant a/(1+|x|^b)$ with $b \in [0,2)$, $a > 0$, and let $k = b/(p-1)$. Then there exist $\delta > 0$ and $C \geqslant 1$ such that for all $u_0 \in L^\infty_k$ satisfying $\|u_0\|_{\infty,k} \leqslant \delta$, the corresponding solution u of problem* (1.2) *is global in time and satisfies*

$$\sup_{t\geqslant 0} \|u(t)\|_{\infty,k} \leqslant C\|u_0\|_{\infty,k}.$$

Moreover if in addition $u_0 \in L^\infty_{k,0}$, then

$$\lim_{t\to\infty} \|u(t)\|_{\infty,k} = 0. \tag{1.3}$$

(b) (Uniform a priori estimate for global solutions.) *Suppose $p < p_S$, $u_0 \in L^\infty \cap H^1$ and $u_0 \geqslant 0$. Assume that the corresponding solution u of* (1.2) *is global. Then u satisfies the estimate*

$$\sup_{t\geqslant 0} \|u(t)\|_\infty \leqslant C(\|u_0\|_{H^1} + \|u_0\|_\infty),$$

where $C(s)$ is bounded for s bounded.

(c) (ω-limit sets containing positive equilibria.) *Assume $V = V(|x|)$ is radially symmetric, $p < p_S$ and*

$$\frac{a_1}{1+|x|^b} \leqslant V(x) \leqslant a_2, \quad n \geqslant 2,\ 0 \leqslant b < \frac{2(n-1)(p-1)}{p+3}, \tag{1.4}$$

with $a_1, a_2 > 0$. Let $k = b/(p-1)$ and let $\phi \in L^\infty_{k,0} \cap H^1$, with $\phi \geqslant 0$, ϕ radially symmetric, $\phi \not\equiv 0$. There exists $\lambda > 0$ such that $T(\lambda\phi) = \infty$ and $\omega_k(\lambda\phi)$ contains a positive equilibrium of (1.2).

Concerning problem (1.1), our result is the following theorem, whose existence part is an immediate consequence of part (c) of Theorem 1.1.

THEOREM 1.2. – *Assume $1 < p < p_S$ and $V = V(x)$ is a radially symmetric, locally Hölder continuous function. Assume that*

$$\frac{a_1}{1+|x|^b} \leqslant V(x) \leqslant a_2, \quad n \geqslant 2,\ 0 \leqslant b < \frac{2(n-1)(p-1)}{p+3}, \tag{1.5}$$

with $a_1, a_2 > 0$. *Then* (1.1) *has a global positive solution* u. *Moreover,* u *is radial and, if* $n \geqslant 3$, u *satisfies*

$$u(x) \leqslant c_1 \exp\left(-c_2|x|^{(2-b)/2}\right)$$

for some $c_1, c_2 > 0$.

Remarks 1.1. – (a) Note that the upper bound for b in Theorem 1.1(c) increases to 2 as p goes to p_S (or to ∞ if $n = 2$). On the other hand, when $b > 2$ and $0 \leqslant V(x) \leqslant \frac{a}{1+|x|^b}$, $a > 0$, Corollary 1.2 in [33] shows that all positive solutions to (1.2) blow up in finite time. Thus in some sense the range for V in Theorem 1.1 is sharp.

(b) The method employed in this paper gives in particular a unified approach to the existence of ground states when the mass is bounded between two positive constants. Theorem 1.2 seems to contain all the known results in the radial case for Eq. (1.1). In particular it gives a different proof of the classical result that $\Delta u - u + u^p = 0$ has a ground state solution. By Theorem 11.1 in [16], when V is a decreasing radial function, the concentrated compactness method does not apply directly. However such kind of V poses no problem to our method. In addition, we do not require that V converges at infinity or has a local minimum. The conclusion of Theorem 1.2 can probably be obtained by a variational method. However besides the intrinsic interest of Theorem 1.1 for the parabolic problem, the dynamical proof provides an interesting alternative approach to the existence of ground states.

Another advantage of this approach is that it covers more general nonlinearities $f(x,u)$ as indicated in Remark 4.1 at the end of Section 4 (see the examples in Remark 4.1(f)). In this case, the traditional variational method may be hard to apply.

(c) All the conclusions of Theorem 1.1 remain true if, in Eq. (1.2), $\mathbb{R}^n$ is replaced by the exterior domain $\Omega = \mathbb{R}^n \setminus \overline{B}_R = \{x \in \mathbb{R}^n\colon\ |x| > R\}$ for some $R > 0$, and (1.2) is complemented by the Dirichlet boundary conditions $u = 0$ on $\partial\Omega \times (0,\infty)$.

The conclusions of Theorem 1.2 remain true if, in Eq. (1.1), $\mathbb{R}^n$ is replaced by the exterior domain $\Omega = \mathbb{R}^n \setminus \overline{B}_R = \{x \in \mathbb{R}^n\colon\ |x| > R\}$ for some $R > 0$, and (1.1) is complemented by the Dirichlet boundary conditions $u_{|\partial\Omega} = 0$. Related problems in exterior domains for $V \equiv 1$ have been considered in, e.g., [1].

(d) At this time we do not know whether the conclusion of Theorem 1.2 still holds if V is not radial.

Remark 1.2. – In the paper [8], Ding and Ni obtained important existence results on the related equation $\Delta u - u + Qu^p = 0$. For instance, they show that this equation possesses a radial solution if $0 \leqslant Q(x) = Q(|x|) \leqslant C|x|^\sigma$ with $\sigma < (p-1)(n-1)/2$ and $1 < p < p_S$. We mention that the situation for (1.2) is quite different from the case in [8]. For example, we have shown that V cannot decay "too fast" near infinity for (1.2) to have any positive solution. In contrast there is no such restriction for Q.

In the more restricted range $1 < p < 1 + \frac{2}{n}$, we show that global solutions of (1.2) satisfy some stronger a priori estimates. In particular, we have a *universal bound,* i.e. independent of initial data, away from $t = 0$.

THEOREM 1.3. – *Assume* $1 < p < 1+\frac{2}{n}$ *and* $V = V(x)$ *is locally Hölder continuous, nonnegative and bounded.*

(a) (Universal bounds for global solutions.) *Suppose* $V \in L^\infty$ *and let* u *be a nonnegative global solution to* (1.2). *For any* $T > 0$, *there exists a universal constant* $C = C(V, T)$, *independent of* u_0, *such that* $u(x, t) \leqslant C$ *for all* x *and* $t \geqslant T$.

(b) (Spatial decay of global solutions.) *Suppose* $0 \leqslant V(x) \leqslant \frac{a}{1+|x|^b}$ *with* $b \in [0, 2)$, $a > 0$. *Assume also* u_0 *is supported in the ball* $B_{R_0}(0)$ *for some* $R_0 > 0$. *Let* u *be a nonnegative global solution to* (1.2). *There exists a constant* C *depending on* R_0 *but otherwise independent of* u_0 *such that*

$$u(x, t) \leqslant \frac{C}{1 + |x|^{b/(p-1)}},$$

for all x *such that* $|x| \geqslant 2R_0$ *and all* $t > 0$.

