Ann. Inst. Henri Poincaré,

Vol. 12, n° 6, 1995, p. 673-725. *Analyse non linéaire*

Gevrey regularizing effect for the (generalized) Korteweg-de Vries equation and nonlinear Schrödinger equations

by

Anne DE BOUARD

C.N.R.S. et Université de Paris-Sud, Mathématique,
Bâtiment 425, 91405 Orsay, France.

Nakao HAYASHI

Department of Mathematics, Faculty of Engineering,
Gunma University, Kiryu 376, Japan.

and

Keiichi KATO

Department of Mathematics, Faculty of Science,
Osaka University, Toyonaka 560, Japan.

ABSTRACT. – This paper is concerned with regularizing effects of solutions to the (generalized) Korteweg-de Vries equation

(gKdV) $$\begin{cases} \partial_t u + \partial_x^3 u = \lambda u^{p-1} \partial_x u, & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0) = \phi, & x \in \mathbb{R}, \end{cases}$$

and nonlinear Schrödinger equations in one space dimension

(NLS) $$\begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u = G(u,\bar{u}), & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0) = \psi, & x \in \mathbb{R}, \end{cases}$$

where p is an integer satisfying $p \geq 2$, $\lambda \in \mathbb{C}$ and G is a polynomial of $(u,\bar{u})$. We prove that if the initial function ϕ is in a Gevrey class of order 3 defined in Section 1, then there exists a positive time T such that the solution of (gKdV) is analytic in space variable for $t \in [-T,T]\backslash\{0\}$, and if the initial function ψ in a Gevrey class of order 2, then there exists a

Vol. 12/95/06/$ 4.00/

positive time T such that the solution of (NLS) is analytic in space variable for $t \in [-T,T]\backslash\{0\}$.

RÉSUMÉ. – Nous étudions, dans cet article, certains effets régularisants pour les solutions de l'équation de Korteweg-de Vries (généralisée)

$$\text{(gKdV)} \qquad \begin{cases} \partial_t u + \partial_x^3 u = \lambda u^{p-1}\partial_x u, & (t,x) \in \mathbb{R}\times\mathbb{R}, \\ u(0) = \phi, \quad x \in \mathbb{R}, \end{cases}$$

et des équations de Schrödinger monodimensionnelles

$$\text{(NLS)} \qquad \begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u = G(u,\bar{u}), & (t,x) \in \mathbb{R}\times\mathbb{R}, \\ u(0) = \psi, \quad x \in \mathbb{R}, \end{cases}$$

où p est un entier supérieur ou égal à 2, $\lambda \in \mathbb{C}$ et G est un polynôme en $(u,\bar{u})$. Nous montrons que, lorsque la donnée initiale ϕ appartient à la classe de Gevrey définie dans la première partie, il existe un temps T tel que la solution de (gKdV) est analytique en espace pour $t \in [-T,T]\backslash\{0\}$; de même, lorsque la donnée initiale ψ appartient à une certaine classe de Gevrey d'ordre 2, il existe un temps T tel que la solution de (NLS) est analytique en espace pour $t \in [-T,T]\backslash\{0\}$.

---

1. INTRODUCTION

In this paper we study regularizing effects of solutions to the (generalized) Korteweg-de Vries equation

$$\text{(gKdV)} \qquad \begin{cases} \partial_t u + \partial_x^3 u = \lambda u^{p-1}\partial_x u, & (t,x) \in \mathbb{R}\times\mathbb{R}, \\ u(0) = \phi, \quad x \in \mathbb{R}, \end{cases}$$

and nonlinear Schrödinger equations in one space dimension

$$\text{(NLS)} \qquad \begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u = G(u,\bar{u}), & (t,x) \in \mathbb{R}\times\mathbb{R}, \\ u(0) = \psi, \quad x \in \mathbb{R}, \end{cases}$$

where p is an integer satisfying $p \geq 2$, $\lambda \in \mathbb{C}$ and G is a polynomial of $(u,\bar{u})$. We prove that if the initial function ϕ is in a Gevrey class of order 3 defined below, then there exists a positive time T such that the solution of (gKdV) is analytic in space variable for $t \in [-T,T]\backslash\{0\}$, and if the initial function ψ is in a Gevrey class of order 2, then there exists a positive time T such that the solution of (NLS) is analytic in space variable for $t \in [-T, T]\backslash\{0\}$. In other words, the singularities of the data go to infinity at once. We call this property the Gevrey regularizing effect.

We also prove analyticity in time of solutions of (gKdV) which is the same result as that in [H-K.K] in the case of (NLS).

To state our results precisely we introduce the basic function space used in this paper. We define a Gevrey class of order σ as follows :

$$G_\sigma^{A_1 A_2 \cdots A_N}(P_1, P_2, \cdots, P_N; X) = \left\{ f \in X; \|f\|_{G_\sigma^{A_1 A_2 \cdots A_N}(P_1, P_2, \cdots, P_N; X)} = \sum_{j_1, \cdots, j_N = 0}^{\infty} \frac{A_1^{j_1}}{(j_1!)^\sigma} \cdots \frac{A_N^{j_N}}{(j_N!)^\sigma} \|P_1^{j_1} P_2^{j_2} \cdots P_N^{j_N} f\|_X < \infty \right\},$$

where $A_1, \cdots, A_N$ are positive constants, $P_1, \cdots, P_N$ are vector fields with analytic coefficients and X is a Banach space of functions on an open set in $\mathbb{R}^n$ with norm $\|\cdot\|_X$ and $\sigma \geq 1$. The above function space is used to define several function spaces :

$$Y_T = \{f \in C([-T, T]; L^2); |||f|||_{Y_T} < \infty\},$$

where

$$|||f|||_{Y_T} = \max\{\gamma(T, P, 3, f), \sigma(T, P, 3, f), \gamma(T, P, 0, \partial_t f), \sigma(T, P, 0, \partial_t f)\}$$

with

$$P = x\partial_x + 3t\partial_t,$$

$$\gamma(T, P, m, f) = \sum_{k,l=0}^{\infty} \frac{A_1^l}{(l!)^\sigma} \frac{A_2^k}{(k!)^\sigma} \sup_{t \in [-T,T]} \|P^l \partial_x^k f(t)\|_{H^m}$$

and

$$\sigma(T, P, m, f) = \sum_{j=0}^{m} \sum_{k,l=0}^{\infty} \frac{A_1^l}{(l!)^\sigma} \frac{A_2^k}{(k!)^\sigma} \sup_x \left(\int_{-T}^{T} |\partial_x^{j+1} P^l \partial_x^k f|^2 dt \right)^{1/2},$$

$$Z_\infty = \{f \in C(\mathbb{R}; L^2); |||f|||_{Z_\infty} < \infty\},$$

where

$$|||f|||_{Z_\infty} = \max\{|||f|||_{Y_\infty}, \mu(f)\}$$

with

$$\mu(f) = \sum_{l,k=0}^{\infty} \frac{A_1^l}{(l!)^\sigma} \frac{A_2^k}{(k!)^\sigma} \sup_{t\in\mathbb{R}}(1+|t|)^{1/3}\|P^l \partial_x^k f(t)\|_{H^{1,\infty}}.$$

$$H^{m,p}(-R,R) = \{f \in L^p(-R,R); \|f\|_{H^{m,p}(-R,R)} = \sum_{0\le j\le m} \|\partial_x^j f\|_{L^p(-R,R)} < \infty\}.$$

For simplicity we let $H^m(-R,R) = H^{m,2}(-R,R), H^{m,p} = H^{m,p}(\mathbb{R})$ and $H^m = H^{m,2}$. We also define the closed balls in Y_T and Z_∞ as follows.

$$Y_{T,\rho} = \{f \in Y_T; |||f|||_{Y_T} \le \rho\}, Z_{\infty,\rho} = \{f \in Z_\infty; |||f|||_{Z_\infty} \le \rho\}.$$

Throughout this paper we assume that $A_1 < 1$ since when $\sigma = 1$ Aikawa's result [A,Theorem 3] says that If $A_1 > 1$, then $G_1^{A_1}(x\partial_x; L^2) = \{0\}$ which implies that the solutions constructed in the theorems below are identically zero when $A_1 > 1$ and $\sigma = 1$. Different positive constants are denoted by $C, C_0, C_1, \cdots, A_1, A_2, \cdots$ in Sections 1 and 4.

We now state our main results.

THEOREM 1.1. (gKdV). – *We assume that* $\sigma \ge 1$ *and*

$$\phi \in G_\sigma^{A_1 A_2}(x\partial_x, \partial_x; H^3).$$

Then there exist positive constants A_3, A_4, T *and a unique solution* u *of (gKdV) such that*

$$u \in C([-T,T]; H^3)$$

and

$$u \in G_\sigma^{|t|A_3}(\partial_t; H^3(-R,R)) \cap G_{\max(\sigma/3,1)}^{|t|^{1/3}A_4}(\partial_x; H^3(-R,R))$$
$$\textit{for} \quad t \in [-T,T]$$

where

$$A_3 < \min\left\{\frac{3A_1}{1+3A_1}, \frac{3A_2}{4A_2 + e^{1/3}(1+R)}\right\},$$

A_4 *depends on* A_3 *and*

$$\|u\|_{G_\sigma^{|t|A_3}(\partial_t; H^3(-R,R))}.$$

THEOREM 1.1'. (gKdV). – *We assume that* $\sigma \geq 1$ *and*

$$\phi \in G_\sigma^{A_1 A_2}(x\partial_x, \partial_x; H^3) \cap G_\sigma^{A_1 A_2}(x\partial_x, \partial_x; H^{1,1}).$$

Furthermore we assume that

$$\epsilon_1 = \|\phi\|_{G_\sigma^{A_1 A_2}(x\partial_x, \partial_x; H^3)} + \|\phi\|_{G_\sigma^{A_1 A_2}(x\partial_x, \partial_x; H^{1,1})}$$

is sufficiently small and p *is an integer satisfying* $p \geq 6$. *Then there exist positive constants* A_3, A_4 *and a unique solution* u *of* (gKdV) *such that*

$$u \in C(\mathbb{R}; H^3),$$

$$u \in G_\sigma^{|t|A_3}(\partial_t; H^3(-R, R)) \cap G_{\max(\sigma/3,1)}^{|t|^{1/3} A_4}(\partial_x; H^3(-R, R)) \quad \textit{for any} \quad t \in \mathbb{R},$$

$$\|u(t)\|_{G_\sigma^{|t|A_3}(\partial_t; H^{1,\infty}(-R,R))} \leq \text{const. } \epsilon_1 (1 + |t|)^{-1/3} \quad \textit{for any} \quad t \in \mathbb{R},$$

where

$$A_3 < \min\left\{ \frac{3A_1}{1 + 3A_1}, \frac{3A_2}{4A_2 + e^{1/3}(1 + R)} \right\},$$

A_4 *depends on* A_3 *and*

$$\|u\|_{G_\sigma^{|t|A_3}(\partial_t; H^3(-R,R))}.$$

Remark 1.1. – The positive constants appearing in Theorems 1.1 and 1.1' are important numbers which determine the domain on which the data and the solution of (gKdV) have analytic continuations when $\sigma = 1$ or 3. We explain this point herein.

(I) ([H-K.K]). If ϕ has an analytic continuation Φ on the complex domain

$$\begin{aligned} \Gamma_{\sqrt{2}A_1, A_2} = \{ & z \in \mathbb{C}; z = x + iy, -\infty < x < \infty, \\ & - A_2 - (\tan\alpha)|x| < y < A_2 + (\tan\alpha)|x|, A_2 > 0\}, \end{aligned}$$

where $0 < \alpha = \sin^{-1} \sqrt{2} A_1 < \pi/2$ and

$$\int_{\Gamma_{\sqrt{2}A_1 A_2}} |\Phi(z)|^2 dx dy < \infty,$$

then

$$\phi \in G_1^{A_1 A_2}(x\partial_x, \partial_x; H^3).$$

(II) In the case $\sigma = 1$, Theorems 1.1-1.1' say that

$$u \in G_1^{|t|A_3}(\partial_t; H^3(-R, R))$$

which implies

$$\sum_{l=0}^{\infty} \frac{(|t|A_3)^l}{l!} |\partial_t^l u(t,x)| < \infty \quad \text{for} \quad x \in (-R, R).$$

Hence u has an analytic continuation $U(z_0, x)$ on the complex plane

$$\{z_0 \in \mathbb{C}; z_0 = t + i\tau; |arg z_0| < \sin^{-1} A_3\} \quad \text{for} \quad x \in (-R, R).$$

From Theorems 1.1-1.1' we see that A_3 is bounded from above by an upper limit $3/4$ ($A_1 = 1, A_2 = \infty$). On the other hand in the case of the nonlinear heat equation

$$\begin{cases} \partial_t u - \partial_x^2 u = u^{p+1}, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \quad p \in \mathbb{N}, \\ u(0) = \phi, & x \in \mathbb{R}, \end{cases}$$

it is well known that if $\phi \in H^1(\mathbb{R})$, then u has an analytic continuation $U(z_0, x)$ on the complex domain

$$\{z_0 \in \mathbb{C}; z_0 = t + i\tau; |arg z_0| < \pi/2\} \quad \text{for} \quad x \in \mathbb{R}.$$

This last result corresponds to the case $A_3 = 1$. Hence the following question arises. Can we improve the upper limit on A_3 ?

(III) In the case $\sigma = 3$, Theorems 1.1-1.1' say that if the initial function ϕ belongs to a Gevrey class of order 3, the solution $u(t, x)$ of (gKdV) has an analytic continuation $U(t, z)$ on the complex domain

$$\{z \in \mathbb{C}; z = x + iy, |x| < R, -|t|^{1/3} A_4 < y < |t|^{1/3} A_4, t \neq 0\}.$$

In the case of the KdV equation ($p = 2$), the end of the proof of Theorem 1.1 (Section 4 below) shows that it is sufficient to take A_4 such that

$$A_4 < 3 \left(\frac{A_3}{2(1 + (1 + 2^4 \cdot 3^{18} |\lambda|^2 C^2 C_0^2 C_1) A_3)} \right)^{1/3},$$

where

$$C_0 = \|u\|_{G_\sigma^{|t|A_3}(\partial_t; H^3(-R,R))}, \quad C_1 = 1 + \sum_{m=1}^{\infty} m^{-4/3}$$

and C is the best constant arising in the Sobolev's inequality

$$\|\cdot\|_{L^\infty(-R,R)} \leq \frac{C}{4}\|\cdot\|_{H^1(-R,R)}.$$

(The constant C can be given explicitly since we have

$$\|\cdot\|_{L^\infty(-R,R)} \leq \left(2R + \frac{1}{2R}\right)^{1/2} \|\cdot\|_{H^1(-R,R)}.)$$

We now give a typical example of the function in $G_3^{A_1 A_2}(x\partial_x, \partial_x; H^3)$, but is not an analytic function, which may help the readers to understand the Gevrey regularizing effect.

We put

$$\varphi(x) = \begin{cases} \exp(-x^{-\frac{1}{s-1}}), & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Then the function $\varphi(x)$ belongs to Gevrey class of order s but not belong to Gevrey class of order r, where $1 \leq r < s$ (see [Ko, p41, Lemma 2.1] for the proof). We let $\psi(x) = \varphi(1 - x^2)$ and $s = 3$. Then there exist positive constants A_1 and A_2 such that $\psi(x) \in G_3^{A_1 A_2}(x\partial_x, \partial_x; H^3)$, but $\psi(x)$ is not analytic in the neighborhoods of ± 1. More precisely $\psi(x)$ is not in $G_r^{A_1 A_2}(x\partial_x, \partial_x; H^3)$ for $1 \leq r < 3$ and any $A_1, A_2 > 0$.

