

Multiplicity of periodic solution with prescribed energy to singular dynamical systems

by

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ABSTRACT. — We prove the existence of multiple solutions with prescribed energy for a class of dynamical systems with singular potentials.

Key words : Singular Hamiltonian Systems, critical points theory, index theory.

RÉSUMÉ. — Nous démontrons l'existence d'une multiplicité de solutions périodiques à énergie fixée pour une classe de systèmes dynamiques dont les potentiels sont singuliers.

INTRODUCTION AND STATEMENT OF THE RESULTS

For a given potential $F \in \mathcal{C}^1(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ and a fixed energy E , we look for multiple solutions to the problem

$$(P_E) \quad \left\{ \begin{array}{l} -\ddot{x} = \nabla F(x) \\ \frac{1}{2} |\dot{x}|^2 + F(x) = E \\ x(t+\lambda) = x(t), \quad \forall t \in \mathbb{R} \\ x(t) \neq 0, \quad \forall t \in \mathbb{R}, \end{array} \right.$$

where the unknowns are both the function x and its period λ .

The potential F presents a singularity at the origin and it behaves like a potential of the form $F(x) = \frac{-a}{|x|^\alpha}$ for some $a, \alpha > 0$, behavior that will be clear from the assumptions of the theorems.

In this work we shall deal separately with the two cases:

- (a) $\alpha > 2$ (strong force case);
- (b) $1 < \alpha < 2$ (weak force case).

As far as the existence of one solution to (P_E) is concerned, results have been obtained for the case (a) by Benci and Giannoni, [9], where they also treated cases of weak forces, but with assumptions strongly different from the ones here. For both cases (a) and (b), existence results have been obtained by Ambrosetti and Coti Zelati, [4]; however, in the case (b) they obtained the existence of generalized (*i. e.* possibly crossing the singularity). In order to avoid the collisions in the case (b), we shall make use of the approach introduced by the author in [22], where the existence of one noncollision solution to (P_T) was proved under assumptions similar to the ones here.

Concerning existence and multiplicity of solutions, we refer also to the work of Moser, [20], where the case $\alpha = 1$ is treated when F presents some symmetry properties.

In the last years, a considerable amount of papers has appeared about the corresponding problem with the fixed period; we refer to [1], [2], [5], [13], [17], [21].

Concerning the case (a) we shall prove the following result:

THEOREM 1. — *Let $F \in \mathcal{C}^1(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ satisfy:*

- (H1) $\exists b \geq a > 0, \exists \alpha_2 \geq \alpha \geq \alpha_1 > 2$, such that
$$\frac{a}{|x|^\alpha} \leq -F(x) \leq \frac{b}{|x|^\alpha}, \quad \forall x \in \mathbb{R}^N \setminus \{0\}$$
- (H2) $-\alpha_1 F(x) \leq \nabla F(x) \cdot x \leq -\alpha_2 F(x), \quad \forall x \in \mathbb{R}^N \setminus \{0\}$
- (H3) $|\nabla F(x)| \leq \frac{a\alpha_2}{|x|^{\alpha+1}}, \quad \forall x \in \mathbb{R}^N \setminus \{0\},$

and assume moreover that $a, b, \alpha_2, \alpha, \alpha_1$ satisfy

$$(1) \quad \frac{b^{2/\alpha}}{a^{2/\alpha}} \frac{\alpha}{\alpha_1} \left(\frac{\alpha_2 - 2}{\alpha - 2} \right)^2 \left(\frac{\alpha - 2}{\alpha_1 - 2} \right)^{(\alpha+2)/\alpha} < 4.$$

Then, for every positive energy E , (P_E) has at least $N - 1$ geometrically distinct solutions having minimal period in the interval

$$(2) \quad \left[\left(\frac{(\alpha_1 - 2)\alpha}{(\alpha - 2)\alpha_1} \right) \frac{2\pi}{\alpha^{1/2}} a^{1/\alpha} \left(\frac{\alpha - 2}{2E} \right)^{(\alpha+2)/2\alpha}, \right. \\ \left. \left(\frac{(\alpha_2 - 2)\alpha}{(\alpha - 2)\alpha_2} \right) \frac{2\pi}{\alpha^{1/2}} b^{1/\alpha} \left(\frac{\alpha - 2}{2E} \right)^{(\alpha+2)/2\alpha} \right].$$

We say that two λ -periodic functions x, y are geometrically distinct if, setting $T_s(x)(t) = x(s+t)$ and $P_s(x)(t) = x(s-t)$, $T_s(x) \neq y$ and $P_s(y) \neq y$, for every $s \in \mathbb{R}$.

Remark. – It is not difficult to see that, when (H1) and (H2) hold, the solutions of (P_E) are constrained in the ball of radius $\left(\frac{(\alpha_2 - 2)b}{2E}\right)^{1/\alpha}$, so that the following Corollary easily follows from Theorem 1:

COROLLARY 1. – Let $U \in \mathcal{C}^1(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ satisfy

$$\begin{aligned} &\exists \alpha > 2 \text{ such that} \\ &\lim_{x \rightarrow 0} |x|^{\alpha+1} |\nabla U(x)| = 0, \end{aligned}$$

and let $F(x) = \frac{-a}{|x|^\alpha} + U(x)$ for some $a > 0$. Then there exists $\bar{E} > 0$ such that, for every $E > \bar{E}$, then (P_E) has at least $N - 1$ geometrically distinct solutions having minimal period in the interval (2).

When dealing with the case (b) some more care is needed. It is a matter of fact that, under assumptions similar to the ones of Theorem 1, the variational approach to the limiting case $\alpha = 1$ fails [10]. It seems then quite natural to introduce a further pinching condition which becomes more and more restrictive as $\alpha > 1$ converges to one. Similar pinching conditions have been used to prove existence and multiplicity results for the fixed period problem: [13], and [21].

We shall prove the following result:

THEOREM 2. – Let $F \in \mathcal{C}^1(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ satisfy:

$$\exists b \geq a > 0, \exists 2 > \alpha_2 \geq \alpha \geq \alpha_1 > 1, \text{ such that}$$

$$(H4) \quad \frac{a}{|x|^\alpha} \leq -F(x) \leq \frac{b}{|x|^\alpha}, \quad \forall x \in \mathbb{R}^N \setminus \{0\}$$

$$(H5) \quad -\alpha_1 F(x) \leq \nabla F(x) \cdot x \leq -\alpha_2 F(x), \quad \forall x \in \mathbb{R}^N \setminus \{0\}$$

$$(H6) \quad |\nabla F(x)| \leq \frac{a\alpha_2}{|x|^{\alpha+1}}, \quad \forall x \in \mathbb{R}^N \setminus \{0\}.$$

Then there exist three functions $\Psi^*(\alpha)$ and $\sigma_1\left(\frac{b}{a}, \alpha_2, \alpha, \alpha_1\right)$, $\sigma_2\left(\frac{b}{a}, \alpha_2, \alpha, \alpha_1\right)$ such that, when

$$(3) \quad \frac{b}{a} \left(\frac{(2-\alpha_1)\alpha}{(2-\alpha)\alpha_1}\right)^\alpha \frac{2-\alpha}{2-\alpha_2} < \Psi^*(\alpha),$$

$$(4) \quad \sigma_2\left(\frac{b}{a}, \alpha, \alpha_1, \alpha_2\right) \left(\frac{2-\alpha_1}{2}\right)^{1/\alpha} < 1,$$

and

$$(5) \quad \frac{b^{2/\alpha}}{a^{2/\alpha}} \frac{\alpha \alpha_2^2}{\alpha_1^3} \left(\frac{2-\alpha_1}{2-\alpha} \right)^2 \left(\frac{2-\alpha}{2-\alpha_2} \right)^{(2+\alpha)/\alpha} \frac{1}{\sigma_1^{2+\alpha}} < 4,$$

then, for every negative energy E , (P_E) has at least $N-1$ geometrically distinct solutions having minimal period in the interval

$$(6) \quad \left[\pi a^{1/\alpha} \sqrt{\alpha} \left(\frac{2-\alpha}{-2E} \right)^{((2-\alpha)/2)\alpha} \left(\frac{2-\alpha_2}{-\alpha_2 E} \right), \right. \\ \left. \pi b^{1/\alpha} \sqrt{\alpha} \left(\frac{2-\alpha}{-2E} \right)^{((2-\alpha)/2)\alpha} \left(\frac{2-\alpha_1}{-\alpha_1 E} \right) \right].$$

Moreover Ψ^* , σ_i ($i=1, 2$) enjoy the following properties:

$$\begin{aligned} \Psi^*(\alpha) &> 1, & \forall 1 < \alpha < 2 \\ \Psi^*(1) &= 1 \\ \lim_{\alpha \rightarrow 2} \Psi^*(\alpha) &= +\infty \\ \sigma_i \left(\frac{b}{a}, \alpha_2, \alpha, \alpha_1 \right) &> 0 \\ \lim_{(2-\alpha_1)b/(2-\alpha_2)a \rightarrow 1} \sigma_i \left(\frac{b}{a}, \alpha_2, \alpha, \alpha_1 \right) &= 1, & \forall \text{ fixed } \alpha \\ \lim_{\alpha \rightarrow 2} \sigma_i \left(\frac{b}{a}, \alpha_2, \alpha, \alpha_1 \right) &= 1 & \text{ if } \frac{(2-\alpha_1)b}{(2-\alpha_2)a} \text{ remains bounded.} \end{aligned}$$

The properties of Φ^* , σ_1 and σ_2 simply mean that, for each fixed $1 < \alpha < 2$, the field of conditions (3), (4) and (5) is non-empty, and that, when $\alpha \rightarrow 2$ they converge to the limit condition:

$$\frac{b^{2/\alpha}}{a^{2/\alpha}} \frac{\alpha \alpha_2^2}{\alpha_1^3} \left(\frac{2-\alpha_1}{2-\alpha} \right)^2 \left(\frac{2-\alpha}{2-\alpha_2} \right)^{(2+\alpha)/\alpha} \frac{1}{\sigma_1^{2+\alpha}} < 4.$$

We wish to point out that in both the situations of Theorems 1 and 2, the conditions on the sign of the energy are necessary conditions for the solvability of (P_E) .

Remark. — It is immediate to check that, when (H4) holds, the motion of each possible solution of (P_E) is constrained in the ball of radius $\left(\frac{b}{-E} \right)^{1/\alpha}$. Therefore all the hypotheses of Theorem 2 can be assumed to hold true just for every x in this ball. Therefore, the following result directly follows from Theorem 2:

COROLLARY 2. — Let $U \in \mathcal{C}^1(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ satisfy

$$\begin{aligned} &\exists \alpha, 1 < \alpha < 2 \text{ such that} \\ &\lim_{x \rightarrow 0} |x|^{\alpha+1} |\nabla U(x)| = 0, \end{aligned}$$

and let $F(x) = \frac{-a}{|x|^\alpha} + U(x)$ for some $a > 0$. Then there exists $\bar{E} < 0$ such that, for every $E < \bar{E}$, then (P_E) has at least $N - 1$ geometrically distinct solutions having minimal period in the interval (6).

In this paper we shall look for solutions of (P_E) as critical points of the functional

$$I(x) = \left(\frac{1}{2} \int_0^1 |\dot{x}|^2 \right) \left(\int_0^1 E - F(x) \right)$$

over its natural domain (taking into account the singularity of F)

$$\Lambda = \{ x \in H / x(t) \neq 0, \forall t \in \mathbb{R} \}$$

where H is the Sobolev space of all the H_{loc}^1 1-periodic functions. Indeed, to each critical point of I at a positive level there corresponds, up to the rescaling of the period, a solution to (P_E) having period λ , where

$$\lambda^2 = \frac{1/2 \int_0^1 |\dot{x}|^2}{\int_0^1 E - F(x)}$$

In order to obtain multiplicity of critical points, we shall exploit the invariance of the functional under the group of symmetries $G = \{ T_s, P_s \}$ where $T_s(x)(t) = x(s + t)$ and $P_s(x)(t) = x(s - t)$. In this order, a convenient tool to treat the problem in the case of Theorem 1 will turn out to be the homotopical index related to the group G , which has been introduced by the author in [21]. As a matter of fact, the homotopical index has been introduced in order to obtain multiple critical points for positive functionals having a singularity and a lack of coercivity at the level of the large constant functions. First of all, a suitable geometrical index i related to the group G is defined; the homotopical index (related to the geometrical index i) of a given set A measures (in terms of i) how big is the set that one has to remove from A in order to make it contractible, in a continuous and symmetry preserving way, into a set of large constant functions, without crossing the boundary of Λ .

As far as Theorem 2 is concerned, an additional difficulty arises from the fact that, when the energy is negative (and it is the natural choice when $\alpha < 2$), then the associated functional is unbounded (from below and from above). To overcome this, a new homotopical pseudo-index theory

is introduced. Roughly speaking, the homotopical pseudo-index of a given set is the homotopical index of its intersection with any deformation of a given closed set of the function space. This approach will turn out to be profitable to treat the case (b), since the restriction of the associated functional to each set of the type $\left\{x \mid \int_0^1 |\dot{x}|^2 = \rho^2\right\}$ is bounded from below and it presents a lack of compactness only at the level of the large constant functions.

Concerning set functions (indices, category and related topics) and applications to the search of multiple critical points of functionals we also quote [6], [7], [18] and [19].

This paper is organized as follows:

1. Definitions of the homotopical indices.
2. Abstract multiplicity results.
3. Computations of the homotopical indices.
4. The strong force case (Proof of Theorem 1).
5. The weak force case (Proof of Theorem 2).
6. Appendix (Proof of proposition 5.6).

NOTATIONS. — Throughout this paper we shall make use of the following notations:

$$\begin{aligned} \mathbf{B}^N &= \{x \in \mathbb{R}^N \mid |x| \leq 1\}; \\ \mathbf{S}^{N-1} &= \{x \in \mathbb{R}^N \mid |x| = 1\}. \end{aligned}$$

H denotes the Sobolev space of the 1-periodic functions:

$$\mathbf{H} = \{y \in \mathbf{H}_{\text{loc}}^1(\mathbb{R}; \mathbb{R}^N) \mid y(t+1) = y(t), \forall t \in \mathbb{R}\},$$

endowed with the Hilbertian norm

$$\|x\|^2 = \int_0^1 |\dot{x}|^2 + \int_0^1 |x|^2.$$

We shall consider the open subset of H defined by

$$\Lambda = \{y \in \mathbf{H} \mid y(t) \neq 0, \forall t \in \mathbb{R}\},$$

and we shall denote

$$\begin{aligned} \partial\Lambda &= \{y \in \mathbf{H} \mid \exists t \in \mathbb{R}, y(t) = 0\}, \\ 2^\Lambda &= \{A \subset \mathbf{H} \mid A \subseteq \Lambda\}. \end{aligned}$$

E_N denotes the subspace of H of all the constant functions. Moreover, in any metric space X, and for every subset $K \subseteq X$,

$$\begin{aligned} V_\varepsilon(K) &= \{x \in X \mid \text{dist}(x, K) < \varepsilon\} \\ N_\varepsilon(K) &= \{x \in X \mid \text{dist}(x, K) \leq \varepsilon\}. \end{aligned}$$

We shall denote by $\text{cl}(A)$ the closure of any subset A of X.