Remarks 1.3. – (a) We point out that in Theorem 1.1(c), the whole trajectory $\{u(t);\ t \geqslant t_0 > 0\}$ need not be precompact (unlike in the case of bounded spatial domains, if u is a bounded solution of (1.2)). However, we have been able to prove (see Proposition 3.1) that at least *some subsequence* $\{u(t_n)\}$ with $t_n \to \infty$ is precompact in L^∞_k for appropriate values of k.

(b) For Eq. (1.2) in bounded domains with homogeneous Dirichlet boundary conditions and $V \equiv 0$, a universal bound, such as in Theorem 1.3(a), was proved in [10] for global positive solutions when $(n-1)p < n+1$. This result was extended to $p < p_S$ when $n \leqslant 3$ in [25]. Here the method of proof is different from both [10] and [25].

(c) The stability result of Theorem 1.1(a) is reminiscent of some results concerning Eq. (1.2) for $V \equiv 0$ in other function spaces. Namely, for $p > 1 + \frac{2}{n}$ and $q_c = n(p-1)/2$, initial data which are small in L^{q_c} yield global solutions (see [31]). Moreover, these solutions are bounded and decay to 0 in L^{q_c} (see [27]). This is related to the fact that Eq. (1.2) for $V \equiv 0$ is invariant under the self-similar rescaling $u_\alpha(x, t) := \alpha^{2/(p-1)} u(\alpha x, \alpha^2 t)$ and that the L^{q_c} norm is preserved by this rescaling. Similar results are known for other equations, e.g. Navier–Stokes (see [15]) and nonlinear Schrödinger equations (see [6]). The phenomenon observed in Theorem 1.1(a) seems different in nature since for $V \not\equiv 0$, Eq. (1.2) does not admit the self-similar invariance unless $V = C|x|^{-2}$.

(d) Some results on convergence of solutions of (1.2) to a ground state for $V \equiv 1$ and different nonlinearities (typically, $u^p - u^q$ with $1 < q < p \leqslant n/(n-2)$) can be found in [9]. The method there is different from ours. In particular the proof uses the *existence* of the ground state and its uniqueness (up to translation).

(e) One can show that $u \equiv 0$ is an isolated solution of (1.1) in L^∞_k for $k = b/(p-1)$ and V as in Theorem 1.1(a). (This follows easily from the maximum principle.)

The rest of the paper is organized as follows. In Sections 2 and 3, we establish preliminary results which will play a crucial role in the proof of Theorems 1.1 and 1.2. In Section 2, we prove some estimates concerning the linear part of Eq. (1.2), namely estimates on the semigroup $\mathrm{e}^{t(\Delta - V)}$ acting on the spaces L^∞_k. These estimates, which may be of some independent interest, rely on the construction of suitable supersolutions, also used later in the proof of Theorem 1.1(a). In Section 3, we derive a key compactness property in the space L^∞_k along some subsequence for global solutions of (1.2), which is based on an energy argument from [26]. We also give some continuous dependence properties of solutions in that space. Section 4 is then devoted to the proof of Theorems 1.2 and 1.1, and of Proposition 1.1. Finally, Theorem 1.3 on universal bounds is proved in Section 5.

The main results of this paper have been announced in [28].

2. Linear estimates

We denote by $e^{t(\Delta-V)}$ the semigroup (on L^∞) associated with the linear part of Eq. (1.2),

$$u_t - \Delta u + V(x)u = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty). \tag{2.1}$$

Namely, for all $\phi \in L^\infty$, $u(x,t) = (e^{t(\Delta-V)}\phi)(x)$ denotes the unique solution of (2.1) with initial data ϕ.

PROPOSITION 2.1. – *Suppose* $V(x) \geqslant \frac{a}{1+|x|^b}$ *with* $b \in [0,2)$, $a > 0$ *and let* $k \geqslant 0$.

(a) *There exists* $C \geqslant 1$ *such that for all* $\phi \in L_k^\infty$,

$$\left\|e^{t(\Delta-V)}\phi\right\|_{\infty,k} \leqslant C\|\phi\|_{\infty,k}, \quad t \geqslant 0. \tag{2.2}$$

(b) *For all* $\phi \in L_{k,0}^\infty$, *it holds*

$$\lim_{t\to\infty} \left\|e^{t(\Delta-V)}\phi\right\|_{\infty,k} = 0.$$

Proof of Proposition 2.1. – (a) For $A > 0$, put

$$U(x) = \left(A + |x|^2\right)^{-k/2}. \tag{2.3}$$

A straightforward calculation shows that

$$\begin{aligned}\Delta U &= -\frac{k}{(A+|x|^2)^{(k/2)+1}} + \frac{k(k+2)|x|^2}{(A+|x|^2)^{(k/2)+2}} - \frac{(n-1)k}{(A+|x|^2)^{(k/2)+1}} \\ &\leqslant \frac{k(k+2-n)}{(A+|x|^2)^{(k/2)+1}} = \frac{k(k+2-n)}{A+|x|^2}U.\end{aligned} \tag{2.4}$$

For $A = A(a,b,k)$ large enough, we have

$$k(k+2-n)\left(1+|x|^b\right) \leqslant a|x|^2 + C_1(a,b,k) \leqslant a\left(A+|x|^2\right),$$

hence

$$\Delta U \leqslant \frac{a}{1+|x|^b}U \leqslant V(x)U \quad \text{in } \mathbb{R}^n.$$

It thus follows from the maximum principle that $e^{t(\Delta-V)}U \leqslant U$. Since, for all $\phi \in L_k^\infty$,

$$|\phi(x)| \leqslant \frac{\|\phi\|_{\infty,k}}{1+|x|^k} \leqslant C_2(A,k)\|\phi\|_{\infty,k}U(x),$$

we deduce, using the maximum principle again, that

$$\left|e^{t(\Delta-V)}\phi\right| \leqslant C_2(A,k)\|\phi\|_{\infty,k} e^{t(\Delta-V)}U \leqslant C_2(A,k)\|\phi\|_{\infty,k} U$$
$$\leqslant \frac{C_3(A,k)\|\phi\|_{\infty,k}}{1+|x|^k}, \quad t \geqslant 0.$$

Estimate (2.2) is proved.

