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A logarithmic Gauss curvature flow and the Minkowski problem

by

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ABSTRACT. – Let X_0 be a smooth uniformly convex hypersurface and f a postive smooth function in S^n. We study the motion of convex hypersurfaces $X(\cdot, t)$ with initial $X(\cdot, 0) = \theta X_0$ along its inner normal at a rate equal to $\log(K/f)$ where K is the Gauss curvature of $X(\cdot, t)$. We show that the hypersurfaces remain smooth and uniformly convex, and there exists $\theta^* > 0$ such that if $\theta < \theta^*$, they shrink to a point in finite time and, if $\theta > \theta^*$, they expand to an asymptotic sphere. Finally, when $\theta = \theta^*$, they converge to a convex hypersurface of which Gauss curvature is given explicitly by a function depending on $f(x)$.
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INTRODUCTION

Let f be a positive smooth function defined in the n-dimensional sphere S^n and let $X_0 : S^n \to \mathbf{R}^{n+1}$ be a parametrization of a smooth, uniformly convex hypersurface M_0. In this paper we are concerned with the motion of the convex hypersurfaces $M(t)$ satisfying the equation

$$\frac{\partial X}{\partial t} = -\log \frac{K(\nu)}{f(\nu)} \nu, \tag{0.1}$$

with $X(p, 0) = X_0(p)$. Here for each t $X(\cdot, t)$ parametrizes $M(t)$, $K(\nu(p, t))$ is the Gauss curvature of $M(t)$ and $\nu(p, t)$ is the unit outer normal at $X(p, t)$. Notice that by strict convexity the Gauss curvature can be regarded as a function of the normal. Recall that a uniformly convex hypersurface is a hypersurface with positive Gaussian curvature and hence it is stricly convex.

Our study on (0.1) is motivated by the search for a variational proof of the classical Minkowski problem in the smooth category. Recall that for a convex hypersurface the inverse of its Gauss map induces a Borel measure on the unit sphere called the area measure of the hypersurface. Naturally one asks when a given Borel measure on S^n is the area measure of some convex hypersurface. This problem was formulated and solved by Minkowski [13] for polytopes in 1897 by a variational argument. Later he extended his result to cover all Borel measures which are of the form $1/f \, d\sigma$ where f is continuous and $d\sigma$ is the standard Lebsegue measure on S^n [14]. The regularity of the convex hypersurface realizing the area measure was not considered by Minkowski. Thus it led to the Minkowski problem in the smooth category, namely, when is a positive, smooth function in S^n the Gauss curvature of a smooth convex hypersurface? There are two approaches for this problem. On one hand, the method of continuity was used by Lewy [12], Miranda [15], Nirenberg [16], and Cheng and Yau [3]. On the other hand, a regularity theory was developed for the generalized solution (see Pogorelov [17]).

Let M be a convex hypersurface and $V(M)$ its enclosed volume. We have

$$V(M) = \frac{1}{n+1} \int_{S^n} \frac{H(x)}{K(x)} \, d\sigma(x),$$

where H and K are respectively the support function and Gauss curvature of M. When expressed in the smooth category, Minkowski's original proof is to show that the solution is the convex hypersurface which minimizes the functional $\int H(x)/f(x)\,d\sigma(x)$ over all convex hypersurfaces of the same enclosed volume. In view of this we may consider the functional

$$J(M) = -V(M) + \int_{S^n} \frac{H}{f}\,d\sigma.$$

It is not hard to see that (0.1) is a negative gradient flow for J. By a careful study of this flow, we shall give another proof of the Minkowski problem in the smooth category.

THEOREM A. – *Let X_0 be a smooth uniformly convex hypersurface. For $\theta > 0$, consider* (0.1) *subject to*

$$X(\cdot, 0) = \theta X_0. \tag{0.2}$$

There exists $\theta^* > 0$ such that the flow $X(\cdot, t)$ beginning at $\theta^* X_0$ tends to a smooth uniformly convex hypersurface X^* in the sense that

$$X(\cdot, t) - \xi t \to X^*,$$

smoothly as $t \to \infty$ where ξ is uniquely determined by

$$\int_{S^n} \frac{x_i}{\mathrm{e}^{\xi\cdot x} f(x)}\,d\sigma(x) = 0, \quad i = 1, \ldots, n+1.$$

Furthermore, the Gauss curvature of X^*, when regarded as a function of the normal, is equal to $\mathrm{e}^{\xi\cdot x} f(x)$.

THEOREM B. – *Let θ^* be as in Theorem* A. *If $\theta \in (0, \theta^*)$, the solution of* (0.1), (0.2) *shrinks to a point in finite time. If $\theta \in (\theta^*, \infty)$, the solution expands to infinity as t goes to infinity. In the latter case, the hypersurface $X(\cdot, t)/r(t)$ where $r(t)$ is the inner radius of $X(\cdot, t)$ converges to a unit sphere uniformly.*

As a direct consequence of Theorem A we have

COROLLARY (Minkowski problem). – *A positive, smooth function f in S^n is the Gauss curvature of a uniformly convex hypersurface if and only if it satisfies*

$$\int_{S^n} \frac{x_i}{f(x)}\,d\sigma(x) = 0, \quad i = 1, \ldots, n+1.$$

Theorems A and B will be proved in the following sections by an approach similar to that used in [4], namely, by introducing the support function of $X(\cdot,t)$ and reducing (0.1) to a single parabolic equation of Monge–Ampère type for its support function. In Section 1 we collect some facts on the support function of a convex hypersurface. In Section 2 *a priori* estimates for the support function, in particular upper and lower bounds for the second derivatives, will be derived. They are used in Section 3 to establish Theorems A and B.