The analyticity of solutions of (gKdV) was studied first by T. Kato and K. Masuda [Ka-M] (see also [H]). They proved that if the initial function ϕ is in $G_1^{A_1}(\partial_x; H^3)$, then there exist $A_1' > 0$, $T > 0$ and a unique solution u of (gKdV) such that

$$u \in C([-T, T]; H^3)$$

and

$$u \in G_1^{A_1'}(\partial_x; H^3) \quad \text{for} \quad t \in [-T, T] \quad \text{and} \quad A_1' < A_1.$$

Their method requires the condition $\sigma = 1$ to treat the nonlinear term involving the space derivative of the solution u. Hence their method is not applicable to the general case $\sigma \geq 1$. To overcome this difficulty we use the local smoothing property of solutions to the Airy equation $\partial_t u + \partial_x^3 u = 0$ which was shown by [Ke-P-V 1] first. Local smoothing property enables us to handle (gKdV) by using the contraction mapping principle (see [Ke-P-V 5]). In fact, in the same way as in the proof of Proposition 3.1 in

Section 3 we can prove that there exist $T > 0$ and a unique solution u of (gKdV) such that

$$u \in C([-T, T]; H^3)$$

and

$$u \in G_\sigma^{A_1}(\partial_x; H^3) \quad \text{for} \quad t \in [-T, T], \tag{1.1}$$

when $\phi \in G_\sigma^{A_1}(\partial_x; H^3)$ $(\sigma \geq 1)$. From the above result we can prove that there exists $A_2 > 0$ such that

$$u \in G_{3\sigma}^{A_2}(\partial_t; H^3) \quad \text{for} \quad t \in [-T, T]. \tag{1.2}$$

However analyticity in time of solutions to (gKdV) does not come from (1.2) since $\sigma \geq 1$ is needed to obtain (1.1).

In [Ka], it was shown that if $\phi \in L_b^2 \equiv H^2 \cap L^2(e^{bx}dx)(b > 0)$, then the solution of (gKdV) becomes $C^\infty(-R, R)$ for $t > 0$ in space variable. His proof is based on the fact that the unitary group $\exp(-t\partial_x^3)$ in L_b^2 is equivalent to

$$\exp(-t(\partial_x - b)^3) = \exp(-t\partial_x^3)\exp(-3bt\partial_x^2)\exp(-3b^2t\partial_x)\exp(-b^3t)$$

in L^2 when $t > 0$. Hence the method is not valid for the negative time and it is not clear whether or not the solution of (KdV) becomes analytic for $t > 0$.

In [Cr-Kap-St], the authors studied a fully nonlinear equation of KdV type in one space dimension :

$$\partial_t u + f(\partial_x^3 u, \partial_x^2 u, \partial_x u, u, x, t) = 0, \quad x \in \mathbb{R},$$

where $f \in C^\infty$ and

$$\partial f/\partial(\partial_x^3 u) \geq C > 0 \quad \text{and} \quad \partial f/\partial(\partial_x^2 u) \leq 0.$$

They showed that if the initial function decays faster than any polynomial on $\mathbb{R}^+$, and possesses certain minimal regularity, then the solution $u(t, \cdot) \in C^\infty(-R, R)$ for $t > 0$. Their result ([Cr-Kap-St, Theorem 2.1]) is considered as a generalization of results of [Ka] and [Kr-F].

We prove the regularity in time result for solutions to (gKdV) implying analyticity by using the operator $P = x\partial_x + 3t\partial_t$ (which almost commutes with the operator $\partial_t + \partial_x^3$) and the local smoothing property. We also prove a global existence in time result for solutions in a Gevrey class implying analyticity by using a similar method to that of W. A. Strauss [St].

By making use of the regularity in time result obtained in Section 3 and an induction argument, we prove regularity in space of solutions to (gKdV) yielding analyticity in space variable.

We next state the results concerning (NLS).

THEOREM 1.2. (NLS). – *We assume that* $\sigma \geq 1$ *and*

$$\psi \in G_\sigma^{A_1 A_2}(x\partial_x, \partial_x; H^2).$$

Then there exist positive constants A_3, A_4, T *and a unique solution* u *of* (NLS) *such that*

$$u \in C([-T,T]; H^2),$$

$$u \in G_\sigma^{|t|A_3}(\partial_t; H^2(-R,R)) \cap G_{\max(\sigma/2,1)}^{|t|^{1/2}A_4}(\partial_x; H^2(-R,R))$$
$$for \quad t \in [-T,T],$$

where

$$A_3 < \min\left\{\frac{2A_1}{1+2A_1}, \frac{2A_2}{3A_2 + e^{1/2}(1+R)}\right\},$$

A_4 *depends on* A_3 *and*

$$\|u\|_{G_\sigma^{|t|A_3}(\partial_t; H^2(-R,R))}.$$

THEOREM 1.2'. (NLS). – *We assume that* $\sigma \geq 1$ *and*

$$\psi \in G_\sigma^{A_1 A_2}(x\partial_x, \partial_x; H^2) \cap G_\sigma^{A_1 A_2}(x\partial_x, \partial_x; L^1).$$

Furthermore we assume that

$$\epsilon_1 = \|\psi\|_{G_\sigma^{A_1 A_2}(x\partial_x, \partial_x; H^2)} + \|\psi\|_{G_\sigma^{A_1 A_2}(x\partial_x, \partial_x; L^1)}$$

is sufficiently small and $G(u, \bar{u})$ *satisfies the following growth condition*

$$|G(s, \bar{s})| \leq const. |s|^p \quad for \quad |s| < 1,$$

where p *is an integer satisfying* $p \geq 5$. *Then there exist* A_3, A_4 *and a unique solution* u *such that*

$$u \in C(\mathbb{R}; H^2),$$

$$u \in G_\sigma^{|t|A_3}(\partial_t; H^2(-R,R)) \cap G_{\max(\sigma/2,1)}^{|t|^{1/2}A_4}(\partial_x; H^2(-R,R))$$
$$for\ any \quad t \in \mathbb{R},$$

$$\|u(t)\|_{G_\sigma^{|t|A_3}(\partial_t; L^\infty(-R,R))} \leq const. \epsilon_1 (1+|t|)^{-1/2} \quad for\ any \quad t \in \mathbb{R},$$

where

$$A_3 < \min\left\{\frac{2A_1}{1+2A_1}, \frac{2A_2}{3A_2 + e^{1/2}(1+R)}\right\},$$

A_4 depends on A_3 and

$$\|u\|_{G_\sigma^{|t|A_3}(\partial_t;H^2(-R,R))}.$$

Remark 1.2.

(I) By the same argument as in Remark 1.1 (I), we see that the solution u of (NLS) has an analytic continuation $U(z_0, x)$ on the complex plane

$$\{z_0 \in \mathbb{C}; z_0 = t + i\tau; |arg z_0| < \sin^{-1} A_3\} \quad \text{for} \quad x \in (-R, R)$$

and A_3 has the upper limit $2/3$. The same question as in Remark 1.1 (I) arises concerning this upper limit.

(II) In the case $\sigma = 2$, Theorems 1.2-1.2' say that if the initial function ψ is in a Gevrey class of order 2, the solution $u(t,x)$ of (NLS) has an analytic continuation $U(t,z)$ on the complex domain

$$\{z \in \mathbb{C}; z = x + iy, |x| < R, -|t|^{1/2}A_4 < y < |t|^{1/2}A_4, t \neq 0\}.$$

(III) In the case $G(u,\bar{u}) = u^2$, the end of the proof of Theorem 1.2 given in Section 5 shows that it is sufficient to take A_4 such that

$$A_4 < 2\left(\frac{A_3}{2(1 + 2^9 \cdot 3CC_0A_3)}\right)^{1/2}$$

where

$$C_0 = \|u\|_{G_\sigma^{|t|A_3}(\partial_t;H^2(-R,R))}$$

and C is the best constant arising in the Sobolev's inequality

$$\|\cdot\|_{L^\infty(-R,R)} \leq \frac{C}{4}\|\cdot\|_{H^1(-R,R)}.$$

The smoothing property of solutions to (NLS) implying analyticity in space variable was studied in [H-Sai] in the case of $G(u,\bar{u}) = |u|^{2k}u$, $k \in \mathbb{N}$. More precisely, the main result of [H-Sai] is as follows.

If the initial function ϕ satisfies

$$||(\cosh A_4 x)\phi||_{H^1} < \infty,$$

then the solution $u(t,x)$ of (NLS) has an analytic continuation $U(t,x)$ on the complex plane

$$\{z \in \mathbb{C}; z = x + iy, -|t|A_4 < y < |t|A_4, t \neq 0\}.$$

The result stated in Remark 1.2 (I) was already obtained in [H-K.K] but the result stated in Remark 1.2 (II) is new. It is interesting to compare the result of [H-Sai] given above and Remark 1.2 (II). Our result is new even in the case $G(u,\bar{u}) = |u|^{2k}u, k \in \mathbb{N}$ since we do not assume that the initial function decays at infinity.

For a general class of equations which includes a large number of models arising in the context of water waves, analytic solutions were obtained in [B]. We note here that the methods used in [H-Sai], [H-K.K] are not sufficient to treat the nonlinear Schrödinger equation with a nonlinear term involving the derivative of the unknown function :

$$i\partial_t u + \frac{1}{2}\partial_x^2 u = G(u, \partial_x u, \bar{u}, \partial_x \bar{u}), \tag{1.3}$$

where G is also a polynomial of $(u, \partial_x u, \bar{u}, \partial_x \bar{u})$.

The gauge transformation techniques used in [H-O] are applicable to prove a time regularity result similar to Theorem 1.2 for (1.3) and for general space dimension the method of [Ke-P-V 2] based on [Ke-P-V 2, Theorem 2.3 (2.9)] is applicable for (1.3) with a smallness condition on the data. However these methods cause undesireble complexities, and so we do not go into the problem (1.3) in this paper. The difficulty to handle (1.3) arises from the fact that the smoothing property of solutions to the linear homogeneous Schrödinger equation is not sufficient compared with the Airy equation (see , [Ke-P-V 2, Theorem 2.1]). Local smoothing properties in the usual Sobolev spaces for the linear Schrödinger equation were studied by [Co-Sau], [Sj] and [V] simultaneously. A sharp version of the local smoothing property was obtained in [Ke-P-V 1, Section 4]. Moreover they proved a sharp inhomogeneous version of the local smoothing property in [Ke-P-V 2, Theorem 2.3] for Schrödinger equations which was used to study several higher order models arising in both physics and mathematics in [Ke-P-V 3,4].

G. Ponce [P, Theorem 3.2, Theorem 4.2] studied the regularity of solutions to nonlinear dispersive equations including (gKdV) and (NLS) as examples. Roughly speaking, his results are as follows. If $\phi \in H^{2k}$ and $(x\partial_x)^j\phi \in L^2$ $(j = 1, \cdots, k)$, then the solution of (gKdV) is in $H^{3k}(-R, R)$ for any R in some time interval. If $\psi \in H^k$ and $(x\partial_x)^j\psi \in L^2$ $(j = 1, \cdots, k)$, then the solution of (NLS) is in $H^{2k}(-R, R)$ for any R in some time interval. His methods are based on the classical energy method and the facts that the linear operator $\partial_t + \partial_x^3$ commutes with the operator $x + 3t\partial_x^2$ and the linear operator $i\partial_t + \frac{1}{2}\partial_x^2$ commutes with the operator $x + it\partial_x$. However our results do not follow from the methods in [P]. This commutation property was used in [G-Vel] to study the scattering theory for nonlinear Schrödinger equations with power nonlinearity satisfying the gauge condition. The iterative use of the operator $x + it\partial_x$ in [H-N-T] allowed the authors to study the smoothing property of solutions to nonlinear Schrödinger equations satisfying the gauge condition.

In Section 5 we prove regularity in space of solutions to (NLS) by using regularity in time of solutions. However the proof in Section 5 is not applicable directly to the case of arbitrary dimension.

Our method in this paper can be applied to the system of nonlinear equations

$$\begin{cases} i\partial_t\tilde{u} + A\partial_x^2\tilde{u} = F(\tilde{u}), \\ \tilde{u}(0) = \tilde{u}_0, \end{cases} \tag{1.4}$$

where

$$\tilde{u} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad F(\tilde{u}) = \begin{pmatrix} f_1(u,v) \\ f_2(u,v) \end{pmatrix} \quad \text{and} \quad \tilde{u}_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}.$$

When

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} f_1(u,v) \\ f_2(u,v) \end{pmatrix} = \begin{pmatrix} -iv \\ 0 \end{pmatrix}$$

(1.4) is written as

$$\begin{cases} i\partial_t u - (\Delta v - v) = 0, \\ i\partial_t v - \Delta u = 0, \\ u(0) = u_0, v(0) = v_0, \end{cases}$$

which is equivalent to the (linearized) Boussinesq equation

$$\begin{cases} \partial_t^2 u + \partial_x^4 u - \partial_x^2 u = 0, \\ u(0) = u_0, \quad u_t(0) = i(-\partial_x^2 + 1)^{-1} v_0. \end{cases}$$

We can obtain the same result as that of Theorem 1.2 for (1.4) since the operator $P = x\partial_x + 2t\partial_t$ has the commutation relation

$$[P, i\partial_t + A\partial_x^2] = 2(i\partial_t + A\partial_x^2).$$

Our method is also applicable to a diagonal system

$$(1.5) \quad \begin{cases} \partial_t \omega^k + \partial_x^{2j+1}\omega^k + Q_k(\omega^1, \cdots, \omega^m, \partial_x \omega^1, \cdots, \partial_x^{2j-1}\omega^m) = 0, \\ (k = 1, \cdots, m = m(j)), \end{cases}$$

where $Q_k's$ are polynomials having no constants or linear terms, since the operator $x\partial_x + (2j+1)t\partial_t$ almost commutes with the linear part of (1.5) and the local smoothing property for the linear part of (1.5) proved in [Ke-P-V 1,Section 4] works well for (1.5) thanks to the nonlinear terms $Q_k's$ which are independent of the derivatives of order $2j$. The system (1.5) was studied in [Ke-P-V 3] to obtain local well-posedness of the initial value problem for higher order nonlinear dispersive equations of the form

$$(1.6) \quad \begin{cases} \partial_t u + \partial_x^{2j+1} u + P(u, \partial_x u, \cdots, \partial_x^{2j} u) = 0, \\ u(0) = u_0, \end{cases}$$

where P is a polynomial having no constant or linear terms. We notice here that P has the highest derivative $\partial_x^{2j} u$. By using a gauge transformation we can write (1.6) as a diagonal system (1.5) (see [Ke-P-V 3] for details), and so local well posedness of (1.6) can be treated.

If we add a smallness condition on the data, our method can apply to a system

$$\partial_t u^k + \partial_x^{2j+1} u^k + P_k(u^1, \cdots, u^m, \cdots, \partial_x^{2j} u^1, \cdots, \partial_x^{2j} u^m)$$

by using the sharp inhomogeneous version of local smoothing property given in [Ke-P-V 3, Theorem 2.1 (2.2)].

The main tool in the proofs of our main results is a first order differential operator which almost commutes with the linear part of the target nonlinear evolution equation and which has only derivatives with respect to the space variables for $t = 0$ and has a derivative with respect to the time variable for $t \neq 0$. Hence the following question arises. For linear dispersive systems of the type

$$(1.7) \quad \begin{cases} \partial_t \tilde{u} + iP(D)\tilde{u} = 0, \\ \tilde{u}(0) = \tilde{u}_0, \end{cases}$$

where

$$\tilde{u} = \begin{pmatrix} u \\ v \end{pmatrix} \quad P(\xi) = \begin{pmatrix} p_{11}(\xi) & p_{12}(\xi) \\ p_{21}(\xi) & p_{22}(\xi) \end{pmatrix} \quad P(D) = \mathcal{F}^{-1}\mathcal{P}(\xi)\mathcal{F},$$

do there exist such first order differential operators ?

Under some conditions on $P(D)$ which includes many applications in the theory of dispersive long waves of small amplitude, local smoothing effects for (1.7) was proved in [Co-Sau]. Laurey [L] studied the Cauchy problem for a third order nonlinear Schrödinger equations

$$(1.8) \qquad \begin{cases} i\partial_t u + ia\partial_x^3 u + b\partial_x^2 u = \alpha|u|^2 u + i\beta(\partial_x|u|^2)u + i\gamma|u|^2\partial_x u, \\ u(0) = u_0, \end{cases}$$

which is introduced by A. Hasegawa and Y. Kodama (see references cited in [L]), where $a, b, \alpha, \beta, \gamma$ are given real parameters. The linear part of (1.8) satisfies the condition of [Co-Sau] and so it has the local smoothing effects. As can be seen in [P], the differential operator

$$(1.9) \qquad x - 3at\partial_x^2 - 2ibt\partial_x$$

commutes with the linear part of (1.8). However it is difficult to construct a first order differential operator which almost commutes with the linear part of (1.8) from (1.9).

By combining the induction method in [P] and the local smoothing property for the linear part of (1.8) in [L, Proposition 2.1] we can obtain the similar result as [P, Theorem 3.2] for (1.8). However the result in a Gevrey class function space is still open. More precisely, assuming that u_0 is in some Gevrey class, is the solution of (1.8) analytic in space variable or not ?