(b) First assume that $\phi \in L^\infty_m$ for some $m > k$. We claim that for all $\varepsilon > 0$,

$$\left\|e^{t(\Delta-V)}\phi\right\|_{\infty,k} \leqslant C_\varepsilon t^{-\frac{m-k}{2}+\varepsilon}\|\phi\|_{\infty,m}, \quad t > 0. \tag{2.5}$$

If $1+|x| \geqslant t^{1/2}$, then by (2.2), we have

$$\left|e^{t(\Delta-V)}\phi\right|(x) \leqslant \frac{C\|\phi\|_{\infty,m}}{1+|x|^m} \leqslant \frac{C\|\phi\|_{\infty,m} t^{-\frac{m-k}{2}}}{(1+|x|^k)}. \tag{2.6}$$

On the other hand, note that for any $q > n/m$, we have $\|\phi\|_q \leqslant C_q\|\phi\|_{\infty,m}$. Therefore, if $1+|x| \leqslant t^{1/2}$, then

$$\left|e^{t(\Delta-V)}\phi\right|(x) \leqslant \left|e^{t\Delta}\phi\right|(x) \leqslant C\|\phi\|_q t^{-n/2q} \leqslant \frac{C_q\|\phi\|_{\infty,m} t^{-\frac{(n/q)-k}{2}}}{1+|x|^k}. \tag{2.7}$$

The claim follows by combining (2.6) and (2.7), since n/q can be made arbitrarily close to m.

Now, since L^∞_m is dense in $L^\infty_{k,0}$ (consider the sequence $\phi_j(x) := \phi(x)\,1_{\{|x|<j\}}$, $j = 1, 2, \ldots$), the property follows from (2.5) and (2.2). □

Remark 2.1. – An alternate proof of Proposition 2.1 can be deduced from the estimates of Schrödinger heat kernels obtained in [32].

3. Energy and compactness properties

To begin with, let us recall some well-known facts related to the existence of an energy functional for Eq. (1.2).

For $u_0 \in L^\infty \cap H^1$ it is well known that $u \in C([0, T(u_0)); H^1)$ and that the energy $E(t)$, defined as

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^n} Vu^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^n} u^{p+1}\,dx$$

satisfies the identity

$$E(0) - E(t) = \int_0^t \int_{\mathbb{R}^n} |u_t(x,s)|^2\,dx\,ds.$$

We will use the following two classical lemmas.

LEMMA 3.1. – *Let $u_0 \in L^\infty \cap H^1$. If $T(u_0) = \infty$, then $E(t) \geqslant 0$ for all $t \geqslant 0$, hence in particular*

$$\int_0^\infty \int |u_t(x,s)|^2\,dx\,ds \leqslant E(0) \leqslant C\|u_0\|_{H^1}^2.$$

Proof. – This is a consequence of the classical concavity argument of Levine (see [17]). □

LEMMA 3.2. – *Let $u_0 \in L^\infty \cap H^1$. If $T(u_0) = \infty$, then for each $k \geqslant 0$ the ω-limit set $\omega_k(u_0)$ consists of equilibria (i.e., of solutions of* (1.1)).

Proof. – Assume $u(t_j) \to v$ in L_k^∞ and fix $t > 0$. By continuous dependence of solutions of (1.2) over initial data in L^∞, it follows that $u(t + t_j) \to S(t)v$ in L^∞. For each $R > 0$, we have

$$\int_{|x|<R} |u(x,t+t_j) - u(x,t_j)|^2\,dx \leqslant C(R) \int_{t_j}^{t+t_j} \int_{|x|<R} |u_t(x,s)|^2\,dx\,ds$$
$$\leqslant C(R) \int_{t_j}^{\infty} \int_{\mathbb{R}^n} |u_t(x,s)|^2\,dx\,ds.$$

Since the RHS goes to 0 as $j \to \infty$ in view of Lemma 3.1, we deduce that

$$\int_{|x|<R} |(S(t)v)(x) - v(x)|^2\,dx = 0,$$

hence $S(t)v \equiv v$ for all $t > 0$, which means that v is an equilibrium. □

The following compactness property is an essential ingredient to the proof of Theorem 1.2(c) and Theorem 1.1.

PROPOSITION 3.1. – *Let $u_0 \in L^\infty \cap H^1$ and assume that $T(u_0) = \infty$.*
(a) *Then there exists $t_j \to \infty$ such that*

$$\|u(t_j)\|_{p+1} + \|\nabla u(t_j)\|_2 + \left\|V^{1/2}u(t_j)\right\|_2 \leqslant C(\|u_0\|_{H^1}), \quad j = 1, 2, \ldots.$$

(b) *Assume in addition that $n \geqslant 2$ and $p < p_S$, that V and u_0 are radially symmetric and that V satisfies*

$$V(x) \geqslant \frac{a}{1+|x|^b} \quad \text{with } b \in [0,2) \text{ and } a > 0. \tag{3.1}$$

Let k satisfy $0 \leqslant k < \frac{n-1}{2} - \frac{b}{4}$. Then the sequence $u(t_j + \tau)$ is precompact in L_k^∞ for some $\tau > 0$. In particular, $\omega_k(u_0) \neq \emptyset$.

Proof. – (a) We use an energy argument from [26, Theorem 2] (given there for $V \equiv 0$). Let $f(t) \equiv \int_{\mathbb{R}^n} u^2(x,t)\,dx$, then by Lemma 3.1,

$$f(t)-f(0)=2\int_0^t\int_{\mathbb{R}^n} uu_s \leqslant 2\left(\int_0^t\int_{\mathbb{R}^n} u_s^2\,dx\,ds\right)^{1/2}\left(\int_0^t\int_{\mathbb{R}^n} u^2\,dx\,ds\right)^{1/2}$$
$$\leqslant 2E(0)^{1/2}\left(\int_0^t f(s)\,ds\right)^{1/2}.$$

This easily implies

$$f(t)\leqslant C\big(E(0)\big)\big(f(0)+t\big)\leqslant C(\|u_0\|_{H^1})(t+1),\quad t\geqslant 0.$$

Multiplying both sides of (1.2) by u and integrating, we obtain, for $T>0$,

$$\int_0^T\int_{\mathbb{R}^n} u^{p+1}(x,t)\,dx\,dt=\int_0^T\int_{\mathbb{R}^n}\big(|\nabla u(x,t)|^2+V(x)u^2(x,t)\big)\,dx\,dt$$
$$+\frac{1}{2}\int_{\mathbb{R}^n}\big(u^2(x,T)-u^2(x,0)\big)\,dx$$
$$=2\int_0^T E(t)\,dt+\frac{2}{p+1}\int_0^T\int_{\mathbb{R}^n} u^{p+1}(x,t)\,dx\,dt+\frac{1}{2}\big(f(T)-f(0)\big).$$

Hence

$$\frac{1}{T}\int_0^T\int_{\mathbb{R}^n} u^{p+1}(x,t)\,dx\,dt\leqslant\frac{p+1}{p-1}\left(\frac{f(T)-f(0)}{2T}+\frac{2}{T}\int_0^T E(t)\,dt\right)$$
$$\leqslant\frac{p+1}{p-1}\left(\frac{f(T)}{2T}+2E(0)\right)\leqslant C(\|u_0\|_{H^1}),\quad T\geqslant 1.$$

In particular, for each integer $j\geqslant 1$, there exists $t_j\in[j,2j]$ such that $\int_{\mathbb{R}^n} u^{p+1}(x,t_j)\,dx \leqslant 2C(\|u_0\|_{H^1})$. Since $E(t_j)\leqslant E(0)$, we also have $\int_{\mathbb{R}^n}(|\nabla u|^2+Vu^2)(x,t_j)\,dx \leqslant C'(\|u_0\|_{H^1})$. The conclusion follows.