Motion of convex hypersurfaces driven by functions of Gauss curvature of the form

$$\frac{\partial X}{\partial t} = \Phi(\nu, K)\nu$$

has been studied by several authors including Andrews [1], Chou [4], Chow [7], Frey [8], Gerhardt [10] and Urbas [18]. When $\Phi = -K^{\sigma}$, $\sigma > 0$, it was proved in [7] that $M(t)$ exists and shrinks to a point in finite time. Moreover, it becomes asymptotically round when σ is equal to $1/n$. In [1] it was shown that $M(t)$ becomes an asymptotic ellipsoid when σ is equal to $1/(n+2)$. Expanding flows rather than contracting ones were studied in [10] and [18]. For a class of curvature functions including $\Phi = K^{-1/n}$ it was proved that $M(t)$ expands to infinity like a sphere in infinite time. In all these results Φ is independent of ν. For anisotropic flows very little is known. We mention the works Andrew [2], Chou and Zhu [6], and Gage and Li [9].

1. THE SUPPORT FUNCTION

In this section we collect some basic facts concerning a convex hypersurface and its support function. Details can be found in Cheng and Yau [3] and Pogorelov [17].

Let M be a closed convex hypersurface in $\mathbf{R}^{n+1}$. Its support function H is defined on S^n by

$$H(x) = \sup\{x \cdot p:\ p \in M\},$$

where $x \cdot p$ is the inner product in $\mathbf{R}^{n+1}$. We extend H to a homogenuous function of degree 1 in $\mathbf{R}^{n+1}$. So H is convex and satisfies

$$\sup_{S^n} |\nabla H| \leqslant \sup_{S^n} |H|, \tag{1.1}$$

since it is the supremum of linear functions. If M is strictly convex, that is, for each x in S^n there is a unique point p on M whose unit outer normal is x, H is differentiable at x and

$$p_i = \frac{\partial H}{\partial x_i}, \quad i = 1, \ldots, n+1.$$

Thus the map $x \mapsto p(x)$ gives a parametrization of M by its normal. In fact, it is nothing but the inverse of the Gauss map.

Geometric quantities of M can now be expressed through H. Let $e_1, \ldots, e_n$ be an orthonormal frame fields on S^n. By a direct computation one sees that the principal radii of curvature at $p(x)$ are precisely the eigenvalues of the matrix $(\nabla_\beta \nabla_\alpha H + H\delta_{\alpha\beta})_{,\alpha,\beta=1,\ldots,n}$, where ∇_α is the covariant differentiation with respect to e_α. In particular, the Gauss curvature at $p(x)$ is given by

$$K(x) = 1/\det(\nabla_\beta \nabla_\alpha H + H\delta_{\alpha\beta}). \tag{1.2}$$

When H is viewed as a homogeneous function over $\mathbf{R}^{n+1}$, the principal radii of curvature of M are also equal to the non-zero eigenvalues of the Hessian matrix $(\partial^2 H/\partial x_i \partial x_j)_{i,j=1,\ldots,n+1}$.

Now we can reduce the problem (0.1), (0.2) to an initial value problem for the support function. In fact, let $H(x,t)$ be the support function of $M(t)$. By definition we have

$$x \cdot \frac{\partial X}{\partial t}\big(p(x), t\big) = -\frac{\partial H}{\partial t}(x,t).$$

From (0.1) and (0.2) it follows that H satisfies

$$\frac{\partial H}{\partial t} = \log\det(\nabla_\beta \nabla_\alpha H + H\delta_{\alpha\beta}) f, \tag{1.3}$$

$$H(x,0) = \theta H_0(x), \tag{1.4}$$

where H_0 is the support function for M_0. Conversely, if $X(\cdot,t)$ is a family of convex hypersurfaces determined by a solution of (1.3) and (1.4), it is not hard to see that $X(\cdot,t)$ does solve (0.1) and (0.2). See, for instance, [4] for details. Notice from (1.3) $H(x,t)$ must determine a uniformly convex hypersurface.

Eq. (1.3) has a variational structure. Consider the enclosed volume of a uniformly convex hypersurface M,

$$V(M) = \frac{1}{n+1} \int_{S^n} \frac{H(x)}{K(x)}\, d\sigma(x)$$
$$= \frac{1}{n+1} \int_{S^n} H \det(\nabla_\beta \nabla_\alpha H + H\delta_{\alpha\beta})\, d\sigma.$$

Regarding V as a functional on support functions, we find that the first variation of V is

$$\delta V(H)h = \int_{S^n} h \det(\nabla_\beta \nabla_\alpha H + H\delta_{\alpha\beta})\, d\sigma,$$

where h is any smooth function. Let's consider the functional J defined on all uniformly convex hypersurfaces

$$J(H) = -V(H) + \int_{S^n} \frac{H}{f}\, d\sigma,$$

where f is positive. When H is a solution of (1.3),

$$\frac{d}{dt} J\bigl(H(\cdot, t)\bigr) = -\int_{S^n} \left[\det(\nabla_\beta \nabla_\alpha H + H\delta_{\alpha\beta}) - \frac{1}{f}\right] \frac{\partial H}{\partial t}\, d\sigma$$
$$= -\int_{S^n} \frac{1}{f}\bigl(e^{H_t} - 1\bigr) H_t\, d\sigma$$
$$\leqslant 0. \tag{1.5}$$

Hence (1.3) is a negative gradient flow for J. (1.5) will be used in the proof of Theorem A. This variational approach to the problem of prescribed Gauss curvature was first adopted in Chou [5].