This paper is organized as follows. In Section 2 we prove useful lemmas which are needed to obtain the main results. Section 3 is devoted to study the existence of solutions to (gKdV) when the initial function is in some Gevrey class of order σ, which yields that the solution of (gKdV) is in the (usual) Gevrey class of order σ in time variable. By using the existence results of Section 3, we prove Theorems 1.1 and 1.1' in Section 4. In Section 5 we state the results of [H-K.K, Propositions 3.3-3.4] that the solution of (NLS) is in the (usual) Gevrey class of order σ in time when the initial function is in some Gevrey class of order σ and using these results we show Theorems 1.2 and 1.2'.

2. PRELIMINARIES

In Sections 2 and 3, we denote positive constants by C and C may change from line to line. We let $S(t)$ be the unitary group associated with the linear equation $\partial_t u + \partial_x^3 u = 0$. We first state well known estimates and local smoothing property of $S(t)\phi$ obtained by [Ke-P-V 1].

LEMMA 2.1. – *For any $\phi \in L^2$ and any $t > 0$*

$$\|S(t)\phi\|_{L^2} = \|\phi\|_{L^2},$$

$$\sup_x \int_{\mathbf{R}} |\partial_x S(t)\phi(x)|^2 dt \leq C\|\phi\|_{L^2}^2$$

and for any $\phi \in L^1$,

$$\|S(t)\phi\|_{L^\infty} \leq C(1+|t|)^{-\frac{1}{3}}\|\phi\|_{L^1}.$$

LEMMA 2.2. – *We have for $P = x\partial_x + 3t\partial_t$*

$$\partial_x P^l = \sum_{k=0}^{l} \binom{l}{k} P^{l-k}\partial_x, \quad P^l \partial_x = \sum_{k=0}^{l} \binom{l}{k} (-1)^k \partial_x P^{l-k}, \tag{2.1}$$

$$\partial_t P^l = \sum_{k=0}^{l} \binom{l}{k} 3^k P^{l-k}\partial_t, \quad P^l \partial_t = \sum_{k=0}^{l} \binom{l}{k} (-3)^k \partial_t P^{l-k}. \tag{2.2}$$

Proof. – We prove (2.1) and (2.2) by induction. When $l = 1$, it is clear that the first equality of (2.1) is valid. We assume that the first one of (2.1) holds true for any l. Then we have by assumption

$$\begin{aligned} \partial_x P^{l+1} &= (P\partial_x + \partial_x)P^l \\ &= (P+1)\partial_x P^l = (P+1)\sum_{k=0}^{l} \binom{l}{k} P^{l-k}\partial_x. \end{aligned} \tag{2.3}$$

By a direct calculation the right hand side of (2.3) equals

$$
\begin{aligned}
(2.4)\quad & \sum_{k=0}^{l}\binom{l}{k}(P^{l+1-k}\partial_x + P^{l-k}\partial_x)\\
&= \sum_{k=1}^{l}\binom{l}{k}P^{l+1-k}\partial_x + \binom{l}{0}P^{l+1}\partial_x + \sum_{k=0}^{l}\binom{l}{k}P^{l-k}\partial_x\\
&= (\sum_{k=0}^{l-1}\binom{l}{k+1} + \sum_{k=0}^{l}\binom{l}{k})P^{l-k}\partial_x + \binom{l}{0}P^{l+1}\partial_x\\
&= \sum_{k=0}^{l-1}(\binom{l}{k+1} + \binom{l}{k})P^{l-k}\partial_x + \partial_x + P^{l+1}\partial_x\\
&= \sum_{k=0}^{l-1}\binom{l+1}{k+1}P^{l-k}\partial_x + \partial_x + P^{l+1}\partial_x\\
&= \sum_{k=0}^{l}\binom{l+1}{k+1}P^{l-k}\partial_x + P^{l+1}\partial_x\\
&= \sum_{k=1}^{l+1}\binom{l+1}{k}P^{l+1-k}\partial_x + P^{l+1}\partial_x = \sum_{k=0}^{l+1}\binom{l+1}{k}P^{l+1-k}\partial_x.
\end{aligned}
$$

From (2.3) and (2.4) we have

$$\partial_x P^{l+1} = \sum_{k=0}^{l+1}\binom{l+1}{k}P^{l+1-k}\partial_x.$$

Hence we obtain the first part of (2.1). The second part of (2.1) and (2.2) are obtained in the same way as in the proof of the first equality of (2.1).

Q.E.D.

LEMMA 2.3. – *We let* $\sigma \geq 1$ *and* $P = x\partial_x + 3t\partial_t$. *Then we have*

$$\sum_{l=0}^{\infty}\frac{A^l}{(l!)^\sigma}\|\partial_x P^l g\|_X \leq e^A \sum_{l=0}^{\infty}\frac{A^l}{(l!)^\sigma}\|P^l\partial_x g\|_X,$$

$$\sum_{l=0}^{\infty}\frac{A^l}{(l!)^\sigma}\|P^l\partial_x g\|_X \leq e^A \sum_{l=0}^{\infty}\frac{A^l}{(l!)^\sigma}\|\partial_x P^l g\|_X,$$

$$\sum_{l=0}^{\infty}\frac{A^l}{(l!)^\sigma}\|\partial_t P^l g\|_X \leq e^{3A} \sum_{l=0}^{\infty}\frac{A^l}{(l!)^\sigma}\|P^l\partial_t g\|_X,$$

and

$$\sum_{l=0}^{\infty} \frac{A^l}{(l!)^\sigma} \|P^l \partial_t g\|_X \le e^{3A} \sum_{l=0}^{\infty} \frac{A^l}{(l!)^\sigma} \|\partial_t P^l g\|_X.$$

Proof. – Since every inequality in the lemma is proved in the same way by using Lemma 2.2, we only prove the third one. By the first equality of (2.2) in Lemma 2.2 we have

$$\begin{aligned}
&\sum_{l=0}^{\infty} \frac{A^l}{(l!)^\sigma} \|\partial_t P^l g\|_X \\
&\le \sum_{l=0}^{\infty} \frac{A^l}{(l!)^\sigma} \sum_{k=0}^{l} \binom{l}{k} 3^k \|P^{l-k} \partial_t g\|_X \\
&\le \sum_{l=0}^{\infty} \sum_{k=0}^{l} \binom{l}{k}^{1-\sigma} \frac{A^{l-k}}{((l-k)!)^\sigma} \frac{(3A)^k}{(k!)^\sigma} \|P^{l-k} \partial_t g\|_X \\
&\le \sum_{l=0}^{\infty} \sum_{k=0}^{l} \frac{(3A)^k}{(k!)^\sigma} \frac{A^{l-k}}{((l-k)!)^\sigma} \|P^{l-k} \partial_t g\|_X \\
&\qquad\qquad \left(\text{by } \binom{l}{k}^{1-\sigma} \le 1 \quad \text{for} \quad 0 \le k \le l, \sigma \ge 1 \right) \\
&\le \left(\sum_{k=0}^{\infty} \frac{(3A)^k}{(k!)^\sigma} \right) \left(\sum_{l=0}^{\infty} \frac{A^l}{(l!)^\sigma} \|P^l \partial_t g\|_X \right) \\
&\le e^{3A} \sum_{l=0}^{\infty} \frac{A^l}{(l!)^\sigma} \|P^l \partial_t g\|_X.
\end{aligned}$$

Q.E.D.

LEMMA 2.4. – *We have*

$$\begin{aligned}
&\int_{-T}^{T} \|f(t) g(t)\|_{L^2} dt \\
&\le C T^{1/2} \sup_{x \in \mathbb{R}} \left(\int_{-T}^{T} |g(t,x)|^2 dt \right)^{1/2} \left(\int_{\mathbb{R}} \sup_{t \in [-T,T]} |f(t,x)|^2 dx \right)^{1/2},
\end{aligned}$$

provided that the right hand side is finite.

Proof. – By Schwarz' inequality we have

$$\int_{-T}^{T} \|f(t)g(t)\|_{L^2} dt \leq C\left(\int_{-T}^{T} \|f(t)g(t)\|_{L^2}^2 dt\right)^{1/2} T^{1/2}$$
$$= C\left(\int_{\mathbb{R}}\int_{-T}^{T} |f(t,x)g(t,x)|^2 dt dx\right)^{1/2} T^{1/2}$$
$$\leq C\left(\int_{\mathbb{R}}\int_{-T}^{T} |g(t,x)|^2 dt \cdot \sup_{t\in[-T,T]} |f(t,x)|^2 dx\right)^{1/2} T^{1/2}$$

from which the lemma follows.

In the same way as in the proof of Lemma 2.4 we have

LEMMA 2.4'.

$$\left(\int_{\mathbb{R}} \|f(t)g(t)\|_{L^2}^2 dt\right)^{1/2} \leq C \sup_{x\in\mathbb{R}}\left(\int_{\mathbb{R}} |g(t,x)|^2 dt\right)^{1/2} \times \left(\int_{\mathbb{R}} \sup_{t\in\mathbb{R}} |f(t,x)|^2 dx\right)^{1/2},$$

provided that the right hand side is finite.

In order to prove Lemmas 2.5-2.5' we need [H-K.K, Proposition 2.1] and since we also use it in several stages of the proofs of the results in Section 3, we state the proposition without proof.

[H-K.K, PROPOSITION 2.1.]. – *We let*

$$f,\ g \in G_\sigma^{A_1 A_2}(P, \partial_x; H^m) \cap G_\sigma^{A_1 A_2}(P, \partial_x; H^{[m/2],\infty}), \qquad \sigma \geq 1$$

and $P = x\partial_x + 3t\partial_t$. *Then we have*

$$\|f \cdot g\|_{G_\sigma^{A_1 A_2}(P+C_1, \partial_x; H^m)} \leq e^{A_1 C_1}\big(\|f\|_{G_\sigma^{A_1 A_2}(P,\partial_x;H^{[m/2],\infty})}\|g\|_{G_\sigma^{A_1 A_2}(P,\partial_x;H^m)} + \|g\|_{G_\sigma^{A_1 A_2}(P,\partial_x;H^{[m/2],\infty})}\|f\|_{G_\sigma^{A_1 A_2}(P,\partial_x;H^m)}\big),$$

where $[s]$ *is the largest integer less than or equal to* s.

The following lemma is needed to show the local existence in time of solutions to (gKdV).

LEMMA 2.5. – *We let* $f \in Y_T$, $F(f) = f^{p-1}\partial_x f, \sigma \geq 1$ *and* $P = x\partial_x + 3t\partial_t$. *Then we have*

$$\int_0^T (\|F(f)\|_{G_\sigma^{A_1A_2}(P,\partial_x;H^3)} + \|\partial_t F(f)\|_{G_\sigma^{A_1A_2}(P,\partial_x;L^2)}d\tau \leq C(T|||f|||_{Y_T}^p + T^{1/2}|||f|||_{Y_T}\|f(0)\|^{p-1}_{G_\sigma^{A_1A_2}(x\partial_x,\partial_x;H^3)}).$$

Proof. – We have by lemma 2.3

$$(2.5)\quad \|F(f)\|_{G_\sigma^{A_1A_2}(P,\partial_x;H^3)} = \sum_{l,k=0}^{\infty}\frac{A_1^l}{(l!)^\sigma}\frac{A_2^k}{(k!)^\sigma}\sum_{j=0}^{3}\|\partial_x^j P^l\partial_x^k F(f)\|_{L^2} \leq e^{3A}\sum_{l,k=0}^{\infty}\frac{A_1^l}{(l!)^\sigma}\frac{A_2^k}{(k!)^\sigma}\sum_{j=0}^{3}\|P^l\partial_x^k\partial_x^j F(f)\|_{L^2} \leq e^{3A}\sum_{j=0}^{3}\|\partial_x^j F(f)\|_{G_\sigma^{A_1A_2}(P,\partial_x;L^2)}.$$

A direct calculation gives

$$(2.6)\quad \begin{cases} \partial_x F(f) = (p-1)f^{p-2}(\partial_x f)^2 + f^{p-1}\partial_x^2 f, \\ \partial_x^2 F(f) = (p-1)(p-2)f^{p-3}(\partial_x f)^3 \\ \qquad +3(p-1)f^{p-2}\partial_x f\cdot\partial_x^2 f + f^{p-1}\partial_x^3 f, \\ \partial_x^3 F(f) = (p-1)(p-2)(p-3)f^{p-4}(\partial_x f)^4 \\ \qquad +6(p-1)(p-2)f^{p-3}(\partial_x f)^2\partial_x^2 f \\ \qquad +3(p-1)f^{p-2}(\partial_x^2 f)^2 + (p-1)f^{p-2}\partial_x f\cdot\partial_x^3 f \\ \qquad +f^{p-1}\partial_x^4 f. \end{cases}$$

By the same argument as in the proof of [H-K.K, Proposition 2.1], Lemma 2.3 and Sobolev's inequality, $G_\sigma^{A_1A_2}(P,\partial_x;L^2)$ norm of each term in the right hand side of (2.6) is estimated from above by

$$(2.7)\quad C(\|f\|^p_{G_\sigma^{A_1A_2}(P,\partial_x;H^3)} + \|f^{p-1}\partial_x^4 f\|_{G_\sigma^{A_1A_2}(P,\partial_x;L^2)}).$$

Hence we have by (2.7)

$$(2.8)\quad \int_0^T \|F(f)\|_{G_\sigma^{A_1A_2}(P,\partial_x;H^3)}d\tau \leq C\left(T|||f|||^p_{Y_T} + \int_0^T \|f^{p-1}\partial_x^4 f\|_{G_\sigma^{A_1A_2}(P,\partial_x;L^2)}d\tau\right).$$

By Lemma 2.4 and a simple calculation

$$(2.9)\quad \int_0^T \|g_1 \cdot g_2\|_{G_\sigma^{A_1 A_2}(P,\partial_x;L^2)} d\tau$$

$$= \sum_{l,k=0}^{\infty} \frac{A_1^l}{(l!)^\sigma} \frac{A_2^k}{(k!)^\sigma} \times \int_0^T \|P^l \partial_x^k (g_1 \cdot g_2)\|_{L^2} d\tau$$

$$= \sum_{l,k=0}^{\infty} \frac{A_1^l}{(l!)^\sigma} \frac{A_2^k}{(k!)^\sigma} \sum_{j=0}^{l} \sum_{n=0}^{k} \binom{l}{j} \binom{k}{n} \times \int_0^T \|P^{l-j} \partial_x^{k-n} g_1 \cdot P^j \partial_x^n g_2\|_{L^2} d\tau$$

$$\leq \sum_{l,k=0}^{\infty} \frac{A_1^l}{(l!)^\sigma} \frac{A_2^k}{(k!)^\sigma} \sum_{j=0}^{l} \sum_{n=0}^{k} \binom{l}{j} \binom{k}{n} T^{1/2} \sup_{x \in \mathbf{R}} \left(\int_0^T |P^j \partial_x^n g_2|^2 d\tau \right)^{1/2} \times \left(\int_{\mathbf{R}} \sup_{t \in [0,T]} |P^{l-j} \partial_x^{k-n} g_1|^2 dx \right)^{1/2}$$

$$\leq T^{1/2} \left(\sum_{l,k=0}^{\infty} \frac{A_1^l}{(l!)^\sigma} \frac{A_2^k}{(k!)^\sigma} \sup_{x \in \mathbf{R}} \left(\int_0^T |P^l \partial_x^k g_2|^2 d\tau \right)^{1/2} \right) \times \left(\sum_{l,k=0}^{\infty} \frac{A_1^l}{(l!)^\sigma} \frac{A_2^k}{(k!)^\sigma} \left(\int_{\mathbf{R}} \sup_{t \in [0,T]} |P^l \partial_x^k g_1|^2 dx \right)^{1/2} \right).$$

Since

$$|g_3(t)|^2 = |g_3(0)|^2 + \int_0^t \partial_\tau |g_3(\tau)|^2 d\tau$$

we have by (2.9)

$$(2.10)\quad \int_0^T \|g_1 \cdot g_2\|_{G_\sigma^{A_1 A_2}(P,\partial_x;L^2)} d\tau$$

$$\leq T^{1/2} \left(\sum_{l,k=0}^{\infty} \frac{A_1^l}{(l!)^\sigma} \frac{A_2^k}{(k!)^\sigma} \sup_{x \in \mathbf{R}} \left(\int_0^T |P^l \partial_x^k g_2|^2 d\tau \right)^{1/2} \right) \times \left(\sum_{l,k=0}^{\infty} \frac{A_1^l}{(l!)^\sigma} \frac{A_2^k}{(k!)^\sigma} \left(\left(\int_{\mathbf{R}} |(x\partial_x)^l \partial_x^k g_1(0,x)|^2 dx \right)^{1/2} + 2 \left(\int_{\mathbf{R}} \int_0^T |\partial_\tau P^l \partial_x^k g_1(\tau,x)| |P^l \partial_x^k g_1(\tau,x)| d\tau dx \right)^{1/2} \right) \right)$$

$$\leq T^{1/2}\left(\sum_{l,k=0}^{\infty}\frac{A_1^l}{(l!)^\sigma}\frac{A_2^k}{(k!)^\sigma}\sup_{x\in\mathbf{R}}\left(\int_0^T|P^l\partial_x^k g_2|^2d\tau\right)^{1/2}\right)$$
$$\times(\|g_1(0)\|_{G_\sigma^{A_1A_2}(x\partial_x,\partial_x;L^2)}$$
$$+T^{1/2}\gamma(T,P,0,g_1)^{1/2}\gamma(T,P,0,\partial_t g_1)^{1/2})$$

by Schwarz' inequality and Lemma 2.3.