(b) Since $p+1>n(p-1)/2$ by assumption, it follows from well-known smoothing properties of semilinear heat equations (see [29,30]) that for all $K>0$, there exist $\tau=\tau(K)>0$ and $M(K)>0$ such that for all $t\geqslant 0$, $\|u(t)\|_{p+1}\leqslant K$ implies

$$\|u(t+s)\|_{p+1}\leqslant 2K,\quad 0\leqslant s\leqslant\tau\quad\text{and}\quad\|u(t+\tau)\|_{W^{1,q}}\leqslant M(K),$$

for $p+1\leqslant q\leqslant\infty$, where $\|.\|_{W^{1,q}}$ denotes the norm in the Sobolev space $W^{1,q}=W^{1,q}(\mathbb{R}^n)$. From part (a) and the fact that $E(t)\leqslant E(0)$, we then obtain

$$\|\nabla u(t_j+\tau)\|_2+\big\|V^{1/2}u(t_j+\tau)\big\|_2+\|u(t_j+\tau)\|_{W^{1,q}}\leqslant C(\|u_0\|_{H^1}),\tag{3.2}$$

for $p+1\leqslant q\leqslant\infty$ and $j=1,2,\ldots.$

Now, letting $r = |x|$ and using the fact that $u(., t)$ is radial and $\lim_{r\to\infty} u(r,t) = 0$, we have

$$u^2(x,t) = \int_r^\infty 2u\partial_\rho u(\rho,t)\,d\rho = 2\int_r^\infty \left(\rho^{(n-1)/2}\partial_\rho u\right)\left(\rho^{(n-1-b)/2}u\right)\rho^{-(n-1-(b/2))}\,d\rho.$$

Since $\gamma := n - 1 - (b/2) > 0$, we get

$$u^2(x,t) \leqslant 2r^{-\gamma}\left(\int_r^\infty (\partial_\rho u)^2\rho^{n-1}\,d\rho\right)^{1/2}\left(\int_r^\infty u^2\rho^{n-1-b}\,d\rho\right)^{1/2}.$$

On the other hand, (3.1) implies that

$$\int_r^\infty u^2(\rho,t)\rho^{n-1-b}\,ds \leqslant C\int_{\mathbb{R}^n} V(x)u^2(x,t)\,dx, \quad r \geqslant 1.$$

It follows that

$$|u(x,t)| \leqslant C\|\nabla u(t)\|_2^{1/2}\left\|V^{1/2}u(t)\right\|_2^{1/2} r^{-\gamma/2}, \quad r \geqslant 1. \tag{3.3}$$

Since $0 \leqslant k < \gamma/2$ by assumption, (3.2) and (3.3) imply that the sequence $\{u(t_j + \tau)\}$ is precompact in L_k^∞. □

We end this section with an auxiliary lemma concerning persistence and continuous dependence of local solutions of (1.2) in L_k^∞, which will be useful in the proof of Theorem 1.2(b) and (c).

LEMMA 3.3. – *Suppose $V(x)$ is a locally Hölder continuous and bounded function* (*not necessarily nonnegative*). *Let $k \geqslant 0$ and assume $u_0 \in L_k^\infty$.*

(a) *For all $0 < \tau < T(u_0)$, it holds*

$$\sup_{t\in[0,\tau]} \|u(t)\|_{\infty,k} < \infty. \tag{3.4}$$

(b) *For all $0 < \tau < T(u_0)$, if $\overline{u}_0 \in L_k^\infty$ and $\|u_0 - \overline{u}_0\|_{\infty,k}$ is sufficiently small, then $T(\overline{u}_0) > \tau$, and we have*

$$\sup_{t\in[0,\tau]} \|u(t;u_0) - u(t;\overline{u}_0)\|_{\infty,k} \to 0, \quad \textit{as } \|u_0 - \overline{u}_0\|_{\infty,k} \to 0.$$

Proof. – (a) We may assume $u_0 \geqslant 0$ and $u \geqslant 0$ without loss of generality. Since $\tau < T(u_0)$, we have $M := \sup_{t\in[0,\tau]} \|u(t)\|_\infty < \infty$. Putting $K = \|V\|_\infty$, we observe that u satisfies

$$u_t - (\Delta - 1)u = u^p + u - V(x)u \leqslant \left(M^{p-1} + 1 + K\right)u \quad \text{in } \mathbb{R}^n \times (0,\tau].$$

Letting $\alpha = M^{p-1} + 1 + K$ and $z(x,t) = \mathrm{e}^{-\alpha t} u(x,t)$, we obtain

$$z_t - (\Delta - 1)z \leqslant 0 \quad \text{in } \mathbb{R}^n \times (0,\tau],$$

so that $z(x,t) \leqslant \mathrm{e}^{t(\Delta-1)} u_0$ in $\mathbb{R}^n \times (0,\tau]$ by the maximum principle. It then follows from Proposition 2.1(a) (applied with $V \equiv 1$), that

$$\|u(t)\|_{\infty,k} \leqslant C\mathrm{e}^{\alpha\tau} \|u_0\|_{\infty,k}, \quad 0 \leqslant t \leqslant \tau,$$

which proves (3.4).

(b) Let $u(t) = u(t; u_0)$, $\overline{u}(t) = u(t; \overline{u}_0)$, $w(t) = u(t) - \overline{u}(t)$ and

$$M = \sup_{t\in[0,\tau]} \|u(t)\|_\infty.$$

By continuous dependence in L^∞, which is well known, if $\|u_0 - \overline{u}_0\|_\infty$ is sufficiently small, then $T(\overline{u}_0) > \tau$ and $\sup_{t\in[0,\tau]} \|\overline{u}(t)\|_\infty \leqslant M + 1$. Since w satisfies

$$w_t - (\Delta - 1)w = u^p - \overline{u}^p + w - V(x)w = a(x,t)w \quad \text{in } \mathbb{R}^n \times (0,\tau],$$

with $|a(x,t)| \leqslant p(M+1)^{p-1} + K + 1$, a calculation similar to that in part (a) with $\alpha = p(M+1)^{p-1} + K + 1$ shows that

$$\|w(t)\|_{\infty,k} \leqslant C\mathrm{e}^{\alpha\tau} \|u_0 - \overline{u}_0\|_{\infty,k}, \quad 0 \leqslant t \leqslant \tau,$$

and the conclusion follows. □

4. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1, part (a). – In view of the comparison principle, we can assume $u \geqslant 0$ without loss of generality.

