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Construction of entropy solutions for one dimensional elastodynamics via time discretisation

by

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ABSTRACT. – It is shown that the variational approximation scheme for one-dimensional elastodynamics given by time discretisation converges, subsequentially, weakly and a.e. to a weak solution which satisfies the entropy inequalities. We also prove convergence under the restriction of positive spatial derivative (for longitudinal motions).

RÉSUMÉ. – Pour les équations d'élasticité dynamiques à une variable d'espace on démontre que la méthode de discrétisation en la variable temps, résolue variationellement, produit une suite de solutions approchées convergeant, par une sous suite, faiblement et presque partout vers une solution faible classique qui satisfait les inégalités d'entropie. Nous prouvons aussi que le même résultat reste valable sous la restriction de positivité des dérivées partielles en la variable spatiale.

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1. STATEMENT OF THE PROBLEM AND THE RESULTS

Global weak solutions to the system of one-dimensional elastodynamics

$$\begin{aligned} u_t - v_x &= 0, \\ v_t - \sigma(u)_x &= 0, \end{aligned} \tag{1.1}$$

for $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$, $(u, v) \in \mathbb{R}^2$ and σ strictly increasing and convex (or σ with one inflection point) are obtained in a well known article of DiPerna [7]. There, following a general conjecture of Tartar, the theory of Young measures and Tartar's commutation relation in [18] are successfully exploited to establish a compactness framework for approximate solutions to the equations of elasticity. In turn, this framework is also used to construct global weak solutions to (1.1) via viscosity approximations [7,12,17], relaxation approximations [19,15], and numerical schemes (see, e.g., [7,8]). Further references to related work are given in the article [2], in which large time behaviour of solutions is analysed.

Here we investigate an alternative approximation scheme, appropriate for certain evolutionary systems (of parabolic or hyperbolic type) with an associated variational structure. The scheme proceeds by solving variationally the time discretised version of the equations with time step $h > 0$ for $j \geqslant 1$. This admits a variational formulation: assuming that the first $j-1$ iterates are known, the jth are the solutions of the minimisation problem

$$\inf_{(u,v)\in H^1(\mathbb{R})^2} \int \left(W(u) + \frac{(v - v^{h,j-1})^2}{2} \right) dx$$

on the set $\{(u, v)\colon\ u = hv_x + u^{h,j-1}\}$ (restricting to periodic functions). Solutions of (1.1) are constructed as limits of this iteration process as $h \to 0$. This scheme is also useful in higher dimensional problems [6] and is potentially useful in the construction of new computational schemes for conservation laws. It produces a regularisation which is *admissible* in the sense that it obeys *entropy inequalities*; the emerging approximate solutions converge subsequentially almost everywhere to a classical weak solution (u, v) of (1.1) which decreases all convex entropies $\eta(u, v)$, that is,

$$\partial_t \eta(u,v) + \partial_x q(u,v) \leqslant 0 \tag{1.3}$$

for all η strictly convex with $\nabla\eta\nabla f = \nabla q$ and $f(u,v) = (-v, -\sigma(u))^{\mathrm{t}}$.

Our analysis using time-discretisation also partially justifies recent approaches to the existence of measure-valued solutions to non-convex evolutionary equations for which classical weak solutions do not, or are not known to exist, for example the forward-backward heat equation in [10], the non-monotone case of (1.1) in arbitrary spatial dimension in [4], and the equations of three dimensional polyconvex elastodynamics in [6]. The technique has also been recently used in [5] and [14] to study systems with a combination of hyperbolic and dissipative character.

STATEMENT OF RESULTS

By time discretisation we show the existence of a classical weak solution to (1.1), under various sets of assumptions and constitutive relations on σ, W, corresponding to physically appropriate *stress-strain laws* of elastic models: for *hardening elastic response* corresponding to assumptions (2.1)–(2.4), for *longitudinal motions* under assumptions (3.1)–(3.5) and finally for *shearing motions with softening response* under assumptions (3.20)–(3.23). In the first case we prove:

THEOREM 1.1 (Weak solutions for 1-d elastodynamics with hardening response). – *Assume that* $(u_0, v_0) \in L^\infty(\mathbb{R})^2$ *are periodic with period* Π*, of zero mean, and that* $\sigma \in C^3(\mathbb{R})$ *satisfies* (2.1)–(2.4)*. Then there exists a sequence* (u^h, v^h) *obtained uniquely via time discretisation* (1.2) *and there is* $(u,v) \in L^\infty(\mathbb{R}^2)^2$*, (periodic in space), such that subsequentially* $(u^h, v^h) \to (u,v)$ *a.e. and in* L^p*, for all* $1 \leqslant p < \infty$*, as* $h \to 0$*. Furtheremore,* (u,v) *satisfies* (1.1) *and the entropy inequalities*

$$\partial_t \eta(u,v) + \partial_x q(u,v) \leqslant 0 \quad \textit{in } \mathcal{D}', \tag{1.4}$$

for all entropy pairs η, q *with* η *convex. The initial data are attained in* $H^{-1}(\mathbb{R})^2$*, as* $t \to 0^+$*.*

The method of proof consists first of solving variationally the discretised equations in L^p. We show that the iterates satisfy discretised entropy inequalites for all convex entropies and, using a class of exponentially growing entropies constructed by Dafermos [3], we strengthen the estimates to uniform ones in L^∞. Then this allows application of the theorem of DiPerna which yields a classical weak solution. Although the theory of approximate solutions of DiPerna applies to general systems of two conservation laws, $z_t + f(z)_x = 0$ (*i.e.* the vector components of f need not be potential gradients), the variational approach to time discretisation adopted here requires additionally that the system possess a variational structure. The proof of Theorem 1.1 is the content of Section 2.