We apply (2.10) with $g_1 = f^{p-1}$ and $g_2 = \partial_x^4 f$, and Lemma 2.3 to the right hand side of (2.8) to obtain

$$\text{(2.11)}\quad \int_0^T \|F(f)\|_{G_\sigma^{A_1A_2}(P,\partial_x;H^3)}d\tau$$
$$\leq CT|||f|||_{Y_T}^p + CT^{1/2}\sigma(T,P,3,f)$$
$$\times(\|f(0)^{p-1}\|_{G_\sigma^{A_1A_2}(x\partial_x,\partial_x;L^2)}$$
$$+T^{1/2}\gamma(T,P,0,f^{p-1})^{1/2}\gamma(T,P,0,\partial_t f^{p-1})^{1/2}).$$

By using [H-K.K, Proposition 2.1] and Sobolev's inequality in the second term of the right hand side of (2.11) we get

$$\text{(2.12)}\quad \int_0^T \|F(f)\|_{G_\sigma^{A_1A_2}(P,\partial_x;H^3)}d\tau$$
$$\leq C(T|||f|||_{Y_T}^p + T^{1/2}|||f|||_{Y_T}(\|f(0)\|^{p-1}_{G_\sigma^{A_1A_2}(x\partial_x,\partial_x;H^3)}$$
$$+T^{1/2}|||f|||_{Y_T}^{p-1}))$$
$$\leq C(T|||f|||_{Y_T}^p + T^{1/2}|||f|||_{Y_T}\|f(0)\|^{p-1}_{G_\sigma^{A_1A_2}(x\partial_x,\partial_x;H^3)}).$$

In the same way as in the proof of (2.8) we have

$$\int_0^T \|\partial_t F(f)\|_{G_\sigma^{A_1A_2}(P,\partial_x;L^2)}d\tau$$
$$\leq C(T|||f|||_{Y_T}^p + \int_0^T \|f^{p-1}\partial_t\partial_x f\|_{G_\sigma^{A_1A_2}(P,\partial_x;L^2)}d\tau).$$

We again use (2.10) with $g_1 = f^{p-1}$ and $g_2 = \partial_t\partial_x f$, and Lemma 2.3 to the right hand side of the above. Then we have

$$(2.13) \quad \int_0^T \|\partial_t F(f)\|_{G_\sigma^{A_1 A_2}(P,\partial_x;L^2)} d\tau$$
$$\leq CT|||f|||_{Y_T}^p + CT^{1/2}\sigma(T,P,0,\partial_t f)$$
$$\times (\|f(0)^{p-1}\|_{G_\sigma^{A_1 A_2}(x\partial_x,\partial_x;L^2)}$$
$$+ T^{1/2}\gamma(T,P,0,f^{p-1})^{1/2}\gamma(T,P,0,\partial_t(f^{p-1}))^{1/2}).$$

By the same argument as in the proof of (2.12), (2.13) yields

$$(2.14) \quad \int_0^T \|\partial_t F(f)\|_{G_\sigma^{A_1 A_2}(P,\partial_x;L^2)} d\tau \leq \text{The right hand side of (2.12).}$$

The lemma follows from (2.12) and (2.14) immediately.

Q.E.D.

In the same way as in the proof of Lemma 2.5 we have

LEMMA 2.6. – *We let F, σ, A_1 and P be the same as those in Lemma 2.5, and we let $f, g \in Y_T$ and $f(0) = g(0)$. Then we have*

$$\int_0^T (\|F(f) - F(g)\|_{G_\sigma^{A_1 A_2}(P,\partial_x;H^3)} + \|\partial_t(F(f) - F(g))\|_{G_\sigma^{A_1 A_2}(P,\partial_x;L^2)}) d\tau$$
$$\leq CT|||f-g|||_{Y_T}(|||f|||_{Y_T}^{p-1} + |||g|||_{Y_T}^{p-1}).$$

The following lemma is needed to show the global existence in time of solutions to (gKdV).

LEMMA 2.5'. – *We let $f \in Z_\infty$, $F(f) = f^{p-1}\partial_x f$, $\sigma \geq 1$ and $P = x\partial_x + 3t\partial_t$, where p is an integer satisfying $p \geq 6$. Then we have*

$$\int_{\mathbf{R}} (\|F(f)\|_{G_\sigma^{A_1 A_2}(P,\partial_x;H^3)} + \|\partial_t F(f)\|_{G_\sigma^{A_1 A_2}(P,\partial_x;L^2)}) d\tau$$
$$\leq C(|||f|||_{Z_\infty}^p + |||f|||_{Z_\infty}^3 \|f(0)\|_{G_\sigma^{A_1 A_2}(x\partial_x,\partial_x;H^3)}^{p-3}).$$

Proof. – We have by (2.5), (2.6) and [H-K.K, Proposition 2.1]

$$\begin{aligned}
&\|F(f)\|_{G_\sigma^{A_1A_2}(P,\partial_x;H^3)}\\
&\leq e^{3A_1}\sum_{j=0}^{3}\|\partial_x^j F(f)\|_{G_\sigma^{A_1A_2}(P,\partial_x;L^2)}\\
&\leq C(\|f\|^{p-1}_{G_\sigma^{A_1A_2}(P,\partial_x;H^{1,\infty})}\|f\|_{G_\sigma^{A_1A_2}(P,\partial_x;H^3)}\\
&\quad+\|f\|^{p-2}_{G_\sigma^{A_1A_2}(P,\partial_x;L^\infty)}\|(\partial_x^2 f)^2\|_{G_\sigma^{A_1A_2}(P,\partial_x;L^2)}\\
&\quad+\|f^{p-1}\partial_x^4 f\|_{G_\sigma^{A_1A_2}(P,\partial_x;L^2)})\\
&\leq C(\|f\|^{p-2}_{G_\sigma^{A_1A_2}(P,\partial_x;H^{1,\infty})}\|f\|^2_{G_\sigma^{A_1A_2}(P,\partial_x;H^3)}+\|f^{p-1}\partial_x^4 f\|_{G_\sigma^{A_1A_2}(P,\partial_x;L^2)})
\end{aligned}$$

(by Sobolev's inequality and Lemma 2.3).

Hence

$$\begin{aligned}
&\int_{\mathbf{R}}\|F(f)\|_{G_\sigma^{A_1A_2}(P,\partial_x;H^3)}d\tau\\
&\quad\leq C\int_{\mathbf{R}}\|f\|^{p-2}_{G_\sigma^{A_1A_2}(P,\partial_x;H^{1,\infty})}\|f\|^2_{G_\sigma^{A_1A_2}(P,\partial_x;H^3)}d\tau\\
&\qquad+C\int_{\mathbf{R}}\|f^{p-1}\partial_x^4 f\|_{G_\sigma^{A_1A_2}(P,\partial_x;L^2)}d\tau\\
&\quad\leq C\gamma(\infty,P,3,f)^2|||f|||^{p-2}_{Z_\infty}\int_{\mathbf{R}}(1+|\tau|)^{-\frac{p-2}{3}}d\tau\\
&\qquad+C\int_{\mathbf{R}}\|f^{p-1}\partial_x^4 f\|_{G_\sigma^{A_1A_2}(P,\partial_x;L^2)}d\tau
\end{aligned}$$

and since $(p-2)/3>1$, the integral in the preceeding term is convergent and we get

$$\begin{aligned}
(2.15)\qquad &\int_{\mathbf{R}}\|F(f)\|_{G_\sigma^{A_1A_2}(P,\partial_x;H^3)}d\tau\\
&\quad\leq C\left(|||f|||^{p}_{Z_\infty}+\int_{\mathbf{R}}\|f^{p-1}\partial_x^4 f\|_{G_\sigma^{A_1A_2}(P,\partial_x;L^2)}d\tau\right).
\end{aligned}$$

We have

$$\begin{aligned}
(2.16)\qquad &\int_{\mathbf{R}}\|f^{p-1}\partial_x^4 f\|_{G_\sigma^{A_1A_2}(P,\partial_x;L^2)}d\tau\\
&\quad\leq\int_{\mathbf{R}}\|f\|^2_{G_\sigma^{A_1A_2}(P,\partial_x;L^\infty)}\|f^{p-3}\partial_x^4 f\|_{G_\sigma^{A_1A_2}(P,\partial_x;L^2)}d\tau
\end{aligned}$$

(by [H-K.K, Proposition 2.1])

$$\leq \int_{\mathbf{R}} \|f\|^2_{G_\sigma^{A_1A_2}(P,\partial_x;L^\infty)}$$
$$\times \sum_{l,k=0}^{\infty} \frac{A_1^l}{(l!)^\sigma}\frac{A_2^k}{(k!)^\sigma}\sum_{j=0}^{l}\sum_{n=0}^{k}\binom{l}{j}\binom{k}{n}$$
$$\times \|P^{l-j}\partial_x^{k-n}(f^{p-3})\cdot P^j\partial_x^n\partial_x^4 f\|_{L^2}d\tau$$
$$\leq \sum_{l,k=0}^{\infty} \frac{A_1^l}{(l!)^\sigma}\frac{A_2^k}{(k!)^\sigma}\sum_{j=0}^{l}\sum_{n=0}^{k}\binom{l}{j}\binom{k}{n}$$
$$\times \left(\int_{\mathbf{R}} \|f\|^4_{G_\sigma^{A_1A_2}(P,\partial_x;L^\infty)}d\tau\right)^{1/2}$$
$$\times \left(\int_{\mathbf{R}} \|P^{l-j}\partial_x^{k-n}(f^{p-3})\cdot P^j\partial_x^n\partial_x^4 f\|^2_{L^2}d\tau\right)^{1/2}$$

(by Schwarz' inequality)

$$\leq C\sum_{l,k=0}^{\infty} \frac{A_1^l}{(l!)^\sigma}\frac{A_2^k}{(k!)^\sigma}\sum_{j=0}^{l}\sum_{n=0}^{k}\binom{l}{j}\binom{k}{n}|||f|||^2_{Z_\infty}$$
$$\times \sup_{x\in\mathbf{R}}\left(\int_{\mathbf{R}}|P^j\partial_x^n\partial_x^4 f|^2dt\right)^{1/2}$$
$$\times \left(\int_{\mathbf{R}}\sup_{t\in\mathbf{R}}|P^{l-j}\partial_x^{k-n}(f^{p-3})|^2dx\right)^{1/2} \quad \text{(by Lemma 2.4')}$$
$$\leq |||f|||^2_{Z_\infty}\left(\sum_{l,k=0}^{\infty} \frac{A_1^l}{(l!)^\sigma}\frac{A_2^k}{(k!)^\sigma}\sup_{x\in\mathbf{R}}\left(\int_{\mathbf{R}}|P^l\partial_x^k\partial_x^4 f|^2dt\right)^{1/2}\right)$$
$$\times \left(\sum_{l,k=0}^{\infty} \frac{A_1^l}{(l!)^\sigma}\frac{A_2^k}{(k!)^\sigma}\left(\int_{\mathbf{R}}\sup_{t\in\mathbf{R}}|P^l\partial_x^k(f^{p-3})|^2dx\right)^{1/2}\right)$$
$$\leq |||f|||^3_{Z_\infty}\sum_{l,k=0}^{\infty} \frac{A_1^l}{(l!)^\sigma}\frac{A_2^k}{(k!)^\sigma}\left(\int_{\mathbf{R}}\sup_{t\in\mathbf{R}}|P^l\partial_x^k(f^{p-3})|^2dx\right)^{1/2}$$

(by Lemma 2.3).

In the same way as in the proof of Lemma 2.5 we use the fact that

$$\sup_{t\in\mathbf{R}}|P^l\partial_x^k(f^{p-3})|^2 \leq |(x\partial_x)^l\partial_x^k(f(0)^{p-3})|^2$$
$$+\int_{\mathbf{R}}|P^l\partial_x^k(f^{p-3})||\partial_t P^l\partial_x^k(f^{p-3})|dt$$

to obtain

$$\sum_{l,k=0}^{\infty} \frac{A_1^l}{(l!)^\sigma} \frac{A_2^k}{(k!)^\sigma} \left(\int_{\mathbf{R}} \sup_{t \in \mathbf{R}} |P^l \partial_x^k (f^{p-3})|^2 dx \right)^{1/2}$$

$$\leq \sum_{l,k=0}^{\infty} \frac{A_1^l}{(l!)^\sigma} \frac{A_2^k}{(k!)^\sigma} \left(\int_{\mathbf{R}} |(x\partial_x)^l \partial_x^k (f(0)^{p-3}) \right) |^2 dx$$

$$+ C \int_{\mathbf{R}} \int_{\mathbf{R}} |\partial_t P^l \partial_x^k (f^{p-3})| \, |P^l \partial_x^k (f^{p-3})| dt dx)^{1/2}$$

$$\leq C \| f(0)^{p-3} \|_{G_\sigma^{A_1 A_2}(x\partial_x, \partial_x; L^2)}$$

$$+ C \sum_{l,k=0}^{\infty} \frac{A_1^l}{(l!)^\sigma} \frac{A_2^k}{(k!)^\sigma} \left(\int_{\mathbf{R}} \left(\int_{\mathbf{R}} |P^l \partial_x^k (f^{p-3})|^2 dx \right)^{1/2} \right.$$

$$\left. \times \left(\int_{\mathbf{R}} |\partial_t P^l \partial_x^k (f^{p-3})|^2 dx \right)^{1/2} dt \right)^{1/2}$$

(by Schwarz' inequality),

and by the same arguments as in [H-K.K, Proposition 2.1],

$$\leq C \| f(0) \|_{G_\sigma^{A_1 A_2}(x\partial_x, \partial_x; H^3)}^{p-3}$$

$$+ C \left(\int_{\mathbf{R}} \| f \|_{G_\sigma^{A_1 A_2}(P, \partial_x; L^\infty)}^{p-4} \| f \|_{G_\sigma^{A_1 A_2}(P, \partial_x; L^2)} \right.$$

$$\left. \times \| \partial_t (f^{p-3}) \|_{G_\sigma^{A_1 A_2}(P, \partial_x; L^2)} dt \right)^{1/2}$$

$$\leq C \| f(0) \|_{G_\sigma^{A_1 A_2}(x\partial_x, \partial_x; H^3)}^{p-3}$$

$$+ C \left(\int_{\mathbf{R}} \| f \|_{G_\sigma^{A_1 A_2}(P, \partial_x; L^\infty)}^{2(p-4)} \| f \|_{G_\sigma^{A_1 A_2}(P, \partial_x; L^2)} \right.$$

$$\left. \times \| \partial_t f \|_{G_\sigma^{A_1 A_2}(P, \partial_x; L^2)} dt \right)^{1/2}$$

$$\leq C \| f(0) \|_{G_\sigma^{A_1 A_2}(x\partial_x, \partial_x; H^3)}^{p-3}$$

$$+ C ||| f |||_{Z_\infty}^{p-4} \gamma(\infty, P, 0, f)^{1/2} \gamma(\infty, P, 0, \partial_t f)^{1/2}.$$

Hence we have

$$\sum_{l,k=0}^{\infty} \frac{A_1^l}{(l!)^\sigma} \frac{A_2^k}{(k!)^\sigma} \left(\int_{\mathbf{R}} \sup_{t \in \mathbf{R}} |P^l \partial_x^k (f^{p-3})|^2 dx \right)^{1/2} \leq C(\|f(0)\|^{p-3}_{G_\sigma^{A_1 A_2}(x\partial_x, \partial_x; H^3)} + |||f|||^{p-3}_{Z_\infty}). \tag{2.17}$$

From (2.15), (2.16) and (2.17) it follows that

$$\int_{\mathbf{R}} \|F(f)\|_{G_\sigma^{A_1 A_2}(P, \partial_x; H^3)} dt \leq C(|||f|||^p_{Z_\infty} + |||f|||^3_{Z_\infty} \|f(0)\|^{p-3}_{G_\sigma^{A_1 A_2}(x\partial_x, \partial_x; H^3)}). \tag{2.18}$$

On the other hand, using the fact that

$$\partial_t F(f) = f^{p-1} \partial_t \partial_x f + (p-1) f^{p-2} \partial_t f \cdot \partial_x f$$

and the same arguments as in the proof of (2.15), we can easily show that

$$\int_{\mathbf{R}} \|\partial_t F(f)\|_{G_\sigma^{A_1 A_2}(P, \partial_x; L^2)} dt \leq C(|||f|||^p_{Z_\infty} + \int_{\mathbf{R}} \|f^{p-1} \partial_t \partial_x f\|_{G_\sigma^{A_1 A_2}(P, \partial_x; L^2)} dt). \tag{2.19}$$

In the same way as we proved (2.16), we have

$$\int_{\mathbf{R}} \|f^{p-1} \partial_t \partial_x f\|_{G_\sigma^{A_1 A_2}(P, \partial_x; L^2)} dt \leq |||f|||^3_{Z_\infty} \left(\sum_{l,k=0}^{\infty} \frac{A_1^l}{(l!)^\sigma} \frac{A_2^k}{(k!)^\sigma} \left(\int_{\mathbf{R}} \sup_{t \in \mathbf{R}} |P^l \partial_x^k (f^{p-3})|^2 dx \right) \right)^{1/2}$$

and this, together with (2.19) and (2.17), gives

$$\int_{\mathbf{R}} \|\partial_t F(f)\|_{G_\sigma^{A_1 A_2}(P, \partial_x; L^2)} dt \leq C(|||f|||^p_{Z_\infty} + |||f|||^3_{Z_\infty} \|f(0)\|^{p-3}_{G_\sigma^{A_1 A_2}(x\partial_x, \partial_x; H^3)}). \tag{2.20}$$

From (2.18) and (2.19) the lemma follows.