Let U be defined by (2.3). For $k = b/(p-1)$ and $A = A(a,b) > 0$ large enough, we have

$$V(x) \geqslant \frac{k(k+2-n)}{A + |x|^2} + \frac{a_1/2}{(A + |x|^2)^{k(p-1)/2}}$$

hence, by (2.4),

$$-(\Delta - V)U \geqslant \frac{a_1/2}{(A + |x|^2)^{k(p-1)/2}} U.$$

Letting $W = \varepsilon U$ with $\varepsilon = (a_1/2)^{1/(p-1)}$, it follows that

$$-(\Delta - V)W \geqslant W^p \quad \text{in } \mathbb{R}^n.$$

The comparison principle thus implies that if $|u_0| \leqslant W$, then $|u(x,t)| \leqslant W(x)$ for all x and all $t < T(u_0)$. Therefore, if $\|u_0\|_{\infty,k} \leqslant \delta < \delta_0$ sufficiently small, we deduce that

$$\sup_{t\in[0,T(u_0))} \|u(t)\|_{\infty,k} \leqslant C' \|u_0\|_{\infty,k} < \infty,$$

hence in particular $T(u_0)=\infty$. The first part follows.

Now, since $b=k(p-1)$, we have, for all $x\in\mathbb{R}^n$ and $t>0$,

$$u_t-\Delta u+\frac{1}{2}Vu=u^p-\frac{1}{2}Vu\leqslant\left(\frac{C''\delta^{p-1}}{1+|x|^{k(p-1)}}-\frac{a_1/2}{1+|x|^b}\right)u\leqslant 0,$$

provided $\delta<\min(\delta_0,(a_1/(2C''))^{1/(p-1)})$. It follows from the maximum principle that $0\leqslant u(t)\leqslant \mathrm{e}^{t(\Delta-\frac{1}{2}V)}u_0$. If in addition $u_0\in L^\infty_{k,0}$, Proposition 2.1(b) then implies that (1.3) holds.

Proof of Theorem 1.1, part (b). – The proof is based on rescaling and is modeled after [13] (where the case $V\equiv 0$ was considered for Eq. (1.2) in a bounded domain). First, by standard local theory, there exists $\tau=\tau(\|u_0\|_\infty)>0$ such that

$$\|u(t;u_0)\|_\infty\leqslant\|u_0\|_\infty+1,\quad 0\leqslant t\leqslant\tau. \tag{4.1}$$

We argue by contradiction and assume that there exist a sequence u_j of global solutions and $s_j\geqslant 0$ such that

$$\|u_j(0)\|_{H^1}+\|u_j(0)\|_\infty\leqslant C\quad\text{and}\quad\sup_{t\in[0,s_j]}\|u_j(t)\|_\infty\to\infty. \tag{4.2}$$

Choose $(x_j,t_j)\in\mathbb{R}^n\times[0,s_j]$ such that

$$M_j:=u_j(x_j,t_j)\geqslant\frac{1}{2}\sup_{t\in[0,s_j]}\|u_j(t)\|_\infty.$$

By (4.1), we may assume $t_j\geqslant\tau$. We put $\lambda_j=M_j^{-(p-1)/2}\to 0$ and rescale u_j about the point (x_j,t_j) as follows:

$$v_j(y,s)=\lambda_j^{2/(p-1)}u_j\big(x_j+\lambda_j y,t_j+\lambda_j^2 s\big),\quad (y,s)\in Q_j:=\mathbb{R}^n\times\big[-\lambda_j^{-2}t_j,0\big].$$

The function v_j satisfies

$$\partial_s v_j-\Delta_y v_j=v_j^p-\lambda_j^2V(x_j+\lambda_j y)v_j\quad\text{in }Q_j$$

and

$$v_j(0,0)=1,\quad 0\leqslant v_j\leqslant 2\quad\text{in }Q_j.$$

By using interior L^q parabolic estimates (see [19, Theorem 7.13]), standard imbedding and a diagonal procedure, it follows that (some subsequence of) v_j converges, uniformly on compact subsets of $Q=\mathbb{R}^n\times(-\infty,0]$, to a (bounded) solution $v\geqslant 0$ of

$$\partial_s v-\Delta_y v=v^p\quad\text{in }Q,$$

such that $v(0,0)=1$. On the other hand, for each $m>0$, using Lemma 3.1 and (4.2), for all j large enough,

$$\int_{-m}^{0}\int_{|y|<m}(\partial_s v_j)^2\,dy\,ds \leqslant \lambda_j^{2(p+1)/(p-1)-n}\int_{t_j-m\lambda_j^2}^{t_j}\int_{\mathbb{R}^n}(\partial_t u_j)^2\,dx\,dt$$
$$\leqslant C\lambda_j^{2(p+1)/(p-1)-n}\|u_j(0)\|_{H^1}^2 \leqslant C\lambda_j^{2(p+1)/(p-1)-n}.$$

Since $2(p+1)/(p-1)-n>0$ by assumption, the RHS goes to 0 as $j\to\infty$. It follows that for all $Q'\subset\subset Q$, $\|\partial_s v\|_{L^2(Q')}\leqslant \liminf_j \|\partial_s v_j\|_{L^2(Q')}=0$, hence $\partial_s v\equiv 0$. Therefore $-\Delta_y v=v^p$ in $\mathbb{R}^n$, with $v\geqslant 0$, $v(0)=1$, which contradicts a Liouville theorem from [12]. The conclusion follows.

Proof of Theorem 1.1, part (c). – Define

$$D_0=\left\{u_0\in L^\infty_{k,0};\ T(u_0)=\infty \text{ and } u(t;u_0)\to 0 \text{ in } L^\infty_k \text{ as } t\to\infty\right\}.$$

By Theorem 1.2(a), it follows that D_0 contains an open neighborhood W of 0 in $L^\infty_{k,0}$ and that

$$D_0=\left\{u_0\in L^\infty_{k,0};\ T(u_0)=\infty \text{ and } 0\in\omega_k(u_0)\right\}. \tag{4.3}$$

We claim that D_0 is open in $L^\infty_{k,0}$. Indeed, if $u_0\in D_0$, there exists $t>0$ such that $u(t;u_0)\in W$. But by continuous dependence of solutions of (1.2) in L^∞_k (Lemma 3.3(b)), if $\|\overline{u}_0-u_0\|_{\infty,k}$ is sufficiently small, then $u(t;\overline{u}_0)\in W\subset D_0$, so that $\overline{u}_0\in D_0$. The claim follows.

Let now

$$\lambda^*=\sup\left\{\lambda>0;\ \lambda\phi\in D_0\right\}.$$

We have just seen that $\lambda\phi\in D_0$ when $\lambda>0$ is small, and it is well known that $T(\lambda\phi)<\infty$ if λ is large. Therefore, $0<\lambda^*<\infty$.