Finally, since we shall deal with critical points of the functional I, we shall denote

$$K_c = \{x \in H / I(x) = c, dI(x) = 0\}.$$

1. DEFINITION OF THE HOMOTOPICAL INDICES

In this section we define the homotopical indices and we state their basic properties. The homotopical index j has been defined in [21], in order to search multiple critical points for *positive* singular functionals having a lack of compactness at the level zero.

We are going to introduce the notion of homotopical pseudo-index in order to study critical points of *indefinite* singular functionals.

All the indices defined here are related to the group

$$G = \{P_s, T_s\}_{s \in [0, 1]},$$

where P_s and T_s are the unitary transformations of H respectively defined by:

$$(1.1) \quad \begin{cases} y = P_s(x) & \Leftrightarrow & y(t) = x(s-t), & \forall t \in [0, 1], \\ y = T_s(x) & \Leftrightarrow & y(t) = x(s+t), & \forall t \in [0, 1]. \end{cases}$$

Remark that the P_s and T_s are actually defined by periodicity for all $s \in \mathbb{R}$.

Consider the relations of H defined as

$$(1.2) \quad \begin{cases} y P x & \Leftrightarrow & \exists s \in [0, T] & \text{such that } y = P_s(x) \\ y T x & \Leftrightarrow & \exists s \in [0, T] & \text{such that } y = T_s(x). \end{cases}$$

We need to fix up some more notations. If $A \subseteq H$, we denote $P(A) = \{x \in H / \exists y \in A, x P y\}$ and $T(A) = \{x \in H / \exists y \in A, x T y\}$. A set A is P -invariant if $P(A) = A$, and it is T -invariant if $T(A) = A$.

A function $h: H \rightarrow H$ is said to be G -equivariant if $h \circ g = g \circ h, \forall g \in G$; for any set X , a function $h: H \rightarrow X$ is G -invariant if $h \circ g = h, \forall g \in G$. A set $A \subseteq H$ is G -invariant if $g(A) = A, \forall g \in G$. Two functions x, y in H are geometrically distinct if $y \neq P_s(x)$ and $y \neq T_s(x), \forall s \in [0, 1]$. F_0 denotes the set of all the fixed points of P :

$$(1.3) \quad F_0 = \{x \in H / x P x\}.$$

Remark that, by definition,

$$\begin{aligned} P_s &= T_s \circ P_0, & \forall s \in [0, 1] \\ T_s &= P_s \circ P_0, & \forall s \in [0, 1] \\ P_0^2 &= \text{id}. \end{aligned}$$

We are going to define the geometrical index as a function defined on a class of admissible sets and taking values in $\mathbb{N} \cup \{+\infty\}$. Let

$$(1.4) \quad \mathcal{B} = \{A \subseteq H \text{ closed} / A \text{ is } G\text{-invariant, } A \cap F_0 = \emptyset\},$$

and, for a fixed integer k , let

$$(1.5) \quad \mathcal{F}_k = \{f: H \rightarrow \mathbb{R}^k \text{ continuous} / xPy \Rightarrow f(x) = -f(y)\}.$$

DEFINITION 1.1. — *The index i is the function $i: \mathcal{B} \rightarrow \mathbb{N} \cup \{+\infty\}$ defined by:*

$$(1.6) \quad i(\emptyset) = 0$$

$$(1.7) \quad i(A) = +\infty \Leftrightarrow \forall k \in \mathbb{N}, \forall f \in \mathcal{F}_k, 0 \in f(A)$$

$$(1.8) \quad i(A) = k \Leftrightarrow k \text{ is the smallest integer such that } \exists f \in \mathcal{F}_k, 0 \notin f(A).$$

It has been shown in [21] that i satisfies the following properties:

PROPOSITION 1.1. — *Let $A, B \in \mathcal{B}$, then*

(i) *if $A \subseteq B$ then $i(A) \leq i(B)$;*

(ii) *$i(A \cup B) \leq i(A) + i(B)$;*

(iii) *if $h: A \rightarrow B$ is continuous and G equivariant then $i(A) \leq i(B)$;*

(iv) *if A is compact then $i(A) < +\infty$;*

(v) *if A is compact then $\exists \varepsilon > 0$ such that $i(N_\varepsilon(A)) = i(A)$;*

(vi) *if A contains only a finite number of geometrically distinct orbits then $i(A) = 1$.*

Let us consider

$$(1.9) \quad \mathcal{H} = \{h: H \times [0, 1] \rightarrow H \text{ continuous such that } \forall \sigma \in [0, 1], \forall g \in G\}.$$

For any $A \in \mathcal{B} \cap 2^\Lambda$ we define

$$(1.10) \quad \mathcal{H}_0(A) = \{h \in \mathcal{H} / h(A, 1) \subseteq E_N\}.$$

DEFINITION 1.2. — *Let $A \in 2^\Lambda$ be G -invariant. We say that A is G -contractible if there exists $h \in \mathcal{H}_0(A)$ such that $h(x, \sigma) \in \Lambda, \forall x \in A, \forall \sigma \in [0, 1]$.*

The homotopical index measures (in the sense of the geometrical index) how big is the set that one has to remove from A to make A be G -contractible.

DEFINITION 1.3. — *For a given compact $A \in \mathcal{B} \cap 2^\Lambda$, we say that the homotopical index of A is k ($j(A) = k$) if*

$$(1.11) \quad k = \min_{h \in \mathcal{H}_0(A)} i(\{x \in A / h(x, [0, 1]) \cap \partial\Lambda \neq \emptyset\}).$$

In the sequel we shall make use of a suitable version of Dugundji's extension theorem which has been proved in [21]:

LEMMA 1.1. — *Let $A \in \mathcal{B}$ be closed and let $h \in \mathcal{H}_0(A)$. Then $h|_{A \times [0, 1]}$ admits a continuous extension \bar{h} such that $\bar{h} \in \mathcal{H}_0(H)$.*

The main properties of the homotopical index can be summarized in the following proposition (we refer to [21] for proofs and comments):

PROPOSITION 1.2. — *Let $A, B \in \mathcal{B} \cap 2^\Lambda$, then*

- (i) *if $A \subseteq B$ then $j(A) \leq j(B)$;*
- (ii) *$j(A \cup B) \leq j(A) + i(B)$;*
- (iii) *if $h \in \mathcal{H}$ is such that $h(A \times [0, 1]) \cap \partial\Lambda = \emptyset$ then $j(A) \leq j(h(A, 1))$;*
- (iv) *if A is compact then $j(A) < +\infty$;*
- (v) *if A is compact then $\exists \varepsilon > 0$ such that $j(N_\varepsilon(A)) = j(A)$.*

Now we are in a position to introduce the homotopical pseudo-index. The homotopical pseudo-index of a set A measures, in terms of the homotopical pseudo-index, the intersection properties of A with respect to a given closed G -invariant subset of H and a class of G -equivariant homotopies. This notion is quite similar to the notion of geometrical pseudo-index of [6].

More precisely, let Σ_1 and Σ_2 be closed G -invariant subsets of H such that

$$\Sigma_1 \cap \Sigma_2 = \emptyset.$$

Consider

$$(1.12) \quad \mathcal{H}_{\Sigma_1, \Sigma_2}^* = \{h \in \mathcal{H} \mid h(\cdot, s) \text{ is a } G\text{-equivariant homeomorphism.} \\ h(x, s) = x, \forall x \in (\Sigma_1 \cap \partial\Lambda) \cup \Sigma_2, \forall s \in [0, 1]\},$$

DEFINITION 1.4. — *Let $A \in \mathcal{B}$, $A \cap (\Sigma_1 \cap \partial\Lambda) = \emptyset$. We say that the homotopical pseudo-index of A is k [and we write $j^*(A) = k$] if*

$$(1.13) \quad k = \min_{h \in \mathcal{H}_{\Sigma_1, \Sigma_2}^*} j(h(A, 1) \cap \Sigma_1).$$

Remark 1.1. — This definition makes sense; indeed, from (1.12) one has that $h(A, 1) \cap (\Sigma_1 \cap \partial\Lambda) = \emptyset$, for each $h \in \mathcal{H}_{\Sigma_1, \Sigma_2}^*$.

PROPOSITION 1.3. — *Let $A, B \in \mathcal{B}$ be such that $(A \cup B) \cap (\Sigma_1 \cap \partial\Lambda) = \emptyset$, then*

- (i) *if $A \subseteq B$ then $j^*(A) \leq j^*(B)$;*
- (ii) *$j^*(A \cup B) \leq j^*(A) + i(B)$;*
- (iii) *if $h \in \mathcal{H}_{\Sigma_1, \Sigma_2}^*$ then $j^*(A) = j^*(h(A, 1))$;*
- (iv) *if A is compact then $j^*(A) < +\infty$;*
- (v) *if A is compact then $\exists \varepsilon > 0$ such that $j^*(N_\varepsilon(A)) = j^*(A)$.*

Proof: (i) Indeed, for any $h \in \mathcal{H}_{\Sigma_1, \Sigma_2}^*$ one has that $h(A, 1) \subseteq h(B, 1)$, so that by proposition 1.2(i) one deduces that

$$j(h(A, 1) \cap \Sigma_1) \leq j(h(B, 1) \cap \Sigma_1).$$

One concludes just by taking the minimum of both sides for every $h \in \mathcal{H}_{\Sigma_1, \Sigma_2}^*$.

(ii) Let $h \in \mathcal{H}_{\Sigma_1, \Sigma_2}^*$ be fixed. From Propositions 1.1(ii), 1.1(iii) and 1.2(ii) one has that

$$j(h(A \cup B, 1) \cap \Sigma_1) \leq j((h(A, 1) \cap \Sigma_1) \cup h(B, 1))$$

$$j(h(A, 1) \cap \Sigma_1) + i(h(B, 1)) = j(h(A, 1) \cap \Sigma_1) + i(B).$$

As before, one concludes by minimizing for every $h \in \mathcal{H}_{\Sigma_1, \Sigma_2}^*$.

(iii) easily follows from the fact that $\mathcal{H}_{\Sigma_1, \Sigma_2}^*$ is a group with respect to the juxtaposition of homotopies.

(iv) is a direct consequence of the following inequality:

$$j^*(A) \leq j(A \cap \Sigma_1) \leq i(A \cap \Sigma_1).$$

(v) easily follows from Proposition 1.2(v).

2. ABSTRACT MULTIPLICITY THEOREMS

In this section the homotopical indices are shown to be profitable tools in order to find a multiplicity of critical points for singular functionals. The homotopical index j has been introduced in [21] for the search of multiple solutions in the case of *positive* singular functionals having a lack of coercivity at the level of large constant functions. This is, for example, the situation in the setting of Theorem 1. On the other hand, the homotopical pseudo-index j^* is introduced in order to treat *unbounded* (above and below) functionals, whose restrictions on a given closed subset of H present a lack of coercivity only at the level of the large constant functions. This will be the abstract setting in the proof of Theorem 2.

In the setting of the following Theorem 2.1, I is a G -invariant functional defined on the set Λ of all the noncollision functions of H :

$$\Lambda = \{x \in H / x(t) \neq 0, \forall t \in \mathbb{R}\}.$$

We denote, for any $c \in \mathbb{R}$,

$$K_c^\Lambda = \{x \in \Lambda / I(x) = c, dI(x) = 0\}.$$

Let us recall the Palais-Smale compactness condition (at the level c):

$$(PS_c) \left\{ \begin{array}{l} \text{every sequence } (x_n)_n \text{ in } \Lambda \text{ such that} \\ \lim_{n \rightarrow +\infty} I(x_n) = c \\ \lim_{n \rightarrow +\infty} dI(x_n) = 0 \\ \text{possesses a subsequence converging to some limit in } H. \end{array} \right.$$

In the situation of Theorem 1 (definite functionals), we shall make use of the following result. This theorem has been proved in [21]:

THEOREM 2.1. — *Let $I \in C^1(\Lambda; \mathbb{R})$ be a G invariant functional, admitting a lower semicontinuous extension $\bar{I}: H \rightarrow \mathbb{R} \cup \{+\infty\}$. Assume that*

$$(2.1) \quad \bar{I} \geq 0, \quad \forall x \in H$$

$$(2.2) \quad \begin{cases} \exists A \in \mathcal{B} \cap 2^\Lambda, \exists k \geq 1 \text{ such that} \\ j(A) \geq k, \quad \sup_A I \leq c_0 = \inf_{\partial \Lambda} \bar{I}. \end{cases}$$

Hence, for $1 \leq r \leq k$, the classes

$$\Gamma_r = \{ A \in \mathcal{B} \cap 2^\Lambda \text{ compact} / j(A) \geq r, \sup_A I \leq c_0 \}$$

are nonempty and we define

$$c_r = \inf_{A \in \Gamma_r} \sup_A I, \quad r = 1, \dots, k;$$

then one has that $0 \leq c_1 \leq \dots \leq c_k \leq c_0$.

Assume moreover that

$$(2.3) \quad (PS_{c_r}) \quad (r = 1, \dots, k)$$

$$(2.4) \quad K_{c_r} \cap F_0 = \emptyset \quad (r = 1, \dots, k)$$

hold. Then I has at least k geometrically distinct critical points in $\{x \in \Lambda / c_1 \leq I(x) \leq c_k\}$.

We refer to [21] for the proof.

A weaker condition than the Palais-Smales's, but still sufficient to obtain critical points is the Cerami condition (at the level c):

$$(C_c) \quad \left\{ \begin{array}{l} \text{every sequence } (x_n)_n \text{ in } H \text{ such that} \\ \lim_{n \rightarrow +\infty} I(x_n) = c \\ \lim_{n \rightarrow +\infty} (1 + \|x_n\|) \|dI(x_n)\| = 0 \\ \text{possesses a subsequence converging to some limit in } H. \end{array} \right.$$

Condition (C_c) will turn out to be easier to verify in the setting of Theorem 2.

In the following, we shall denote

$$K_c = \{x \in H / I(x) = c, dI(x) = 0\}.$$

According with the notations of section 1, let Σ_1 and Σ_2 be closed G -invariant subsets of H such that

$$\Sigma_1 \cap \Sigma_2 = \emptyset,$$

and consider

$$\mathcal{H}_{\Sigma_1, \Sigma_2}^* = \{h \in \mathcal{H}/h(\cdot, s) \text{ is a G-equivariant homeomorphism,} \\ h(x, s) = x, \forall x \in (\Sigma_1 \cap \partial\Lambda) \cup \Sigma_2, \forall s \in [0, 1]\}.$$

Now we state the suitable version of the Deformation Lemma:

LEMMA 2.2. — *Let $I \in \mathcal{C}^1(H; \mathbb{R})$ be a G-invariant functional. Let $c^* \in \mathbb{R}$ such that*

$$(2.5) \quad \inf_{x \in \Sigma_1 \cap \partial\Lambda} I(x) > c^* > \sup_{x \in \Sigma_2} I(x).$$

Assume that, for some $\gamma > 0$, I satisfies the condition $(C)_c$, for all $c \in [c^ - \gamma, c^* + \gamma]$. Then there exists ε_0 such that, for every $\varepsilon \leq \varepsilon_0$, there are $\delta > 0$ and $\eta \in \mathcal{H}_{\Sigma_1, \Sigma_2}^*$ such that, if $N = N_\varepsilon(K_{c^*})$, then:*

$$(2.6) \quad \sup_A I \leq c^* + \delta \Rightarrow \sup_{\eta(A \setminus N, 1)} I \leq c^* - \delta.$$

Moreover, if $K_{c^} = \emptyset$, then*

$$(2.7) \quad \sup_A I \leq c^* + \delta \Rightarrow \sup_{\eta(A, 1)} I \leq c^* - \delta.$$

Proof. — The proof is standard (see for instance [6]).