Q.E.D.

The following three lemmas are needed to show analyticity of solutions to (gKdV) in space variable when the solutions are in a Gevrey class of order 3 in time variable.

LEMMA 2.7. – *We have for any* $m \in \mathbb{N}$

$$\sum_{l'=0}^{l} \binom{l}{l'}^{-m} \leq 3.$$

Proof. – By an elementary calculation

$$\sum_{l'=0}^{l} \binom{l}{l'}^{-m} = 2 + \sum_{l'=1}^{l-1} \binom{l}{l'}^{-m} \leq 2 + \left(\frac{l-1}{l}\right)^{m} \leq 3.$$

Q.E.D.

LEMMA 2.8. – *We have*

$$\binom{l}{l'}\binom{m}{m'}\binom{l+m}{l'+m'}^{-1} \leq 1 \quad for \quad l' \leq l, m' \leq m.$$

Proof. – We expand both sides of the equality $(1+t)^{l+m} = (1+t)^{l}(1+t)^{m}$. Then we have

$$(1+t)^{l+m} = \sum_{k=0}^{l+m} \binom{l+m}{k} t^{k},$$

$$(1+t)^{l}(1+t)^{m} = \left(\sum_{k_1=0}^{l} \binom{l}{k_1} t^{k_1}\right)\left(\sum_{k_2=0}^{m} \binom{m}{k_2} t^{k_2}\right)$$

$$= \sum_{k_1=0}^{l} \sum_{k_2=0}^{m} \binom{l}{k_1}\binom{m}{k_2} t^{k_1+k_2} = \sum_{k=0}^{l+m} \left(\sum_{\substack{k_1+k_2=k \\ 0\leq k_1 \leq l \\ 0 \leq k_2 \leq m}} \binom{l}{k_1}\binom{m}{k_2}\right) t^{k}.$$

Hence

$$\sum_{\substack{k_1+k_2=k \\ 0\leq k_1 \leq l \\ 0 \leq k_2 \leq m}} \binom{l}{k_1}\binom{m}{k_2} = \binom{l+m}{k}.$$

Since every term of the left hand side is positive, we have the result.

Q.E.D.

LEMMA 2.9. – *We have*

$$\binom{3m}{3m_1}\binom{m}{m_1}^{-3} \leq 3^6 \frac{(\max(1,m_1))^{2/3}(\max(1,m-m_1))^{2/3}}{m^{2/3}}$$
$$for \quad m \in \mathbb{N}, 0 \leq m_1 \leq m.$$

Proof. – We first prove

$$\frac{1}{3}m^{1/3} \leq a(m)^{-1} \leq b(m)^{-1} \leq 3m^{1/3} \quad \text{for} \quad m \geq 1, \tag{2.20}$$

where

$$a(m) = \prod_{k=1}^{m} \frac{3k-1}{3k}, \quad b(m) = \prod_{k=1}^{m} \frac{3k-2}{3k}.$$

For $m = 1, 2$, it is clear that (2.20) is valid, and so we consider the case $m \geq 2$. We have

$$\log(a(m)^{-1}) = \sum_{k=1}^{m} \log\left(\frac{3k}{3k-1}\right) = \sum_{k=1}^{m} \log\left(1 + \frac{1}{3k-1}\right).$$

Since $x - x^2 \leq \log(1+x) \leq x$ for $x \geq 0$, the above equality gives

$$\sum_{k=1}^{m}\left(\frac{1}{3k-1} - \frac{1}{(3k-1)^2}\right) \leq \log(a(m)^{-1}) \leq \sum_{k=1}^{m} \frac{1}{3k-1}.$$

By an elementary calculation

$$\frac{1}{3}\int_1^m \frac{1}{x}dx \leq \sum_{k=1}^{m} \frac{1}{3k-1} \leq \frac{1}{2} + \frac{1}{3}\int_1^m \frac{1}{x}dx$$

from which it follows

$$\frac{1}{3}\log m \leq \sum_{k=1}^{m} \frac{1}{3k-1} \leq \frac{1}{2} + \frac{1}{3}\log m$$

and so

$$\log e^{-1/3}m^{1/3} \leq \log(a(m)^{-1}) \leq \log e^{1/2}m^{1/3} \quad \text{for} \quad m \geq e.$$

This implies

$$\frac{1}{3}m^{1/3} \leq a(m)^{-1} \leq 3m^{1/3} \quad \text{for} \quad m \geq 3. \tag{2.21}$$

Similarly, we have

$$\frac{1}{3}m^{1/3} \leq b(m)^{-1} \leq 3m^{1/3} \quad \text{for} \quad m \geq 3. \tag{2.22}$$

Therefore (2.20) comes from (2.21) and (2.22). We note that

$$\frac{(3m)!}{(m!)^3} = 3^{3m}a(m)b(m).$$

From this we see that if $m_1 \neq 0$, $m_1 \neq m$

$$\binom{3m}{3m_1}\binom{m}{m_1}^{-3} = \frac{(3m)!}{(m!)^3} \cdot \frac{(m_1!)^3}{(3m_1)!} \cdot \frac{((m-m_1)!)^3}{(3m-3m_1)!} \tag{2.23}$$

$$= a(m)b(m)a(m_1)^{-1}b(m_1)^{-1}a(m-m_1)^{-1}b(m-m_1)^{-1}$$
$$\leq 3^6 m^{-2/3} m_1^{2/3}(m-m_1)^{2/3}.$$

When $m_1 = 0$ or $m_1 = m$,

$$\binom{3m}{3m_1}\binom{m}{m_1}^{-3} = 1. \tag{2.24}$$

From (2.23) and (2.24) the lemma follows.

Q.E.D.

The following lemma is needed to show analyticity in space variable of solutions to (NLS) which are in a Gevrey class of order 2 in time variable, and is proved in the same way as Lemma 2.9.

LEMMA 2.10. – *We have*

$$\binom{2m}{2m_1}\binom{m}{m_1}^{-2} \leq 2^6 \frac{(\max(1,m_1))(\max(1,m-m_1))}{m^{1/2}}$$
$$\textit{for} \quad m \in \mathbb{N}, 0 \leq m_1 \leq m.$$

3. EXISTENCE OF SOLUTIONS TO (gKdV)

In this section we prove

PROPOSITION 3.1. – *Let $\phi \in G_\sigma^{A_1 A_2}(x\partial_x, \partial_x, H^3)$. Then there exist a unique solution u of* (gKdV) *and a positive constant T such that $u \in Y_T$.*

PROPOSITION 3.1'. – *In addition to the assumptions on Proposition 3.1, we assume that*

$$\phi \in G_\sigma^{A_1 A_2}(x\partial_x, \partial_x; H^{1,1}),$$
$$\epsilon_1 = \|\phi\|_{G_\sigma^{A_1 A_2}(x\partial_x, \partial_x; H^{1,1})} + \|\phi\|_{G_\sigma^{A_1 A_2}(x\partial_x, \partial_x; H^3)}$$

is sufficiently small and the nonlinear term satisfies the same growth conditions as those in Theorem 1.1'. Then there exists a unique global solution u of (gKdV) such that $u \in Z_\infty$.

In what follows we only consider positive time, since the case of negative time is treated similarly. We let u_n be the solution of

$$\begin{cases} Lu_n = \lambda u_{n-1}^{p-1} \partial_x u_{n-1} \equiv F(u_{n-1}), \\ u_n(0) = \phi \end{cases} \tag{3.1}$$

for $n \geq 1$, and u_0 be the solution of

$$Lu_0 = 0, \quad u_0(0) = \phi, \tag{3.2}$$

where $L = \partial_t + \partial_x^3$. In the same way as in the proof of [H-K.K, (3.3)] we have by induction

$$L\partial_x^j P^l \partial_x^k u_n(t) = \partial_x^j (P+3)^l \partial_x^k F(u_{n-1}(t)). \tag{3.3}$$

The integral equation associated with (3.3) is written as

$$\begin{aligned} \partial_x^j P^l \partial_x^k u_n(t) =& S(t)\partial_x^j (x\partial_x)^l \partial_x^k \phi \\ &+ \int_0^t S(t-\tau)\partial_x^j (P+3)^l \partial_x^k F(u_{n-1}(\tau)) d\tau, \end{aligned} \tag{3.4}$$

where $S(t)$ is the unitary group associated with the linear equation $Lu = 0$.
By using formula (3.4) and Lemma 2.1 we prove Propositions 3.1-3.1'.

Proof of Proposition 3.1. – It is sufficient to prove that $\{u_n\}$ is a Cauchy sequence in Y_T when T is sufficiently small. Taking L^2 norm in (3.4), multiplying both sides of the resulting inequality by $A_1^l A_2^k/(l!)^\sigma (k!)^\sigma$, making a summation with respect to l and k, and using Lemma 2.1, we obtain

$$(3.5)\quad \gamma(T,P,3,u_n)+\sigma(T,P,3,u_n) \leq C(\|\phi\|_{Y_0}+\int_0^T \|F(u_{n-1}(\tau))\|_{G_\sigma^{A_1A_2}(P+3,\partial_x;H^3)}d\tau),$$

where and in what follows we let for simplicity

$$\|\cdot\|_{Y_0}=\|\cdot\|_{G_\sigma^{A_1A_2}(x\partial_x,\partial_x;H^3)}.$$

In the same way as in the proof of (3.5) we have

$$\begin{aligned}(3.6)\quad &\gamma(T,P,0,\partial_t u_n)+\sigma(T,P,0,\partial_t u_n)\\ &\leq C\left(\|\partial_t u_n(0)\|_{Y_0}+\int_0^T \|\partial_\tau F(u_{n-1}(\tau))\|_{G_\sigma^{A_1A_2}(P+3,\partial_x;L^2)}d\tau\right)\\ &\leq C\left(\|\phi\|_{Y_0}+\|F(\phi)\|_{G_\sigma^{A_1A_2}(x\partial_x,\partial_x;L^2)}\right.\\ &\quad\left.+\int_0^T \|\partial_\tau F(u_{n-1}(\tau))\|_{G_\sigma^{A_1A_2}(P+3,\partial_x;L^2)}d\tau\right).\end{aligned}$$

We have by [H-K.K, Proposition 2.1] and Sobolev's inequality

$$(3.7)\qquad \|F(\phi)\|_{G_\sigma^{A_1A_2}(x\partial_x,\partial_x;L^2)}\cdot\leq C\|\phi\|_{Y_0}^p.$$

From (3.5)-(3.7) it follows that

$$\begin{aligned}(3.8)\quad |||u_n|||_{Y_T} &\leq C\Big\{\|\phi\|_{Y_0}(1+\|\phi\|_{Y_0}^{p-1})\\ &\quad+\int_0^T(\|F(u_{n-1}(\tau))\|_{G_\sigma^{A_1A_2}(P+3,\partial_x;H^3)}\\ &\quad+\|\partial_\tau F(u_{n-1}(\tau))\|_{G_\sigma^{A_1A_2}(P+3,\partial_x;L^2)})d\tau\Big\}\\ &\leq C\Big\{\|\phi\|_{Y_0}(1+\|\phi\|_{Y_0}^{p-1})\end{aligned}$$

$$+ \int_0^T (\|F(u_{n-1}(\tau))\|_{G_\sigma^{A_1 A_2}(P,\partial_x;H^3)}$$
$$+ \|\partial_\tau F(u_{n-1}(\tau))\|_{G_\sigma^{A_1 A_2}(P,\partial_x;L^2)}) d\tau \Big\}$$

(by [H-K.K, Lemma 2.1])

$$\leq C\{\|\phi\|_{Y_0}(1 + \|\phi\|_{Y_0}^{p-1}) + T|||u_{n-1}|||_{Y_T}^p$$
$$+ T^{1/2}|||u_{n-1}|||_{Y_T}\|\phi\|_{Y_0}^{p-1}\}$$

(by Lemma 2.5)

$$\leq C\{\|\phi\|_{Y_0}(1 + \|\phi\|_{Y_0}^{p-1})$$
$$+ (T + T^{p/2})|||u_{n-1}|||_{Y_T}^p\}.$$

From (3.2) and energy estimates it follows that

$$|||u_0|||_{Y_T} \leq \|\phi\|_{Y_0}. \tag{3.9}$$

We take ρ such that

$$C\|\phi\|_{Y_0}(1 + \|\phi\|_{Y_0}^{p-1}) \leq \frac{\rho}{2}$$

and T such that

$$C(T + T^{p/2}) \leq \frac{1}{2}$$

in the right hand side of (3.8). Then we have by (3.8) and (3.9)

$$|||u_n|||_{Y_T} \leq \rho \quad \text{for any} \quad n \in \mathbb{N}. \tag{3.10}$$

Proposition 3.1 is obtained by showing $\{u_n\}$ is a Cauchy sequence in Y_T. In the same way as in the proof of the second inequality of (3.8)

$$|||u_{n+1} - u_n|||_{Y_T} \tag{3.11}$$
$$\leq C \int_0^T (\|F(u_n) - F(u_{n-1})\|_{G_\sigma^{A_1 A_2}(P,\partial_x;H^3)}$$
$$+ \|\partial_\tau(F(u_n) - F(u_{n-1}))\|_{G_\sigma^{A_1 A_2}(P,\partial_x;L^2)}) d\tau$$
$$\leq CT|||u_n - u_{n-1}|||_{Y_T}(|||u_n|||_{Y_T}^{p-1} + |||u_{n-1}|||_{Y_T}^{p-1})$$

(by Lemma 2.6)

$$\leq CT\rho^{p-1}|||u_n - u_{n-1}|||_{Y_T}$$

(by (3.10)).

If we take T satisfying

$$CT\rho^{p-1} \leq \frac{1}{2}$$

in the right hand side of (3.11) we have by (3.10)

$$|||u_{n+1} - u_n|||_{Y_T} \leq \frac{1}{2}|||u_n - u_{n-1}|||_{Y_T} \leq \left(\frac{1}{2}\right)^n |||u_1 - u_0|||_{Y_T} \leq \left(\frac{1}{2}\right)^n \rho.$$

This means that $\{u_n\}$ is a Cauchy sequence in Y_T. This completes the proof of Proposition 3.1.

Q.E.D.

