Ann. Inst. Henri Poincaré,

Vol. 13, n° 5, 1996, p. 553-565 *Analyse non linéaire*

Asymptotics for L^2 minimal blow-up solutions of critical nonlinear Schrödinger equation

by

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ABSTRACT. – In this note, we describe the behavior of a sequence $v_n : \mathbb{R}^N \to \mathbb{C}$ minimal in L^2 such that $\frac{1}{2}\int |\nabla v_n|^2 - \frac{1}{\frac{4}{N}+2}\int |v_n|^{\frac{4}{N}+2} \leq E_0$ and $|v_n|_{H^1} \to +\infty$.

RÉSUMÉ. – Dans cette note, on explicite de façon optimale le comportement d'une suite $v_n : \mathbb{R}^N \to \mathbb{C}$ de norme L^2 minimale telle que $\frac{1}{2}\int |\nabla v_n|^2 - \frac{1}{\frac{4}{N}+2}\int |v_n|^{\frac{4}{N}+2} \leq E_0$ et $|v_n|_{H^1} \to +\infty$.

In the present note, we are interested in the behavior of a sequence $v_n : \mathbb{R}^N \to \mathbb{C}$ of H^1 functions such that

$$\int |v_n|^2 = \int Q^2, \tag{1}$$

$$E(v_n) = \frac{1}{2}\int |\nabla v_n|^2 - \frac{1}{\frac{4}{N}+2}\int |v_n|^{\frac{4}{N}+2} \leq E_0, \tag{2}$$

$$\int |\nabla v_n|^2 \to +\infty, \tag{3}$$

where Q is the radial positive symetric solution of the equation

$$\Delta v + |v|^{\frac{4}{N}} v = v. \tag{4}$$

(*See* references [1], [4], [7] for existence and uniqueness of Q.)

This problem is related to the asymptotics of minimal blow-up solutions in H^1 of the equation

$$iu_t = -\Delta u - k(x)|u|^{\frac{4}{N}}u \text{ and } u(0) = \varphi, \tag{5}$$

where

$$\operatorname*{Max}_{x\in\mathbf{R}^N} k(x) = 1. \tag{6}$$

Indeed, for all $\varphi \in H^1$, there is a unique solution in H^1 on $[0,T]$ ([2], [4]) and

$$T = +\infty \text{ or } \lim_{t\to T} \int |\nabla u(t,x)|^2 = +\infty.$$

In addition, $\forall\ t$

$$\int |u(t,x)|^2 dx = \int |\varphi(x)|^2 dx \tag{7}$$

$$E_k(u(t)) = E_k(\varphi) \tag{8}$$

where

$$E_k(v) = \frac{1}{2}\int |\nabla v|^2 - \frac{1}{\frac{4}{N}+2}\int k(x)|v|^{\frac{4}{N}+2}.$$

From [9]

$$\forall\ v \in H^1,\ \frac{1}{\frac{4}{N}+2}\int |v|^{\frac{4}{N}+2} \le \frac{1}{2}\left(\frac{\int |v|^2}{\int |Q|^2}\right)^{\frac{2}{N}} \int |\nabla v|^2 \tag{9}$$

and it follows from (6)-(9) ([6]) that

$$\text{if } |\varphi|_{L^2} < |Q|_{L^2},\ \text{then } T = +\infty.$$

Moreover under some conditions on $k(x)$, for any $\varepsilon > 0$ there are blow-up solutions $u_\varepsilon(t)$ such that

$$|u_\varepsilon(0)|_{L^2}^2 = |\varphi|_{L^2}^2 + \varepsilon \quad ([6]).$$

Thus the questions are about existence of minimal blow-up solution (that is such that $u(t)$ blows up in finite time and $\int |\varphi|^2 = \int Q^2$ and on the form of these solutions. In the case where $k(x) \equiv 1$, the question has been completely solved (*see* Merle [5]). The general case is still open. We remark that from (6)-(9), if $u(t)$ is a blow up solution, the sequences $v_n = u(t_n)$ as $t_n \to T$ satisfies (1)-(3) and we ask about the constrains it imply on v_n.