Now we are in a position to prove the main goal of this section: for every integer r we denote

$$\Gamma_r^* = \{A \in \mathcal{B} \text{ compact, } A \cap (\Sigma_1 \cap \partial\Lambda) = \emptyset / j^*(A) \geq r\}.$$

THEOREM 2.2. — *Let $I \in C^1(H; \mathbb{R})$. Assume that there exists $\Sigma_1, \Sigma_2 \subseteq H$ closed, $k \in \mathbb{N}$, and $c > 0$ such that*

$$(2.8) \quad \Gamma_k^* \neq \emptyset.$$

$$(2.9) \quad \sup_A I \geq c > 0, \quad \forall A \subseteq \Sigma_1, \quad j(A) \geq 1.$$

Then the numbers

$$c_r^* = \inf_{A \in \Gamma_r^*} \sup_A I$$

are well defined. Assume moreover that

$$(2.10) \quad \inf_{\Sigma_1 \cap \partial\Lambda} I > c_r^* > \sup_{\Sigma_2} I \quad (r = 1, \dots, k)$$

and that

$$(2.11) \quad (C)_{c_r^*} \quad (r = 1, \dots, k)$$

$$(2.12) \quad K_{c_r^*} \cap F_0 = \emptyset \quad (r = 1, \dots, k)$$

hold. Then I has at least k geometrically distinct critical points in $\{x \in H / c_1^ \leq I(x) \leq c_k^*\}$.*

Proof. — First of all, we claim that the c_r^* 's are critical levels for every $r = 1, \dots, k$. Indeed, the classes Γ_r^* are invariant under all the homotopies of $\mathcal{H}_{\Sigma_1, \Sigma_2}^*$. Hence, assuming that $K_{c_r^*} = \emptyset$ for some r , one deduces from the second part of Lemma 2.2 [which can be applied because of (2.10) and (2.11)] that there is $A \in \Gamma_r^*$ such that $\sup_A I < c_r^*$, in contradiction with the definition of c_r^* .

Hence, if $c_1^* < \dots < c_k^*$ the proof is complete. Now we assume that, for some $r, h \geq 1, c_r^* = \dots = c_{r+h}^* = c^*$ holds, and we claim that $i(K_{c^*}) \geq h + 1$ [observe that, by (2.12), the index i of K_{c^*} is well defined]. This fact will end the proof, taking into account of Proposition 1.1 (vi).

Let us fix $\varepsilon > 0$ such that Lemma 2.1 holds, and let $A \in \Gamma_{r+h}^*$ be such that $\sup_A I \leq c^* + \delta$. Let $N = N_\varepsilon(K_{c^*})$ and η be as in Lemma 2.1: then one deduces from (2.6) that $j^*(\text{cl}(\eta(A \setminus N, \sigma))) \leq r - 1$. Remark that, from Proposition 2.3 (iii), one has that $j^*(\text{cl}(\eta(A \setminus N, 1))) = j^*(\text{cl}(A \setminus N))$, since $\eta \in \mathcal{H}_{\Sigma_1, \Sigma_2}^*$. On the other hand, it follows from Proposition 2.3 (ii) that $j^*(A) \leq j^*(\text{cl}(A \setminus N)) + i(A \cap N)$, so that $i(N) \geq i(A \cap N) \geq h + 1$. The proof is then complete by virtue of Proposition 2.1 (v) [indeed, for ε small, one has that $i(N) = i(K_{c^*})$].

3. COMPUTATION OF THE INDICES

In this section we provide examples of sets having nontrivial (*i.e.* larger than 1) homotopical indices. To this end we shall make use of some results of [21] about the homotopical index; we refer to that paper for the proof of Theorem 3.1.

We denote by C_N the set of all the great circles of S^{N-1} :

$$(3.1) \quad C_N = \{z \in H / z(t) = x \sin 2\pi t + y \cos 2\pi t, |x| = |y| = 1, x \cdot y = 0\}.$$

The following results have been proved in [21].

THEOREM 3.1. — *For every $N \in \mathbb{N}$,*

$$(3.2) \quad j(C_N) = N - 1.$$

Proof. — The proof is contained in [21]. We recall here the main steps. The results of the Step 2 will be also used in proving the following Theorem 3.2.

We first prove the inequality $j(C_N) \geq N - 1$ (the reversed inequality is easier to prove).

Step 1. — In order to prove that $j(C_N) \geq N - 1$, by Definitions 1.1, 1.2 and 1.3, we have to prove that

$$(3.3) \quad \begin{cases} \forall f \in \mathcal{F}_{N-2}, \quad \forall h \in \mathcal{H}_0(C_N), \\ 0 \in f(\{z \in C_N / h(z, [0, 1]) \cap \partial\Lambda \neq \emptyset\}). \end{cases}$$

Thus let $f \in \mathcal{F}_{N-2}$ and $h \in \mathcal{H}_0(C_N)$ be fixed. Let $e_1 \in S^{N-1}$ and consider $S^{N-2} = \{x \in S^{N-1} / x \cdot e_1 = 0\}$. Consider the continuous parametrization $F' : S^{N-2} \times S^{N-2} \rightarrow \Lambda$ of C_N defined by

$$(3.4) \quad F'(x, y)(t) = x \cos 2\pi t + ((x \cdot y)e_1 + y - (y \cdot x)x) \sin 2\pi t,$$

and denote by F the associated continuous function

$$F : S^{N-2} \times S^{N-2} \times S^1 \rightarrow \mathbb{R}^N$$

defined by

$$(3.5) \quad \begin{cases} F(x, y, e^{2\pi it}) = F'(x, y)(t), \\ \forall (x, y, \tau e^{2\pi it}) \in S^{N-2} \times S^{N-2} \times S^1. \end{cases}$$

To each $h \in \mathcal{H}_0(C_N)$ we associate the continuous extension of F , $\Phi : S^{N-2} \times S^{N-2} \times B^2 \rightarrow \mathbb{R}^N$ given by

$$(3.6) \quad \begin{cases} \Phi(x, y, \tau e^{2\pi it}) = h(F'(x, y), 1 - \tau)(t), \\ \forall (x, y, \tau e^{2\pi it}) \in S^{N-2} \times S^{N-2} \times B^2. \end{cases}$$

Notice that the definition of $\mathcal{H}_0(C_N)$ (1.10) provides the continuity of Φ .

Now define $\varphi : S^{N-2} \times S^{N-2} \times B^2 \rightarrow \mathbb{R}^{N-2}$ as

$$(3.7) \quad \begin{cases} \varphi(x, y, \tau e^{2\pi it}) = f(F'(x, y)), \\ \forall (x, y, \tau e^{2\pi it}) \in S^{N-2} \times S^{N-2} \times B^2. \end{cases}$$

Then one immediately checks that, when

$$(3.8) \quad 0 \in \Phi \times \varphi(S^{N-2} \times S^{N-2} \times B^2)$$

holds, then (3.3) is satisfied.

Step 2. – From the definition, one has that

$$(3.9) \quad \begin{cases} \Phi(x, y, e^{2\pi it}) = x \cos 2\pi t + ((x \cdot y)e_1 + y - (y \cdot x)x) \sin 2\pi t, \\ \forall (x, y, e^{2\pi it}) \in (S^{N-2} \times S^{N-2} \times S^1) \end{cases}$$

so that

$$(3.10) \quad 0 \notin \Phi \times \varphi(S^{N-2} \times S^{N-2} \times S^1).$$

Moreover, the G -equivariance of F' leads to

$$(3.11) \quad \begin{cases} \Phi(x, -y, \tau e^{i2\pi t}) = \Phi(x, y, \tau e^{-i2\pi t}) \\ \Phi(-x, y - 2(y - (y \cdot x)x), \tau e^{i2\pi t}) = \Phi(x, y, \tau e^{i2\pi(1/2+t)}) \\ \forall (x, y, \tau e^{i2\pi t}) \in S^{N-2} \times S^{N-2} \times B^2, \end{cases}$$

and (3.7) together with (1.5) implies

$$(3.12) \quad \begin{cases} \varphi(x, -y, \tau e^{i2\pi t}) = -\varphi(x, y, \tau e^{i2\pi t}) \\ \varphi(-x, y - 2(y - (y \cdot x)x), \tau e^{i2\pi t}) = \varphi(x, y, \tau e^{i2\pi(1/2+t)}) \\ \forall (x, y, \tau e^{i2\pi t}) \in S^{N-2} \times S^{N-2} \times B^2. \end{cases}$$

From these symmetry properties one deduces that the zeroes of $\Phi \times \varphi$ appear in 4-ples of the type

$$(3.13) \quad \left\{ (x, y, \tau e^{i2\pi t}), (x, -y, \tau e^{-i2\pi t}), \right. \\ \left. (-x, -y + 2(y - (y \cdot x)x), e^{i2\pi(1/2-t)}), \right. \\ \left. (x, y - 2(y - (y \cdot x)x), e^{i2\pi(1/2+t)}) \right\}.$$

By the perturbation arguments used in [21], next Proposition is easily proved:

PROPOSITION 3.1. — *Let*

$$\Phi : S^{N-2} \times S^{N-2} \times B^2 \rightarrow \mathbb{R}^N \quad \text{and} \quad \varphi : S^{N-2} \times S^{N-2} \times B^2 \rightarrow \mathbb{R}^{N-2}$$

be continuous satisfying (3.9), (3.11) and (3.12). Then

(i) $0 \in \Phi \times \varphi (S^{N-2} \times S^{N-2} \times B^2)$,

(ii) $\forall |z| < 1, \forall \delta > 0 \exists z_\delta$ and $\exists \varphi_\delta$ satisfying (3.12) such that $|z_\delta - z| < \delta, \|\varphi_\delta - \varphi\|_\infty < \delta$; and $((\Phi - z_\delta) \times \varphi_\delta)^{-1}(0)$ consists in an odd number of 4-ples of the type of (3.13).

Hence the inequality $j(C_N) \geq N - 1$ is proved.

To prove the reversed inequality, consider the homotopy $h_1 : C_N \times [0, 1] \rightarrow H$,

$$(3.14) \quad h_1(z, \sigma) = \begin{cases} -8\sigma e_1 + z, & \text{if } 0 \leq \sigma \leq \frac{1}{2} \\ -4e_1 + 2(1 - \sigma)z, & \text{if } \frac{1}{2} \leq \sigma \leq 1, \end{cases}$$

and extend it by Lemma 1.1 to $\bar{h}_1 \in \mathcal{H}_0(H)$. Then the set $\{z \in C_N / \bar{h}_1(z, [0, 1]) \cap \partial\Lambda \neq \emptyset\}$ can be parametrized by means of the following continuous function $g : S^{N-2} \times S^1 \rightarrow C_N$:

$$(3.15) \quad g(x, \theta) = e_1 \cos 2\pi(t + \theta) + x \sin 2\pi(t + \theta).$$

One then concludes by Borsuk's Theorem that

$$i(\{z \in C_N / \bar{h}_1(z, [0, 1]) \cap \partial\Lambda \neq \emptyset\}) = N - 1.$$

Finally, from Definition 1.3 and the first part of this proof, this fact implies that $j(C_N) = N - 1$.

Remark 3.1. — Of course $j(\tau C_N) = N - 1$, for every $\tau > 0$. Moreover, Proposition 3.1 holds, for every $\tau \Phi \times \varphi$.

Let us fix up some more notations: let $\rho > 0$ and $0 < \varepsilon < 1$ be fixed and consider

$$(3.16) \quad \Sigma_1 = \left\{ x \in H \mid \frac{1}{2} \int_0^1 |\dot{x}|^2 = 2\pi^2 \rho^2 \right\},$$

$$(3.17) \quad \Sigma_2 = \left\{ x \in H / \frac{1}{2} \int_0^1 |\dot{x}|^2 = 2\pi^2 \rho^2 \varepsilon^2 \right\} \cup \left\{ x \in H / \frac{1}{2} \int_0^1 |\dot{x}|^2 = 2\pi^2 \rho^2 \varepsilon^{-2} \right\},$$

and let

$$(3.18) \quad C_N^* = [\varepsilon, \varepsilon^{-1}] \rho C_N.$$

The following theorem holds:

THEOREM 3.2. — *For every $0 < \varepsilon < 1$, $\rho > 0$ and $N \in \mathbb{N}$,*

$$(3.19) \quad j^*(C_N^*) = N - 1.$$

Proof. — It is not difficult to see that $j^*(C_N^*) \leq j(C_N)$, and hence that $j^*(C_N^*) \leq N - 1$. Thus we have to prove the reversed inequality. From Definitions 1.1, 1.2, 1.3 and 1.4, and from Lemma 1.1, proving it is equivalent to prove that

$$(3.20) \quad \begin{cases} \forall h_1 \in \mathcal{H}_{\Sigma_1, \Sigma_2}^*, \quad \forall h_2 \in \mathcal{H}_0(H), \quad \forall f \in \mathcal{F}_{N-2}, \\ \mathbf{0} \in f(\{z \in h_1(C_N^*, 1) \cap \Sigma_1 / h_2(z, [0, 1]) \cap \partial\Lambda \neq \emptyset\}). \end{cases}$$

Let $h_1 \in \mathcal{H}_{\Sigma_1, \Sigma_2}^*$, $h_2 \in \mathcal{H}_0(H)$ and $f \in \mathcal{F}_{N-2}$ be fixed. Let $e_1 \in S^{N-1}$ be fixed and consider $S^{N-2} = \{x \in S^{N-1} / x \cdot e_1 = 0\}$, and consider the continuous $F^* : S^{N-2} \times S^{N-2} \times S^1 \times [\varepsilon\rho, \varepsilon^{-1}\rho] \rightarrow \mathbb{R}^N$, defined by

$$(3.21) \quad F^*(x, y, e^{2\pi i t}, \lambda) = \lambda F(x, y, e^{2\pi i t}),$$

where F is defined in (3.5). F^* induces a parametrization of C_N^* ,

$$\bar{F}^* : S^{N-2} \times S^{N-2} \times [\varepsilon\rho, \varepsilon^{-1}\rho] \rightarrow C_N^*.$$