Proof of Proposition 3.1'. – In the same way as in the proof of the second inequality in (3.8) we obtain by (3.4)

$$\begin{aligned} |||u_n|||_{Y_\infty} \leq C\Bigg\{ & \|\phi\|_{Y_0}(1 + \|\phi\|_{Y_0}^{p-1}) \\ & + \int_0^\infty (\|F(u_{n-1}(t))\|_{G_\sigma^{A_1 A_2}(P,\partial_x;H^3)} \\ & + \|\partial_t F(u_{n-1}(t))\|_{G_\sigma^{A_1 A_2}(P,\partial_x;L^2)})dt \Bigg\}. \end{aligned}$$

We apply Lemma 2.5' to the above to have

$$\begin{aligned} (3.12) \quad & |||u_n|||_{Y_0} \\ & \leq C\{\|\phi\|_{Y_0}(1 + \|\phi\|_{Y_0}^{p-1}) + |||u_{n-1}|||_{Z_\infty}^p + |||u_{n-1}|||_{Z_\infty}^3 \|\phi\|_{Y_0}^{p-3}\} \\ & \leq C\{\|\phi\|_{Y_0}(1 + \|\phi\|_{Y_0}^{p-1}) + |||u_{n-1}|||_{Z_\infty}^p\} \end{aligned}$$

(by Young's inequality).

On the other hand, by Lemma 2.1

$$\|S(t)f\|_{H^{1,\infty}} \leq C(1+t)^{-1/3}(\|f\|_{H^{1,1}} + \|f\|_{H^2}).$$

Hence we have by (3.4) and [H-K.K, Propositions 2.1-2.3]

$$
\begin{aligned}
\|u_n(t)\|_{G_\sigma^{A_1A_2}(P,\partial_x;H^{1,\infty})} &\le C\epsilon_1(1+t)^{-1/3} + C\int_0^t (1+(t-\tau))^{-1/3} \\
&\times (\|F(u_{n-1}(\tau))\|_{G_\sigma^{A_1A_2}(P+3,\partial_x;H^{1,1})} \\
&\quad + \|F(u_{n-1}(\tau))\|_{G_\sigma^{A_1A_2}(P+3,\partial_x;H^{2})})d\tau \\
&\le C\Bigg(\epsilon_1(1+t)^{-1/3} + |||u_{n-1}|||_{Y_\infty}^2 \int_0^t (1+(t-\tau))^{-1/3} \\
&\quad \times \|u_{n-1}(\tau)\|^{p-2}_{G_\sigma^{A_1A_2}(P,\partial_x;H^{1,\infty})} d\tau\Bigg) \\
&\le C\left(\epsilon_1(1+t)^{-1/3} + |||u_{n-1}|||_{Y_\infty}^p \int_0^t (1+(t-\tau))^{-\frac{1}{3}}(1+\tau)^{-\frac{p-2}{3}} d\tau\right)
\end{aligned}
$$

which gives

$$
(3.13) \qquad (1+t)^{1/3}\|u_n(t)\|_{G_\sigma^{A_1A_2}(P,\partial_x;H^{1,\infty})} \le C(\epsilon_1 + |||u_{n-1}|||_{Z_\infty}^p).
$$

By (3.12) and (3.13)

$$
(3.14) \qquad |||u_n|||_{Z_\infty} \le C(\epsilon_1 + |||u_{n-1}|||_{Z_\infty}^p).
$$

In the same way as in the proof of (3.14) we have

$$
|||u_0|||_{Z_\infty} \le C\epsilon_1,
$$

this with (3.14) gives

$$
(3.15) \qquad |||u_n|||_{Z_\infty} \le C\epsilon_1, \quad \text{for any} \quad n \in \mathbb{N}.
$$

Similarly, we have by Lemma 2.6' and (3.15)

$$
(3.16) \qquad |||u_{n+1} - u_n|||_{Z_\infty} \le C\epsilon_1^{p-1}|||u_n - u_{n-1}|||_{Z_\infty} \le C\left(\frac{1}{2}\right)^n \epsilon_1.
$$

By (3.15) and (3.16) we have the result.

Q.E.D.

In the same way as in the proofs of [H-K.K, Proposition 3.3-3.4] we have by Proposition 3.1-3.1'

PROPOSITION 3.2. – *Let u be the solution of (gKdV) constructed in Proposition* 3.1. *Then*

$$u \in G_\sigma^{|t|A_3}(\partial_t; H^3(-R,R)) \quad for \quad |t| \leq T,$$

where

$$A_3 \leq \min\left\{\frac{3A_1}{1+3A_1}, \frac{3A_2}{4A_2 + e^{1/3}(1+R)}\right\}.$$

PROPOSITION 3.2'. – *Let u be the solution of (gKdV) constructed in Proposition* 3.1'. *Then*

$$u \in G_\sigma^{|t|A_3}(\partial_t; H^3(-R,R)) \quad for\ any \quad t \in \mathbb{R},$$

and

$$\|u(t)\|_{G_\sigma^{|t|A_3}(\partial_t; H^{1,\infty}(-R,R))} \leq C\epsilon_1(1+|t|)^{-1/3} \quad for\ any \quad t \in \mathbb{R},$$

where R and A_3 are the same as in Proposition 3.2.

4. PROOFS OF THEOREMS 1.1-1.1' (gKdV)

By Propositions 3.1, 3.1', 3.2, 3.2', to obtain Theorems 1.1,1.1' it is sufficient to prove

PROPOSITION 4.1. – *Let u be the solution of (gKdV) satisfying*

$$u \in G_\sigma^{|t|A_3}(\partial_t, H^3(-R,R)).$$

Then there exists a positive constant A_4 such that

$$u \in G_{\max(\sigma/3,1)}^{|t|^{1/3}A_4}(\partial_x; H^3(-R,R)).$$

Proof of Proposition 4.1. – We divide the proposition into the following two lemmas.

LEMMA 4.1. – *Let u be the solution of (gKdV) satisfying*

$$u \in G_\sigma^{|t|A_3}(\partial_t, H^3(-R,R)).$$

Then there exists a positive constant A_5 such that

$$u \in G_{\max(\sigma,3)}^{|t|A_5}(\partial_x^3; H^3(-R,R)).$$

LEMMA 4.2. – *Let u be a function satisfying*

$$u \in G^{|t|A_5}_{\max(\sigma,3)}(\partial_x^3; H^3(-R,R)).$$

Then there exists a positive constant A_4 such that

$$u \in G^{|t|^{1/3}A_4}_{\max(\sigma/3,1)}(\partial_x; H^3(-R,R)).$$

Proof of Lemma 4.1. – It suffices to prove the result for $t = 1$. By the assumption we have

$$C_0 = \sum_{l=0}^{\infty} \frac{A_3^l}{(l!)^\sigma} \|\partial_t^l u\|_{H^3(-R,R)} < \infty$$

which implies

$$\frac{A_3^l}{(l!)^\sigma} \|\partial_t^l u\|_{H^3(-R,R)} \le C_0. \tag{4.1}$$

We prove by induction with respect to m that there exists a positive constant A_6 such that

$$\frac{A_3^{l+m}}{((l+m)!)^{\sigma'}} \|\partial_t^l \partial_x^{3m} u\|_{H^1(-R,R)} \le C_0 A_6^m (\max(1,m))^{-4/3}, \tag{4.2}$$

$$\frac{A_3^{l+m}}{((l+m)!)^{\sigma'}} \|\partial_t^l \partial_x^{3m+k} u\|_{H^1(-R,R)} \le A C_0 A_6^m, \tag{4.3}$$

for $l, m \in \mathbb{N} \cup \{0\}$, $k = 1, 2$, where $\sigma' = \max(\sigma, 3)$, A is a positive constant determined later. If (4.2) is valid, then taking $l = 0$ in (4.2) and (4.3) we have Lemma 4.1 with

$$A_5 = \frac{A_3}{A_6} - \epsilon, \quad (\epsilon > 0).$$

It is clear that (4.2) and (4.3) hold for all l and $k = 1, 2$ when $m = 0$. We assume that (4.2) and (4.3) are valid for all l, m and $k = 1, 2$. For simplicity we denote

$$\|\cdot\|_{H^1(-R,R)} = \|\cdot\|_1$$

until the end of of the proof of Proposition 4.1.

Since

$$\partial_x^3 u = -\partial_t u - \lambda(\partial_x(u^p))/p,$$

we have

$$\partial_t^l M^{m+1}\partial_x^k u = -\partial_t^{l+1} M^m \partial_x^k u - \frac{\lambda}{p}\partial_t^l M^m \partial_x^{k+1}(u^p),$$

where and in what follows

$$M = \partial_x^3.$$

Hence

$$\text{(4.4)} \quad \begin{aligned} \|\partial_t^l M^{m+1}\partial_x^k u\|_1 &\le \|\partial_t^{l+1} M^m \partial_x^k u\|_1 + |\lambda| \|\partial_t^l M^m \partial_x^k(u^{p-1}u_x)\|_1 \\ &\equiv I_{1,k} + |\lambda| I_{2,k}. \end{aligned}$$

The crucial term is $I_{2,k}$ because it is easy to handle $I_{1,k}$ by the induction assumption. Indeed we have

$$\text{(4.5)} \quad I_{1,k}\cdot\frac{A_3^{l+m+1}}{((l+m+1)!)^{\sigma'}} \le \begin{cases} C_0 A_6^m(\max(1,m))^{-4/3} & \text{for} \quad k=0, \\ AC_0A_6^m \quad \text{for} \quad k=1,2. \end{cases}$$

We shall prove that

$$\text{(4.6)} \quad |\lambda| I_{2,k}\cdot\frac{A_3^{l+m+1}}{((l+m+1)!)^{\sigma'}} \le \begin{cases} \frac{1}{2}C_0A_6^{m+1}(1+m)^{-4/3} & \text{for} \quad k=0, \\ \frac{1}{2}AC_0A_6^{m+1} \quad \text{for} \quad k=1,2. \end{cases}$$

For simplicity, we show (4.6) in the case $p=2$, since the general case $p \ge 3$ can be proved by induction. We have by Sobolev's inequality

$$\|\cdot\|_{L^\infty(-R,R)} \le \frac{C}{4}\|\cdot\|_{H^1(-R,R)}$$

$$\text{(4.7)} \qquad I_{2,k} = \|\partial_t^l \partial_x^{3m+k}(u\cdot u_x)\|_1$$

$$\le C\sum_{l'=0}^{l}\sum_{m'=0}^{3m+k}\binom{l}{l'}\binom{3m+k}{m'}\|\partial_t^{l'}\partial_x^{m'}u\|_1\|\partial_t^{l-l'}\partial_x^{3m+k+1-m'}u\|_1.$$

First we treat the term $I_{2,0}$. We have by (4.7)

$$\text{(4.8)}\quad \frac{1}{C}I_{2,0} \leq \sum_{l'=0}^{l}\sum_{m'=0}^{3m}\binom{l}{l'}\binom{3m}{m'}\|\partial_t^{l'}\partial_x^{m'}u\|_1\|\partial_t^{l-l'}\partial_x^{3m+1-m'}u\|_1$$

$$=\sum_{l'=0}^{l}\sum_{m_1=0}^{m}\binom{l}{l'}\binom{3m}{3m_1}\|\partial_t^{l'}M^{m_1}u\|_1\|\partial_t^{l-l'}M^{m-m_1}\partial_x u\|_1$$

$$+\sum_{l'=0}^{l}\sum_{m_1=0}^{m-1}\binom{l}{l'}\binom{3m}{3m_1+1}\|\partial_t^{l'}M^{m_1}\partial_x u\|_1\|\partial_t^{l-l'}M^{m-m_1}u\|_1$$

$$+\sum_{l'=0}^{l}\sum_{m_1=0}^{m-1}\binom{l}{l'}\binom{3m}{3m_1+2}\|\partial_t^{l'}M^{m_1}\partial_x^2 u\|_1\|\partial_t^{l-l'}M^{m-m_1-1}\partial_x^2 u\|_1$$

$$\equiv I_{2,0,1}+I_{2,0,2}+I_{2,0,3}.$$

Hence

$$\text{(4.9)}\qquad I_{2,0,1}\cdot\frac{A_3^{l+m+1}}{((l+m+1)!)^{\sigma'}}$$

$$\leq \sum_{l'=0}^{l}\sum_{m_1=0}^{m}\binom{l}{l'}\binom{3m}{3m_1}\binom{l+m}{l'+m_1}^{-\sigma'}\cdot\frac{A_3}{(l+m+1)^{\sigma'}}$$

$$\times\frac{A_3^{l'+m_1}}{((l'+m_1)!)^{\sigma'}}\|\partial_t^{l'}M^{m_1}u\|_1\frac{A_3^{l+m-l'-m_1}}{((l+m-l'-m_1)!)^{\sigma'}}\|\partial_t^{l-l'}M^{m-m_1}\partial_x u\|_1$$

$$\leq \sum_{l'=0}^{l}\sum_{m_1=0}^{m}\binom{l}{l'}^{1-\sigma'}\binom{3m}{3m_1}\binom{m}{m_1}^{-\sigma'}\left[\binom{l}{l'}\binom{m}{m_1}\binom{l+m}{l'+m_1}^{-1}\right]^{\sigma'}$$

$$\times\frac{A_3}{(l+m+1)^{\sigma'}}C_0A_6^{m_1}(\max(1,m_1))^{-4/3}AC_0A_6^{m-m_1}$$

(by the induction assumption)

$$\leq 3\sum_{m_1=0}^{m}\binom{3m}{3m_1}\binom{m}{m_1}^{-3}\frac{A_3}{(1+m)^3}AC_0^2A_6^{m}(\max(1,m_1))^{-4/3}$$

(by Lemmas 2.7-2.8)

$$\leq 3^7 \sum_{m_1=0}^{m} \frac{(\max(1,m_1))^{2/3}(\max(1,m-m_1))^{2/3}}{m^{2/3}} \frac{A_3}{(1+m)^3} \times AC_0^2 A_6^m (\max(1,m_1))^{-4/3} \quad \text{(by Lemma 2.9)}$$

$$\leq 3^7 AC_0^2 \sum_{m_1=0}^{m} \frac{(\max(1,m_1))^{-2/3}(\max(1,m-m_1))^{2/3}}{m^{2/3}} \frac{A_3}{(1+m)^3} A_6^m$$

$$\leq 3^7 AC_0^2 \frac{A_3}{(1+m)^2} A_6^m.$$

In the same way as in the proof of (4.9) we have

$$I_{2,0,2} \cdot \frac{A_3^{l+m+1}}{((l+m+1)!)^{\sigma'}} \tag{4.10}$$

$$\leq \sum_{l'=0}^{l} \sum_{m_1=0}^{m-1} \binom{l}{l'}^{1-\sigma'} \binom{3m}{3m_1+1} \binom{m}{m_1}^{-\sigma'} \left[\binom{l}{l'} \binom{m}{m_1} \binom{l+m}{l'+m_1}^{-1} \right]^{\sigma'} \times \frac{A_3}{(l+m+1)^{\sigma'}} AC_0 A_6^{m_1} (\max(1,m_1))^{-4/3} C_0 A_6^{m-m_1}$$

(by the induction assumption)

$$\leq 3 \sum_{m_1=0}^{m-1} \binom{3m}{3m_1+1} \binom{m}{m_1}^{-3} \frac{A_3}{(1+m)^3} AC_0^2 A_6^m (\max(1,m_1))^{-4/3}$$

(by Lemmas 2.7-2.8)

$$= 3 \sum_{m_1=0}^{m-1} \binom{3m}{3m_1} \binom{m}{m_1}^{-3} \frac{3(m-m_1)}{3m_1+1} \frac{A_3}{(1+m)^3} AC_0^2 A_6^m (\max(1,m_1))^{-4/3}$$

$$\leq 3^8 A C_0^2 \sum_{m_1=0}^{m-1} \frac{(\max(1,m_1))^{2/3}(\max(1,m-m_1))^{-1/3}}{m^{2/3}} \times \frac{A_3}{(1+m)^3} A_6^m \frac{1}{3m_1+1}$$