To obtain apriori estimates for the higher derivatives for H it is convenient to express Eq. (1.3) locally in the Euclidean space. Thus let $u(y, t)$ be the restriction of $H(x, t)$ to the hypersurface $x_{n+1} = -1$, i.e., $u(y, t) = H(y, -1, t)$. Then u is convex in $\mathbf{R}^n$ and we have

$$\det \nabla^2 u(y, t) = \bigl(1 + |y|^2\bigr)^{-\frac{n+2}{2}} \det(\nabla_\beta \nabla_\alpha H + H\delta_{\alpha\beta})(x, t)$$

and

$$\frac{\partial u}{\partial t}(y, t) = \sqrt{1 + |y|^2}\, \frac{\partial H}{\partial t}(x, t)$$

for $x = (y, -1)/\sqrt{1+|y|^2}$. Extend f to be a homogenuous function of degree 0 in $\mathbf{R}^{n+1}$. We get

$$\frac{\partial u}{\partial t} = \sqrt{1+|y|^2}\log\det\nabla^2 u + g(y), \quad y \in \mathbf{R}^n, \tag{1.6}$$

where

$$g(y) = \sqrt{1+|y|^2}\left[\frac{n+2}{2}\log\bigl(1+|y|^2\bigr) + \log f(y,-1)\right].$$

2. A PRIORI ESTIMATION

First of all we note that the uniqueness of solution to (1.3), (1.4) follows from the following comparison principle which is a direct consequence of the maximum principle.

LEMMA 2.1. – *For $i = 1, 2$, let f_i be two positive C^2-functions on S^n and H_i $C^{2,1}$-solutions of*

$$\frac{\partial H}{\partial t} = \log\det(\nabla_\beta\nabla_\alpha + H\delta_{\alpha\beta}) f_i.$$

Suppose that $H_1(x,0) \leqslant H_2(x,0)$ and $f_1(x) \leqslant f_2(x)$ on S^n. Then $H_1 \leqslant H_2$ for all $t > 0$ and $H_1 < H_2$ unless $H_1 \equiv H_2$.

In the following we shall always assume $H \in C^{4,2}(S^n \times [0,T])$ is a solution of (1.3), (1.4). Let $R(t)$ and $r(t)$ be the outer and inner radii of the hypersurface $X(\cdot,t)$ determined by $H(x,t)$ respectively. We set

$$R_0 = \sup\bigl\{R(t)\colon\ t \in [0,T]\bigr\}$$

and

$$r_0 = \inf\bigl\{r(t)\colon\ t \in [0,T]\bigr\}.$$

We shall estimate the principal radii of curvatures of $X(\cdot,t)$ from both side in terms of r_0^{-1}, R_0, and initial data.

LEMMA 2.2. – *Let r and R be the inner and outer radii of a uniformly convex hypersurface X respectively. Then there exists a dimensional constant C such that*

$$\frac{R^2}{r} \leqslant C\sup\bigl\{R(x,\xi)\colon\ x,\xi \in S^n\bigr\},$$

where $R(x,\xi)$ is the principal radius of curvature of X at the point with normal x and along the direction ξ.

Proof. – For any given $t>0$, let

$$h=\inf\{H(x)+H(-x)\colon x\in S^n\}.$$

Then X is pinched between two parallel hyperplanes with distance h. Suppose the infimum is attained at $x=(1,0,\ldots,0)$. By convexity we can choose a direction perpendicular to the x_1-axis, say, the x_2-axis such that

$$H(0,1,0,\ldots,0)+H(0,-1,0,\ldots,0)\geqslant\frac{1}{2}R.$$

Let F be the projection of X on the plane $x_3=\cdots=x_{n+1}=0$. Then F is a convex set and its diameter is larger than $\frac{1}{2}R$. By a proper choice of the origin we may assume F is contained in $\{-h<x_1<h\}$ and $\{0,\pm\frac{1}{8}R\}$ belongs to F. By projection we see that the supremum of the principal radii of curvatures of the boundary of F cannot exceed that of X.