Define the continuous

$$\Phi^* : S^{N-2} \times S^{N-2} \times B^2 \times [\varepsilon\rho, \varepsilon^{-1}\rho] \rightarrow \mathbb{R}^N,$$

and

$$\begin{aligned} \varphi^* &: S^{N-2} \times S^{N-2} \times B^2 \times [\varepsilon\rho, \varepsilon^{-1}\rho] \rightarrow \mathbb{R}^{N-2}, \\ \xi^* &: S^{N-2} \times S^{N-2} \times B^2 \times [\varepsilon\rho, \varepsilon^{-1}\rho] \rightarrow \mathbb{R} \end{aligned}$$

as

$$(3.22) \quad \Phi^*(x, y, \tau e^{2\pi i t}, \lambda) = \begin{cases} h_1(\bar{F}^*(x, y, \lambda), 2 - 2\tau)(t) & \text{if } \frac{1}{2} \leq \tau \leq 1 \\ h_2(h_1(\bar{F}^*(x, y, \lambda), 1), 1 - 2\tau)(t) & \text{if } 0 \leq \tau \leq \frac{1}{2}. \end{cases}$$

$$(3.23) \quad \varphi^*(x, y, \tau e^{2\pi i t}, \lambda) = f(h_1(\bar{F}^*(x, y, \lambda), 1)),$$

and

$$(3.24) \quad \xi^*(x, y, \tau e^{i 2 \pi t}, \lambda) = \begin{cases} \int_0^1 \left| \frac{d}{dt} (h_1(\bar{F}^*(x, y, \lambda), 2 - 2\tau)) \right|^2 dt & \text{if } \frac{1}{2} \leq \tau \leq 1, \\ \int_0^1 \left| \frac{d}{dt} (h_1(\bar{F}^*(x, y, \lambda), 1)) \right|^2 dt & \text{if } 0 \leq \tau \leq \frac{1}{2}. \end{cases}$$

It is immediate to check the continuity of φ^* and ξ^* ; the continuity of Φ^* follows from (1.10) and (1.12). We claim that if

$$(3.25) \quad (0, 0, 2\pi^2 \rho^2) \in \Phi^* \times \varphi^* \times \xi^* (S^{N-2} \times S^{N-2} \times B^2 \times [\varepsilon\rho, \varepsilon^{-1}\rho]),$$

then (3.20) is satisfied. Indeed, let

$$\Phi^* \times \varphi^* \times \xi^* (x_0, y_0, \tau_0 e^{2\pi i t_0}, \lambda_0) = (0, 0, 2\pi^2).$$

Then, from (1.12) and (3.22) (remember that $h_1 \in \mathcal{H}_{\Sigma_1, \Sigma_2}^*$), we obtain that $0 \leq \tau_0 \leq \frac{1}{2}$. Setting

$$z_0 = h_1(\bar{F}^*(x_0, y_0, \lambda_0), 1),$$

from (3.22), (3.23) and (3.24) we deduce

$$\begin{aligned} f(z_0) &= 0 \\ z_0 &\in h_1(C_N^*, [0, 1]) \cap \Sigma_1 \\ h_2(z_0, 1 - 2\tau_0)(t_0) &= 0; \end{aligned}$$

that is (3.20) holds.

In order to prove (3.25), we first remark that, by the G-equivariance of h_1 and h_2 , and from (1.5), the following symmetry properties hold:

$$(3.26) \quad \begin{cases} \Phi^*(x, -y, \tau e^{i 2 \pi t}, \lambda) = \Phi(x, y, \tau e^{-i 2 \pi t}, \lambda) \\ \Phi^*(-x, y - 2(y \cdot x)x, \tau e^{i 2 \pi t}, \lambda) = \Phi^*(x, y, \tau e^{i 2 \pi (1/2+t)}, \lambda) \\ \forall (x, y, \tau e^{i 2 \pi t}, \lambda) \in S^{N-2} \times S^{N-2} \times B^2 \times [\varepsilon\rho, \varepsilon^{-1}\rho], \end{cases}$$

and

$$(3.27) \quad \begin{cases} \varphi^*(x, -y, \tau e^{i 2 \pi t}, \lambda) = -\varphi^*(x, y, \tau e^{i 2 \pi t}, \lambda) \\ \varphi^*(-x, y - 2(y \cdot x)x, \tau e^{i 2 \pi t}, \lambda) = \varphi^*(x, y, \tau e^{i 2 \pi (1/2+t)}, \lambda) \\ \forall (x, y, \tau e^{i 2 \pi t}, \lambda) \in S^{N-2} \times S^{N-2} \times B^2 \times [\varepsilon\rho, \varepsilon^{-1}\rho]. \end{cases}$$

By the same perturbations arguments used in [21] in proving Theorem 3.1, for every $\delta > 0$, one can find z_δ and φ_δ^* with $z_\delta < \delta$, $|\varphi_\delta^* - \varphi^*|_\infty < \delta$ and φ_δ^* still satisfying (3.27) such that:

- (a) $(0, 0)$ is a regular value for $(\Phi^* - z_\delta) \times \varphi_\delta^*$;
- (b) $(0, 0)$ is a regular value for both

$$((\Phi^* - z_\delta) \times \varphi_\delta^*)|_{S^{N-2} \times S^{N-2} \times B^2 \times \{\varepsilon\rho\}}$$

and

$$((\Phi^* - z_\delta) \times \varphi_\delta^*)|_{S^{N-2} \times S^{N-2} \times B^2 \times \{\varepsilon^{-1} \rho\}}.$$

Indeed, one is in a position to apply Proposition 3.1.

Hence $((\Phi^* - z_\delta) \times \varphi_\delta^*)^{-1}(0, 0)$ consists in a finite number of 1-manifolds. From the symmetry of the problem, these 1-manifolds appear in 4-ples of the type

$$(3.28) \quad \left\{ \begin{array}{l} (x(\sigma), y(\sigma), \tau(\sigma) e^{i 2 \pi t(\sigma)}, (x(\sigma), -y(\sigma), \tau(\sigma) e^{-i 2 \pi t(\sigma)}), \\ (-x(\sigma), -y(\sigma) + 2(y(\sigma) - (y(\sigma) \cdot x(\sigma)) x(\sigma)), \tau(\sigma) e^{i 2 \pi (1/2 - t(\sigma))}), \\ (-x(\sigma), y(\sigma) - 2(y(\sigma) - (y(\sigma) \cdot x(\sigma)) x(\sigma)), e^{i 2 \pi (1/2 + t(\sigma))}) \end{array} \right\}_{\sigma \in [0, 1]}.$$

It is a well-known fact that a compact 1-manifold imbedded in a compact manifold is either homeomorphic to S^1 or starts and dies on the boundary. From (3.22), each 1-manifold can intersect the boundary only in $S^{N-2} \times S^{N-2} \times B^2 \times \{\varepsilon \rho, \varepsilon^{-1} \rho\}$.

Moreover both

$$((\Phi^* - z_\delta) \times \varphi_\delta^*)^{-1}(0) \cap (S^{N-2} \times S^{N-2} \times B^2 \times \{\varepsilon \rho\})$$

and

$$((\Phi^* - z_\delta) \times \varphi_\delta^*)^{-1}(0) \cap (S^{N-2} \times S^{N-2} \times B^2 \times \{\varepsilon^{-1} \rho\})$$

consist in an odd number of 4-ples of the type of (3.13). One can assume without loss of generality that the 1-manifolds starting at the boundary intersect it transversally.

Since two symmetric 1-manifolds (i. e. belonging to the same 4-ple) can never intersect, one can conclude that at least one 4-ple of solutions has to connect $S^{N-2} \times S^{N-2} \times B^2 \times \{\varepsilon \rho\}$ with $S^{N-2} \times S^{N-2} \times B^2 \times \{\varepsilon^{-1} \rho\}$.

Let us denote by $(x(\sigma), y(\sigma), \rho(\sigma) e^{2 \pi i t(\sigma)}, \lambda(\sigma))_{\sigma \in [0, 1]}$ one of such 1-manifolds: then one has that $\lambda(0) = \varepsilon \rho$ and $\lambda(1) = \varepsilon^{-1} \rho$, so that from (3.24) one deduces that

$$\begin{aligned} \xi^*(x(0), y(0), \rho(0) e^{2 \pi i t(0)}, \lambda(0)) \\ = 2 \pi^2 \varepsilon^2 \rho^2 < 2 \pi^2 \rho^2 < 2 \pi^2 \varepsilon^{-2} \rho^2 \xi^*(x(1), y(1), \rho(1) e^{2 \pi i t(1)}, \lambda(1)). \end{aligned}$$

Hence a $\sigma_0 \in [0, 1]$ exists such that $\xi^*(x(\sigma_0), y(\sigma_0), \rho(\sigma_0) e^{2 \pi i t(\sigma_0)}, \lambda(\sigma_0)) = 2 \pi^2 \rho^2$.

Therefore

$$\Phi^* \times \varphi^* \times \xi^*(x(\sigma_0), y(\sigma_0), \rho(\sigma_0) e^{2 \pi i t(\sigma_0)}, \lambda(\sigma_0)) = (0, 0, 2 \pi^2 \rho^2),$$

that is, (3.25) is satisfied.

4. THE STRONG FORCE CASE (PROOF OF THEOREM 1)

Throughout this section, the energy E is fixed positive. Then the solutions of (P_E) correspond to the critical points of the functional

$$I(x) = \left(\frac{1}{2} \int_0^1 |\dot{x}|^2 \right) \left(\int_0^1 E - F(x) \right)$$

in the set $\{x \in \Lambda / I(x) > 0\}$.

To carry out the proof of Theorem 1, we are first going to add to the functional I a term inducing the strong force. Then we shall prove some preliminary propositions, that will allow the application of Theorem 2.1. By virtue of an *a priori* estimate (Proposition 4.1) we shall be in a position to conclude that the critical points found by Theorem 2.1 are actually (up to a rescaling) solutions of (P_E) .

To this end, we start with the following:

DEFINITION 4.1. — For any $\varepsilon > 0$, $V_\varepsilon \in \mathcal{C}^2(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ denotes a function such that

$$(4.1) \quad V_\varepsilon(x) \geq \frac{1}{|x|^2} \quad \text{if } 0 < |x| \leq \varepsilon$$

$$(4.2) \quad V_\varepsilon(x) > 0 \quad \text{if } 0 < |x| < 2\varepsilon$$

$$(4.3) \quad V_\varepsilon(x) = 0 \quad \text{if } |x| \geq 2\varepsilon$$

$$(4.4) \quad \nabla V_\varepsilon(x) \cdot x \leq -2V_\varepsilon(x), \quad \forall x \in \mathbb{R}^N - \{0\}.$$

Define

$$I_\varepsilon(x) = I(x) + \int_0^1 V_\varepsilon(x).$$

PROPOSITION 4.1 (A priori estimate). — Assume (H1), (H2) hold and let $x \in \Lambda$ be a critical point of I_ε , such that $I_\varepsilon(x) > 0$. If

$$(4.5) \quad (2\varepsilon)^\alpha \leq \frac{(\alpha_1 - 2)a}{2E}$$

then

$$(4.6) \quad |x(t)| \geq 2\varepsilon, \quad \forall t \in [0, 1].$$

Proof. — Let x be such a critical point. We have

$$(4.7) \quad \left(\frac{1}{2} \int_0^1 |\dot{x}|^2 \right) \left(\int_0^1 E - F(x) \right) + \int_0^1 V_\varepsilon(x) = c > 0$$

$$(4.8) \quad - \left(\int_0^1 E - F(x) \right) \ddot{x} = \left(\frac{1}{2} \int_0^1 |\dot{x}|^2 \right) \nabla F(x) - \nabla V_\varepsilon(x).$$

First, we deduce from (4.7), (4.8), (4.2) and (4.4) that x is not constant. From (4.8) we have the energy integral

$$(4.9) \quad \frac{1}{2} |\dot{x}|^2 \left(\int_0^1 E - F(x) \right) + F(x) \left(\frac{1}{2} \int_0^1 |\dot{x}|^2 \right) - V_\varepsilon(x) = h,$$

and by integrating (4.9) we find

$$(4.10) \quad h = E \left(\frac{1}{2} \int_0^1 |\dot{x}|^2 \right) - \int_0^1 V_\varepsilon(x) \leq E \left(\frac{1}{2} \int_0^1 |\dot{x}|^2 \right),$$

since $V_\varepsilon(x) \geq 0$.

Let $t_m \in \mathbb{R}$ such that $|x(t_m)| = \min_{t \in \mathbb{R}} |x(t)|$, we have

$$(4.11) \quad \frac{d^2}{dt^2} \frac{1}{2} |x(t_m)|^2 = |\dot{x}(t_m)|^2 + x(t_m) \cdot \ddot{x}(t_m) \geq 0.$$

From (H2), (4.4), (4.8) and (4.11) we obtain

$$\begin{aligned} 0 \leq |\dot{x}(t_m)|^2 - \frac{\left(\frac{1}{2} \int_0^1 |\dot{x}|^2 \right)}{\left(\int_0^1 E - F(x) \right)} \nabla F(x(t_m)) \cdot x(t_m) \\ + \frac{1}{\left(\int_0^1 E - F(x) \right)} \nabla V_\varepsilon(x(t_m)) \cdot x(t_m) \\ \leq |\dot{x}(t_m)|^2 + \alpha_1 \frac{\left(\frac{1}{2} \int_0^1 |\dot{x}|^2 \right)}{\left(\int_0^1 E - F(x) \right)} F(x(t_m)) \\ - \frac{2}{\left(\int_0^1 E - F(x) \right)} V_\varepsilon(x(t_m)), \end{aligned}$$

and by substituting in (4.9) we obtain,

$$(4.12) \quad h \geq - \left(\frac{\alpha_1}{2} - 1 \right) \left(\frac{1}{2} \int_0^1 |\dot{x}|^2 \right) F(x(t_m)).$$

Taking into account of (H1) and (4.10), (4.12) leads to

$$\frac{a}{|x(t_m)|^\alpha} \leq \left(\frac{2}{\alpha_1 - 2} \right) E,$$

that is [from (4.5)],

$$|x(t)| \geq |x(t_m)| \geq \left(\frac{(\alpha_1 - 2)a}{2(E)} \right)^{1/\alpha} \geq 2\varepsilon.$$

By virtue of Proposition 4.1, looking for critical points of I is equivalent to looking for critical points of I_ε , provided that (4.5) holds. We are going to show that I_ε satisfies the assumptions of Theorem 2.1.

PROPOSITION 4.2. — *Assume that (H1) holds. Then*

$$(4.13) \quad \liminf_{x \rightarrow \partial\Lambda} I_\varepsilon(x) = +\infty.$$

Proof. — Indeed

$$(4.14) \quad I_\varepsilon(x) \geq \frac{E}{2} \int_0^1 |\dot{x}|^2 + \int_0^1 V_\varepsilon(x),$$

and, from (4.1), V_ε satisfies the “strong force” condition. This fact (see [16]) implies (4.13).