The first result in this direction was obtained by Weinstein in [9]. Using the concentration compactness method, he showed that there is a $\theta_n \in \mathbb{R}$, $x_n \in \mathbb{R}^N$ such that

$$v_n = \lambda_n^{\frac{N}{2}} e^{i\theta_n} Q\Big(\lambda_n^{\frac{N}{2}}(x - x_n)\Big) + \varepsilon_n, \tag{10}$$

where

$$\lambda_n = \frac{|\nabla v_n|_{L^2}}{|\nabla Q|_{L^2}}, \tag{11}$$

$$|\varepsilon_n|_{L^2} \xrightarrow[n \to +\infty]{} 0 \quad \text{and} \quad \frac{|\nabla \varepsilon_n|_{L^2}}{\lambda_n} \xrightarrow[n \to +\infty]{} 0. \tag{12}$$

In [5], Merle then showed that for all $R > 0$, there is a $c > 0$ such that

$$\int_{|x - x_n| > R} |\nabla v_n|^2 \leq c. \tag{13}$$

We now claim the following result

THEOREM. – *Let (v_n) be a sequence of H^1 functions satisfying* (1)-(3) *and $\theta_n(x)$ be such that $v_n = |v_n| e^{i\theta_n}$.*

i) *Phase estimates. There is a $c > 0$ such that*

$$\forall\, n, \quad \int |v_n|^2 |\nabla \theta_n|^2 \leq c.$$

ii) *asymptotics on the modulus.*

There is a $\varepsilon_n(x)$, $x_n \in \mathbb{R}^N$, and $c > 0$ such that

$$\forall\, x, \quad |v_n(x)| = \lambda_n^{\frac{N}{2}} Q(\lambda_n (x - x_n)) + \varepsilon_n(x)$$

where

$$|\nabla \varepsilon_n|_{L^2} \leq c, \quad |\varepsilon_n|_{L^2} \leq \frac{c}{\lambda_n} \quad \text{and} \quad \lambda_n \left(\frac{|\nabla Q|_{L^2}}{|\nabla v_n|_{L^2}} \right) \xrightarrow[n \to +\infty]{} 1.$$

Remark. – This Theorem simplifies some proofs in [5], [6]. The case where v_n is real valued is also related to similar problems for the generalized Korteweg-de Vries equation with critical nonlinearity.

Remark. – The Theorem implies in particular for blow-up solution of equation (5) $u(t,x) = |u(t,x)|e^{i\theta(t,x)}$ and $\int |u(t)|^2 = \int Q^2$ the phase gradient is uniformly bounded at the blow-up: there is a $c > 0$ such that

$$\forall\ 0 < t < T, \quad \int |u(t,x)|^2 |\nabla\theta(t,x)|^2 dx \leq c.$$

(Of course, we still have $\int |\nabla |u||^2(t,x) \underset{t\to T}{\longrightarrow} +\infty$.)

Remark. – It is easy to check that the result is optimal. We remark that the residual term in the theorem $\varepsilon_n = O(1)$ (compared to $o(|\nabla v_n|_{L^2})$ in [9]).

Proof of the Theorem. – Let (v_n) a sequence of H^1 function satisfying (1)-(3) and $\theta_n(x)$ such that $v_n = |v_n|e^{i\theta_n}$. We have that

$$(14) \qquad \frac{1}{2}\int |\nabla v_n|^2 = \frac{1}{2}\left\{ |\nabla |v_n||^2 + \int |v_n|^2 |\nabla\theta_n|^2 \right\}$$

and

$$(15) \qquad E(v_n) = \frac{1}{2}\int |v_n|^2 |\nabla\theta_n|^2 + E(|v_n|).$$

The idea is to apply the variational identity (9) not with v_n but with $|v_n|$.