Let E be the ellipse given by

$$\frac{x_1^2}{b^2}+\frac{x_2^2}{(R/16)^2}=1$$

where b is chosen so that $E\subset F$ and $\partial E\cap\partial F$ is non-empty. Then $h/4\leqslant b\leqslant h/2$ provided $R\gg r$. For any $(\bar{x}_1,\bar{x}_2)\in\partial E\cap\partial F$, since $(0,\pm\frac{1}{8}R)\in F$, we have $|\bar{x}_1|\geqslant b/2$. Hence $|\bar{x}_2|\leqslant\sqrt{3}R/32$. Simple computation shows that the principal radius of curvature of the boundary of F at $(\bar{x}_1,\bar{x}_2)$ is larger than $R^2/8^3b$. Hence by noticing $b\leqslant r$ we obtain

$$\frac{R^2}{r}\leqslant C\frac{R^2}{b}\leqslant C\sup_{x,\xi}R(x,\xi).\qquad\square$$

LEMMA 2.3. – *Suppose that $a(t),b(t)\in C^1([0,T])$ and $a(t)<b(t)$ for all t. Then there exists $h(t)\in C^{0,1}([0,T])$ such that*

(1) $a(t)-2M\leqslant h(t)\leqslant b(t)+2M$;

(2) $\sup\{\frac{|h(t_1)-h(t_2)|}{|t_1-t_2|}\colon t_1,t_2\in[0,T]\}\leqslant 2\max\{\sup_t b'(t),\sup_t(-a'(t))\}$,

where $M=\sup_t(b(t)-a(t))$.

Proof. – We define $h(t)$ step by step. Let $t_0=0$, and $h_0=(a(0)+b(0))/2$. For $j\geqslant 1$, let

$$t_j = \sup\bigl\{\tau \in (t_{j-1}, T)\colon\ a(t) \geqslant h_{j-1} - M,\ b(t) \leqslant h_{j-1} + M,\ \forall t \in (t_{j-1}, \tau)\bigr\},$$

$$h_j = \frac{1}{2}\bigl(a(t_j) + b(t_j)\bigr),$$

and

$$h(t) = h_{j-1} + \frac{h_j - h_{j-1}}{t_j - t_{j-1}}(t - t_{j-1}) \quad \text{for } t \in (t_{j-1}, t_j).$$

Then $h(t)$ is the desired function. □

Now we give an upper estimate for the principal radii of curvature.

LEMMA 2.4. – *For any $\gamma \in (1, 2]$ there exists a constant C_γ, which may depend on initial data, such that*

$$\sup\bigl\{H_{\xi\xi}(x, t)\colon\ \xi \text{ tangential to } S^n\bigr\} \leqslant C_\gamma\bigl(1 + D^\gamma\bigr),$$

where $D = \sup\{d(t)\colon\ t \in [0, T]\}$ and $d(t)$ is the diameter of $X(\cdot, t)$.

Proof. – Applying Lemma 2.3 to the functions $-H(-e_i, t)$ and $H(e_i, t)$ where $\pm e_i$ are the intersection points of S^n with the x_i-axis, $i = 1, \ldots, n+1$, we obtain $p_i(t)$ so that

$$-H(-e_i, t) - 2D \leqslant p_i(t) \leqslant H(e_i, t) + 2D$$

and

$$\sup\left\{\frac{|p_i(t_1) - p_i(t_2)|}{|t_1 - t_2|}\colon\ t_1, t_2 \in [0, T]\right\} \leqslant 2\sup\bigl\{H_t(x, t)\colon\ (x, t) \in S^n \times [0, T]\bigr\}. \tag{2.1}$$

Henceforth

$$\left|H(x, t) - \sum_{i=1}^{n+1} p_i(t)x_i\right| \leqslant 2D \quad \text{for } (x, t) \in S^n \times [0, T], \tag{2.2}$$

and by (1.1)

$$\sum_{i=1}^{n+1} \bigl|H_i(x, t) - p_i\bigr|^2 \leqslant 4D^2. \tag{2.3}$$

Let

$$\Phi(x,t) = H_{\xi\xi}(x,t) + \left[1 + \sum_{i=1}^{n+1} \left|H_i(x,t) - p_i(t)\right|^2\right]^{\gamma/2}$$

where $\gamma \in (1,2]$. Suppose that the supremum

$$\sup\left\{\Phi(x,t)\colon\ (x,t) \in S^n \times [0,T],\ \xi \text{ tangential to } S^n,\ |\xi| = 1\right\}$$

is attained at the south pole $x = (0,\dots,0,-1)$ at $t = \bar{t} > 0$ and in the direction $\xi = e_1$. For any x on the south hemisphere, let

$$\xi(x) = \left(\sqrt{1 - x_1^2}, -\frac{x_1 x_2}{\sqrt{1 - x_1^2}}, \dots, -\frac{x_1 x_{n+1}}{\sqrt{1 - x_1^2}}\right).$$

Let u be the restriction of H on $x_{n+1} = -1$. Using the homogenity of H we obtain, after a direct computation,

$$\begin{aligned}
&\sum_{i=1}^{n+1} (H_i - p_i)^2(x,t) \\
&\quad = \sum_{i=1}^{n} \bigl(u_i(y,t) - p_i(t)\bigr)^2 + \left|u(y,t) + p_{n+1} - \sum_{i=1}^{n} y_i u_i(y,t)\right|^2
\end{aligned}$$

and

$$H_{\xi\xi}(x,t) = u_{11}(y,t)\frac{(1 + y_1^2 + \cdots + y_n^2)^{3/2}}{1 + y_2^2 + \cdots + y_n^2},$$