PROPOSITION 4.3. — *Assume (H1) and (H2) hold. Then, for any $c > 0$, I_ε satisfies the (PS)_c condition.*

Proof. — Let $(x_n)_n$ be a Palais-Smale sequence in Λ , that is

$$(4.15) \quad x_n \in \Lambda, \quad I_\varepsilon(x_n) = c_n \rightarrow c > 0$$

$$(4.16) \quad \left\{ \begin{array}{l} - \left(\int_0^1 E - F(x_n) \right) \ddot{x}_n \\ = \left(\frac{1}{2} \int_0^1 |\dot{x}_n|^2 \right) \nabla F(x_n) - \nabla V_\varepsilon(x_n) + h_n \\ \text{with } h_n \rightarrow 0 \text{ in } H^{-1}. \end{array} \right.$$

From Proposition 4.2, we deduce that

$$(4.17) \quad \exists \gamma > 0 \quad \text{such that } d(x_n, \partial\Lambda) \geq \gamma \quad (\forall n \in \mathbb{N}).$$

From (4.14) and (4.15) we then obtain that a positive constant C exists such that

$$(4.18) \quad \int_0^1 |\dot{x}_n|^2 \leq \frac{4C}{E} \quad \text{for } n \text{ large.}$$

Let $m_n = \min_{t \in \mathbb{R}} |x_n(t)|$. If $(m_n)_n$ is bounded, then, up to a subsequence, $(x_n)_n$ converges uniformly to some limit x , and from (4.16) we deduce that $(x_n)_n$ converges to x strongly in H , so from (4.17) we conclude that $x \in \Lambda$.

Assuming by the contrary that $(m_n)_n$ is unbounded, then (up to a subsequence) we can assume that

$$\lim_{n \rightarrow +\infty} m_n = +\infty,$$

so that both $(\nabla F(x_n))_n$ and $(\nabla V_\varepsilon(x_n))_n$ converge uniformly (and hence in H^{-1}) to zero. Moreover, since

$$\lim_{n \rightarrow +\infty} \int_0^1 E - F(x_n) = E > 0,$$

(4.16) implies that $(-\ddot{x}_n)_n$ converges to zero in H^{-1} . Hence $\|\dot{x}_n\|_{L^2} \rightarrow 0$ and therefore

$$I(x_n) \rightarrow 0$$

[indeed, for n large $\int_0^1 V_\varepsilon(x_n) = 0$], which contradicts (4.15).

PROPOSITION 4.4. — Assume (H1), (H2) and (4.5) hold. Let $x \in \Lambda$ be a critical point of I_ε at a positive level. Then $x \notin F_0$.

Proof. — Let x be a such a critical point; by Proposition 4.1, we know that x is actually a critical point of I , so that it satisfies

$$\begin{aligned} -\ddot{x} &= \lambda^2 \nabla F(x), \\ |\dot{x}|^2 + \lambda^2 F(x) &= \lambda^2 E, \end{aligned}$$

where

$$\lambda^2 = \frac{1/2 \int_0^1 |\dot{x}|^2}{\int_0^1 E - F(x)} > 0$$

since $I(x) > 0$. Now assume that $x \in F_0$, that is that there exists s such that $x(s-t) = x(t)$, $\forall t \in \mathbb{R}$. Therefore $\dot{x}(s/2) = 0$. One then deduce from the conservation of the energy that $F(x(s/2)) = E$, and hence, from (H1), that $E < 0$.

PROPOSITION 4.5. — Assume (H1), (H2), (H3) and (1) hold. Let x be a critical point of I such that

$$(4.19) \quad 0 < I(x) \leq \pi^2 b^{2/\alpha} \alpha \left(\frac{2E}{\alpha - 2} \right)^{(\alpha - 2)/\alpha}.$$

Then x has 1 as its minimal period.

Proof. – First, arguing as in the proof of Proposition 4.1, we deduce from (H1) and (H2) that

$$(4.20) \quad |x(t)|^\alpha \geq \frac{a(\alpha_1 - 2)}{2E}, \quad \forall t \in \mathbb{R}.$$

Since x is a critical point of I at positive level, we have

$$(4.21) \quad -\ddot{x} = \frac{1/2 \int_0^1 |\dot{x}|^2}{\int_0^1 E - F(x)} \nabla F(x)$$

and taking the L^2 product of (4.21) by x , from (H1) and (H2) we obtain

$$\int_0^1 |\dot{x}|^2 \leq \frac{1/2 \int_0^1 |\dot{x}|^2}{\int_0^1 E - F(x)} \int_0^1 -\alpha_2 F(x),$$

and hence

$$(4.22) \quad \int_0^1 E - F(x) \geq \left(\frac{\alpha_2}{\alpha_2 - 2} \right) E.$$

Setting

$$(4.23) \quad \lambda^2 = \frac{1/2 \int_0^1 |\dot{x}|^2}{\int_0^1 E - F(x)} = \frac{I(x)}{\left(\int_0^1 E - F(x) \right)^2},$$

from (4.22) we deduce

$$(4.24) \quad \lambda^2 \leq \left(\frac{\alpha_2 - 2}{\alpha_2 E} \right)^2 I(x).$$

(4.21) and (H3) lead to

$$\int_0^1 |\ddot{x}|^2 \leq \lambda^4 a^2 \alpha_2^2 \int_0^1 \frac{1}{|x|^{2(\alpha+1)}};$$

hence, from (4.20) and (H1) we obtain

$$(4.25) \quad \int_0^1 |\ddot{x}|^2 \leq \lambda^4 a \alpha_2^2 \left[\frac{2E}{(\alpha_1 - 2)a} \right]^{(\alpha+2)/\alpha} \int_0^1 -F(x).$$

Since \dot{x} has zero mean value, Wirtinger inequality says that

$$(4.26) \quad \int_0^1 |\ddot{x}|^2 \geq (2k\pi)^2 \int_0^1 |\dot{x}|^2,$$

where $1/k$ is the minimal period of x ; therefore, from (4.21) and (H2) we find

$$(4.27) \quad \int_0^1 |\ddot{x}|^2 \geq (2k\pi)^2 \lambda^2 \int_0^1 -\alpha_1 F(x).$$

(4.24) together with (4.27) lead to

$$k^2 \leq \frac{1}{\alpha_1} \frac{1}{(2\pi)^2} \lambda^2 a \alpha_2^2 \left[\frac{2E}{(\alpha_1 - 2)a} \right]^{(\alpha + 2)/\alpha},$$

and therefore, from (4.23) and (4.19),

$$k^2 \leq \frac{b^{2/\alpha}}{a^{2/\alpha}} \frac{\alpha}{\alpha_1} \left(\frac{\alpha_2 - 2}{\alpha - 2} \right)^2 (\alpha - 2 \alpha_1 - 2)^{\alpha_2/\alpha}.$$

Hence (1) implies that $k^2 < 4$, and therefore (k is an integer), $k = 1$.

Now we turn to the proof of Theorem 1.

Proof of Theorem 1. — To carry out the proof, we first replace the functional I with I_ε as in Definition 4.3, with ε so small that (4.5) holds. Then, by virtue of Proposition 4.1, the critical points of I_ε at positive levels are solutions of (P_ε) , up to the rescaling of the period. In order to find multiple critical points of I_ε , we shall apply the results of sections 1, 2 and 3.

We claim that I_ε satisfy all the assumptions of Theorem 3.1. Indeed, from Proposition 4.2, I_ε admits a lower semicontinuous extension to the whole of H as $\bar{I}_\varepsilon(x) = +\infty$, when $x \in \partial\Lambda$. Then (2.1) holds. Therefore, by Theorem 3.1, (2.2) is satisfied. By Proposition 4.4, (2.4) is fulfilled too. In order to check (2.3), from Proposition 4.3 we have to prove that the critical levels are positive.

To do this, we assume on the contrary that, for some r , $c_r = 0$. Then there is a sequence $(A_n)_n$ in Γ_r such that $\sup_{A_n} I_\varepsilon \rightarrow 0$. We are then going to find a contradiction proving that, for large values of n , the A_n s are G -contractible sets. Indeed, from (4.14) we deduce that $\sup_{A_n} \int_0^1 |\dot{x}|^2 \rightarrow 0$ and $\inf_{A_n} \min_t |x(t)| \rightarrow +\infty$. Therefore, for large values on n , the G -equivariant

continuous homotopy $h(x, \sigma) = (1 - \sigma)x + \sigma \int_0^1 x$ can be performed, contracting the A_n s into a set of large constant functions, without crossing the boundary of Λ .

Hence Theorem 2.1 can be applied, providing the existence of at least $N - 1$ geometrically distinct critical points.

We remark that, taking $R_b = \left[\frac{(\alpha - 2)/b}{2E} \right]^{1/\alpha}$, from (H1) we have

$$\begin{aligned} \sup_{R_b C_N} I_\varepsilon &= \sup_{R_b C_N} I \leq \sup_{R_b C_N} \left\{ \left(\frac{1}{2} \int_0^1 |\dot{x}|^2 \right) \left(\int_0^1 E + \frac{b}{|x|^\alpha} \right) \right\} \\ &= \pi^2 b^{2/\alpha} \alpha \left(\frac{2E}{\alpha - 2} \right)^{(\alpha - 2)/\alpha}. \end{aligned}$$

Therefore, from Theorem 3.1, $c_{N-1} \leq \pi^2 b^{2/\alpha} \alpha \left(\frac{2E}{\alpha - 2} \right)^{(\alpha - 2)/\alpha}$, so that from Proposition 4.5 we deduce that the minimal period of these critical points is exactly one.

Now we point out that the same method apply to the functional corresponding to the potential $F(x) = \frac{-a}{|x|^\alpha}$. It is not difficult to see that, in that case $c_1 = \dots = c_{N-1} = \pi^2 a^{2/\alpha} \alpha \left(\frac{2E}{\alpha - 2} \right)^{(\alpha - 2)/\alpha}$ (see also [23]). Hence, from (H1) we deduce that

$$\pi^2 a^{2/\alpha} \alpha \left(\frac{2E}{\alpha - 2} \right)^{(\alpha - 2)/\alpha} \leq c_r \leq \pi^2 b^{2/\alpha} \alpha \left(\frac{2E}{\alpha - 2} \right)^{(\alpha - 2)/\alpha}, \quad r = 1, \dots, N - 1.$$

Finally, arguing as in the proof of Proposition 4.5, from (H2) the periods of the solutions satisfy

$$\left(\frac{\alpha_1 - 2}{\alpha_1 E} \right)^2 I(x_r) \leq \lambda_r^2 \leq \left(\frac{\alpha_2 - 2}{\alpha_2 E} \right)^2 I(x_r);$$

hence we deduce from the above discussion that the minimal period of these solutions belong to the interval

$$\left[\left(\frac{(\alpha_1 - 2)\alpha}{(\alpha - 2)\alpha_1} \right) \frac{2\pi}{\sqrt{\alpha}} a^{1/\alpha} \left(\frac{\alpha - 2}{2E} \right)^{(\alpha + 2)/2\alpha}, \left(\frac{(\alpha_2 - 2)\alpha}{(\alpha - 2)\alpha_2} \right) \frac{2\pi}{\sqrt{\alpha}} b^{1/\alpha} \left(\frac{\alpha - 2}{2E} \right)^{(\alpha + 2)/2\alpha} \right].$$

5. THE WEAK FORCE CASE (PROOF OF THEOREM 2)

In this section we turn to the proof of Theorem 2. To this aim, we shall apply the results of the previous sections related to the homotopical pseudo-index. First of all, we are going to replace the “singular” F with a regular F_ε , which is defined and smooth even at the origin. Next we shall provide estimates which will allow us to apply Theorem 2.2. Finally from the *a priori* estimate (Proposition 5.1) we shall conclude that the critical points found in this way do not interact with the truncation.

We start by defining the suitable truncation of F :

DEFINITION 5.1. — Assume that F satisfies (H4), (H5). For any $\varepsilon > 0$, F_ε denotes a function such that

$$\begin{aligned}
 (5.1) \quad & F_\varepsilon \in \mathcal{C}^2(\mathbb{R}^N; \mathbb{R}) \\
 (5.2) \quad & F_\varepsilon(x) = F(x) \quad \text{if } |x| \geq \varepsilon \\
 (5.3) \quad & 0 \leq \nabla F_\varepsilon(x) \cdot x \leq -\alpha_2 F_\varepsilon(x), \quad \forall x \in \mathbb{R}^N \\
 (5.4) \quad & -F_\varepsilon(x) \leq \frac{b}{|x|^\alpha}, \quad \forall x \in \mathbb{R}^N
 \end{aligned}$$

there is a non increasing $f: \mathbb{R} \rightarrow \mathbb{R}$, with

$$\begin{aligned}
 (5.5) \quad & f(|x|) \geq \frac{a}{\varepsilon^\alpha} \quad \text{if } |x| \leq \varepsilon \\
 & \text{such that } -F_\varepsilon(x) \geq f(|x|), \quad \forall x \in \mathbb{R}^N.
 \end{aligned}$$

We set

$$I_\varepsilon(x) = \left(\frac{1}{2} \int_0^1 |\dot{x}|^2 \right) \left(\int_0^1 E - F_\varepsilon(x) \right), \quad \forall x \in H.$$

The following proposition provides the above mentioned *a priori* estimate:

PROPOSITION 5.1 (A priori estimate). — Let F satisfy (H4), (H5), and let $F_\varepsilon, I_\varepsilon$ as in Definition 5.1. Let $c_1 > 0$ be fixed. There exist $\bar{\varepsilon} > 0$ and a function $\Psi_1: [1, 2) \rightarrow [1, +\infty)$ such that if

$$(5.6) \quad \frac{b}{a} \left(\frac{(2-\alpha_1)\alpha}{(2-\alpha)\alpha_1} \right)^{\alpha/2} \frac{2-\alpha}{2-\alpha_2} < \Psi_1(\alpha);$$

then, for every $0 < \varepsilon \leq \bar{\varepsilon}$, each critical point of I_ε at level

$$(5.7) \quad c_1 \leq I_\varepsilon(x) \leq c_2 \leq \pi^2 b^{2/\alpha} \alpha \left(\frac{2-\alpha}{-2E} \right)^{(2-\alpha)/\alpha},$$

satisfies

$$|x(t)| \geq \varepsilon, \quad \forall t \in \mathbb{R}.$$

Moreover, Ψ_1 fulfills the following properties:

$$\begin{aligned} \Psi_1(1) &= 1; \\ \Psi_1 &\text{ is increasing;} \\ \lim_{\alpha \rightarrow 2} \Psi_1(\alpha) &= +\infty. \end{aligned}$$

Proof. — The proof is contained in [22] (Section 4). For the reader's convenience, we recall here the main steps.

Step 1. — Assume that (H4), (H5) holds and let c_1, c_2 be fixed such that $0 < c_1 \leq c_2$. Then there is a function $\delta(\varepsilon)$ with $\lim_{\varepsilon \rightarrow 0} \delta(\varepsilon) = 0$, such that every critical point x of I_ε at level $c_1 \leq I_\varepsilon(x) \leq c_2$ satisfies

$$(5.8) \quad \left(\int_0^1 E - F_\varepsilon(x) \right) \left(\int_0^1 |\dot{x}|^2 \right) \geq \left(\frac{1}{2} \int_0^1 |\dot{x}|^2 \right) \left(\int_0^1 -\alpha_1 F_\varepsilon(x) \right) - \delta(\varepsilon).$$

The proof is contained in [22], Appendix.