Indeed, since $v_n \in H^1$ we have that $|v_n| \in H^1$. From (9) (applied with $|v_n|$)

$$(16) \quad \frac{1}{\frac{4}{n}+2}\int |v_n|^{\frac{4}{n}+2} \leq \frac{1}{2}\left(\frac{\int |v_n|^2}{\int Q^2}\right)^{\frac{2}{N}} \int |\nabla |v_n||^2 \leq \frac{1}{2}\int |\nabla |v_n||^2,$$

or equivalently

$$(17) \qquad E(|v_n|) \geq 0.$$

Thus (2), (15), (17) imply that

$$(18) \qquad \frac{1}{2}\int |v_n|^2 |\nabla\theta_n|^2 \leq E_0$$

$$(19) \qquad E(|v_n|) \leq E_0.$$

Part i). – It is implied by (18).

Part ii). – We claim that it is as a consequence of (18)-(19). We prove it in three steps:

– from Weinstein's results, we first obtain rough estimates on $|v_n|$,

– we then choose appropriate approximations parameters,

– we conclude the proof using a convexity property in certain directions of E near Q (and use in a crucial way that $|v_n|$ is a real-valued function).

Step 1: First asymptotics. – Since

$$\int |\nabla v_n|^2 = \int |\nabla |v_n||^2 + \int |v_n|^2 |\nabla \theta_n|^2 \xrightarrow[n \to +\infty]{} +\infty, \tag{20}$$

and

$$\forall\, n, \int |v_n|^2 |\nabla \theta_n|^2 \le c,$$

we have

$$\int |\nabla |v_n||^2 \xrightarrow[n \to +\infty]{} +\infty. \tag{21}$$

Moreover,

$$\int |v_n|^2 = \int Q^2 \text{ and } E(|v_n|) \le E_0. \tag{22}$$

We conclude from Weinstein's result on the existence of $\hat{x}_n, \hat{\varepsilon}_n$ such that

$$|v_n|(x) = \hat{\lambda}_n^{N/2} Q\left(\hat{\lambda}_n x - \hat{x}_n\right) + \hat{\varepsilon}_n(x) \tag{23}$$

where

$$\hat{\lambda}_n = \frac{|\nabla |v_n||_{L^2}}{|\nabla Q|_{L^2}}, \tag{24}$$

$$|\nabla \hat{\varepsilon}_n|_{L^2} = o\left(\hat{\lambda}_n\right), \quad |\hat{\varepsilon}_n|_{L^2} = o(1). \tag{25}$$

In order to obtain better estimates on the rest (that is $|\nabla \varepsilon_n|_{L^2} \le c$), we have to choose appropriate parameters λ_n, x_n and use the structure of the functional $E(\cdot)$ near Q.

Step 2: Choice of the parameters of approximation. – Let us first renormalize the problem. We consider

$$w_{n,\lambda_1,x_1}(x) = \left(\frac{\lambda_1}{\hat{\lambda}_n}\right)^{\frac{N}{2}} |v_n| \left((\lambda_1 x + \hat{x}_n + x_1)\frac{1}{\hat{\lambda}_n}\right) \tag{26}$$

$$(27) \qquad \tilde{\varepsilon}_{n,\lambda_1,x_1}(x) = \left(\frac{\lambda_1}{\hat{\lambda}_n}\right)^{\frac{N}{2}} \hat{\varepsilon}_n\left((\lambda_1 x + \hat{x}_n + x_1)\frac{1}{\hat{\lambda}_n}\right).$$

We have from (23),

$$(28) \qquad w_{n,\lambda_1,x_1}(x) = \lambda_1^{N/2} Q(\lambda_1 x + x_1) + \tilde{\varepsilon}_{n,\lambda_1,x_1}(x)$$

where

$$(29) \qquad \frac{|\nabla\tilde{\varepsilon}_n|_{L^2}}{\lambda_1} + |\tilde{\varepsilon}_n|_{L^2} \xrightarrow[n\to+\infty]{} 0.$$