where $y = -(x_1,\dots,x_n)/x_{n+1}$ in $\mathbf{R}^n$. Thus the function

$$\begin{aligned}
\varphi(y,t) = {}& u_{11}\frac{(1 + y_1^2 + \cdots + y_n^2)^{3/2}}{1 + y_2^2 + \cdots + y_n^2} \\
&+ \left[1 + \sum (u_i - p_i)^2 + \left|u + p_{n+1} - \sum y_i u_i\right|^2\right]^{\gamma/2}
\end{aligned}$$

attains its maximum at $(y,t) = (0,\bar{t})$. Without loss of generality we may further assume that the Hessian of u at $(0,\bar{t})$ is diagonal. Hence at $(0,\bar{t})$ we have, for each k,

$$\begin{aligned}
0 \leqslant \varphi_t &= u_{11t} + \gamma\bigl[(u_i - p_i)(u_{it} - p_{i,t}) \\
&\qquad + (u + p_{n+1})(u_t + p_{n+1,t})\bigr] Q^{(\gamma-2)/2}, \\
0 = \varphi_k &= u_{11k} + \gamma(u_i - p_i)u_{ik} Q^{(\gamma-2)/2},
\end{aligned}$$

and

$$0 \geqslant \varphi_{kk} = u_{kk11} + \tau_k u_{11} + \gamma\left[u_{kk}^2 + (u_i - p_i)u_{ikk} - (u + p_{n+1})u_{kk}\right]Q^{(\gamma-2)/2} + \gamma(\gamma-2)(u_i - p_i)^2 u_{ik}^2 Q^{(\gamma-4)/2},$$

where $Q = 1 + \sum(u_i - p_i)^2 + (u + p_{n+1})^2$, $\tau_k = 3$ if $k > 1$ and $\tau_1 = 1$, and $p_{i,t} = dp_i/dt$. On the other hand, differentiating Eq. (1.6) gives

$$u_{kt} = \sum_i u^{ii} u_{iik} + g_k,$$
$$u_{kkt} = \sum_i u^{ii} u_{iikk} - \sum_{i,j} u^{ii} u^{jj} u_{ijk}^2 + \log\det\nabla^2 u + g_{kk},$$

where $\{u^{ij}\}$ is the inverse matrix of $\{u_{ij}\}$. Hence at $(0, \bar{t}\,)$ we have

$$\begin{aligned}
0 &\geqslant \sum_k u^{kk}\varphi_{kk} - \varphi_t \\
&\geqslant \sum_k u^{kk} u_{kk11} - u_{11t} + u_{11}u^{kk} \\
&\quad + \gamma\Bigg\{\sum_k u_{kk}\left[1 + \frac{(\gamma-2)(u_k - p_k)^2}{1 + \sum(u_i - p_i)^2 + (u + p_{n+1})^2}\right] \\
&\quad + (u_i - p_i)\left(\sum_k u^{kk} u_{ikk} - u_{it}\right) - n(u + p_{n+1}) \\
&\quad - (u + p_{n+1})(u_t + p_{n+1,t}) + (u_i - p_i)p_{i,t}\Bigg\}Q^{(\gamma-2)/2} \\
&\geqslant u_{11}u^{kk} - \log\det\nabla^2 u - g_{11} + \gamma\big[(\gamma-1)u_{kk} - (u_i - p_i)g_i \\
&\quad - n(u + p_{n+1}) - (u + p_{n+1})(u_t + p_{n+1,t}) \\
&\quad + (u_i - p_i)p_{i,t}\big]Q^{(\gamma-2)/2}.
\end{aligned}$$

To proceed further let's assume $u_{11} > 1$. By (2.2) we have $|u + p_{n+1}| \leqslant 2D$ and $|u_i - p_i| \leqslant 2D$. From the inequality above we therefore obtain, in view of (2.1),

$$\begin{aligned}
&u_{kk} + u^{kk} \\
&\quad \leqslant C\big(1 + |u_t|\big)Q^{(2-\gamma)/2} + C\big(1 + |u + p_{n+1}|\big)\big(1 + |u_t| + |p_{n+1,t}|\big) \\
&\quad \leqslant C\Big[1 + D\log\big(u_{kk} + u^{kk}\big) + D\sup_{t\leqslant T} H_t(x,t)\Big].
\end{aligned}$$

From Eq. (1.3),

$$\sup_{t\leqslant T} H_t(x,t) \leqslant C + \log\Big[\sup_{t<T}\big\{H^n_{\xi\xi}(x,t);\, x\in S^n,\, \xi \text{ tangential to } S^n\big\}\Big].$$

It follows

$$u_{kk} + u^{kk} \leqslant C\big(1 + D\log\big(u_{kk} + u^{kk}\big)\big).$$

Hence $u_{11} \leqslant C(1 + D|\log^2 D|)$. This completes the proof of the lemma. □

By combining Lemmas 2.2 and 2.4 we deduce the following important corollary.

LEMMA 2.5. – *For any given* $\gamma \in (1,2]$, *there exists* $\delta = \delta(\gamma) > 0$ *such that*

$$r(t) \geqslant \frac{\delta R^2(t)}{1 + \sup_{\tau\leqslant t} R^\gamma(\tau)}.$$

Next we give a positive lower bound for the principal radii of the curvature. In view of Lemma 2.4 and Eq. (1.3) it suffices to give a lower bound on H_t.