Step 2. — Let us define, for any $\rho > 0$

$$(5.9) \quad S_\rho = \left\{ x \in H \mid \frac{1}{2} \int_0^1 |\dot{x}|^2 = \rho^2 \right\};$$

$$(5.10) \quad J(\alpha, \rho) = \inf_{x \in S_\rho \cap \partial \Lambda} \int_0^1 \frac{1}{|x|^\alpha};$$

$$(5.11) \quad \Psi_1(\alpha) = \frac{J(\alpha, 1)}{(\sqrt{2}\pi)^\alpha}.$$

It was proved in [22] (Proposition 3.1) that J and Ψ_1 enjoy the following properties:

$$(5.12) \quad \begin{aligned} J(\alpha, \rho) &= J(\alpha, 1) \rho^{-\alpha}; \\ \Psi_1(1) &= 1; \\ \Psi_1 &\text{ is increasing;} \\ \lim_{\alpha \rightarrow 2} \Psi_1(\alpha) &= +\infty. \end{aligned}$$

Moreover, for every $\rho_2 \geq \rho_1 > 0$ and $\gamma > 0$, there exists $\varepsilon(\gamma)$ such that, for every $0 \leq \varepsilon \leq \varepsilon(\gamma)$ and for every $\rho \in [\rho_1, \rho_2]$, then

$$(5.13) \quad \inf_{\substack{1/2 \int_0^1 |\dot{x}|^2 \leq \rho^2 \\ |x(0)| = |x(1)| = \varepsilon}} \int_0^1 -F_\varepsilon(x) \geq (J(\alpha, 1) - \gamma) \frac{a}{\rho^\alpha}.$$

Step 3. — Let x be a critical point of I_ε . Then x satisfies

$$(5.14) \quad - \left(\int_0^1 E - F_\varepsilon(x) \right) \ddot{x} = \left(\frac{1}{2} \int_0^1 |\dot{x}|^2 \right) \nabla F_\varepsilon(x).$$

Taking the L^2 product of (5.14) by x , from (5.3) and (5.7) one deduces

$$(5.15) \quad \int_0^1 E - F_\varepsilon(x) \leq \frac{-\alpha_2 E}{2 - \alpha_2},$$

and therefore

$$\frac{1}{2} \int_0^1 |\dot{x}|^2 \geq c_1 \left(\frac{2 - \alpha_2}{-\alpha_2 E} \right).$$

Therefore assuming that (5.7) holds and setting

$$\delta^*(\varepsilon) = \delta(\varepsilon) \left(-E \int_0^1 |\dot{x}|^2 \right)^{-1};$$

one obtains from (5.8) that

$$(5.16) \quad \frac{1}{2} \int_0^1 |\dot{x}|^2 \leq I_\varepsilon(x) \left(\frac{2 - \alpha_1}{-\alpha_1 E + E \delta^*(\varepsilon)} \right) \\ \leq 2\pi^2 \left(\frac{(2 - \alpha)b}{-2E} \right)^{2/\alpha} \left(\frac{(2 - \alpha_1)\alpha}{(2 - \alpha)(\alpha_1 - \delta^*(\varepsilon))} \right),$$

with $\lim_{\varepsilon \rightarrow 0} \delta^*(\varepsilon) = 0$. Now by the step 1, taking into account of (5.6) and (5.11), we can fix $\gamma > 0$ and $\varepsilon > 0$ sufficiently small that

$$(5.17) \quad \frac{b}{a} \frac{\alpha_2}{\alpha_1} \left(\frac{(2 - \alpha_1)\alpha}{(2 - \alpha)(\alpha_1 - \delta^*(\varepsilon))} \right)^{\alpha/2} \frac{2 - \alpha}{2 - \alpha_2} < \frac{J(\alpha, 1) - \gamma}{(\sqrt{2}\pi)^\alpha},$$

moreover, ε is fixed sufficiently small that (5.13) holds for every

$$\rho \in \left[c_1 \left(\frac{2 - \alpha_2}{-\alpha_2 E} \right), c_2 \left(\frac{2 - \alpha_1}{-\alpha_1 E + E \delta^*(\varepsilon)} \right) \right].$$

Therefore, assuming that $\min_t |x(t)| < \varepsilon$, it follows from (5.13) and (5.16) that

$$(5.18) \quad \int_0^1 -F(x) \geq (J(\alpha, 1) - \gamma) \frac{a}{\left(\int_0^1 |\dot{x}|^2 \right)^{\alpha/2}} \\ \geq \frac{(J(\alpha, 1) - \gamma)}{(\sqrt{2}\pi)^\alpha} \frac{a}{b} \left(\frac{(2 - \alpha)(\alpha_1 - \delta^*(\varepsilon))}{(2 - \alpha_1)\alpha} \right)^{\alpha/2} \left(\frac{-2E}{(2 - \alpha)} \right).$$

and hence, from (1.17) that

$$\int_0^1 E - F(x) > \frac{-\alpha_2 E}{2 - \alpha_2},$$

which contradicts (5.15). Therefore $\min_t |x(t)| \geq \varepsilon$

It is clear from Proposition 5.1 that finding critical points of I_ε such that (5.7) holds is in fact equivalent to finding critical points of I .

In order to apply Theorem 2.2, let us prove some preliminary estimates. According with the notations of Sections 2 and 3, let us define

$$(5.19) \quad \rho_b = \left[\frac{(2-\alpha)b}{-2E} \right]^{1/\alpha};$$

$$(5.20) \quad \Sigma_1 = \left\{ x \in H \mid \frac{1}{2} \int_0^1 |\dot{x}|^2 = 2\pi^2 \rho_b^2 \right\};$$

$$(5.21) \quad \Sigma_2 = \varepsilon \rho_b C_N \cup \varepsilon^{-1} \rho_b C_N;$$

$$(5.22) \quad C_N^* = [\varepsilon \rho_b, \varepsilon^{-1} \rho_b] C_N.$$

PROPOSITION 5.2. — Let Ψ_1 be defined in (5.11) and assume that (H4) holds with $\frac{b}{a} < \Psi_1(\alpha)$. Then there exists $\bar{\varepsilon} > 0$ such that, for every $0 < \varepsilon \leq \bar{\varepsilon}$, then

$$(5.23) \quad \inf_{\Sigma_1 \cap \partial\Lambda} I_\varepsilon > \sup_{C_N^*} I_\varepsilon.$$

Proof. — Let $\gamma > 0$ be fixed such that $\frac{b}{a} < \frac{J(\alpha, 1) - \gamma}{(\sqrt{2}\pi)^\alpha}$, and let $\bar{\varepsilon}$ be associated to γ so that (5.13) holds with $\rho = \rho_b$. Then, for every $\varepsilon \leq \bar{\varepsilon}$, one has

$$\begin{aligned} \inf_{\Sigma_1 \cap \partial\Lambda} I_\varepsilon &\geq 2\pi^2 \rho_a^2 \left(E + a \inf_{\substack{x \in \Sigma_1 \\ |x(0)| = |x(1)| = \varepsilon}} \int_0^1 \frac{1}{|x|^\alpha} \right) \\ &\geq 2\pi^2 \rho_b^2 \left(E + a \frac{J(\alpha, 1) - \gamma}{(\sqrt{2}\pi \rho_b)^\alpha} \right) \\ &> 2\pi^2 \rho_b^2 \left(E + \left(\frac{-2E}{2-\alpha} \right) \right) = \pi^2 b^{2/\alpha} \alpha \left(\frac{2-\alpha}{-2E} \right)^{(2-\alpha)/\alpha}. \end{aligned}$$

On the other hand, when ε is sufficiently small

$$\sup_{C_N^*} I_\varepsilon = 2\pi^2 \rho_b^2 \left(E + \frac{-2E}{2-\alpha} \right) = \pi^2 b^{2/\alpha} \alpha \left(\frac{2-\alpha}{-2E} \right)^{(2-\alpha)/\alpha}.$$

PROPOSITION 5.3. — Let $F(x) = \frac{-a}{|x|^\alpha}$, for some $a > 0$ and $1 < \alpha < 2$. Then the smallest positive critical level of the associated functional I in Λ is $\pi^2 a^{2/\alpha} \alpha \left(\frac{2-\alpha}{-2E} \right)^{(2-\alpha)/\alpha}$.

Proof. — Indeed, by the radial symmetry of the associated Euler equation, one deduces that every critical point is planar; moreover, the conservation of the angular momentum also implies that the topological degree of every solution with respect to the origin is different from zero. Now, to each critical point $x(t)$ of the functional there is associated a solution $y(\lambda^{-1}t)$ of (P_E) having period $\lambda^2 = I(x) \left(\frac{2-\alpha}{-\alpha E} \right)^2$. One easily deduces from the form of the potential that

$$(5.24) \quad \int_0^\lambda \frac{1}{2} |\dot{y}|^2 + \frac{a}{|y|^\alpha} = \left(\frac{2+\alpha}{\alpha} \right) \sqrt{I(x)}.$$

Following the ideas of [15] and [13], one can prove that the minimum of the integral in the left hand side, over the set of all the planar H^1 λ -periodic functions having non zero topological degree with respect to the origin is attained by a circular function (that is a function having constant modulus). Therefore, it is not difficult to prove that this minimum is equal to

$$(5.25) \quad \left(\frac{2+\alpha}{2\alpha} \right) (2\pi)^{2\alpha/(2+\alpha)} \alpha^{2/(2+\alpha)} a^{2/(2+\alpha)} \lambda^{(2-\alpha)/(2+\alpha)}.$$

Hence, from (5.24) we have

$$\left(\frac{2+\alpha}{\alpha} \right) \sqrt{I(x)} \geq \left(\frac{2+\alpha}{2\alpha} \right) (2\pi)^{2\alpha/(2+\alpha)} \alpha^{2/(2+\alpha)} a^{2/(2+\alpha)} \lambda^{(2-\alpha)/(2+\alpha)}.$$

From the last inequality and the value of $\lambda \left(\lambda^2 = I(x) \left(\frac{2-\alpha}{-\alpha E} \right)^2 \right)$ we finally deduce that $I(x) \geq \pi^2 a^{2/\alpha} \alpha \left(\frac{2-\alpha}{-2E} \right)^{(2-\alpha)/\alpha}$.

PROPOSITION 5.4. — Let $F(x) = \frac{-a}{|x|^\alpha}$, and let I_ε be as in Definition 4.1. For any $\rho > 0$, consider the class

$$\mathcal{A}_\rho^* = \{ A \in \mathcal{B} \cap 2^\Lambda / j(A \cap S_\rho) \geq 1 \};$$

and let us define

$$(5.26) \quad K(\alpha, \rho) = \inf_{A \in \mathcal{A}_\rho^*} \sup_A \int_0^1 F_\varepsilon(x).$$

Then, for every fixed interval $[\rho_1, \rho_2]$ there is $\bar{\varepsilon} > 0$ such that, for every $\varepsilon \leq \bar{\varepsilon}$,

$$(5.27) \quad K(\alpha, \rho) = K(\alpha, 1) \rho^{-\alpha}, \quad \forall \rho \in [\rho_1, \rho_2], \quad \forall 1 \leq \alpha < 2;$$

$$(5.28) \quad K(\alpha, 1) = a(\sqrt{2}\pi)^\alpha, \quad \forall 1 \leq \alpha < 2.$$

Proof. — In order to prove (5.28) we first claim that $K(\alpha, 1) > 0$, $\forall 1 \leq \alpha < 2$. Indeed, assume on the contrary that $K(\alpha, 1) = 0$ for some α , and let $(A_n)_n$ be a sequence of non G -contractible sets of S_1 such that $\sup_{A_n} \int_0^1 \frac{1}{|x|^\alpha} \rightarrow 0$. Since, for every n , $A_n \subseteq S_1$, one then deduces (cf. (5.9)) that

$$\lim_{n \rightarrow \infty} \min_{x \in A_n} \min_{t \in \mathbb{R}} |x(t)| = +\infty,$$

and therefore that, for n large, the G -equivariant homotopy $h(x, \sigma) = (1 - \sigma)x + \sigma \int_0^1 x$ contracts A_n into sets of constant functions, without crossing $\partial\Lambda$. Therefore $j(A_n) = 0$. Now we prove (5.28). First of all, one easily deduces that

$$(5.29) \quad K(\alpha, 1) \leq a(\sqrt{2}\pi)^\alpha,$$

just by taking $A = C_N$. Now we prove the reversed inequality: from the fact that $K(\alpha, 1) > 0$, by applying the min-max principle (note that the Palais-Smale condition holds at any positive level), one finds a Lagrangean multiplier λ and a function $x \in \Lambda \cap S_1$ satisfying

$$(5.30) \quad \int_0^1 \frac{a}{|x|^\alpha} = K(\alpha, 1),$$

and

$$(5.31) \quad -\lambda \ddot{x} = \frac{a \alpha x}{|x|^{\alpha+2}}.$$

Indeed, although the functional $\int_0^1 \frac{1}{|x|^\alpha}$ is singular, from (5.11) and the properties of Ψ_1 , (5.13) and (5.29) one deduces that

$$\inf_{S_1 \cap \partial\Lambda} \int_0^1 \frac{a}{|x|^\alpha} \geq a(\sqrt{2}\pi)^\alpha \geq K(\alpha, 1).$$

It follows from (5.30) and (5.31) that

$$(5.32) \quad \lambda = \frac{\alpha}{2} K(\alpha, 1),$$

and hence

$$(5.33) \quad \int_0^1 \frac{1}{2} |\dot{x}|^2 + \frac{a}{\lambda} \frac{1}{|x|^\alpha} = \frac{2+\alpha}{\alpha}.$$

The radial symmetry of the equation (5.31) implies that x is planar. Moreover, since its angular momentum is constant, one easily deduces that the topological degree of x with respect to the origin is not zero. Let us denote by $\theta_0\left(\alpha, \frac{a}{\lambda}\right)$ the infimum of the functional in the left hand side of (5.33) over the set of all the planar 1-periodic functions having non zero topological degree, with respect to the origin. Following the ideas of [15] one can prove that, whenever $\alpha \geq 1$, $\theta_0\left(\alpha, \frac{a}{\lambda}\right)$ is attained on the circular solutions of the equation $-\ddot{x} = \frac{a\alpha x}{\lambda|x|^{2+\alpha}}$ having minimal period 1. Then (5.28) follows from the direct computation of $\theta_0(\alpha, \lambda)$ [cf. also (5.25)], taking into account of (5.32).

Formula (5.27) easily follows from the definition and the fact that $K(\alpha, \rho)$ is attained in $S_\rho \cap \Lambda$.

PROPOSITION 5.5. — Assume (H4), (H5) hold and let I_ε be defined in Definition 5.1. If ε is sufficiently small, then, for every $0 > c_1 \leq c_2$ and every $c^* \in [c_1, c_2]$, I_ε satisfies condition (C) $_{c^*}$.

Proof. — The proof is contained in [22]. For completeness we recall it here.