We write (28) as follows

$$w_{n,\lambda_1,x_1}(x) = Q(x) + \varepsilon_{n,\lambda_1,x_1}(x)$$

where

$$(30) \qquad \varepsilon_{n,\lambda_1,x_1}(x) = \left[\lambda_1^{N/2} Q(\lambda_1 x + x_1) - Q(x)\right] + \tilde{\varepsilon}_{n,\lambda_1,x_1}(x).$$

From the implicit function Theorem, we derive easily for $|\tilde{\varepsilon}_n|_{H^1}$ small enough the existence of $\lambda_{1,n}, x_{1,n}$ such that

$$(31) \qquad \forall\, i = 1, ..., N, \quad \int \varepsilon_{n,\lambda_{1,n},x_{1,n}} x_i Q = 0$$

$$(32) \qquad \int \varepsilon_{n,\lambda_{1,n},x_{1,n}} |x|^2 Q = 0.$$

Moreover, from (29)

$$(33) \qquad (\lambda_{1,n}, x_{1,n}) \xrightarrow[n\to+\infty]{} (1,0).$$

Indeed, let us note

$$\text{for } i = 1, ..., N, \quad \rho_i(\lambda_1, x_1) = \int \varepsilon_{n,\lambda_1,x_1} x_i Q,$$
$$\rho_{N+1}(\lambda_1, x_1) = \int \varepsilon_{n,\lambda_1,x_1} |x|^2 Q.$$

From (30), we have

$$\frac{\partial \varepsilon_{n,1,0}}{\partial x_{1,i}} = \partial_i Q + \partial_i\, \tilde{\varepsilon}_{n,1,0}$$
$$\frac{\partial \varepsilon_{n,1,0}}{\partial \lambda_1} = \frac{N}{2} Q + x.\nabla Q + \left(\frac{N}{2}\tilde{\varepsilon}_{n,1,0} + x.\nabla\tilde{\varepsilon}_{n,1,0}\right),$$

where $x_1 = (x_{1,1}, ..., x_{1,N})$.

Therefore, from (29) and integration by parts,
– for $i = 1, ..., N$, and $j = 1, ..., N$,

$$\frac{\partial \rho_i}{\partial x_{1,j}}(1,0) = \int \partial_j Q \; x_i Q + o(1) = -2\delta_{i,j} \int Q^2 + o(1),$$

$$\frac{\partial \rho_i}{\partial \lambda_1}(1,0) = \int \left(\frac{N}{2}Q + x.\nabla Q\right) x_i Q + o(1) = o(1),$$

$$\frac{\partial \rho_{N+1}}{\partial x_{1,j}}(1,0) = \int \partial_j Q |x|^2 Q + o(1) = o(1),$$

$$\begin{aligned}\frac{\partial \rho_{N+1}}{\partial \lambda_1}(1,0) &= \int \left(\frac{N}{2}Q + x.\nabla Q\right)|x|^2 Q + o(1) \\ &= \frac{N}{2}\int |x|^2 Q^2 - \frac{N}{2}\int |x|^2 Q^2 - \frac{1}{2}\int x.x \; Q^2 + o(1) \\ &= -\frac{1}{2}\int |x|^2 Q^2 + o(1).\end{aligned}$$

Therefore, the implicit function theorem implies the existence of $(\lambda_{1,n}, x_{1,n})$ such that (31)-(33) hold.

In conclusion, we have proved the following. There exist $(\lambda_{1,n}, x_{1,n}) \xrightarrow[n \to +\infty]{} (1,0)$ such that

$$w_{n,\lambda_{1,n},x_{1,n}}(x) = Q(x) + \varepsilon_{n,x_{1,n},x_{1,n}}(x) \tag{34}$$

where

$$\forall \, i = 1, ..., n, \quad \int \varepsilon_{n,\lambda_{1,n},x_{1,n}} x_i Q = 0, \tag{35}$$

$$\int \varepsilon_{n,\lambda_{1,n},x_{1,n}} |x|^2 Q = 0, \tag{36}$$

$$\left|\varepsilon_{n,\lambda_{1,n},x_{1,n}}\right|_{H^1} \xrightarrow[n \to +\infty]{} 0. \tag{37}$$

We now note

$$w_n = w_{n,\lambda_{1,n},x_{1,n}},$$
$$\varepsilon_n = \varepsilon_{n,\lambda_{1,n},x_{1,n}}.$$

Step 3 : Conclusion of the proof. – Geometry of energy functions at Q.