(by Lemma 2.9)

$$\leq 3^8 A C_0^2 \tfrac{A_3}{(1+m)^2} A_6^m,$$

and we also have

$$(4.11) \qquad I_{2,0,3} \cdot \frac{A_3^{l+m+1}}{((l+m+1)!)^{\sigma'}}$$
$$\leq \sum_{l'=0}^{l} \sum_{m_1=0}^{m-1} \binom{l}{l'} \binom{3m}{3m_1+2} \binom{l+m}{l'+m_1}^{-\sigma'}$$
$$\times \frac{A_3}{(l+m-l'-m_1)^{\sigma'}} \cdot \frac{A_3}{(l+m+1)^{\sigma'}}$$
$$\times \frac{A_3^{l'+m_1}}{((l'+m_1)!)^{\sigma'}} \|\partial_t^{l'} M^{m_1} \partial_x^2 u\|_1$$
$$\times \frac{A_3^{l+m-l'-m_1-1}}{((l+m-l'-m_1-1)!)^{\sigma'}} \|\partial_t^{l-l'} M^{m-m_1-1} \partial_x^2 u\|_1$$
$$\leq \sum_{l'=0}^{l} \sum_{m_1=0}^{m-1} \binom{l}{l'}^{1-\sigma'} \binom{3m}{3m_1+2} \binom{m}{m_1}^{-\sigma'}$$
$$\times \left[\binom{l}{l'}\binom{m}{m_1}\binom{l+m}{l'+m_1}^{-1}\right]^{\sigma'}$$
$$\times \frac{A_3}{(l+m-l'-m_1)^{\sigma'}} \cdot \frac{A_3}{(l+m+1)^{\sigma'}} A^2 C_0^2 A_6^{m-1}$$

(by the induction assumption)

$$\leq 3 \sum_{m_1=0}^{m-1} \binom{3m}{3m_1+2} \binom{m}{m_1}^{-3} \frac{A_3^2}{(1+m)^3(m-m_1)^3} A^2 C_0^2 A_6^{m-1}$$

(by Lemmas 2.7-2.8)

$$= 3 \sum_{m_1=0}^{m-1} \binom{3m}{3m_1} \binom{m}{m_1}^{-3}$$
$$\times \frac{3(m-m_1)(3m-3m_1-1)}{(3m_1+1)(3m_1+2)} \frac{A_3^2 A^2 C_0^2 A_6^{m-1}}{(1+m)^3(m-m_1)^3}$$
$$\leq 3^9 A^2 C_0^2 \frac{A_3^2}{(1+m)^2} A_6^{m-1} \quad \text{(by Lemma 2.9).}$$

Consequently we have

$$\frac{1}{C} I_{2,0} \cdot \frac{A_3^{l+m+1}}{((l+m+1)!)^{\sigma'}}$$
$$\leq (2 \cdot 3^8 A C_0^2 A_3 A_6^{-1} + 3^9 A^2 C_0^2 A_3^2 A_6^{-2}) \frac{1}{(m+1)^2} A_6^{m+1}.$$

If we take

$$\tilde{C} = C(2 \cdot 3^8 A C_0^2 A_3 A_6^{-1} + 3^9 A^2 C_0^2 A_3^2 A_6^{-2}).$$

Then

$$|\lambda| I_{2,0} \cdot \frac{A_3^{l+m+1}}{((l+m+1)!)^{\sigma'}} \leq |\lambda| \tilde{C} A_6^{m+1} (1+m)^{-4/3} \tag{4.11}$$

which implies (4.6) for $k = 0$ if

$$\frac{1}{4} A_6 = \max\{|\lambda| C 2 \cdot 3^8 C_0 A A_3, (|\lambda| C \cdot 3^9 C_0)^{1/2} A A_3\} \tag{4.12}$$

Next we treat the case $k = 1$. We have by (4.7)

$$\frac{1}{C} I_{2,1} \tag{4.13}$$
$$\leq \sum_{l'=0}^{l} \sum_{m'=0}^{3m+1} \binom{l}{l'} \binom{3m+1}{m'} \|\partial_t^{l'} \partial_x^{m'} u\|_1 \|\partial_t^{l-l'} \partial_x^{3m+2-m'} u\|_1$$
$$= \sum_{l'=0}^{l} \sum_{m_1=0}^{m} \binom{l}{l'} \binom{3m+1}{3m_1} \|\partial_t^{l'} M^{m_1} u\|_1 \|\partial_t^{l-l'} M^{m-m_1} \partial_x^2 u\|_1$$
$$+ \sum_{l'=0}^{l} \sum_{m_1=0}^{m} \binom{l}{l'} \binom{3m+1}{3m_1+1} \|\partial_t^{l'} M^{m_1} \partial_x u\|_1 \|\partial_t^{l-l'} M^{m-m_1} \partial_x u\|_1$$
$$+ \sum_{l'=0}^{l} \sum_{m_1=0}^{m-1} \binom{l}{l'} \binom{3m+1}{3m_1+2} \|\partial_t^{l'} M^{m_1} \partial_x^2 u\|_1 \|\partial_t^{l-l'} M^{m-m_1} u\|_1$$
$$\equiv I_{2,1,1} + I_{2,1,2} + I_{2,1,3}.$$

In the same way as in the proofs of (4.9)-(4.11) we have

$$(4.14)\qquad I_{2,1,1}\cdot\frac{A_3^{l+m+1}}{((l+m+1)!)^{\sigma'}}$$

$$\leq 3\sum_{m_1=0}^{m}\binom{3m}{3m_1}\binom{m}{m_1}^{-3}AC_0^2A_6^mA_3$$
$$\times\frac{1}{(m+1)^3}(\max(1,m_1))^{-4/3}\frac{3m+1}{3m-3m_1+1}$$
$$\leq 3^8AC_0^2A_3A_6^m,$$

$$(4.15)\qquad I_{2,1,2}\cdot\frac{A_3^{l+m+1}}{((l+m+1)!)^{\sigma'}}$$

$$\leq 3\sum_{m_1=0}^{m}\binom{3m}{3m_1}\binom{m}{m_1}^{-3}A^2C_0^2A_6^mA_3\frac{1}{(m+1)^3}\frac{3m+1}{3m_1+1}$$
$$\leq 3^8A^2C_0^2A_3A_6^m,$$

$$(4.16)\qquad I_{2,1,3}\cdot\frac{A_3^{l+m+1}}{((l+m+1)!)^{\sigma'}}$$

$$\leq 3\sum_{m_1=0}^{m-1}\binom{3m}{3m_1}\binom{m}{m_1}^{-3}$$
$$\times\frac{AC_0^2A_6^mA_3}{(m+1)^3}(\max(1,m-m_1))^{-4/3}\frac{3m+1}{3m_1+1}\frac{3m-3m_1}{3m_1+2}$$
$$\leq 3^9AC_0^2A_3A_6^m.$$

From (4.13)-(4.16), (4.6) follows for $k=1$ if

$$(4.17)\qquad \frac{1}{4}A_6=|\lambda|3^8CC_0A_3\max\{6,A\}.$$

Finally we treat the case $k = 2$. We have by (4.7)

$$(4.18)\quad \frac{1}{C}I_{2,2}$$

$$\begin{aligned}
&\le \sum_{l'=0}^{l}\sum_{m'=0}^{3m+2}\binom{l}{l'}\binom{3m+2}{m'}\|\partial_t^{l'}\partial_x^{m'}u\|_1\|\partial_t^{l-l'}\partial_x^{3m+2+1-m'}u\|_1\\
&=\sum_{l'=0}^{l}\sum_{m_1=0}^{m}\binom{l}{l'}\binom{3m+2}{3m_1}\|\partial_t^{l'}M^{m_1}u\|_1\|\partial_t^{l-l'}M^{m+1-m_1}u\|_1\\
&\quad+\sum_{l'=0}^{l}\sum_{m_1=0}^{m}\binom{l}{l'}\binom{3m+2}{3m_1+1}\|\partial_t^{l'}M^{m_1}\partial_x u\|_1\|\partial_t^{l-l'}M^{m-m_1}\partial_x^2 u\|_1\\
&\quad+\sum_{l'=0}^{l}\sum_{m_1=0}^{m}\binom{l}{l'}\binom{3m+2}{3m_1+2}\|\partial_t^{l'}M^{m_1}\partial_x^2 u\|_1\|\partial_t^{l-l'}M^{m-m_1}\partial_x u\|_1\\
&\equiv I_{2,2,1}+I_{2,2,2}+I_{2,2,3}.
\end{aligned}$$

In the same way as in the proof of (4.9) we have by (4.18)

$$(4.19)\, I_{2,2,1}\cdot\frac{A_3^{l+m+1}}{((l+m+1)!)^{\sigma'}}$$

$$\begin{aligned}
&\le \sum_{l'=0}^{l}\sum_{m_1=0}^{m}\binom{l}{l'}\binom{3m+2}{3m_1}\binom{l+m+1}{l'+m_1}^{-\sigma'}\\
&\quad\times\frac{A_3^{l'+m_1}}{((l'+m_1)!)^{\sigma'}}\|\partial_t^{l'}M^{m_1}u\|_1\\
&\quad\times\frac{A_3^{l+m+1-l'-m_1}}{((l+m+1-l'-m_1)!)^{\sigma'}}\|\partial_t^{l-l'}M^{m+1-m_1}u\|_1\\
&\le \sum_{l'=0}^{l}\sum_{m_1=0}^{m}\binom{l}{l'}^{1-\sigma'}\binom{3m+2}{3m_1}\binom{m+1}{m_1}^{-\sigma'}\\
&\quad\times\left[\binom{l}{l'}\binom{m+1}{m_1}\binom{l+m+1}{l'+m_1}^{-1}\right]^{\sigma'}\\
&\quad\times C_0A_6^{m_1}(\max(1,m_1))^{-4/3}C_0A_6^{m+1-m_1}(\max(1,m+1-m_1))^{-4/3}
\end{aligned}$$

(by the induction assumption)

$$\leq 3 \sum_{m_1=0}^{m} \binom{3m+2}{3m_1} \binom{m+1}{m_1}^{-3} C_0^2 A_6^{m+1}$$
$$\times (\max(1, m_1))^{-4/3} (\max(1, m+1-m_1))^{-4/3}$$

(by Lemmas 2.7-2.8)

$$= 3 \sum_{m_1=0}^{m} \binom{3(m+1)}{3m_1} \binom{m+1}{m_1}^{-3}$$
$$\times \frac{3(m-m_1+1)}{3(m+1)} C_0^2 A_6^{m+1} (\max(1, m_1))^{-4/3}$$
$$\times (\max(1, m+1-m_1))^{-4/3}$$

$$\leq 3^7 \sum_{m_1=0}^{m} \frac{(\max(1, m_1))^{2/3} (m+1-m_1)^{2/3}}{(m+1)^{2/3}}$$
$$\times C_0^2 A_6^{m+1} (\max(1, m_1))^{-4/3} (m+1-m_1)^{-4/3} \qquad \text{(by Lemma 2.9)}$$

$$= 3^7 C_0^2 (m+1)^{-2/3} A_6^{m+1} \sum_{m_1=0}^{m} (\max(1, m_1))^{-2/3} (m+1-m_1)^{-2/3}.$$

Using Schwarz' inequality, we have

$$\sum_{m_1=0}^{m} (\max(1, m_1))^{-2/3} (m+1-m_1)^{-2/3}$$
$$= (m+1)^{-2/3} + \sum_{m_1=1}^{m} m_1^{-2/3} (m+1-m_1)^{-2/3}$$
$$\leq (m+1)^{-2/3} + \left(\sum_{m_1=1}^{m} m_1^{-4/3} \right)^{1/2} \left(\sum_{m_1=1}^{m} m_1^{-4/3} \right)^{1/2} \leq C_1.$$

Hence we have by (4.19)

$$I_{2,2,1} \cdot \frac{A_3^{l+m+1}}{((l+m+1)!)^{\sigma'}} \leq 3^7 C_0^2 C_1 A_6^{m+1}. \tag{4.20}$$

In the same way as in the proofs of (4.10)-(4.11) we have by (4.18)

$$(4.21)\quad I_{2,2,2}\cdot\frac{A_3^{l+m+1}}{((l+m+1)!)^{\sigma'}}$$

$$\leq 3\sum_{m_1=0}^{m}\binom{3m}{3m_1}\binom{m}{m_1}^{-3}$$
$$\times A^2C_0^2A_6^mA_3\frac{(3m+2)(3m+1)}{(3m_1+1)(3m-3m_1+1)}\frac{1}{(m+1)^3}$$
$$\leq 3^9A^2C_0^2A_3A_6^m,$$

$$(4.22)\quad I_{2,2,3}\cdot\frac{A_3^{l+m+1}}{((l+m+1)!)^{\sigma'}}$$

$$\leq 3\sum_{m_1=0}^{m}\binom{3m}{3m_1}\binom{m}{m_1}^{-3}A^2C_0^2A_6^mA_3\frac{(3m+2)(3m+1)}{(3m_1+2)(3m_1+1)}\frac{1}{(m+1)^3}$$
$$\leq 3^9A^2C_0^2A_3A_6^m,$$

Hence (4.20)-(4.22) imply that (4.6) holds true for $k=2$ if

$$(4.23)\qquad \frac{1}{4}A_6=2|\lambda|C3^9AC_0A_3\quad\text{and}\quad 3^7CC_0C_1|\lambda|=\frac{A}{4}.$$

From (4.9)-(4.11), (4.14)-(4.16), (4.20)-(4.22), and (4.12), (4.17), (4.23) it follows that (4.6) holds for

$$(4.24)\qquad A_6=2(1+2^43^{18}|\lambda|^2C^2C_0^2C_1)A_3.$$

From (4.5) and (4.6) we have (4.2) and (4.3) under the condition

$$(4.25)\qquad A_5<\frac{A_3}{2(1+(1+2^43^{18}|\lambda|^2C^2C_0^2C_1)A_3)}.$$

This completes the proof of Lemma 4.1.

Q.E.D.

Proof of Lemma 4.2. – For simplicity we denote $\|\cdot\|_3$ by $\|\cdot\|_{H^3(-R,R)}$. Since

$$\frac{3^{3m}}{(3m+2)!}\leq\frac{1}{(m!)^3},$$

we have

$$\sum_{m=0}^{\infty} \frac{(|t|^{1/3}A_4)^m}{(m!)^{\sigma'/3}} \|\partial_x^m u\|_3$$

$$= \sum_{m=0}^{\infty} \Big(\frac{(|t|^{1/3}A_4)^{3m}}{((3m)!)^{\sigma'/3}} \|\partial_x^{3m} u\|_3 + \frac{(|t|^{1/3}A_4)^{3m+1}}{((3m+1)!)^{\sigma'/3}} \|\partial_x^{3m+1} u\|_3$$

$$+ \frac{(|t|^{1/3}A_4)^{3m+2}}{((3m+2)!)^{\sigma'/3}} \|\partial_x^{3m+2} u\|_3 \Big) \quad (\sigma' = \max(\sigma, 3))$$

$$\leq \sum_{m=0}^{\infty} \Bigg(((3m+2)(3m+1))^{\sigma'/3} \cdot \left(\frac{|t|^{1/3}A_4}{3} \right)^{3m} \frac{1}{(m!)^{\sigma'}} \|\partial_x^{3m} u\|_3$$

$$+ 3(3m+2)^{\sigma'/3} \cdot \left(\frac{|t|^{1/3}A_4}{3} \right)^{3m+1} \frac{1}{(m!)^{\sigma'}} \|\partial_x^{3m+1} u\|_3$$

$$+ 3^2 \cdot \left(\frac{|t|^{1/3}A_4}{3} \right)^{3m+2} \frac{1}{(m!)^{\sigma'}} \|\partial_x^{3m+2} u\|_3 \Bigg)$$

$$\leq \text{Const} \cdot \sum_{m=0}^{\infty} \Bigg(\left(\frac{|t|^{1/3}A_4'}{3} \right)^{3m} \frac{1}{(m!)^{\sigma'}} \|\partial_x^{3m} u\|_3$$

$$+ \left(\frac{|t|^{1/3}A_4'}{3} \right)^{3m+1} \frac{1}{(m!)^{\sigma'}} \|\partial_x^{3m+1} u\|_3$$

$$+ \left(\frac{|t|^{1/3}A_4}{3} \right)^{3m+2} \frac{1}{(m!)^{\sigma'}} \|\partial_x^{3m+2} u\|_3 \Bigg) \quad \text{for} \quad A_4' > A_4.$$

$$\leq \text{Const} \cdot \sum_{m=0}^{\infty} \Bigg(\left(1 + \frac{|t|^{1/3}A_4'}{3} + \left(\frac{|t|^{1/3}A_4'}{3} \right)^2 \right) \frac{(|t|A_5')^m}{(m!)^{\sigma'}} \|\partial_x^{3m} u\|_3$$

$$+ \left(\frac{|t|^{1/3}A_4'}{3} + \left(\frac{|t|^{1/3}A_4'}{3} \right)^2 \right) \frac{(|t|A_5')^m}{(m!)^{\sigma'}} \|\partial_x^{3(m+1)} u\|_3 \Bigg) \quad \text{for } A_5' > (A_4'/3)^3.$$

Therefore we have

$$\sum_{m=0}^{\infty} \frac{(|t|^{1/3}A_4)^m}{(m!)^{\sigma'/3}} \|\partial_x^m u\|_3$$

$$\leq \text{Const} \cdot (1 + |t|^{1/3} + |t|^{2/3}) \sum_{m=0}^{\infty} \frac{(|t|A_5)^m}{(m!)^{\sigma'}} \|\partial_x^{3m} u\|_3 < \infty$$

$$\text{for} \quad A_5 > A_5'.$$

Hence, if we take A_4 such that

$$A_4 < 3A_5^{1/3}, \tag{4.26}$$

we have Lemma 4.2. We note here that by (4.25) and (4.26) , when $p = 2$, it is sufficient to take A_4 satisfying

$$A_4 < 3\left(\frac{A_3}{2(1+(1+2^4\cdot 3^{18}|\lambda|^2C^2C_0^2C_1)A_3)}\right)^{1/3}$$

to obtain the result.