Let $(x_n)_n$ be a sequence satisfying

$$(5.34) \quad I_\varepsilon(x_n) \rightarrow c^*;$$

$$(5.35) \quad -\left(\int_0^1 E - F_\varepsilon(x_n)\right)\ddot{x} - \left(\frac{1}{2}\int_0^1 |\dot{x}|^2\right)\nabla F_\varepsilon(x_n) \rightarrow 0, \quad \text{in } H^{-1};$$

$$(5.36) \quad \left(\int_0^1 E - F_\varepsilon(x_n)\right)\left(\int_0^1 |\dot{x}|^2\right) - \left(\frac{1}{2}\int_0^1 |\dot{x}|^2\right)\left(\int_0^1 \nabla F_\varepsilon(x_n) \cdot x_n\right) \rightarrow 0.$$

Assuming that $\|\dot{x}_n\|_{L^2}$ is bounded, by standard compactness arguments one easily deduces the existence of a converging subsequence (remember that F_ε is regular). Now we assume that $\|\dot{x}_n\|_{L^2}$ is unbounded, that is, up to a subsequence, that

$$(5.37) \quad \lim_{n \rightarrow +\infty} \int_0^1 |\dot{x}|^2 = +\infty.$$

Then (5.34) yields

$$(5.38) \quad \lim_{n \rightarrow +\infty} \int_0^1 E - F_\varepsilon(x_n) = 0,$$

and therefore, setting $\Omega_n = \{t \in [0, 1] \mid |x_n(t)| \geq \varepsilon\}$, one deduces from Tchebicev inequality, (H4), (5.2) and (5.5) that

$$(5.39) \quad \liminf_{n \rightarrow +\infty} \text{meas}(\Omega_n) > 0.$$

From (5.38), (H4) and (5.1) one also deduces that there is a constant d_1 such that

$$(5.40) \quad \min_{t \in \mathbb{R}} |x_n(t)| \leq d_1, \quad \forall n \in \mathbb{N}.$$

From (5.37) and (5.40) one deduces the existence of a constant d_2 such that

$$\max_{t \in \mathbb{R}} |x_n(t)| \leq d_2 \left(\int_0^1 |\dot{x}_n|^2 \right)^{1/2}.$$

Then (5.2), (5.3), (H4), (H5) and (5.36) imply that

$$\begin{aligned} 2c^* &= \lim_{n \rightarrow +\infty} 2I_\varepsilon(x_n) = \lim_{n \rightarrow +\infty} \left(\frac{1}{2} \int_0^1 |\dot{x}_n|^2 \right) \left(\int_0^1 \nabla F_\varepsilon(x_n) \cdot x_n \right) \\ &\geq \lim_{n \rightarrow +\infty} \left(\frac{1}{2} \int_0^1 |\dot{x}_n|^2 \right) \left(\int_{\Omega_n} \nabla F_\varepsilon(x_n) \cdot x_n \right) \\ &\geq \lim_{n \rightarrow +\infty} \frac{\alpha_1 a \text{meas}(\Omega_n)}{2 d_2^\alpha} \left(\int_0^1 |\dot{x}_n|^2 \right)^{(2-\alpha)/2} = +\infty, \end{aligned}$$

a contradiction.

Next Proposition will be proved in the Appendix.

PROPOSITION 5.6. — *Let (H4) (H5) and (H6) holds. Then there are functions Ξ, σ_1 and σ_2 , such that, when*

$$(5.41) \quad \frac{b}{a} \left(\frac{(2-\alpha_1)\alpha}{(2-\alpha)\alpha_1} \right)^\alpha \frac{2-\alpha}{2-\alpha_2} < \Xi(\alpha),$$

$$(5.42) \quad \sigma_2 \left(\frac{b}{a}, \alpha, \alpha_1, \alpha_2 \right) \left(\frac{2-\alpha_1}{2} \right)^{1/\alpha} < 1,$$

and

$$(5.43) \quad \frac{b^{2/\alpha}}{a^{2/\alpha}} \frac{\alpha \alpha_2^2}{\alpha_1^3} \left(\frac{2-\alpha_1}{2-\alpha} \right)^2 \left(\frac{2-\alpha}{2-\alpha_2} \right)^{(2+\alpha)/\alpha} \frac{1}{\sigma_1^{2+\alpha}} < 4,$$

then, for every energy $E < 0$, each critical point of I at level

$$(5.44) \quad \pi^2 a^{2/\alpha} \alpha \left(\frac{2-\alpha}{-2E} \right)^{(2-\alpha)/\alpha} \leq I(x) \leq \pi^2 b^{2/\alpha} \alpha \left(\frac{2-\alpha}{-2E} \right)^{(2-\alpha)/\alpha},$$

has 1 as minimal period and can not belong to F_0 .

Moreover, Ξ and σ_i ($i = 1, 2$) enjoy the following properties:

$$\begin{aligned} \Xi(\alpha) &\geq 2, & \forall 1 < \alpha < 2 \\ \lim_{\alpha \rightarrow 2} \Xi(\alpha) &= +\infty; \\ \sigma_i\left(\frac{b}{a}, \alpha, \alpha_1, \alpha_2\right) &> 0 \\ \lim_{(\alpha_2 b)/(\alpha_1 a) \rightarrow 1} \sigma_i\left(\frac{b}{a}, \alpha, \alpha_1, \alpha_2\right) &= 1 \text{ for every fixed } \alpha \\ \lim_{\alpha \rightarrow 2} \sigma_i\left(\frac{b}{a}, \alpha, \alpha_1, \alpha_2\right) &= 1 \quad \text{if } \frac{(2-\alpha_1)b}{(2-\alpha_2)a} \text{ remains bounded.} \end{aligned}$$

Now we are in a position to prove Theorem 2:

Proof of Theorem 2. – To carry out the proof, we are going to apply the results of sections 2 and 3 about the homotopical pseudo-index.

We are going to define $\Psi^*(\alpha) = \min\left(\Psi_1(\alpha), \Xi(\alpha), \frac{2}{2-\alpha}\right)$, in such a way that the assumptions of all the above Propositions are satisfied.

Let us first replace F with F_ε as in Definition 5.1, ε can be taken so small that all the previous Propositions hold true. Then we claim that the associated functional I_ε fulfills all the assumptions of Theorem 2.2. Indeed, let Σ_1, Σ_2 be defined in (5.20), (5.21). Then, from Theorem 3.2, we know that C_N^* as defined in (2.23) belongs to Γ_{N-1}^* . Moreover, we deduce from (H1) and the application Proposition 5.4 (with $\rho_b \in [\rho_1 \rho_2]$) that, for every $A \subseteq \Sigma_1, j(A) \geq 1$, we have

$$\sup_A I_\varepsilon \geq -2\pi^2 \rho_b^2 E \left(\frac{2a}{(2-\alpha)b} - 1 \right) > 0.$$

So that (2.8) and (2.8) hold true. Moreover, by Proposition 5.2, when ε is small, then

$$(5.45) \quad \inf_{\Sigma_1} I_\varepsilon > \pi^2 b^{2/\alpha} \alpha \left(\frac{2-\alpha}{-2E} \right)^{(2-\alpha)/\alpha} \geq c_r^*, \quad r = 1, \dots, N-1,$$

while Proposition 5.4 leads to

$$\begin{aligned} c_r^* &\geq \inf_{\substack{A \subseteq \Sigma_1 \\ j(A) \geq 1}} \geq -2\pi^2 \rho_b^2 E \left(\frac{2a}{(2-\alpha)b-1} \right) > \sup_{\Sigma_2} I_\varepsilon; \\ & \quad r = 1, \dots, N-1, \end{aligned}$$

(indeed, $\sup_{\Sigma_2} I_\varepsilon \leq C_1 \varepsilon^{2-\alpha}$, for some constant C_1 independent of ε). Therefore (2.10) is fulfilled and, by Proposition 5.5, also (2.11) holds true. Finally, from the above inequalities, Proposition 5.6 allows us to exclude

the existence of elements of F_0 at the critical levels (2.12). Thus Theorem 2.2 provides the existence of at least $N - 1$ distinct critical points $x_r (r = 1, \dots, N - 1)$ of I_ε . Let us observe that the same method provides critical point for the potential $\frac{-a}{|x|^\alpha}$, so that from (5.45) and Proposition 5.3 we deduce that the following estimate holds:

$$\pi^2 b^{2/\alpha} \alpha \left(\frac{2 - \alpha}{-2E} \right)^{(2 - \alpha)/\alpha} \geq I_\varepsilon(x_r) \geq \pi^2 a^{2/\alpha} \alpha \left(\frac{2 - \alpha}{-2E} \right)^{(2 - \alpha)/\alpha}.$$

By virtue of Proposition 5.1, we can say that these critical points do not interact with the truncation, that is, they are critical point of I . As such, up to the rescaling of the period, they are solutions of (P_E) having minimal (because of Proposition 5.6) period in the interval

$$\left[\pi a^{1/\alpha} \sqrt{\alpha} \left(\frac{2 - \alpha}{-2E} \right)^{(2 - \alpha)/2\alpha} \left(\frac{2 - \alpha_2}{-\alpha_2 E} \right), \pi b^{1/\alpha} \sqrt{\alpha} \left(\frac{2 - \alpha}{-2E} \right)^{(2 - \alpha)/2\alpha} \left(\frac{2 - \alpha_1}{-\alpha_1 E} \right) \right].$$

6. PROOF OF PROPOSITION 5.6

In this section we turn to the proof of Proposition 5.6. To do this, some preliminaries are needed.

Let x be a critical point of I such that (5.44) holds, and let $y(t) = x(\lambda^{-1} t)$ be the corresponding solution of (P_E) . We shall actually prove that $y \notin F_0$ and that the minimal period of y is λ .

From (H5) and the estimate on the level of x , we deduce the following estimates on the period λ of y :

$$(6.1) \quad \pi^2 a^{2/\alpha} \alpha \left(\frac{2 - \alpha}{-2E} \right)^{(2 - \alpha)/\alpha} \left(\frac{2 - \alpha_2}{-\alpha_2 E} \right)^2 \leq \lambda^2 \leq \pi^2 b^{2/\alpha} \alpha \left(\frac{2 - \alpha}{-2E} \right)^{(2 - \alpha)/\alpha} \left(\frac{2 - \alpha_1}{-\alpha_1 E} \right)^2.$$

As a solution of (P_E) , y satisfies

$$(6.2) \quad -\ddot{y} = \nabla F(y)$$

$$(6.3) \quad \frac{1}{2} |\dot{y}|^2 + F(y) = E.$$

Let us denote

$$(6.4) \quad \rho(t) = |y(t)|, \quad \forall t \in \mathbb{R}$$

and

$$(6.5) \quad c(t) = \frac{1}{a(2-\alpha_2)} |y(t)|^\alpha (-2F(y(t)) - \nabla F(y(t)) \cdot y(t)),$$

$$\forall t \in \mathbb{R}.$$

Then from (H5) we deduce that

$$(6.6) \quad 1 \leq c(t) \leq \frac{(2-\alpha_1)b}{(2-\alpha_2)a}, \quad \forall t \in \mathbb{R}.$$

Moreover, from (6.2) and (6.3) we have

$$(6.7) \quad \left\{ \begin{array}{l} -\frac{1}{2} \ddot{\rho}^2 = -2E - (2-\alpha_2)a \frac{c(t)}{\rho^\alpha}, \quad \forall t \in \mathbb{R} \\ \rho(t+\lambda) = \rho(t), \quad \forall t \in \mathbb{R} \\ \rho(t) > 0, \quad \forall t \in \mathbb{R}. \end{array} \right.$$

PROPOSITION 6.1. — Assume that (H1) and (H4) hold, and assume moreover that $\frac{b}{a} < \frac{2}{2-\alpha_1}$. Assume that $y \in F_0$. Then there are λ_1 and λ_2 with $\lambda_2 - \lambda_1 \leq \frac{\lambda}{2}$, such that $\rho(\lambda_1) = \rho(\lambda_2)$, $\dot{\rho}(\lambda_1) = \dot{\rho}(\lambda_2) = 0$, and $\min_{t \in [\lambda_1, \lambda_2]} |\dot{y}| = 0$.

Proof. — By definition, $y \in F_0$ implies that there exists $s \in [0, 1]$ such that

$$y(s-t) = y(t), \quad \forall t \in \mathbb{R},$$

or, equivalently

$$y\left(\frac{s}{2} - t\right) = y\left(\frac{s}{2} + t\right), \quad \forall t \in \mathbb{R}.$$

One easily deduces that

$$y\left(\frac{s+\lambda}{2} - t\right) = y\left(\frac{s+\lambda}{2} + t\right), \quad \forall t \in \mathbb{R},$$

and therefore that

$$\dot{y}\left(\frac{s}{2}\right) = \dot{y}\left(\frac{s+\lambda}{2}\right) = 0.$$

Hence, it follows from (6.3) that

$$F\left(y\left(\frac{s}{2}\right)\right) = F\left(y\left(\frac{s+\lambda}{2}\right)\right) = E,$$

so that (H4) implies

$$\left| y\left(\frac{s}{2}\right) \right|^\alpha \geq \frac{a}{-E}, \quad \text{and} \quad \left| y\left(\frac{s+1}{2}\right) \right|^\alpha \geq \frac{a}{-E}.$$

From (6.7), we can conclude that both $\left| y\left(\frac{s}{2}\right) \right|$ and $\left| y\left(\frac{s+\lambda}{2}\right) \right|$ are strict local maxima for $|y(t)|$. One then deduces that there is at least one local minimum $|y(t^*)|$, with $t^* \in \left[\frac{s}{2}, \frac{s+\lambda}{2} \right]$. Assuming for example that $t^* - \frac{s}{2} \leq \frac{1}{4}$, one finds that $\rho(t^*) = \rho(s - t^*)$ and $\dot{\rho}(t^*) = \dot{\rho}(s - t^*) = 0$. \diamond

PROPOSITION 6.2. — Assume that the minimal period of y is $\frac{\lambda}{k}$, with $k \geq 2$. Then there are λ_1 and λ_2 with $\lambda_2 - \lambda_1 \leq \frac{\lambda}{2}$, such that $\rho(\lambda_1) = \rho(\lambda_2)$, $\dot{\rho}(\lambda_1) = \dot{\rho}(\lambda_2) = 0$.