We now use convexity properties of a functional (related to E) and the fact $\int w_n^2 = \int Q^2$ to conclude the proof. Let

$$H(v) = \frac{1}{2}\int |\nabla v|^2 - \frac{1}{\frac{4}{N}+2}\int |v|^{\frac{4}{N}+2} + \frac{1}{2}\int v^2 = E(v) + \frac{1}{2}\int v^2,$$

and

$$H_2(v) = \frac{1}{2}\int |\nabla v|^2 - \frac{4}{N}\int Q^{\frac{4}{N}} v^2 + \frac{1}{2}\int v^2.$$

We know that Q is a critical point of H, and H_2 is the quadratic part of H near Q (where v is real-valued). Moreover, it is classical that for $|\varepsilon|_{H^1} \leq 1$,

$$H(Q+\varepsilon) - H(Q) = H_2(\varepsilon) + \tilde{H}_2(\varepsilon) \tag{38}$$

where $\left|\tilde{H}_2(\varepsilon)\right| = o(|\varepsilon|_{H^1}^2)$.

From a result of Weinstein [8] (*see* also Kwong [4]), we have the following convexity property of H_2 at Q.

PROPOSITION 1. *See* [8]. – *(Directions of convexity of H at Q in the set of real-valued functions.)*
There is a constant $c_1 > 0$ such that $\forall\ \varepsilon \in H^1$
If

(i) $$\forall\ i = 1, ..., N, \quad \int \varepsilon\ x_i Q = 0$$

(ii) $$\int \varepsilon |x|^2 Q = 0,$$

(iii) $$\int \varepsilon Q = 0,$$

then

$$H_2(\varepsilon) \geq c_1 \left(\int |\nabla \varepsilon|^2 + \varepsilon^2 \right) = c_1 |\varepsilon|_{H^1}^2.$$

Remark. – We have here a strict convexity property (up to the invariance of the equation) except in the direction Q *which is not true* for the quadratic part of H for complex valued functions (*see* [8]).

Remark. – This proposition is optimal. Other functions can be chosen also.

Using now crucially estimates on the L^2 norm, we obtain the following

PROPOSITION 2 (Control of the Q direction by the L^2 norm).
Assume

(i) $$\forall\, i = 1, \dots, N, \qquad \int \varepsilon x_i Q = 0,$$

(ii) $$\int \varepsilon |x|^2 Q = 0,$$

(iii) $$\int (Q+\varepsilon)^2 = \int Q^2,$$

then there are $c_1 > 0$ *and* $c_2 > 0$ *such that*

$$|\nabla \varepsilon|_{L^2} + |\varepsilon|_{L^2} \leq c_2 \text{ implies } H_2(\varepsilon) \geq c_1\left(|\nabla \varepsilon|_{L^2}^2 + |\varepsilon|_{L^2}^2\right).$$

Remark. – We need control on 3 directions to obtain estimates on $|\varepsilon|_{H^1}$ with $H_2(Q+\varepsilon)$. Two directions can be controlled using the invariance of the equation. The last one is controlled by the condition of minimality on the L^2 norm (among sequence satisfying (2)).