Subsequently we generalise this result in Section 3.1 to longitudinal motions for certain stress-strain laws and then in Section 3.2 to shearing motions with softening elastic response. Regarding longitudinal motions, recall that a particular requirement in elasticity theory is that the energy blows up as the deformation gradient loses invertibility which here corresponds to $u \to 0$. To obtain estimates in this case we show that the entropy construction of Dafermos [3] can be generalised to incorporate such singularities and then show that the same conclusion as in Theorem 1.1 is valid. The precise statements in this case appear in Lemma 3.2 and Theorem 3.1 which we here summarise:

(Weak solutions for longitudinal motions) *Under the hypotheses* (3.1)–(3.5), *there exist convex entropies, defined only on the half line, exponentially growing at zero and infinity, and by time discretisation there exists a classical weak solution to* (1.1) *and the conclusions of Theorem* 1.1 *hold.*

In the last Section 3.2 we outline the extension to the case of shearing motions with softening elastic response.

In closing we note that (1.1) is Hamiltonian; in fact, let $\omega(X,Y) \equiv \langle X, JY\rangle_{L^2}$ be a two-form on the space $(y,v) \in \mathcal{H} = H^1 \oplus L^2$ and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

so that ω is a symplectic structure. Letting

$$H(y,v) = \int_{\mathbb{R}} \left(W(y_x) + \frac{v^2}{2} \right) dx,$$

then

$$dH(X) = \omega(\xi_H, X) = \int \left(-\sigma(y_x)_x X_1 + vX_2 \right) dx$$

for any vector field $X=(X_1,X_2)$, where $\xi_H=(v,\sigma(y_x)_x)$. In fact, considering

$$J:\mathcal{D}(J)\equiv L^2\oplus H^1\subset H^{-1}\oplus L^2\equiv\mathcal{H}^*\to\mathcal{H}$$

we have $\xi_H=J\,dH$, as $dH\in\mathcal{H}^*$. It would be of interest to determine if the Hamiltonian structure affects the convergence of the present scheme.

2. WEAK SOLUTIONS FOR HARDENING RESPONSE

Assumptions. In this section it is assumed that the function $\sigma\in C^3(\mathbb{R})$ satisfies:

$$\sigma'(u)\geqslant\gamma>0\quad\forall u\in\mathbb{R},\tag{2.1}$$

$$u\sigma''(u)>0\quad\forall u\neq 0.\tag{2.2}$$

The latter implies genuine nonlinearity of the conservation law for $u\neq 0$. The model (1.1) with the stress-strain law (2.1)–(2.2) describes shearing motions of elastic materials with hardening response. It is also assumed that the strain energy function $W(u)$, such that $W'=\sigma$, satisfies $W\in C^4(\mathbb{R};\mathbb{R}^+)$ and the growth restrictions

$$\left(c_1|u|^p-c_2\right)^+\leqslant W(u)\leqslant C_1|u|^p+C_2,\tag{2.3}$$

$$|\sigma(u)|+|u||\sigma'(u)|\leqslant C_3+C_4|u|^{p-1},\tag{2.4}$$

for some positive constants c_i, C_i and $p\geqslant 2$.

Notation. Throughout this article we use the following spaces of functions: for $1\leqslant p<\infty$ denote by L^p the Lebesgue space of real valued, measurable, pth-integrable functions on $\mathbb{R}$ which are *Π-periodic* satisfying $u(x+\Pi)=u(x)$ almost everywhere, and of *zero mean, i.e.,*

$$\int_0^{\Pi}u(x)\,dx=0.$$

For $p=\infty$ the space L^∞ is the space of essentially bounded functions of zero mean which satisfy $u(x+\Pi)=u(x)$ almost everywhere. We denote by $W^{1,p}$ the space of absolutely continuous *Π-periodic functions of zero mean* whose almost everywhere derivative lies in L^p, and $W^{-1,p'}=(W^{1,p})'$. Below we denote by H^1 the space $W^{1,2}$. Finally, $Q\equiv\mathbb{R}^+\times\mathbb{R}$ and the spaces $L^p(Q)$, $W^{1,p}(Q)$ are the corresponding spaces of functions of (t, x) which are periodic and of zero mean in the x variable only.

The first and second difference operators δ, δ^2 are linear operators acting on sequences $y \in l^\infty(\mathbb{Z}^+; X)$, X any vector space, given by

$$(\delta y)^j = y^j - y^{j-1} \qquad (\delta^2 y)^j = y^j - 2y^{j-1} + y^{j-2}. \tag{2.5}$$

The parentheses will be omitted henceforth.

2.1. The discretisation and iteration

We now develop the regularisation (1.2) introduced in the introduction and solve it variationally. We work with the equivalent second order equation

$$y_{tt} - \big(\sigma(y_x)\big)_x = 0 \quad (t, x) \in Q. \tag{2.6}$$

If $u = y_x$ and $v = y_t$ this reduces to the system (1.1). Note that starting with Π-periodic data and by subtracting a rigid motion $Y(t) = a_0 t + b_0$ with a_0, b_0 constants we may assume that the initial data of (2.6)

$$y(0) = y_0 \quad \text{and} \quad y_t(0) = v_0$$

are Π-periodic and of zero mean. In accordance, the data $u(0) = u_0$ and $v(0) = v_0$ of (1.1) are also taken to be Π-periodic and of zero mean.