LEMMA 2.6. – *There exists a constant C depending only on* n, r_0, R_0, f, *and initial data such that*

$$\inf\big\{H_t(x,t):\ (x,t)\in S^n\times[0,T]\big\} \geqslant -C.$$

Proof. – Let

$$q(t) = \frac{1}{|S^n|}\int_{S^n} xH(x,t)\,d\sigma(x)$$

be the Steiner point of $X(\cdot,t)$. Then there exists a positive δ which depends only on n, r_0, and R_0 so that $H(x,t) - q(t)\cdot x \geqslant 2\delta$. Let us consider consider the function

$$\Psi(x,t) = \frac{H_t(x,t)}{H(x,t) - x\cdot q(t) - \delta}.$$

Suppose the (negative) infimum of Ψ attains at $x = (0,\dots,0,-1)$ and $\bar t > 0$. Let u be the restriction of H to $x_{n+1} = -1$ as before. Then

$$\psi(y,t) = \frac{u_t(y,t)}{u(y,t) - q(t)\cdot(y,-1) - \delta\sqrt{1+|y|^2}}$$

attains its negative minimum at $(0, \bar{t}\,)$. Hence

$$0 \geqslant \psi_t = \frac{u_{tt}}{u + q_{n+1}(t) - \delta} - \frac{u_t\,(u_t + dq_{n+1}/dt)}{(u + q_{n+1}(t) - \delta)^2},$$
$$0 = \psi_k = \frac{u_{tk}}{u + q_{n+1}(t) - \delta} - \frac{u_t(u_k - q_k(t))}{(u + q_{n+1}(t) - \delta)^2},$$

and

$$0 \leqslant \psi_{kk} = \frac{u_{tkk}}{u + q_{n+1}(t) - \delta} - \frac{u_t u_{kk}}{(u + q_{n+1}(t) - \delta)^2} + \frac{\delta u_t}{(u + q_{n+1}(t) - \delta)^2}.$$

On the other hand, we differentiate (1.3) to get

$$u_{tt} = u^{ij} u_{ijt}.$$

Rotate the axes so that $\{u^{ij}\}$ is diagonal at $(0, \bar{t}\,)$. Then

$$\begin{aligned} 0 &\leqslant \sum u^{kk} \psi_{kk} - \psi_t \\ &\leqslant \frac{\delta u_t \sum u^{kk} - n u_t + u_t\,(u_t + dq_{n+1}/dt)}{(u + q_{n+1} - \delta)^2}. \end{aligned}$$

Since u_t is negative at $(0, \bar{t}\,)$, it follows from Lemma 2.4 that

$$\begin{aligned} \sum u^{kk} &\leqslant \frac{n}{\delta}\left(1 + |u_t| + \left|\frac{dq_{n+1}}{dt}\right|\right) \\ &\leqslant C\frac{n}{\delta}\big(1 + |u_t| + R_0\big) \\ &\leqslant C\frac{n}{\delta}\Big(1 + \log \sum u^{kk} + R_0\Big). \end{aligned}$$

We therefore conclude $\sum u^{kk} \leqslant C\delta^{-2}(1 + R_0)^2$. Hence

$$\begin{aligned} u_t &\geqslant -C - C \log \sum u^{kk} \\ &\geqslant -C\big(1 + \log(1 + R_0) - \log r_0\big) \end{aligned}$$

and the lemma follows. □

Finally by comparing (1.3), (1.4) with the problem

$$\frac{d\rho}{dt} = \log \rho^n M, \qquad \rho(0) = \rho_0$$

where $M = \max\{f(x)\colon x \in S^n\}$ and ρ_0 is sufficiently large, we see that $H(x,t)$ is always bounded in any finite time interval. Furthermore, its gradient is also bounded by (1.1). It follows from the regularity property of fully nonlinear parabolic equations [11] that a $C^{4+\alpha,2+\alpha/2}$-estimate holds for H, provided $H_0 \in C^{4+\alpha}(S^n)$, $0 < \alpha < 1$. By a continuity argument we arrive at

THEOREM 2.1. – *The problem* (1.3), (1.4) *with* $H_0 \in C^{4+\alpha}(S^n)$ *admits a unique* $C^{4+\alpha,2+\alpha/2}$ *solution in a maximal interval* $[0,T^*)$, $T^* \leqslant \infty$. *Moreover,* $\lim_{t\uparrow T^*} R(t) = 0$ *if* T^* *is finite.*

Notice that the last assertion follows from Lemma 2.5.

3. PROOFS OF THEOREMS A AND B

We first prove Theorem A. Let $m = \inf f$ and $M = \sup f$ on S^n. It is readily seen that if the initial hypersurface X_0 is a sphere of raduis $\rho_0 > m^{-1/n}$, the solution $X(\cdot,t)$ to the equation

$$\frac{\partial X}{\partial t} = -\log\frac{K}{m}\nu, \quad X(\cdot,0) = X_0,$$

remains to be spheres and the flow expands to infinity as $t \to \infty$. On the other hand, if X_0 is a sphere of radius less than $M^{-1/n}$, the solution to

$$\frac{\partial X}{\partial t} = -\log\frac{K}{M}\nu, \quad X(\cdot,0) = X_0$$

is a family of spheres which shrinks to a point in finite time. Henceforth by the comparison principle the solution $X(x,t)$ of (1.3), (1.4) will shrink to a point if θ is smalll enough, and will expand to infinity if $\theta > 0$ is large. We put

$$\theta_* = \sup\{\theta > 0\colon\ X(\cdot,t) \text{ shrinks to a point in finite time}\}$$

and

$$\theta^* = \inf\{\theta > 0\colon\ X(\cdot,t) \text{ expands to infinity as } t \to \infty\}.$$

By the results in Section 2, it is easy to see that $X(\cdot,t)$ continuously depends on θ. Hence by the comparison principle $\theta_* \leqslant \theta^*$.