Proof of Proposition 2. – Let us note

$$\tilde{H}_2(v_1, v_2) = \frac{1}{2}\int \nabla v_1 \nabla v_2 - \frac{4}{N}\int Q^{\frac{4}{N}} v_1 v_2 + \frac{1}{2}\int v_1 v_2.$$

We can write

$$\varepsilon = z + aQ + b|x|^2 Q$$

with

$$\int zQ = \int z x_i Q = \int z|x|^2 Q = 0 \quad \text{for } i = 1, \dots, N.$$

Indeed a and b have to satisfy

$$\int \varepsilon Q = a\int Q^2 + b\int |x|^2 Q^2$$
$$o = a\int |x|^2 Q^2 + b\int |x|^4 Q^2$$

or equivalently

$$b = -a\left(\frac{\int |x|^2 Q^2}{\int |x|^4 Q^2}\right)$$

$$a\left(\frac{\int Q^2 \int |x|^4 Q^2 - \left(\int |x|^2 Q^2\right)^2}{\int |x|^4 Q^2}\right) = \int \varepsilon Q$$

(which has always a solution since from the Schwarz inequality and the fact $|x|^2 Q \neq Q$, $\int |x|^2 Q^2 < \left(\int Q^2 \int |x|^4 Q^4\right)^{1/2}$).

On the other hand, we have from $\int (Q+\varepsilon)^2 = \int Q^2$

$$2\int Q\varepsilon = -\int \varepsilon^2$$

$$2\left(a\int Q^2 + b\int |x|^2 Q^2\right) = -\int \varepsilon^2$$

$$2a\left(\frac{\int Q^2 \int |x|^4 Q^2 - \left(\int |x|^2 Q^2\right)^2}{\int |x|^4 Q^2}\right) = -\int z^2 + O(a^2 + b^2)$$

or equivalently,

$$ac_0 = -\int z^2 + O(a^2) \text{ where } c_0 \neq 0$$

which implies that

$$a = O\left(\int z^2\right) \text{ and } b = O\left(\int z^2\right)$$

and for $|\varepsilon|_{H^1}$ small enough

$$|\varepsilon|^2_{H^1} \geq |z|^2_{H^1} \geq \frac{1}{2}|\varepsilon|^2_{H^1}.$$

On the other hand, by bilinearity and Proposition 1, we have for $|\varepsilon|_{H^1}$ small enough

$$\begin{aligned}
H_2(\varepsilon) &= H_2(z) + 2a\tilde{H}_2(z, Q) + 2b\tilde{H}_2\left(z, |x|^2 Q\right) + 2ab\tilde{H}_2\left(Q, |x|^2 Q\right) \\
&\quad + a^2 H_2(Q) + b^2 H_2\left(|x|^2 Q\right) \\
&\geq H_2(z) - c\left(|z|_{H^1}(|a| + |b|) + a^2 + b^2\right) \\
&\geq H_2(z) - c\left(|z|^3_{H^1} + |z|^4_{H^1}\right) \\
&\geq c_1\left(|z|^2_{H^1}\right) - c\left(|z|_{H^1} + |z|^4_{H^1}\right) \\
&\geq \frac{c_1}{2}|z|^2_{H^1} \\
&\geq \frac{c_1}{4}|\varepsilon|^2_{H^1}.
\end{aligned}$$

This concludes the proof of Proposition 2.

As a corollary of Proposition 2 and (38), we have

COROLLARY. – *There are $c_1 > 0$ and $c_2 > 0$ such that if*

(i)
$$\forall\, i = 1, ..., N, \quad \int \varepsilon x_i Q = 0$$

(ii)
$$\int \varepsilon |x|^2 Q = 0$$

(iii)
$$\int (Q+\varepsilon)^2 = \int Q^2$$

(iv)
$$|\nabla \varepsilon|_{L^2} + |\varepsilon|_{L^2} \le c_2$$

then
$$H(Q+\varepsilon) - H(Q) \ge c_1\Big(\int \nabla\varepsilon^2 + \int \varepsilon^2\Big).$$

We now apply the corollary. If $w_n = Q + \varepsilon_n$, we have

- $|\varepsilon_n|_{H^1} \xrightarrow[n\to+\infty]{} 0$, and in particular there is n_0 such that
$$\forall\, n \ge n_0, \quad |\varepsilon_n|_{H^1} \le c_2,$$
- $\forall\, i = 1, ..., N$, $\int \varepsilon_n x_i Q = 0$.