Let $\lambda_j\uparrow\lambda^*$ with $\lambda_j\phi\in D_0$. By Theorem 1.2(b), we have

$$\sup_{t\geqslant 0}\|u(t;\lambda_j\phi)\|_\infty\leqslant C\big(\lambda_j(\|\phi\|_{H^1}+\|\phi\|_\infty)\big)\leqslant C,\quad j=1,2,\ldots.$$

Since by continuous dependence in L^∞, we have, for each $t\in[0,T(\lambda^*\phi))$,

$$\|u(t;\lambda^*\phi)\|_\infty=\lim_j\|u(t;\lambda_j\phi)\|_\infty\leqslant C,$$

it follows that $T(\lambda^*\phi)=\infty$.

On the other hand, by the openness of D_0, $\lambda^*\phi\notin D_0$, and (4.3) thus implies that $0\notin\omega_k(\lambda^*\phi)$. The assumption (1.4) implies that $k=\frac{b}{p-1}<\frac{n-1}{2}-\frac{b}{4}$. Therefore, $\omega_k(\lambda^*\phi)\neq\emptyset$ by Proposition 3.1(b) and we deduce from Lemma 3.2 that $\omega_k(\lambda^*\phi)$ contains a nontrivial nonnegative equilibrium v. The strong maximum principle finally implies that $v>0$ in $\mathbb{R}^n$. The proof is complete. □

Proof of Theorem 1.2. – The existence of a radial positive equilibrium v follows from Theorem 1.1 part (c). Then estimates (3.2) and (3.3) imply that u satisfies

$$v(x)\leqslant\frac{C_0}{1+|x|^{\gamma/2}},\quad \gamma=n-1-(b/2).$$

Since $V \geqslant \frac{a_1}{1+|x|^b}$ and $\gamma/2 > b/(p-1)$, there exists $R_0 > 0$ such that

$$\frac{V(x)}{2}v(x) - v^p(x) = v(x)\left(\frac{V(x)}{2} - v^{p-1}(x)\right)$$
$$\geqslant v(x)\left(\frac{a_1}{2(1+|x|^b)} - C\frac{C_0^{p-1}}{1+|x|^{\gamma(p-1)/2}}\right) > 0$$

when $|x| \geqslant R_0$. Therefore v satisfies

$$\Delta v(x) - V(x)v(x)/2 \geqslant 0, \quad |x| \geqslant R_0; \qquad v(x) \geqslant C > 0, \quad |x| = R_0.$$

Let $u_0(x) = \Gamma_1(0, x)$, where Γ_1 is the Green's function of the operator $\Delta - V/2$. Since both v and Γ_1 vanish near infinity, by the maximum principle, there exists $c_0 > 0$ such that $u(x) \leqslant c_0 u_0(x)$ when $|x| \geqslant R_0$. Since $V/2 \geqslant \frac{a_1}{2(1+|x|^b)}$, by [22] or Corollary 1 in [32], under the assumptions in the theorem, there exist positive constants c_1, c_2 such that, for all x, y and $\alpha = (2-b)/2$,

$$\Gamma_1(x, y) \leqslant c_1 \mathrm{e}^{-c_2[|x-y|/(1+|x|^{b/2})]^\alpha} \mathrm{e}^{-c_2[|x-y|/(1+|y|^{b/2})]^\alpha} \frac{C}{|x-y|^{n-2}}.$$

Taking $y = 0$ in the above inequality, we have

$$v(x) \leqslant c_0\Gamma_1(x, 0) \leqslant Cc_0 \exp\left(-c_2|x|^{(2-b)/2}\right). \qquad \square$$

We close the section by a remark indicating some extension of Theorems 1.1 and 1.2 covering broader nonlinearities. We omit the proof since it is a straightforward generalization of the current one.

Remarks 4.1. – Let $f = f(x, u)$ be C^1 in u and locally Hölder continuous in x.

(a) The result of Theorem 1.1(a) still holds if u^p is replaced by $f(x, u)$ satisfying $|f(x, u)| \leqslant C|u|^p$ for small u.

(b) If u^p is replaced by $f(x, u)$ satisfying $(2+\varepsilon)F(x, u) \leqslant uf(u)$ for $u > 0$, $x \in \mathbb{R}^n$, where $F(x, u) = \int_0^u f(x, s)\,ds$ and $\varepsilon > 0$, then Proposition 3.1(a) remains valid with $\|u(t_j)\|_{p+1}$ replaced by $\|u(t_j)f(., u(t_j))\|_1$.

(c) Proposition 3.1(b) continues to hold if u^p is replaced by $f(x, u) = f(|x|, u)$ such that $|\frac{\partial f}{\partial u}(x, u)| \leqslant C(1 + |u|^{r-1})$, $1 < r < p_S$. The main change in the proof is that one no longer knows that $\|u(t_j)\|_{p+1} \leqslant K$. Instead, one can show that

$$\|u(t_j)\|_m \leqslant K \tag{4.4}$$

for some $m > n(p-1)/2$. If $n \geqslant 3$, (4.4) with $m = 2n/(n-2) > n(p-1)/2$ follows from $\|\nabla u(t_j)\|_2 \leqslant C$ (Proposition 3.1(a)) and the Sobolev inequality. If $n = 2$, we first use (3.3) for $t = t_j,\ r = 1$ and $\|V^{1/2}u(t_j)\|_2 + \|\nabla u(t_j)\|_2 \leqslant C$, along with the Poincaré and Sobolev inequalities, to deduce that $\|u(t_j)\|_{L^m(B_1(0))} \leqslant C_m,\ 2 \leqslant m < \infty$. Applying (3.3) again, we then get (4.4) for all sufficiently large $m < \infty$.

(d) Theorem 1.1(b) is valid when u^p is replaced by $f(x,u)$. Here $f(x,u)$ satisfies $f(x,0) \geqslant 0$, the conditions in (b) and either of the assumptions below.

(i) $\lim_{u\to\infty} \frac{f(x,u)}{u^r} = K > 0$ uniformly in $x \in \mathbb{R}^n$, with $1 < r < p_S$;

(ii) $f(x,u) \geqslant C_1 u^r - C_2,\ u > 0$, with $1 < r \leqslant n/(n-2)_+,\ C_1, C_2 > 0$.

In case (ii) one substitutes to the Liouville Theorem in [12] a Liouville Theorem in [2], which is valid for the elliptic inequality $\Delta u + u^p \leqslant 0$ (see Theorem 2.2 in [2] for $n \geqslant 3$ and the proof of Theorem 2.1 for $n \leqslant 2$).

(e) Theorems 1.1(c) and 1.2 hold when u^p is replaced by $f(x,u) = f(|x|,u)$ satisfying the assumptions in (a), (b), (c) and (d).