By Lemma 2.5 we know that for any $\theta \in [\theta_*, \theta^*]$ the inner radii of $X(\cdot,t)$ have a uniform positive lower bound and the outer radii are unformly bound from above. Hence (1.3) is uniformly parabolic and we have $C^{4+\alpha,2+\alpha/2}$-bound on the solution in $S^n \times [0,\infty)$.

In the following we fix $\theta \in [\theta_*, \theta^*]$. Let $\xi \in \mathbf{R}^{n+1}$ be the point uniquely determined by

$$\int_{S^n} \frac{x_i}{\mathrm{e}^{\xi\cdot x} f(x)}\, d\sigma(x) = 0, \quad i = 1, \ldots, n+1. \tag{3.1}$$

Write $\tilde{X}(x,t) = X(x,t) + \xi \cdot t$. So $\tilde{X}$ is X translated in $\xi/|\xi|$ with speed $|\xi|$. $\tilde{X}$ satisfies

$$\frac{\partial \tilde{X}}{\partial t} = -\log \frac{K}{f}\nu + \xi$$

and the corresponding support function $\tilde{H} = H + \xi \cdot xt$ satisfies

$$\tilde{H}_t = \log\det(\nabla_\beta \nabla_\alpha \tilde{H} + \tilde{H}\delta_{\alpha\beta}) + \log f \mathrm{e}^{\xi\cdot x}.$$

The enclosed volumes of $\tilde{X}$ and X are equal to

$$V(t) = \frac{1}{n+1}\int \tilde{H} \det(\nabla_\beta \nabla_\alpha \tilde{H} + \tilde{H}\delta_{\alpha\beta})$$

and is uniformly bounded. On the other hand, by (3.1)

$$\int \frac{\tilde{H}}{\mathrm{e}^{\xi\cdot x} f} = \int \frac{H - q(t)\cdot x}{f \mathrm{e}^{\xi\cdot x}}$$

is also uniformly bounded for all t. Hence the functional $\tilde{J}(t) = J(\tilde{H}(\cdot,t))$ is uniformly bounded. Moreover, from (1.5) it is non-increasing. By the $C^{4+\alpha,2+\alpha/2}$-regularity of $\tilde{H}$ we also have that

$$\left|\tilde{J}'(t)\right| \leqslant C$$

and

$$\sup \frac{|\tilde{J}'(t+\tau) - \tilde{J}'(t)|}{\tau^{\alpha/2}} \leqslant C.$$

Therefore, we conclude that $\lim_{t\to\infty} \tilde{J}'(t) = 0$.

We claim that $\tilde{H}$ is bounded for all t. In fact, it is sufficient to show that $\int x \frac{\tilde{H}}{f\mathrm{e}^{\xi\cdot x}}\, d\sigma$ is bounded. For, assume $\tilde{H}$ is unbounded. Then we can find $\{t_j\}$, $t_j \to \infty$, such that $\tilde{X}(x,t_j)/d(t_j)$, where $d(t_j)$ is the distance from the origin to $\tilde{X}(\cdot,t_j)$, converges to a point on S^n. Without loss of generality we take this point to be $\boldsymbol{e}_{n+1}$. Then the characteristic functions of $A_j = \{x \in S^n\colon\ x_{n+1} > 0, H(x,t_j) > 0\}$ and $B_j = \{x \in S^n\colon\ x_{n+1} < 0, H(x,t_j) < 0\}$ converges pointwisely to the upper and lower hemispheres. We may also assume that $\tilde{H}(x,t_j)/(f\mathrm{e}^{\xi\cdot x}d(t_j))$ converges uniformly to some function g which is positive on the upper hemi-sphere S^+. Therefore, we have

$$\begin{aligned}\lim_{j\to\infty}\int \frac{x_{n+1}H(x,t_j)}{d(t_j)f\mathrm{e}^{\xi\cdot x}} &= \int \lim_{j\to\infty}\left[\mathcal{X}_{A_j\cup B_j}\frac{|x_{n+1}H(x,t_j)|}{d(t_j)f\mathrm{e}^{\xi\cdot x}}\right]\\ &\geqslant \int\limits_{S^+} x_{n+1}g(x)\\ &> 0.\end{aligned}$$

Hence $\int \frac{x_{n+1}H(x,t_j)}{f\mathrm{e}^{\xi\cdot x}}$ can be arbitrarily large for large t_j.