In addition,
$$\begin{aligned} H(Q) &= \frac{1}{2}\int |\nabla Q|^2 - \frac{1}{\frac{4}{N}+2}\int |Q|^{\frac{4}{N}+2} + \frac{1}{2}\int Q^2 \\ &= E(Q) + \frac{1}{2}\int Q^2 \\ &= \frac{1}{2}\int Q^2 \end{aligned}$$

(since the Pohozaev identity for equation (4) yields $E(Q) = 0$), and
$$\begin{aligned} H(Q+\varepsilon_n) = H(w_n) &= H\bigg(\Big(\frac{\lambda_1}{\hat\lambda_n}\Big)^{\frac{N}{2}} |v_n|\Big(\frac{x\lambda_1}{\hat\lambda_n} + \hat x_n + x_1\Big)\bigg) \\ &= E\bigg(\Big(\frac{\lambda_1}{\hat\lambda_n}\Big)^{\frac{N}{2}} |v_n|\Big(\frac{x\lambda_1}{\hat\lambda_n}\Big)\bigg) + \frac{1}{2}\int |v_n|^2 \\ &= \Big(\frac{\lambda_1}{\hat\lambda_n}\Big)^2 E(|v_n|) + \frac{1}{2}\int Q^2. \end{aligned}$$

Therefore $\forall\ n \geq n_0$

$$(40) \qquad \left(\frac{\lambda_1}{\hat{\lambda}_n}\right)^2 E(|v_n|) > c_3\left(|\varepsilon_n|_{H^1}^2\right)$$

or equivalently from (19), (24) and the fact that $\lambda_1 \to 1$,

$$(41) \qquad |\varepsilon_n|_{H^1}^2 \leq \frac{c}{2}\frac{1}{\int |\nabla |v_n||^2} \leq c\frac{1}{\int |\nabla v_n|^2},$$

where c is independent of n. Thus,

$$w_n = Q + \varepsilon_n,$$

with

$$(42) \qquad |\varepsilon_n|_{H^1}^2 \leq \frac{c}{\int |\nabla v_n|^2}.$$

Therefore from (26), there is x_n such that

$$(43)\ |v_n|(x) = \left(\frac{\hat{\lambda}_n}{\lambda_1}\right)^{\frac{N}{2}} Q\left(x\left(\frac{\hat{\lambda}_n}{\lambda_1}\right) + x_n\right) + \left(\frac{\hat{\lambda}_n}{\lambda_1}\right)^{\frac{N}{2}} \varepsilon_n\left(\frac{\hat{\lambda}_n}{\lambda_1}x + x_n\right).$$

We remark that from (19), (42), the fact that $\lambda_1 \to 1$,

$$\frac{\hat{\lambda}_n}{\lambda_1}\frac{1}{\left(\int \nabla v_n^2\right)^{\frac{1}{2}}} = \frac{1}{\lambda_1}\left(\frac{\int |\nabla |v_n||^2}{\int |\nabla v_n|^2 \int |\nabla Q|^2}\right)^{\frac{1}{2}} \xrightarrow[n \to +\infty]{} 1,$$

$$\left|\left(\frac{\hat{\lambda}_n}{\lambda_1}\right)^{\frac{N}{2}} \varepsilon_n\left(\frac{\hat{\lambda}_n}{\lambda_1}x + x_n\right)\right|_{L^2}^2 = |\varepsilon_n|_{L^2}^2 \leq \frac{c}{\int |\nabla v_n|^2},$$

$$\left|\nabla\left(\frac{\hat{\lambda}_n}{\lambda_1}\right)^{\frac{N}{2}} \varepsilon_n\left(\frac{\hat{\lambda}_n}{\lambda_1}x + x_n\right)\right|_{L^2}^2 \leq c\left(\frac{\hat{\lambda}_n^2}{\int |\nabla v_n|^2}\right) \leq c$$

conclude the proof of the Theorem.

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(Manuscript received October 25, 1994;
revised version received January 26, 1995.)