Q.E.D.

5. PROOFS OF THEOREMS 1.2-1.2' (NLS)

The following two propositions were proved in [H-K.K, Propositions 3.3-3.4].

PROPOSITION 5.1. – *We assume that* $\sigma \geq 1$ *and*

$$\psi \in G_\sigma^{A_1A_2}(x\partial_x, \partial_x; H^2).$$

Then there exist A_3, T *and a unique solution* u *of (NLS) such that*

$$u \in C([-T,T]; H^2),$$

$$u \in G_\sigma^{|t|A_3}(\partial_t; H^2(-R,R)) \quad for \quad t \in [-T,T],$$

where

$$A_3 < \min\left\{\frac{2A_1}{1+2A_1}, \frac{2A_2}{3A_2+e^{1/2}(1+R)}\right\}.$$

PROPOSITION 5.2. – *We assume that* $\sigma \geq 1$ *and*

$$\psi \in G_\sigma^{A_1A_2}(x\partial_x, \partial_x; H^2) \cap G_\sigma^{A_1A_2}(x\partial_x, \partial_x; L^1).$$

Furthermore we assume that

$$\epsilon_1 = \|\psi\|_{G_\sigma^{A_1A_2}(x\partial_x,\partial_x;H^2)} + \|\psi\|_{G_\sigma^{A_1A_2}(x\partial_x,\partial_x;L^1)}$$

is sufficiently small and $G(u, \bar{u})$ *satisfies the following growth condition*

$$|G(s, \bar{s})| \leq \text{Const.}|s|^p \quad for \quad |s| < 1,$$

where p is an integer satisfying $p \geq 5$. Then there exist A_3 and a unique solution u of (NLS) such that

$$u \in C(\mathbb{R}; H^2),$$

$$u \in G_\sigma^{|t|A_3}(\partial_t; H^2(-R,R)) \quad for\ any \quad t \in \mathbb{R},$$

$$\|u(t)\|_{G_\sigma^{A_1 A_2}(x\partial_x, \partial_x; L^\infty(-R,R))} \leq \text{Const.}\epsilon_1 (1+|t|)^{-1/2} \quad for\ any \quad t \in \mathbb{R},$$

where

$$A_3 < \min\left\{\frac{2A_1}{1+2A_1}, \frac{2A_2}{3A_2 + e^{1/2}(1+R)}\right\}.$$

We now prove Theorems 1.2 and 1.2'.

Proofs of Theorems 1.2-1.2'. – From Proposition 5.1-5.2 it is sufficient to prove that there exists a positive constant A_4 such that

$$u \in G^{|t|^{1/2} A_4}_{\max(\sigma/2,1)}(\partial_x; H^2(-R,R)). \tag{5.1}$$

The proof is obtained in the same way as in the proof of Proposition 4.1, and so we only give the outline. We first prove there exists a positive constant A_5 such that

$$u \in G^{|t|A_5}_{\max(\sigma,2)}(\partial_x^2; H^2(-R,R)). \tag{5.2}$$

It suffices to show it for $t = 1$. By Propositions 5.1-5.2 we have

$$C_0 = \sum_{l=0}^{\infty} \frac{A_3^l}{(l!)^\sigma} \|\partial_t^l u\|_{H^2(-R,R)} < \infty,$$

which implies

$$\frac{A_3^l}{(l!)^\sigma} \|\partial_t^l u\|_{H^2(-R,R)} \leq C_0. \tag{5.3}$$

We prove by induction with respect to m that there exists a positive constant A_6 such that

$$\frac{A_3^{l+m}}{((l+m)!)^{\sigma'}} \|\partial_t^l \partial_x^{2m} \partial_x^k u\|_{H^1(-R,R)} \leq C_0 A_6^m, \quad \text{for} \quad k = 0, 1 \tag{5.4}$$

for $l, m \in \mathbb{N} \cup \{0\}$, $\sigma' = \max(\sigma, 2)$. If (5.4) is valid, then taking $l = 0$, we get (5.2) with

$$A_5 = \frac{A_3}{A_6} - \epsilon \quad (\epsilon > 0)$$

In what follows for simplicity we let

$$\|\cdot\|_{H^1(-R,R)} = \|\cdot\|_1 \quad \text{and} \quad M = \partial_x^2.$$

It is clear that (5.4) holds true for all l when $m = 0$ by (5.3). We assume that (5.4) is valid for all l, m and $k = 0, 1$. We only consider the case

$$\text{(NLS2)} \qquad i\partial_t u + \frac{1}{2}\Delta u = u^2,$$

since the general nonlinearity can be treated by induction. We have by an elementary calculation and Sobolev's inequality

$$\text{(5.5)} \quad I_{2,0} \equiv \|\partial_t^l \partial_x^{2m}(u^2)\|_1$$

$$\leq \sum_{l'=0}^{l} \sum_{m'}^{2m} \binom{l}{l'} \binom{2m}{m'} \|\partial_t^{l'} \partial_x^{m'} u \cdot \partial_t^{l-l'} \partial_x^{2m-m'} u\|_1$$

$$= \sum_{l'=0}^{l} \sum_{m_1=0}^{m} \binom{l}{l'} \binom{2m}{2m_1} \|\partial_t^{l'} M^{m_1} u \cdot \partial_t^{l-l'} M^{m-m_1} u\|_1$$

$$+ \sum_{l'=0}^{l} \sum_{m_1=0}^{m-1} \binom{l}{l'} \binom{2m}{2m_1+1} \|\partial_t^{l'} M^{m_1} \partial_x u \cdot \partial_t^{l-l'} M^{m-m_1-1} \partial_x u\|_1$$

$$\leq C \sum_{l'=0}^{l} \sum_{m_1=0}^{m} \binom{l}{l'} \binom{2m}{2m_1} \|\partial_t^{l'} M^{m_1} u\|_1 \|\partial_t^{l-l'} M^{m-m_1} u\|_1$$

$$+ C \sum_{l'=0}^{l} \sum_{m_1=0}^{m-1} \binom{l}{l'} \binom{2m}{2m_1+1} \|\partial_t^{l'} M^{m_1} \partial_x u\|_1 \|\partial_t^{l-l'} M^{m-m_1-1} \partial_x u\|_1$$

$$\equiv C(I_{2,0,1} + I_{2,0,2}).$$

In the same way as in the proof of (4.9)

$$(5.6)\quad I_{2,0,1}\frac{A_3^{l+m+1}}{((l+m+1)!)^{\sigma'}}$$

$$\leq 3\sum_{m_1=0}^{m}\binom{2m}{2m_1}\binom{m}{m_1}^{-2}\frac{C_0^2A_3A_6^m}{(m+1)^2}$$

$$\leq 3\cdot 2^6\sum_{m_1=0}^{m}\frac{(\max(1,m_1))(\max(1,m-m_1))}{m}\cdot\frac{C_0^2A_3A_6^m}{(m+1)^2}$$

(by Lemma 2.10)

$$\leq 3\cdot 2^6C_0^2A_3A_6^m,$$

and in the same way as in the proof of (4.11) we have

$$(5.7)\quad I_{2,0,2}\frac{A_3^{l+m+1}}{((l+m+1)!)^{\sigma'}}$$

$$\leq 3\sum_{m_1=0}^{m-1}\binom{2m}{2m_1+1}\binom{m}{m_1}^{-2}\frac{C_0^2A_3^2A_6^{m-1}}{(m+1)^2(m-m_1)^2}$$

$$=3\sum_{m_1=0}^{m-1}\binom{2m}{2m_1}\binom{m}{m_1}^{-2}\frac{2(m-m_1)}{2m_1+1}\frac{C_0^2A_3^2A_6^{m-1}}{(m+1)^2(m-m_1)^2}$$

$$\leq 3\cdot 2^7C_0^2A_3A_6^m\quad\text{(by Lemma 2.10).}$$

We also have (see (4.13))

$$\frac{1}{C}I_{2,1}\equiv\frac{1}{C}\|\partial_t^l\partial_x^{2m}\partial_x(u^2)\|_1$$

$$\leq\sum_{l'=0}^{l}\sum_{m_1=0}^{m}\binom{l}{l'}\binom{2m+1}{2m_1}\|\partial_t^{l'}M^{m_1}u\|_1\|\partial_t^{l-l'}M^{m-m_1}\partial_xu\|_1$$

$$+\sum_{l'=0}^{l}\sum_{m_1=0}^{m-1}\binom{l}{l'}\binom{2m+1}{2m_1+1}\|\partial_t^{l'}M^{m_1}\partial_xu\|_1\|\partial_t^{l-l'}M^{m-m_1-1}\partial_xu\|_1$$

$$\equiv I_{2,1,1}+I_{2,1,2}.$$

In the same way as we proved (4.14) and (4.15) we obtain by Lemma 2.10

$$(5.8)\qquad (I_{2,1,1}+I_{2,1,2})\frac{A_3^{l+m+1}}{((l+m+1)!)^{\sigma'}}\leq 3\cdot 2^8C_0^2A_3A_6^m.$$

From (5.6)-(5.8) we see that if we take A_6 such that

$$A_6 = 3 \cdot 2^1 0 C_0 C A_3 \tag{5.9}$$

then

$$\|\partial_t^l M^m \partial_x^k (u^2)\|_1 \leq \frac{C_0}{2} A_6^{m+1}. \tag{5.10}$$

On the other hand, the solution of (NLS2) satisfies

$$\|\partial_t^l M^{m+1} \partial_x^k u\|_1 \leq \|\partial_t^{l+1} M^m \partial_x^k u\|_1 + \|\partial_t^l M^m \partial_x^k (u^2)\|_1.$$

By the induction assumption, we see that the first term of the right hand side is estimated from above by

$$C_0 A_6^m = \frac{1}{A_6} C_0 A_6^{m+1} \leq \frac{C_0}{2} A_6^{m+1} \quad \text{if} \quad A_6 \geq 2.$$

Hence, we have (5.4) by (5.9) and (5.10) if

$$A_6 = 2 + 3 \cdot 2^1 0 C_0 C A_3. \tag{5.11}$$

Using (5.4) and the same argument as in Lemma 4.2, we obtain the desired estimate (5.1), provided that A_4 satisfies

$$A_4 < 2\left(\frac{A_3}{2(1 + 3 \cdot 2^9 C C_0 A_3)}\right)^{1/2}.$$

Q.E.D.

ACKNOWLEDGEMENTS

This work was started when the second and third authors stayed at the University of Paris-Sud. The authors (N. H and K. K) would like to express their hearty thanks to Professors T. Cazenave, J. Ginibre, G. Lebeau and J. C. Saut for their kind hospitality at Université de Paris-Sud and Université Pierre-et-Marie-Curie. The second author was partially supported by the Gunma University Foundation for Science and Technology.

REFERENCES

[A] H. AIKAWA, Infinite order Sobolev spaces, analytic continuation and polynomial expansions, *Complex Variables*, Vol. **18**, 1992, pp. 253-266.

[B] A. DE BOUARD, Analytic solutions to nonelliptic nonlinear Schrödinger equations, *J. Diff. Eqs.*, Vol. **104**, 1993, pp. 196-213.

[Co-Sau] P. CONSTANTIN and J. C. SAUT, Local smoothing properties of dispersive equations, *J. Amer. Math. Soc.*, Vol. **1**, 1988, pp. 413-439.

[Cr-Kap-St] W. CRAIG, K. KAPPELER and W. A. STRAUSS, Gain of regularity for solutions of KdV type, *Ann. Inst. Henri Poincaré, Analyse non linéaire*, Vol. **9**, 1992, pp. 147-186.

[G-Vel] J. GINIBRE and G. VELO, On a class of nonlinear Schrödinger equations I, II, *J. Funct. Anal.*, Vol. **32**, 1979, pp. 1-32, 33-71.

[H] N. HAYASHI, Analyticity of solutions of the Korteweg-de Vries equation, SIAM *J. Math. Anal.*, Vol. **22**, 1991, pp. 1738-1745.

[H-N-T] N. HAYASHI, K. NAKAMITSU and M. TSUTSUMI, On solutions of the initial value problem for the nonlinear Schrödinger equation in one space dimension, *Math. Z.*, Vol. **192**, 1986, pp. 637-650.

[H-K.K] N. HAYASHI and K. KATO, Regularity of solutions in time to nonlinear Schrödinger equations, *J. Funct. Anal.*, 1994 (to appear).

[H-O] N. HAYASHI and T. OZAWA, Remarks on nonlinear Schrödinger equations in one space dimensions, *Diff. and Integral Eqs.*, Vol. **7**, 1994, pp. 453-461.

[H-Sai] N. HAYASHI and S. SAITOH, Analyticity and global existence of small solutions to some nonlinear Schrödinger equations, *Commun. Math. Phys.*, Vol. **129**, 1990, pp. 27-42.

[Ka] T. KATO, On the Cauchy problem for the (generalized) Korteweg-de Vries equation, *Stud. Appl. Math. Adv. in Math. Supplementary Studies*, Vol. **18**, 1983, pp. 93-128.

[Ka-M] T. KATO and K. MASUDA, Nonlinear evolution equations and analyticity I, *Ann. Inst. Henri Poincaré Analyse non linéaire*, Vol. **3**, 1986, pp. 455-467.

[Ke-P-V 1] C. E. KENIG, G. PONCE and L. VEGAA, Oscillatory integrals and regularity of dipersive equations, *Indiana Univ. Math. J.*, Vol. **40**, 1991, pp. 33-69.

[Ke-P-V 2] C. E. KENIG, G. PONCE and L. VEGAA, Small solutions to nonlinear Schrödinger equations, *Ann. Inst. Henri Poincaré, Analyse non linéaire*, Vol. **10**, 1993, pp. 255-288.

[Ke-P-V 3] C. E. KENIG, G. PONCE and L. VEGAA, *Higher order non-linear dispersive equations*, preprint, 1992.

[Ke-P-V 4] C. E. KENIG, G. PONCE and L. VEGAA, On the hierarchy of the generalized KdV equations, *Proc. Lyon Workshop on Singular Limits of Dispersive Waves*, 1993 (to appear).

[Ke-P-V 5] C. E. KENIG, G. PONCE and L. VEGAA, Well-posedness and scattering results for the generalized Korteweg-de Vries equations via the contraction principle, *Comm. Pure. Appl. Math.*, Vol. **46**, 1993, pp. 527-620.

[Ko] H. KOMATSU, "*Introduction to distributions, ultradistributions and hyperfunctions* (in Japanese)", Iwanami Kisosugaku, Iwanami shoten, Tokyo, 1978.

[Kr-F] S. N. KRUZHKOV and A. V. FAMINSKII, Generalized solutions to the Cauchy problem for the Korteweg-de Vries equation, *Math. U.S.S.R. Sbornik*, Vol. **48**, 1984, pp. 93-138.

[L] C. LAUREY, *The Cauchy problem for a third order nolinear Schrödinger equations*, preprint, 1993.

[P] G. PONCE, Regularity of solutions to nonlinear dispersive equations, *J. Diff. Eqs.*, Vol. **78**, 1989, pp. 122-135.

[Sj] P. Sjölin, Regularity of solutions to the Schrödinger equations, *Duke Math. J.*, Vol. **55**, 1987, pp. 699-715.

[St] W. Strauss, Dispersion of low energy waves for two conservative equations, *Arch. Rational Mech. Anal.*, Vol. **55**, 1974, pp. 86-92.

[V] L. Vega, Schrödinger equations : pointwise convergence to the initial data, *Proc. Amer. Math. Soc.*, Vol. **102**, pp. 874-878.

(Manuscript received February 17, 1994.)