(f) Here are some examples of $f(|x|,u)$ for which Theorems 1.1(c) and 1.2 hold. The functions $a = a(|x|)$ and $h = h(|x|)$ are assumed to be bounded and locally Hölder continuous. Also, recall that p is the number which appears in assumptions (1.4), (1.5) (note that the behaviors of f at both $u \to 0$ and $u \to \infty$ are important).

$$f(x,u) = u^p - h(|x|)u^q, \quad \text{with } 1 < q < p < p_S,\ h \geqslant 0;$$

$$f(x,u) = a(|x|)u^p - h(|x|)u^q, \quad \text{with } 1 < q < p \leqslant \frac{n}{n-2},\ a \geqslant C_1 > 0,\ h \geqslant 0;$$

$$f(x,u) = a(|x|)u^p + u^q, \quad \text{with } 1 < p < q < p_S;$$

$$f(x,u) = u^p + h(|x|)u^q, \quad \text{with } 1 < p < q \leqslant \frac{n}{n-2},\ h \geqslant C_1 > 0.$$

5. Proof of Theorem 1.3: universal bounds for $1 < p < 1 + \frac{2}{n}$

Throughout this section we let $\phi, \eta \in C^\infty([0,\infty))$ be two functions satisfying

$$\phi(r) = 1, \quad r \in [0, 1/2], \qquad 0 < \phi(r) < 1, \quad r \in (1/2, 3/4),$$

$$\phi(r) = 0, \quad r \in [3/4, \infty), \qquad 0 \leqslant \eta \leqslant 1, \qquad \eta(t) = 1, \quad t \in [0, 1/4],$$

$$\eta(t) = 0, \quad t \in [1, \infty), \qquad -C \leqslant \phi'(r) \leqslant 0, \qquad |\phi''(r)| \leqslant C, \qquad -C \leqslant \eta'(t) \leqslant 0.$$

LEMMA 5.1. – *Suppose* $V(x) \leqslant \frac{a}{1+|x|^b}$ *with* $b \in [0,2)$, $a > 0$ *and let* u *be a nonnegative global solution to* (1.1). *Given any* x *and letting* $R = \max(2, |x|^{b/2})$, *it holds*

$$\int_\tau^{\tau+R^2} \int_{B_R(x)} u^p(y,s)\,dy\,ds \leqslant C R^{n+2-2q}, \tag{5.1}$$

where $q = p/(p-1)$ *and* $\tau \geqslant 0$.

Proof. – Without loss of generality let us assume $\tau = 0$. For any x_0 and any $R > 0$ define $Q_{R,x_0} = B_R(x_0) \times [0, R^2]$. We also need a cut-off function

$$\psi_R(x,t) = \phi_R(|x - x_0|)\eta_R(t),$$

where $\phi_R(r) = \phi(r/R)$ and $\eta_R = \eta(t/R^2)$. Clearly

$$-\frac{C}{R} \leqslant \frac{\partial \phi_R}{\partial r} \leqslant 0, \qquad \left|\frac{\partial^2 \phi_R}{\partial r^2}\right| \leqslant \frac{C}{R^2}, \qquad |\Delta \phi_R| \leqslant \frac{C}{R^2}, \qquad -\frac{C}{R^2} \leqslant \eta_R'(t) \leqslant 0. \quad (5.2)$$

We set

$$I_R \equiv \int_{Q_{R,x_0}} u^p(x,t)\psi_R^q(x,t)\,dx\,dt,$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Since u is a solution of (1.1), we have

$$I_R = \int_{Q_{R,x_0}} \left[u_t(x,t) - \Delta u(x,t) + V(x)u(x,t)\right]\psi_R^q(x,t)\,dx\,dt.$$

Noting that $\phi_R^q \in C^2$, this implies, via integration by parts,

$$\begin{aligned} I_R = &\int_{B_R(x_0)} u(x,.)\psi_R^q(x,.)\big|_0^{R^2}\,dx - \int_{Q_{R,x_0}} u(x,t)\phi_R^q(x)q\eta_R^{q-1}(t)\eta_R'(t)\,dx\,dt \\ &+ \int_0^{R^2}\int_{\partial B_R(x_0)} u(x,t)\frac{\partial \psi_R^q(x,t)}{\partial r}\,dS_x\,dt - \int_0^{R^2}\int_{\partial B_R(x_0)} \psi_R^q \frac{\partial u}{\partial r}(x,t)\,dS_x\,dt \\ &- \int_{Q_{R,x_0}} u(x,t)\Delta\phi_R^q(x)\eta_R^q(t)\,dx\,dt + \int_{Q_{R,x_0}} u(x,t)V(x)\psi_R^q(x,t)\,dx\,dt. \end{aligned}$$

Using $u(x,0) \geqslant 0$, $\psi_R(x,R^2) = 0$ and $\psi_R(x,t) = \frac{\partial \psi_R^q(x,t)}{\partial r} = 0$ on $\partial B_R(x_0) \times [0,R^2]$, we obtain

$$\begin{aligned} I_R \leqslant &- \int_{Q_{R,x_0}} u(x,t)\phi_R^q(x)q\eta_R^{q-1}(t)\eta_R'(t)\,dx\,dt - \int_{Q_{R,x_0}} u(x,t)\Delta\phi_R^q(x)\eta_R^q(t)\,dx\,dt \\ &+ \int_{Q_{R,x_0}} u(x,t)V(x)\psi_R^q(x,t)\,dx\,dt. \end{aligned}$$

Since $\Delta\phi_R^q = q\phi_R^{q-1}\Delta\phi_R + q(q-1)\phi_R^{q-2}|\nabla\phi_R|^2$, the above yields

$$\begin{aligned} I_R \leqslant &- \int_{Q_{R,x_0}} u(x,t)\phi_R^q(x)q\eta_R^{q-1}(t)\eta_R'(t)\,dx\,dt \\ &- \int_{Q_{R,x_0}} u(x,t)q\left(\phi_R^{q-1}\Delta\phi_R\right)(x)\eta_R^q(t)\,dx\,dt + \int_{Q_{R,x_0}} u(x,t)V(x)\psi_R^q(x,t)\,dx\,dt. \end{aligned}$$

Consequently, by (5.2), we have

$$I_R \leqslant \frac{C}{R^2} \int\limits_{Q_{R,x_0}} u(x,t)\phi_R^q(x)\eta_R^{q-1}(t)\,dx\,dt + \frac{C}{R^2} \int\limits_{Q_{R,x_0}} u(x,t)\phi_R^{q-1}\eta_R^q(t)\,dx\,dt + \int\limits_{Q_{R,x_0}} u(x,t)V(x)\psi_R^q(x,t)\,dx\,dt.$$