The time discretisation method amounts to replacing (2.6) by the discrete dynamical system

$$\frac{y^{h,j} - 2y^{h,j-1} + y^{h,j-2}}{h^2} = \big(\sigma\big(y_x^{h,j}\big)\big)_x. \tag{2.7}$$

Fix a time step $h > 0$; assuming the iterates $y^{h,k}$ are known for $k < j$ then $y^{h,j}$ is obtained as the solution to the minimisation problem

$$\begin{aligned} I_{h,j}\big[y^{h,j}\big] &= \min_{y \in H^1} I_{h,j}[y] \\ &= \min_{y \in H^1} \int_{\mathbb{R}} \left(W(y_x) + \frac{(y - 2y^{h,j-1} + y^{h,j-2})^2}{2h^2} \right) dx. \end{aligned} \tag{2.8}$$

We show for each $h > 0$ the existence of a sequence $y^h = \{y^{h,j}\}_{j=0}^\infty$: $\mathbb{Z}^+ \mapsto W^{1,p}$, solutions of (2.7). We derive the basic properties of the iterates, and make use of suitably growing entropy pairs to obtain L^∞

bounds, uniformly in h. It is convenient to define

$$v^{h,j} = \frac{y^{h,j} - y^{h,j-1}}{h} \quad \text{and} \quad u^{h,j} = y_x^{h,j}, \tag{2.9}$$

in terms of which and (2.5), (2.7) may be written as

$$\frac{\delta v^{h,j}}{h} = \frac{\delta^2 y^{h,j}}{h^2} = \sigma\left(y_x^{h,j}\right)_x. \tag{2.10}$$

The initial values $(y^{h,0}, y^{h,-1}) \in W^{1,\infty} \times L^\infty$ are determined from $(u(0), v(0))$ as follows: let y_0 be the periodic function of zero mean such that $(y_0)_x = u(0)$ and set

$$y^{h,0} = y_0, \qquad y^{h,-1} = y^{h,0} - hv(0).$$

This ensures that for all $h > 0$

$$\|u^{h,0}\|_{L^\infty} + \|v^{h,0}\|_{L^\infty} = \|y_x^{h,0}\|_{L^\infty} + \|v^{h,0}\|_{L^\infty} \leqslant M < \infty, \tag{2.11}$$

where $M = \|u(0)\|_{L^\infty} + \|v(0)\|_{L^\infty}$.

2.2. Existence of the iterates

Let $y^{h,j-2}, y^{h,j-1}$ be assumed to be given. By assumptions (2.1)–(2.4) the functional

$$I_{h,j}[y] = \int_0^{\Pi} \left(W(y_x) + \frac{(y - 2y^{h,j-1} + y^{h,j-2})^2}{2h^2} \right) dx \tag{2.12}$$

is strictly convex, co-ercive, and weakly lower semicontinuous on $W^{1,p}$ so that its infimum there is attained at a *unique* minimum $y^{h,j}$. By the growth (2.4) and regularity assumptions on W it follows by direct computation that the functional $y \mapsto I[y]$ is Gateaux differentiable on $W^{1,p}$ so that the minimiser satisfies the Euler–Lagrange equation which in integral form implies

$$\int_0^{\Pi} \left(\sigma\left(y_x^{h,j}\right)\zeta_x + \frac{y^{h,j} - 2y^{h,j-1} + y^{h,j-2}}{h^2}\zeta \right) dx = 0, \tag{2.13}$$

for all $\zeta \in W^{1,p}$ (where the boundary terms vanish by the periodic conditions on $y^{h,j}, \zeta$). In view of this it follows that (2.7) holds in $W^{-1,p'}$. This implies further regularity properties of the iterates as follows.

To start with $y^{h,1}$ lies in $W^{1,p}$ by construction and so for $j \geqslant 1$ we may assume that $y^{h,j-2}, y^{h,j-1}$ both lie in L^p. Express (2.7) as $y^{h,j} - a = \sigma(y_x^{h,j})_x$ where $a \in L^p$. Since $y^{h,j} \in W^{1,p}$, the growth condition (2.4) and Poincare's inequality imply that $\sigma(y_x^{h,j}) \in W^{1,p}$ and hence is continuous. Since σ^{-1} exists and is globally Lipschitz and C^2 by (2.1), it follows that $y_x^{h,j} = \sigma^{-1}(\sigma(y_x^{h,j}))$ is itself in $C^0 \cap W^{1,p}$. This implies that $y^{h,j} \in W^{2,p}$ and in turn that $(u^{h,1}, v^{h,1}) \in W^{1,p} \oplus W^{1,p}$, $(u^{h,j}, v^{h,j}) \in W^{1,p} \oplus W^{2,p}$ for all $j \geqslant 2$ and the Euler–Lagrange equation (2.13) holds a.e. These estimates depend on j and h: this situation will be remedied in Section 2.3 where it is proved that the iterates $(u^{h,j}, v^{h,j})$ satisfy L^∞ estimates uniformly in h, j.