Now we have, by (1.5),

$$\tilde{J}(0)-\tilde{J}(\infty) \geqslant \int\limits_0^t \left|\tilde{J}'(t)\right| dt \geqslant \int\limits_0^t \int\limits_{S^n} \tilde{H}_t^2\,d\sigma\,dt.$$

On the other hand, by the necessary condition for the Minkowski problem, we have

$$\begin{aligned}0 &= \int x\frac{1}{\tilde{K}}\,d\sigma = \int x\frac{1}{f\mathrm{e}^{\xi\cdot x}}\left(1+\tilde{H}_t+\mathrm{O}\left(\tilde{H}_t^2\right)\right)\\ &= \int x\frac{1}{f\mathrm{e}^{\xi\cdot x}}\left(\tilde{H}_t+\mathrm{O}\left(\tilde{H}_t^2\right)\right)\end{aligned}$$

as $\tilde{H}_t$ is uniformly small for large t. Therefore,

$$\begin{aligned}\left|\int\limits_0^t \frac{d}{dt}\left(\int x\frac{\tilde{H}}{f\mathrm{e}^{\xi\cdot x}}\,d\sigma\right)dt\right| &\leqslant C\int\limits_0^t\int\limits_{S^n}\tilde{H}_t^2\,d\sigma\,dt\\ &\leqslant C\left(\tilde{J}(0)-\tilde{J}(\infty)\right).\end{aligned}$$

Hence $\int x\frac{\tilde{H}}{f\mathrm{e}^{\xi\cdot x}}$ is uniformly bounded for all time. Consequently by the Blaschke selection theorem for any sequence $\{t_j\}$, $t_j \to \infty$, we can extract a subsequence $\{t_{j_k}\}$ such that $\{\tilde{H}(x,t_{j_k})\}$ converges uniformly to some $H(x)$ on S^n. Clearly H is a solution of $K = f\mathrm{e}^{\xi\cdot x}$. To show the convergence is actually uniform let's consider another limit H'. Since the curvature of H' is also given by $f e^{\xi\cdot x}$, H and H' differ by a translation. Let $H - H' = l\cdot x$ for some $l \in \mathbf{R}^{n+1}$. Since

$$\left|\int_s^t \frac{d}{dt}\int x\frac{\tilde H}{f e^{\xi\cdot x}}\,d\sigma\,dt\right| \leqslant C\big(\tilde J(t) - \tilde J(s)\big) \to 0$$

as $t, s \to \infty$. So $l = 0$ and $H = H'$.

Finally let's show $\theta_* = \theta^*$. First we observe that by the comparison principle one must have $H_* = H^*$, where H_* (respectively H^*) is the solution of $K = f e^{\xi\cdot x}$ starting from $\theta_* H_0$ (respectively $\theta^* H_0$). However, consider the equation obtained by differentiating (1.3) and (1.4) in θ:

$$\begin{cases} \frac{\partial H'}{\partial t} = A^{\alpha\beta}(\nabla_\beta\nabla_\alpha H' + H'\delta_{\alpha\beta}), \\ H'(0) = H_0(x), \end{cases}$$

where $(A^{\alpha\beta})$ is the inverse of $(\nabla_\beta\nabla_\alpha H + H\delta_{\alpha\beta})$. By the maximum principle $H'(x,t) \geqslant \min H_0 > 0$. Thus

$$\begin{aligned} 0 &= H^*(\cdot) - H_*(\cdot) \\ &= \lim_{t\to\infty}\big(H_{\theta^*}(\cdot,t) - H_{\theta_*}(\cdot,t)\big) \\ &\geqslant (\min H_0)\big(\theta^* - \theta_*\big) \\ &> 0. \end{aligned}$$

So $\theta^* = \theta_*$. The proof of Theorem A is finished.

Proof of Theorem B. – It remains to show that the normalized hypersurface $X(\cdot,t)/r(t)$ converges to a unit sphere in case $\theta > \theta^*$. Let's denote the solution of (1.3), (1.4) by $H(\cdot,t)$ and its hypersurface by $X(\cdot,t)$. Since X is expanding, we may simply assume that it contains the ball $B_{R_1}(0)$ where $R_1 > 1 + m^{-1/n}$ at $t = 0$. On the other hand, we fix R_2 so large that $X(\cdot,0)$ is contained in $B_{R_2}(0)$.

For $i = 1, 2$, let $X_i(\cdot,t)$ be the solution of (1.3), (1.4) where f is replaced by m and M respectively and $X_i(\cdot,0) = \partial B_{R_i}$. Clearly $X_i(\cdot,t)$ are spheres whose radii $R_i(t)$ satisfy

$$\begin{aligned} C^{-1}(1+t)\log(1+t) &\leqslant R_1(t) \\ &\leqslant R_2(t) \\ &\leqslant C\big[1 + (1+t)\log^2(1+t)\big] \end{aligned}$$

for some $C > 0$. Hence

$$\begin{aligned}\frac{d}{dt}\big(R_2(t)-R_1(t)\big) &\leqslant n\log\frac{R_2(t)}{R_1(t)}+C\\ &\leqslant \log\log(1+t)+C\end{aligned}$$

and so

$$R_2(t)-R_1(t)\leqslant C\big[1+t\log\log(1+t)\big].$$

Consequently $\lim_{t\to 0}\frac{R_2(t)-R_1(t)}{R_1(t)}=0$. By the comparision principle $X(\cdot,t)$ is pinched between $X_2(\cdot,t)$ and $X_1(\cdot,t)$. So $X(\cdot,t)/r(t)$ must tend to the unit sphere uniformly.

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