Since $\phi_R, \eta_R \leqslant 1$, by Hölder's inequality we have

$$\begin{aligned} I_R \leqslant{}& \frac{C}{R^2}\Bigg(\int\limits_{Q_{R,x_0}} u^p\psi_R^{p(q-1)}(x,t)\,dx\,dt\Bigg)^{1/p}\Bigg(\int\limits_{Q_{R,x_0}} dx\,dt\Bigg)^{1/q} \\ &+ \frac{C}{R^2}\Bigg(\int\limits_{Q_{R,x_0}} u^p\psi_R^{p(q-1)}(x,t)\,dx\,dt\Bigg)^{1/p}\Bigg(\int\limits_{Q_{R,x_0}} dx\,dt\Bigg)^{1/q} \\ &+ \Bigg(\int\limits_{Q_{R,x_0}} u^p\psi_R^{p(q-1)}(x,t)\,dx\,dt\Bigg)^{1/p}\Bigg(\int\limits_{Q_{R,x_0}} V(x)^q\,dx\,dt\Bigg)^{1/q}. \end{aligned}$$

Therefore

$$I_R \leqslant C I_R^{1/p} R^{\frac{n+2}{q}-2} + C I_R^{1/p}\Bigg(\int_0^{R^2}\int\limits_{B_R(x_0)} V(x)^q\,dx\,dt\Bigg)^{1/q}. \tag{5.3}$$

Take now $R = \max(2, |x_0|^{b/2})$. If $|x_0|^{b/2} \geqslant 2$, as $b < 2$, we have $|x| \geqslant |x_0| - |x - x_0| \geqslant cR^{2/b}$ when $x \in B_R(x_0)$. Since $V(x) \leqslant \frac{a}{1+|x|^b}$, we get

$$V(x) \leqslant CR^{-2}, \quad x \in B_R(x_0). \tag{5.4}$$

If $|x_0|^{b/2} < 2 = R$, then (5.4) is also true since V is bounded. It follows from (5.3) and (5.4) that

$$I_R \leqslant CR^{n+2-2q}. \qquad \square$$

Next we will prove Theorem 1.3 part (b). The proof of part (a), which is similar but easier, will follow shortly.

Proof of Theorem 1.3 part (b). – We will use the standard parabolic Harnack inequality [21] and Lemma 5.1. Fix x and let $R = \max(2, |x|^{b/2})$. Put $Q_R(x,t) = B_R(x) \times [t - R^2, t]$ and $Q_R^+(x,t) = B_R(x) \times [t + R^2, t + 2R^2]$. We need to divide the proof in two cases.

Case 1: $t \geqslant R^2$. Since u is a solution to (1.1), we can recognize it as a solution to the linear equation

$$\Delta u - wu - u_t = 0, \tag{5.5}$$

where $w = V - u^{p-1}$. Using a standard scaling argument on the parabolic Harnack inequality, we have

$$u(x,t) \leqslant C(R,w) \inf_{Q^+_{x,R}} u \tag{5.6}$$

where $C(R,w)$ is a constant depending only on the dimension n and the quantity

$$R^{(-n-2+2q)/q} \| w|_{Q_{2R}(x,t)}\|_q$$

where $q = p/(p-1) > (n+2)/2$. For the reader's convenience we give a short proof of (5.6). For simplicity we take $x = 0,\ t = 0$. It is well known that (5.6) holds when $R = 1$. Suppose $R > 1$, we take the substitution: $y' = y/R,\ \ s' = s/R^2$ and $w' = R^2 w(Ry', R^2 s')$. If u is a solution of (5.5) in $Q_{2R}(0,0)$, then $u' = u'(y', s') = u(Ry', R^2 s')$ is a solution to

$$\Delta u' - w'u' - u'_t = 0$$

in $Q_2(0,0)$. Hence

$$u(x,t) \leqslant C(1,w') \inf_{Q^+_R(x,t)} u.$$

(5.6) is proven by noticing that

$$\|w'|_{Q_2(0,0)}\|_q = R^{(-n-2+2q)/q} \| w|_{Q_{2R}(0,0)}\|_q.$$

If $|x|^{b/2} \geqslant 2$, for all $y \in B_{2R}(x)$, we have $|y| \geqslant |x| - 2R = R^{2/b} - 2R \geqslant cR^{2/b}$. Therefore $V(y) \leqslant C/R^2$. It follows that

$$\|V|_{Q_{2R}(x,t)}\|_q \leqslant CR^{(n+2-2q)/q}. \tag{5.7}$$

Note that (5.7) remains true if $|x|^{b/2} < 2 = R$ since V is bounded. Using (5.7) and Lemma 5.1 we obtain

$$R^{(-n-2+2q)/q} \|w|_{Q_{2R}(x,t)}\|_q \leqslant R^{(-n-2+2q)/q} \left(\|V|_{Q_{2R}(x,t)}\|_q + \left(\int_{Q_{2R}(x,t)} u^{(p-1)q} \right)^{1/q} \right) \leqslant C.$$

Inequality (5.6) then implies

$$u(x,t) \leqslant C \inf_{Q^+_R(x,t)} u \leqslant C \left(\frac{1}{|Q^+_R(x,t)|} \int_{Q^+_R(x,t)} u^p \right)^{1/p}$$

where C is independent of u and R. Applying Lemma 5.1 again, one has

$$u(x,t) \leqslant C\left[R^{-n-2} R^{n+2-2q}\right]^{1/p} = CR^{-2q/p} = CR^{-2/(p-1)} \leqslant \frac{C}{1 + |x|^{b/(p-1)}}. \tag{5.8}$$

This proves the result in case 1.

Case 2: $t \leqslant R^2$. If $|x| \geqslant C(1+R_0)$, then $B_{2R}(x) \cap B_{R_0}$ is empty for $R = \max(2, |x|^{b/2})$. Let us define a function u_1 in $Q_{x,2R}$ in the following manner: $u_1(y,s) = u(y,s)$ when $s > 0$, $u_1(y,s) = 0$ when $s \leqslant 0$. Since u_0 is supported in $B_R(0)$, we know that u_1 can be recognized as a weak solution to the equation

$$\Delta u_1 - w_1 u_1 - \partial_t u_1 = 0$$

in $Q_{x,2R}$. Here $w_1(y,s) = V(y,s) - u^{p-1}(y,s)$ if $s > 0$ and $w_1(y,s) = 0$ if $s \leqslant 0$. Note that w_1 is a bounded function, so the Harnack inequality still applies. Now the same argument as in case 1 yields (5.8).

Finally, if $2R_0 \leqslant |x| \leqslant C(1+R_0)$, take instead $R = R_0/2$ so that $B_{2R}(x) \cap B_{R_0}$ is empty. Observing that for $R' = \max(2R, 2, |x|^{b/2})$, we have $\int_{Q_{2R}(x,t)} u^p \leqslant \int_{Q_{R'}(x,t)} u^p \leqslant C(R_0)$ by Lemma 5.1, one can then obtain (5.8) by arguing as above. □

We close the section by giving

Proof of Theorem 1.3 part (a). – This part can be proven just like part (b). The only change is that we may take $b = 0$ and that (5.1) in Lemma 5.1 is then valid for all $R > 0$. We then choose $R = \sqrt{T}/2$ in the proof of part (b) and only Case 1 occurs (so that we do not need to assume u_0 compactly supported). □

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