Remark (Constrained minimisation). – As an alternative to the minimisation process described above, the successive iterates can be generated in turn as solutions to a constrained minimisation scheme, as appears in the introduction. Thus if $(u^{h,j-1}, v^{h,j-1})$ are known consider the problem of minimising

$$\int \left(W(u) + \frac{(v - v^{h,j-1})^2}{2} \right) dx \tag{2.14}$$

in the space of $(u, v) \in L^p \oplus L^2$ such that the constraint equation holds weakly, *i.e.*, for all $\phi \in C^1$ of zero mean and Π-periodic,

$$\int \left(\frac{(u - u^{h,j-1})}{h} \phi + v\phi_x \right) dx = 0. \tag{2.15}$$

Consider (u^i, v^i) a minimising sequence: by co-ercivity we may assume boundedness in $L^p \oplus L^2$ and so there exists a subsequence, also labelled as (u^i, v^i), converging weakly to $(u, v) \in L^p \oplus L^2$. The condition (2.15) is closed and the functional (2.14) is weakly lower semi-continuous under weak convergence in the space $L^2 \oplus L^p$, so the minimum value is attained at (u, v). Now let $(\tilde{u}, \tilde{v})$ be smooth and related by

$$\tilde{u} = h\tilde{v}_x;$$

then we know that

$$\int \left(\sigma(u)\tilde{u} + (v - v^{h,j-1})\tilde{v} \right) dx = 0, \tag{2.16}$$

and so

$$\int\left(\sigma(u)_x - \frac{(v - v^{h,j-1})}{h}\right)\tilde{v}\,dx = 0 \tag{2.17}$$

for all smooth $\tilde{v}$. Combined with (2.15) this shows that $(u,v) = (u^{h,j}, v^{h,j})$. This observation is used in extending the scheme to higher dimensional problems involving polyconvex energy integrands as in [6] for three dimensional elastodynamics.

2.3. Uniform estimates for the iterates

In this section entropy pairs are used to obtain uniform L^∞ bounds for the iterates whose existence was just proved. An entropy pair for the system (1.1) is a pair of functions (η, q) which are classical solutions of the system for $(u,v) \in \mathbb{R}^2$

$$\begin{aligned} &q_u(u,v) + \sigma'(u)\eta_v(u,v) = 0,\\ &q_v(u,v) + \eta_u(u,v) = 0. \end{aligned} \tag{2.18}$$

If the entropy η is convex (respectively strictly convex) (η, q) will be said to be a convex (respectively strictly convex) entropy pair. An entropy $\eta = \eta(u,v)$ satisfies the equation:

$$\eta_{uu} - \sigma'(u)\eta_{vv} = 0. \tag{2.19}$$

It is straightforward to check that if (u,v) is a smooth solution of the system (1.1) then $(\eta, q) = \big(\eta(u,v), q(u,v)\big)$ satisfies

$$\partial_t \eta(u,v) + \partial_x q(u,v) = 0.$$

An example of a strictly convex entropy pair is provided by

$$E(u,v) = W(u) + \frac{1}{2}v^2, \qquad F(u,v) = -v\sigma(u), \tag{2.20}$$

where E is the energy density and F the power produced by the contact forces. Now given an entropy pair (η, q) define, momentarily supressing the dependence on h,

$$\eta^j = \eta\big(u^j, v^j\big), \qquad q^j = q\big(u^j, v^j\big). \tag{2.21}$$

LEMMA 2.1 (Discrete entropy inequalities). – *Let* (η, q) *be a convex entropy pair and assume the sequences* (u^j, v^j) *lie in* $W^{1,p}$ *for all* $j \geqslant 1$ *and satisfy* (1.2) *with initial data satisfying* (2.11)*. Then pointwise a.e.,*

$$\frac{1}{h}\left(\eta^j - \eta^{j-1}\right) + q_x^j \leqslant 0 \tag{2.22}$$

and consequently,

$$\sup_{j>0} \int \eta^j \, dx \leqslant \int \eta^0 \, dx. \tag{2.23}$$

Furthermore there exists a positive number $c = c(\gamma)$ *such that*

$$\sum_{j=0}^{\infty} \left\|\delta u^j\right\|_{L^2}^2 + \left\|\delta v^j\right\|_{L^2}^2 \leqslant cE\big(u(0), v(0)\big), \tag{2.24}$$

and there exists $\widetilde{M} = \widetilde{M}(M) > 0$ (*independent of* h) *such that*

$$\sup_{j\geqslant 1}\left(\|u^j\|_{L^\infty} + \|v^j\|_{L^\infty}\right) \leqslant \widetilde{M}. \tag{2.25}$$

(Here and henceforth we often write for pairs $(u, v) \in W^{1,p}$ instead of $(u, v) \in W^{1,p} \oplus W^{1,p}$.)

Proof. – Introduce the notation $\mathbf{z}^j = (u^j, v^j)$ so that (1.2) may be written as

$$\frac{1}{h}\delta\mathbf{z}^j = \left(f\left(\mathbf{z}^j\right)\right)_x, \tag{2.26}$$

where $f(z) = f(u, v) = (v, \sigma(u))$, and as explained above this equation holds pointwise a.e. The defining relations for an entropy pair become

$$\nabla_{\mathbf{z}} q = -\nabla_{\mathbf{z}}\eta \cdot \nabla_{\mathbf{z}} f. \tag{2.27}$$

The proof depends upon the following calculation: setting, for $(s, \tau) \in [0, 1]^2$

$$\mathbf{z}_s^j = \mathbf{z}^{j-1} + s\left(\mathbf{z}^j - \mathbf{z}^{j-1}\right),$$

$$\mathbf{z}_{s,\tau}^j = \mathbf{z}^{j-1} + \left(s + \tau(1-s)\right)\left(\mathbf{z}^j - \mathbf{z}^{j-1}\right),$$

then

$$\eta^j-\eta^{j-1}=\int_0^1\nabla_{\mathbf z}\eta(\mathbf z_s^j)\,\delta\mathbf z^j\,ds$$
$$=\nabla_{\mathbf z}\eta(\mathbf z^j)\,\delta\mathbf z^j-\int_0^1\left(\nabla_{\mathbf z}\eta(\mathbf z^j)-\nabla_{\mathbf z}\eta(\mathbf z_s^j)\right)\delta\mathbf z^j\,ds. \tag{2.28}$$

Since $j\geqslant 1$ we have $(u^j,v^j)\in W^{1,p}$, by the chain rule, (2.26) and (2.27) we have the identity, valid pointwise a.e.,

$$\frac{1}{h}\nabla_{\mathbf z}\eta(\mathbf z^j)\,\delta\mathbf z^j=\nabla_{\mathbf z}\eta(\mathbf z^j)\left(f(\mathbf z^j)\right)_x=-q_{\mathbf z}\mathbf z_x^j=-q_x^j.$$

Now apply the fundamental theorem of calculus again, to deduce

$$\frac{1}{h}\left(\eta^j-\eta^{j-1}\right)+q_x^j=-h\int_0^1\int_0^1(1-s)\left(\delta\mathbf z^j\right)^t\nabla_{\mathbf z}^2\eta\left(\mathbf z_{s,\tau}^j\right)\delta\mathbf z^j\,ds\,d\tau. \tag{2.29}$$

Let (η,q) be a convex entropy pair, then (2.29) implies (2.22) immediately. Integration of (2.22) gives (2.23), while the strict convexity of the entropy E, (which holds since $\sigma'\geqslant\gamma$), gives (2.24). □

Next we will apply (2.23) to the entropy pairs described in the following lemma, a generalisation of which is given in Lemma 3.2.

LEMMA 2.2 (Dafermos, [3]). – *Assume that* $\sigma\in C^2(\mathbb R)$ *satisfies* (2.1)–(2.2). *Then for all* $k>0$ *the function* η_k *on* $\mathbb R^2$ *defined by*

$$\eta_k(u,v)=Y_k(u)\cosh kv,$$

where

$$Y_k''(u)=k^2\sigma'(u)Y_k(u),\qquad Y_k(0)=1,\qquad Y_k'(0)=0,$$

is a positive, strictly convex entropy which satisfies:

$$\cosh(kv)\cosh(k\sqrt{\gamma}u)\ \leqslant\ \eta_k(u,v)\ \leqslant\ \cosh(kv)\exp\left[k\left|\int_0^u\sqrt{\sigma'}\right|\right].$$

Substituting these entropies into (2.23) gives

$$\int_0^{\Pi}\cosh\left(kv^j\right)\cosh\left(k\sqrt{\gamma}u^j\right)dx$$

$$\leqslant \int_0^{\Pi} \eta_k^j \, dx \leqslant \int_0^{\Pi} \eta_k^0 \, dx$$
$$\leqslant \int_0^{\Pi} \cosh(kv^0) \exp\left[k\left|\int_0^{u^0} \sqrt{\sigma'}\right|\right] dx$$
$$\leqslant c_1 \mathrm{e}^{kM} \mathrm{e}^{kc_2},$$

where $c_2 = c_2(M)$. Therefore

$$\left(\int_0^{\Pi} \cosh(kv^j) \cosh(k\sqrt{\gamma}u^j) \, dx\right)^{1/k} \leqslant c\mathrm{e}^M \mathrm{e}^{c_2}$$

independent of k, the $k \to +\infty$ limit of which gives (2.25).

2.4. Proof of Theorem 1.1.

The strategy is to construct interpolates from the iterates and then show that the Young measure which they generate is a Dirac measure (whose centre of mass is the desired weak solution). This crucial final step is carried out by the following lemma of DiPerna:

LEMMA 2.3 (DiPerna, [7]). – *Let* $(u^h, v^h) \in L^\infty(Q)$ *and suppose that for all convex entropy pairs* (η, q) *the quantities*

$$\partial_t \eta(u^h, v^h) + \partial_x q(u^h, v^h) \tag{2.30}$$

are precompact in $H^{-1}_{loc}(Q)$. *Then there exists a subsequence* $h_i \to 0$ *along which* (u^{h_i}, v^{h_i}) *converges to a pair* $(u, v) \in L^\infty(Q)$, *almost everywhere and in* L^p_{loc} *for all* $p < \infty$.

As in the case of the scalar law in [18], the proof hinges on the div-curl lemma to show that the Young measure ν generated by (u^h, v^h) satisfies Tartar's [18] commutation relation

$$\langle \nu, \eta_1 q_2 - \eta_2 q_1 \rangle = \langle \nu, \eta_1 \rangle \langle \nu, q_2 \rangle - \langle \nu, \eta_2 \rangle \langle \nu, q_1 \rangle$$

for all entropy pairs. It was conjectured by Tartar that these relations should be effective in proving that the weak limit is in fact a weak solution of the conservation law, and in [18] this was proved to be so for scalar conservation laws. In the present case of the system (1.1) the reduction of the measure ν was effected in [7] when (1.1) is genuinely nonlinear or when σ has one inflection point. The proof is based on a detailed analysis of the above commutation relation for a class of entropies constructed in [11].

Next we construct time-continuous, piecewise linear interpolates $\mathbf{Z}^h$ out of the iterates $\mathbf{Z}^{h,j}$ weakly converging to $\mathbf{z}$, and then show that the Young measure ν generated by $\mathbf{Z}^h$ is a Dirac measure supported on $\mathbf{z}$, the classical weak solution to (1.1). We define,

$$\mathbf{Z}^h(t,x)=\sum_{j=1}^{\infty}\chi^{h,j}(t)\left(\mathbf{z}^{h,j-1}+\frac{t-h(j-1)}{h}\left(\mathbf{z}^{h,j}-\mathbf{z}^{h,j-1}\right)\right), \tag{2.31}$$

where $\chi^{h,j}$ is the characteristic function of the interval $J_j := [(j-1)h, jh)$. Compute for $(t,x)\in J_j\times[0,\Pi]$

$$\partial_t\eta(\mathbf{Z}^h)+\partial_x q(\mathbf{Z}^h)=\nabla_{\mathbf{z}}\eta(\mathbf{Z}^h)\frac{\delta\mathbf{z}^{h,j}}{h}+\nabla_{\mathbf{z}}q(\mathbf{Z}^h)\partial_x\mathbf{Z}^h$$

which by (2.26) and (2.27) is equal to

$$\begin{aligned}&=\nabla_{\mathbf{z}}\eta(\mathbf{Z}^h)\nabla_{\mathbf{z}}f(\mathbf{z}^{h,j})\partial_x\mathbf{z}^{h,j}+\nabla_{\mathbf{z}}q(\mathbf{Z}^h)\partial_x\mathbf{Z}^h\\&=\left(\nabla_{\mathbf{z}}\eta(\mathbf{Z}^h)-\nabla_{\mathbf{z}}\eta(\mathbf{z}^{h,j})\right)\frac{\delta\mathbf{z}^{h,j}}{h}\\&\quad+\partial_x\left(q(\mathbf{Z}^h)-q(\mathbf{z}^{h,j})\right).\end{aligned}$$

Apply the fundamental theorem of calculus and rewrite this as

$$\partial_t\eta(\mathbf{Z}^h)+\partial_x q(\mathbf{Z}^h)=I_1^h+I_2^h,$$

where

$$\begin{aligned}I_1^h&=\sum_{j=1}^{\infty}\chi^{h,j}(t)(t-hj)\\&\quad\times\left(\int_0^1\nabla_{\mathbf{z}}^2\eta\left(\mathbf{z}^{h,j}+\tau\frac{(t-hj)}{h}\delta\mathbf{z}^{h,j}\right)d\tau\right)\left(\frac{\delta\mathbf{z}^{h,j}}{h},\frac{\delta\mathbf{z}^{h,j}}{h}\right),\\I_2^h&=\partial_x\left(\sum_{j=1}^{\infty}\chi^{h,j}(t)\left(q(\mathbf{Z}^h)-q(\mathbf{z}^{h,j})\right)\right).\end{aligned}$$

The first term is bounded in $L^1(Q)$ independently of h, while the second term converges to 0 in $H^{-1}(Q)$ as $h\to 0$ by (2.24) and the fact that

q is globally Lipshitz on the range dictated by (2.24). Also it is clear from (2.24) that the left hand side is bounded in $W^{-1,\infty}(Q)$. Now by the lemma of Murat [13] (see also [18]) this implies that $\partial_t\eta(\mathbf{Z}^h) + \partial_x q(\mathbf{Z}^h)$ is precompact in $H^{-1}_{loc}(Q)$ and so DiPerna's lemma implies the existence of a subsequence $h_n \to 0$ along which $\mathbf{Z}^{h_n}$ generates a Young measure which is Dirac and the sequence converges almost everywhere. The limit $\mathbf{z} = (u, v)$ of this subsequence is a weak solution of (1.1): to see this apply the entropy calculation just given to the cases $\eta(u, v) = u$ and v in which $I_1^h = 0$ and $I_2^h \to 0$ in H^{-1}_{loc}. Furthermore, this weak solution will satisfy the entropy inequalities (1.4) because $(t - hj)\chi^{h,j}(t) \in (-h, 0]$ so that for η convex $I_1^h \leqslant 0$ while again $I_2^h \to 0$. This completes the proof of Theorem 1.1.

3. OTHER MODELS

Time-step discretisation can be used to construct entropy weak solutions for other models of one-dimensional elastic response, always based on (1.1). Recall that under hypotheses (2.1)–(2.4) the system (1.1) describes one-dimensional shearing motions for elastic materials with hardening response and that in this context u describes the shear strain and v the velocity in the direction of shearing. In this section we explain how to generalise the results above to two other types of constitutive relations.

3.1. Longitudinal elastic motions

The same system (1.1) describes *longitudinal motions* of a one-dimensional elastic rod, with v the *velocity* and u the *longitudinal strain*. For this model to be meaningful u must be stricly positive. Typically for solids there is a constant $u_0 > 0$ such that $\sigma''(u) > 0$ for $u > u_0$, and $\sigma''(u) < 0$ for $u < u_0$, manifesting respectively the *tensile* and *compressive* response of the material.

We obtain for this model the same result as in Theorem 1.1 under a modified set of assumptions on σ. To achieve this it is necessary to generalise the entropy construction of Dafermos to produce entropies which blow up at a known (exponential) rate both as $u \to \infty$ and as $u \to 0^+$. The previous assumptions (2.1)–(2.4) on σ are now replaced by

$$(2.3)\text{–}(2.4) \text{ hold for } u > 0, \tag{3.1}$$

$$\sigma \in C^3((0,\infty); \mathbb{R}), \tag{3.2}$$

$$\sigma'(u) \geqslant \gamma > 0 \quad \forall u > 0, \tag{3.3}$$

$$\begin{aligned} \sigma''(u) &> 0 \quad \text{if } u > u_0, \\ \sigma''(u) &< 0 \quad \text{if } 0 < u < u_0, \end{aligned} \tag{3.4}$$

$$\sigma(u) \to -\infty \quad \text{as } u \to 0^+ \text{ in such a way that}$$

$$\int_0^{u_0} \sqrt{\sigma'(s)}\, ds = +\infty \quad \text{and} \quad \limsup_{u \to 0^+} \frac{|((\sigma')^{-1/4})''|}{(\sigma')^{3/4}} < \infty. \tag{3.5}$$

Notice that the last assumption holds if $\sigma = -u^{-\rho}$ for some $\rho > 1$ on an interval $(0, \delta]$ in which case

$$W(u) \to +\infty \quad \text{as } u \to 0^+.$$

THEOREM 3.1 (Weak solutions for longitudinal motions). – *Under the hypotheses* (3.1)–(3.5)*, the conclusions of Theorem* 1.1 *hold.*

For W, σ satisfying (3.2)–(3.5) the minimiser exists and has been shown to be C^1 on its domain, to satisfy the Euler–Lagrange equation and the constraint $y_x > 0$, see [1, Theorems 1 and 2] and references therein. Thus we conclude as in Section 2.2 that $y^{h,j} \in W^{3,p}$. The proof of Theorem 3.1 is carried out in the same way as that of Theorem 1.1, except that we need to obtain *uniform* estimates of the form

$$\min_{x \in \mathbb{R}} u^{j,h}(x) > \theta > 0 \tag{3.6}$$

in addition to the L^∞ estimates. This is done by a construction of entropies analogous to that of Dafermos (see Lemma 2.2 and [3]) for σ satisfying (3.1)–(3.5). Define

$$a(u) = \sqrt{\sigma'(u)}, \qquad A(u) = \int_{u_0}^{u} a(s)\, ds \tag{3.7}$$

then the assumptions (3.4)–(3.5) translate to

$$a'(u) > 0 \quad \text{for } u > u_0, \tag{3.8}$$

$$a'(u) < 0 \quad \text{for } 0 < u < u_0, \tag{3.9}$$

$$A(0) = -\int_0^{u_0} a(s)\, ds = -\infty, \tag{3.10}$$

$$\limsup_{u\to 0^+} \frac{|(a^{-1/2})''|}{a^{3/2}} < \infty. \tag{3.11}$$

Note that if $a(u) = u^{-\beta}$ for some $\beta > 1$ when u is small compared to 1 then (3.10)–(3.11) are satisfied.

LEMMA 3.2 (Construction of convex entropies defined only on the half line). – *For $k > 0$ the functions*

$$\eta_k(u, v) = Y_k(u)\cosh(kv), \tag{3.12}$$

where Y_k is the solution of the initial value problem

$$\begin{aligned} &Y_k''(u) = k^2 a^2(u) Y_k(u),\\ &Y_k(u_0) = 1,\\ &Y_k'(u_0) = 0, \end{aligned} \tag{3.13}$$

are strictly convex entropies. Moreover, η_k satisfy the following bounds:

$$\cosh\big(k\sqrt{\gamma}(u-u_0)\big)\ \cosh(kv) \leqslant \eta_k(u, v) \leqslant \mathrm{e}^{kA(u)}\cosh(kv) \quad \textit{for } u > u_0,\ v \in \mathbb{R}, \tag{3.14}$$

$$\eta_k(u, v) \leqslant \mathrm{e}^{-kA(u)}\cosh(kv) \quad \textit{for } u < u_0,\ v \in \mathbb{R}, \tag{3.15}$$

and for any $\varepsilon > 0$ there is a $k_0(\varepsilon)$ such that if $k \geqslant k_0$

$$\left(\frac{a(u_0)}{a(u)}\right)^{1/2} \cosh\big((k-\varepsilon)A(u)\big)\cosh(kv) \leqslant \eta_k(u, v) \quad \textit{for } u < u_0,\ \ v \in \mathbb{R}. \tag{3.16}$$

Given this lemma, we obtain the bound (3.6) as well as the L^∞ bounds as an immediate consequence of the non-increase of integrals of convex entropies:

$$\int \eta_k\big(u^j, v^j\big)\, dx \leqslant \int \eta\big(u^0, v^0\big)\, dx.$$

Indeed this implies, as in the proof of (2.24), a uniform L^∞ estimate for $\mathrm{e}^{|A(u)|}$ which gives (3.6). The proof of Theorem 3.1 is completed in a way identical to that of Theorem 1.1.

Proof of Lemma 3.2. – The proof follows [3] except for estimate (3.16). Let Y_k be the solution of (3.13) and let η_k be defined as in (3.12).

A direct calculation shows that strict convexity of η is equivalent to the inequalities

$$Y_k(u) > 0, \tag{3.17}$$

$$-ka(u)Y_k(u) \leqslant Y_k'(u) \leqslant ka(u)Y_k(u) \tag{3.18}$$

for $0 < u < \infty$. Clearly solutions Y_k of (3.13) satisfy (3.17). Also, setting $\chi_k = kaY_k - Y_k'$ then

$$\chi_k' + ka\chi_k = ka'Y_k$$

which is positive for $u > u_0$ and negative for $0 < u < u_0$, and

$$\chi_k(u_0) = ka(u_0) > 0.$$

Therefore $\chi_k > 0$ for $0 < u < \infty$. The remaining part of (3.18) follows similarly.

The right side inequalities in (3.14)–(3.15) follow by integrating (3.18). Also by (3.3) and (3.13),

$$Y_k'' \geqslant k^2\gamma Y_k$$

with $Y_k(u_0) = 1$ and $Y_k'(u_0) = 0$, whence

$$Y_k(u) \geqslant \cosh\bigl(k\sqrt{\gamma}(u-u_0)\bigr).$$

To conclude we show (3.16). The change of variables

$$Y_k(u) = a(u)^{-1/2}F(p), \qquad p = A(u)$$

transforms (3.13) to

$$\begin{aligned} &F_{pp} - \bigl(k^2 - q(p)\bigr)F(p) = 0 \quad \text{for } p \in \mathbb{R}, \\ &F(0) = a(u_0)^{1/2}, \\ &F_p(0) = 0, \end{aligned} \tag{3.19}$$

where

$$q(p) = \frac{(a^{-1/2})''}{a^{3/2}} \circ A^{-1}(p).$$

By (3.2) q is a continuous function and by (3.11) is uniformly bounded, $|q(p)| \leqslant M$ for $p \in (-\infty, 0]$. Also, by (3.10), the range of interest

$u \in (0, u_0]$ is mapped onto $p \in (-\infty, 0]$. Fix $\varepsilon > 0$. The function

$$f(p) = a(u_0)^{1/2} \cosh\bigl((k - \varepsilon)p\bigr)$$

satisfies for $k > \frac{(\varepsilon^2 + M)}{2\varepsilon}$ the differential inequality

$$f_{pp} - \bigl(k^2 - q(p)\bigr) f = a(u_0)^{1/2}\bigl(-2k\varepsilon + \varepsilon^2 + q(p)\bigr) \cosh(k - \varepsilon\bigr) p < 0.$$

As a result,

$$(f - F)_{pp} - \bigl(k^2 - q(p)\bigr)(f - F) < 0 \quad \text{for } p \in \mathbb{R},$$
$$(f - F)(0) = (f_p - F_p)(0) = 0.$$

We conclude that

$$f - F < 0 \quad \text{on } (-\infty, 0)$$

which, once transformed to the original variables, implies (3.16). □

3.2. Shearing models with softening behaviour

Consider now the system (1.1) for a stress-strain constitutive relation satisfying, for some $0 < \gamma < \Gamma < +\infty$,

$$\sigma \in C^3(\mathbb{R}), \tag{3.20}$$
$$+\infty > \Gamma > \sigma'(u) \geqslant \gamma > 0 \quad \forall u \in \mathbb{R}, \tag{3.21}$$
$$\sigma''(u) \neq 0 \quad \forall u \in \mathbb{R}, \tag{3.22}$$
$$\sigma'', \sigma''' \in L^2 \cap L^\infty(\mathbb{R}). \tag{3.23}$$

If in addition $\sigma''(u) > 0$ $\forall u \in \mathbb{R}$ then this law describes hardening response for $u > 0$ and softening response for $u < 0$. In [17] the convergence of viscosity approximations of (1.1) is obtained for such a model. (In fact this work applies to a somewhat larger class of constitutive laws; see also [16] for a model with stress-strain relation having one inflection point.)

We briefly describe the steps required to construct a weak solution via time-step discretisation. The construction of the iterates is done in exactly the same way: the problem (2.8) is solved in energy norm, which is $L^2 \times L^2$ under (3.20)–(3.23). Furthermore, energy non-increase (as in (2.22)–(2.23)) implies that the iterates (u^j, v^j) are bounded in this norm uniformly in j. Let (u^h, v^h) be the continuous piecewise linear interpolates given by (2.31). Under the present hypotheses on σ it is no longer possible to obtain L^∞ estimates for (u^h, v^h). This situation is remedied in the following way. In [17] two classes of entropies, with growth controlled by the wave speeds at infinity, are constructed and used to show that the support of the (generalised) Young measure is a point mass. Under the assumption (3.21) of uniform boundedness of the wave speeds, it suffices to establish that

$$\begin{aligned}&\partial_t \eta\big(u^h, v^h\big) + \partial_x q\big(u^h, v^h\big)\\ &\qquad \in \text{a precompact subset } \subset H^{-1}_{loc}([0, \Pi] \times [0, T]),\end{aligned} \tag{3.24}$$

for all entropy pairs with

$$\eta, q, \eta_u, \eta_v, \eta_{uu}, \eta_{uv} \text{ and } \eta_{vv} \in L^\infty(\mathbb{R}^2). \tag{3.25}$$

The same steps as in Section 3 provide (3.24) for entropy pairs (η, q) with growth as in (3.25). This class contains enough entropies to allow the reduction of the generalised Young measure to a point mass and hence to show the existence of a subsequence $h_i \to 0$ along which

$$\big(u^{h_i}, v^{h_i}\big) \to (u, v) \quad \text{a.e. and in } L^p_{loc}\ \forall p < 2. \tag{3.26}$$

The limit (u, v) is a weak solution of (1.1).

Since the convergence is in L^p_{loc}, with $p < 2$, we can only show that the solution (u, v) satisfies a weak form of energy dissipation. Note that by (2.22)–(2.23)

$$\int_0^\Pi E\big(u^h, v^h\big)\, dx \leqslant \int_0^\Pi E\big(u_0(x), v_0(x)\big)\, dx,$$

where $E(u, v) = W(u) + \frac{v^2}{2}$, and thus by Fatou's lemma

$$\int_0^\Pi E\big(u(x,t), v(x,t)\big)\, dx \leqslant \int_0^\Pi E\big(u_0(x), v_0(x)\big)\, dx, \tag{3.27}$$

for all $t > 0$. On the other hand it is easily seen, also from (2.22)–(2.23), that

$$\partial_t \eta(u, v) + \partial_x q(u, v) \leqslant 0 \quad \text{in } \mathcal{D}' \tag{3.28}$$

for any convex entropy pair (η, q) with subquadratic growth as $|u|, |v| \to +\infty$.

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