Proof. — In an obvious consequence of the fact that $\rho\left(t + \frac{\lambda}{k}\right) = \rho(t)$, $\forall t \in \mathbb{R}$. \diamond

PROPOSITION 6.3. — Under the assumptions of Proposition 6.1 (respectively Proposition 6.2), there are three functions $\sigma_1, \sigma_2 : (1, 2)^3 \times [1, +\infty) \rightarrow (0, +\infty)$ and $\Xi : (1, 2) \rightarrow (1, +\infty)$ such that, if

$$\frac{b \left(\frac{(2 - \alpha_1) \alpha}{(2 - \alpha) \alpha_1} \right)^\alpha \frac{2 - \alpha}{2 - \alpha_2} \leq \Xi(\alpha),$$

holds, then

$$\rho(t) \geq \sigma_1\left(\frac{b}{a}, \alpha_2, \alpha, \alpha_1\right) \left(\frac{(2 - \alpha_1) a}{-2E} \right)^{1/\alpha}, \quad \forall t \in [\lambda_1, \lambda_2],$$

and

$$\rho(t) \leq \sigma_2\left(\frac{b}{a}, \alpha_2, \alpha, \alpha_1\right) \left(\frac{(2 - \alpha_2) b}{-2E} \right)^{1/\alpha}, \quad \forall t \in [\lambda_1, \lambda_2].$$

Moreover, the following properties hold:

$$\begin{aligned} \Xi(\alpha) &\geq 1, \quad \forall \alpha \in (1, 2); \\ \Xi &\text{ is increasing} \\ \lim_{\alpha \rightarrow 2} \Xi(\alpha) &= +\infty; \end{aligned}$$

$$\lim_{(\alpha_2 b)/(\alpha_1 a) \rightarrow 1} \sigma_i \left(\frac{b}{a}, \alpha_2, \alpha, \alpha_1 \right) = 1, \quad \forall \alpha \in (1, 2)$$

$$\lim_{\alpha \rightarrow 2} \sigma_i \left(\frac{b}{a}, \alpha_2, \alpha, \alpha_1 \right) = 1, \quad \text{if } \frac{(2-\alpha_1)b}{(2-\alpha_2)a} \text{ remains bounded.}$$

Proof:

Step 1. — By the change of variables $s(t) = \int_0^t \frac{1}{\rho^\alpha}$, and $\mu(s) = \rho^{2-\alpha}(s(t))$, from Proposition 6.1 (resp. Proposition 6.2), (6.7) becomes equivalent to

$$(6.8) \quad \begin{cases} -\mu'' = -2(2-\alpha)E\mu^\beta - (2-\alpha)^2 c(s) \\ \mu(0) = \mu(\omega), \\ \mu'(0) = \mu'(\omega) = 0 \\ \mu(s) > 0, \quad \forall s. \end{cases}$$

Where $(\cdot)'$ denotes the derivation with respect to the variable s

$$(6.9) \quad \beta = \frac{\alpha}{2-\alpha}$$

and

$$(6.10) \quad \omega = \int_{\lambda_1}^{\lambda_2} \frac{1}{\rho^\alpha}.$$

Note that, by Proposition 6.1 (resp. Proposition 6.2), by integrating the equation in (6.7) one obtains

$$(6.11) \quad \omega = \int_{\lambda_1}^{\lambda_2} \frac{1}{\rho^\alpha} \leq \frac{-2E}{(2-\alpha_2)a} (\lambda_2 - \lambda_1) \leq \frac{-2E}{(2-\alpha_2)a} \frac{\lambda}{2}.$$

We set $c(s) = (1 + c_0(s))$, so that, from (6.6) we have

$$(6.12) \quad 0 \leq c_0(s) \leq \frac{(2-\alpha_1)b-1}{(2-\alpha_2)a}, \quad \forall s \in [0, \omega].$$

By derivating the equation in (6.8) and by taking the L^2 inner product with μ' we obtain

$$(6.13) \quad \int_0^\omega (\mu'')^2 ds = -2\alpha E \int_0^\omega \mu^{\beta-1} (\mu')^2 ds + (2-\alpha)(2-\alpha_2)a \int_0^\omega c_0 \mu'' ds,$$

and, from Holder inequality, taking into account of (6.12),

$$(6.14) \quad \int_0^\omega (\mu'')^2 ds - (2-\alpha)(2-\alpha_2)a \times \left(\frac{(2-\alpha_1)b}{(2-\alpha_2)a} - 1 \right) \sqrt{\omega} \left(\int_0^\omega (\mu'')^2 ds \right)^{1/2} \leq -2\alpha E \int_0^\omega \mu^{\beta-1} (\mu')^2 ds.$$

Now, for any $0 < \delta < 1$, the inequality

$$\delta x^2 - \frac{d^2}{4(1-\delta)} \leq x^2 - dx, \quad \forall x, d \in \mathbb{R},$$

together with (6.14) imply

$$(6.15) \quad \delta \int_0^\omega (\mu'')^2 ds - \frac{(2-\alpha)^2(2-\alpha_2)^2 a^2 \left(\frac{(2-\alpha_1)b}{(2-\alpha_2)a} - 1 \right)^2 \omega}{4(1-\delta)} \leq -2\alpha E \int_0^\omega \mu^{\beta-1} (\mu')^2 ds.$$

Step 2. – Let us denote by H_ω^2 the Sobolev space

$$\{u \in H^2(\mathbb{R}; \mathbb{R}) / u(s+\omega) = u(s), \forall s \in \mathbb{R}\}.$$

Let us consider, for every $\beta > 1$

$$(6.16) \quad \Xi^*(\beta, \omega) = \inf_{H_\omega^2} \frac{\left(\int_0^\omega |u|^\beta \right)^{(\beta-1)/\beta} \int_0^\omega |\ddot{u}|^2}{\int_0^\omega |\dot{u}|^2 |u|^{\beta-1}}$$

and

$$(6.17) \quad \Omega^*(\beta, \omega) = \inf_{u \in H_\omega^2} \frac{\int_0^\omega (u'')^2}{\left(\int_0^\omega (u')^{2\beta} \right)^{1/\beta}}.$$

The following properties hold

$$(6.18) \quad \Xi^*(\beta, \omega) = \frac{1}{\omega^{(\beta+1)/\beta}} \Xi^*(\beta, 1), \quad \forall \beta \leq 1, \quad \forall \omega > 0;$$

$$(6.19) \quad \Omega^*(\beta, \omega) = \frac{1}{\omega^{(\beta+1)/\beta}} \Omega^*(\beta, 1), \quad \forall \beta \geq 1, \quad \forall \omega > 0;$$

$$(6.20) \quad \Xi^*(\beta, 1) \geq \Omega^*(\beta, 1), \quad \forall \beta \geq 1;$$

$$(6.21) \quad \Omega^*(\beta, 1) = \frac{1}{\omega^{(1+\beta)/\beta}} \frac{(1+\beta)^{(1+\beta)/\beta}}{\beta} \left\{ \int_0^{2\pi} |\sin \theta|^{(1+\beta)/\beta} d\theta \right\}^2.$$

Taking into account of (6.9), we set

$$(6.22) \quad \left\{ \begin{aligned} (\Xi(\alpha))^{2/\alpha} &= \frac{4}{(2\pi)^2(2-\alpha)} \Xi^* \left(\frac{\alpha}{2-\alpha}, 1 \right), \\ \Omega(\alpha) &= \frac{4}{(2\pi)^2(2-\alpha)} \Omega^* \left(\frac{\alpha}{2-\alpha}, 1 \right). \end{aligned} \right.$$

The proof of (6.18) and (6.19) obviously follows from the definitions. Formula (6.20) follow from the definitions and Holder's inequality. Finally (6.21) was proved in [22].

Remark 6.1. – From (6.22), one deduces that

$$\Omega(\alpha) \geq \frac{4}{\pi^2} \left(\frac{\pi}{2-\alpha} \right)^{2/\alpha};$$

and then that

$$\Xi(\alpha) \geq \left(\frac{2}{\pi} \right)^\alpha \frac{\pi}{2-\alpha}$$

since the right hand side of the above inequality is increasing, we deduce that Ξ fulfills all the properties of the claim of Corollary 6.1

Now, from our change of variables we have

$$(6.23) \quad \int_0^\omega \mu^\beta = \lambda_2 - \lambda_1.$$

Hence the inequality (6.15) can be rewritten as

$$(6.24) \quad \delta \int_0^\omega (\mu'')^2 ds - \frac{(2-\alpha)^2(2-\alpha_2)^2 a(((2-\alpha_1)b/(2-\alpha_2)a) - 1)^2 \omega}{4(1-\delta)} \leq -2\alpha E \left(\frac{\lambda}{2} \right)^{(\beta-1)/\beta} \frac{\int_0^1 (\mu')^2 \mu^{\beta-1}}{\left(\int_0^\omega \mu^\beta \right)^{(\beta-1)/\beta}}.$$

Assuming that μ is not constant, from (6.24), (6.15), (6.18), we deduce

$$(6.25) \quad \delta - \frac{(2-\alpha)^2(2-\alpha_2)^2 a(((2-\alpha_1)b/(2-\alpha_2)a) - 1)^2 \omega}{4(1-\delta) \int_0^\omega (\mu'')^2} \leq -2\alpha E \left(\frac{\lambda}{2} \right)^{(\beta-1)/\beta} \frac{1}{\Xi^*(\beta, 1)} \omega^{(1+\beta)/\beta}$$

and therefore from (6.22), (6.9), (6.23) and (6.1), we obtain

$$(6.26) \quad \delta - \frac{(2-\alpha)^2(2-\alpha_2)^2 a ((2-\alpha_1)b/(2-\alpha_2)a - 1)^2 \omega}{4(1-\delta) \int_0^\omega (\mu'')^2} \\ \leq \left(\frac{b}{a}\right)^{2/\alpha} \left(\frac{(2-\alpha_1)\alpha}{(2-\alpha)\alpha_1}\right)^2 \left(\frac{2-\alpha}{2-\alpha_2}\right)^{2/\alpha} \frac{1}{(\Xi(\alpha))^{2/\alpha}}.$$

Now, according with the claim of Proposition 6.3, we assume that

$$(6.27) \quad \frac{b}{a} \left(\frac{(2-\alpha_1)\alpha}{(2-\alpha)\alpha_1}\right)^\alpha \frac{2-\alpha}{2-\alpha_2} < \Xi(\alpha).$$

Let us denote

$$(6.28) \quad \xi_1\left(\frac{b}{a}, \alpha_2, \alpha, \alpha_1\right) = \left(\frac{b}{a}\right)^{2/\alpha} \left(\frac{(2-\alpha_1)\alpha}{(2-\alpha)\alpha_1}\right)^2 \left(\frac{2-\alpha}{2-\alpha_2}\right)^{2/\alpha}.$$

We set

$$\delta = \frac{1}{2} \left(1 + \frac{\xi_1(b/a, \alpha_2, \alpha, \alpha_1)}{(\Xi(\alpha))^{2/\alpha}}\right);$$

it follows from (6.4) that $0 < \delta < 1$. Now (6.26) becomes

$$(6.29) \quad \int_0^\omega (\mu'')^2 \leq (2-\alpha)^2(2-\alpha_2)^2 a \omega \\ \times \left(\frac{(2-\alpha_1)b}{(2-\alpha_2)a} - 1\right)^2 \left(1 - \frac{\xi_1(b/a, \alpha_2, \alpha, \alpha_1)}{(\Xi(\alpha))^{2/\alpha}}\right)^{-2},$$

and hence, from (6.19),

$$(6.30) \quad \left(\int_0^\omega (\mu')^{2\beta}\right)^{1/\beta} \leq \frac{1}{\pi^2(2-\alpha)\Omega(\alpha)} (2-\alpha)^2(2-\alpha_2)^2 a^2 \omega^{(2\beta+1)/\beta} \\ \times \left(\frac{(2-\alpha_1)b}{(2-\alpha_2)a} - 1\right)^2 \left(1 - \frac{\xi_1(b/a, \alpha_2, \alpha, \alpha_1)}{(\Xi(\alpha))^{2/\alpha}}\right)^{-2}.$$

Now, an easy computation shows that

$$\left(\int_{\lambda_1}^{\lambda_2} \left(\frac{d}{dt} \rho^{(2+\alpha)/2}\right)^{2\beta} dt\right)^{1/\beta} = \left(\frac{(2+\alpha)}{2(2-\alpha)}\right)^2 \left(\int_0^\omega (\mu')^{2\beta} ds\right)^{1/\beta},$$

so that, from (6.30), (6.9) we obtain that

$$(6.31) \quad \left(\int_{\lambda_1}^{\lambda_2} \left(\frac{d}{dt} \rho^{(2+\alpha)/2}\right)^{2\beta} dt\right)^{1/\beta} \\ \leq \xi_2\left(\frac{b}{a}, \alpha_2, \alpha, \alpha_1\right) a^2 \left(\frac{-2E}{(2-\alpha_2)a}\right)^{(2+\alpha)/\alpha} \left(\frac{\lambda}{2}\right)^{(2+\alpha)/\alpha},$$

where

$$\xi_2\left(\frac{b}{a}, \alpha_2, \alpha, \alpha_1\right) = \left(\frac{(2+\alpha)}{2(2-\alpha)}\right)^2 \frac{1}{\pi^2 \Omega(\alpha)} (2-\alpha)(2-\alpha_2)^2 \times \left(\frac{(2-\alpha_1)b}{(2-\alpha_2)a} - 1\right)^2 \left(1 - \frac{\xi_1(b/a, \alpha_2, \alpha, \alpha_1)}{(\Xi(\alpha))^{2/\alpha}}\right)^{-2}$$

Therefore ξ_2 enjoys the following properties (see Remark 6.1):

$$(6.32) \left\{ \begin{array}{l} \lim_{((2-\alpha_1)b/(2-\alpha_2)a) \rightarrow 1} \xi_2\left(\frac{b}{a}, \alpha_2, \alpha, \alpha_1\right) = 0, \quad \forall 1 < \alpha < 2 \\ \lim_{\alpha \rightarrow 2} \xi_2\left(\frac{b}{a}, \alpha_2, \alpha, \alpha_1\right) = 0 \quad \text{if } \frac{(2-\alpha_1)b}{(2-\alpha_2)a} \text{ remains bounded.} \end{array} \right.$$

Let us write $\rho_* = \left(\frac{(2-\alpha_2)a}{-2E}\right)^{1/\alpha}$. Remark that, since from (6.7) every local maximum $\rho(t_0)$ has $\rho(t_0) \geq \rho_*$, then either ρ_* is assumed by the function $\rho(t)$ or $\rho(t) \geq \rho_*$, $\forall t$. From (6.31) we deduce

$$\begin{aligned} |\rho^{(2+\alpha)/2}(t) - \rho_*^{(2+\alpha)/2}| &\leq \int_{\lambda_1}^{\lambda_2} \left| \frac{d}{dt} \rho^{(2+\alpha)/2} \right| dt \\ &\leq \left(\int_{\lambda_1}^{\lambda_2} \left(\frac{d}{dt} \rho^{(2+\alpha)/2} \right)^{2\beta} dt \right)^{1/2\beta} \lambda^{1-(1/2\beta)} \leq \sqrt{\xi_2} \left(\frac{-2E}{(2-\alpha_2)a} \right)^{(2+\alpha)/2\alpha} a \left(\frac{\lambda}{2} \right)^2 \end{aligned}$$

and hence, from (6.1),

$$(6.33) \quad |\rho^{(2+\alpha)/2}(t) - \rho_*^{(2+\alpha)/2}| \leq \xi_3 \left(\frac{(2-\alpha_2)a}{-2E} \right)^{(2+\alpha)/2\alpha} = \xi_3 \rho_*^{(2+\alpha)/2},$$

where

$$(6.34) \quad \xi_3\left(\frac{b}{a}, \alpha_2, \alpha, \alpha_1\right) = \sqrt{\xi_2} \frac{\pi^2}{\alpha} \left(\frac{b}{a}\right)^{2/\alpha} \left(\frac{(2-\alpha_1)\alpha}{(2-\alpha)\alpha_1}\right)^2 \left(\frac{2-\alpha}{2-\alpha_2}\right)^{(2+\alpha)/\alpha}$$

still satisfies

$$(6.35) \left\{ \begin{array}{l} \lim_{((2-\alpha_1)b/(2-\alpha_2)a) \rightarrow 1} \xi_3\left(\frac{b}{a}, \alpha_2, \alpha, \alpha_1\right) = 0, \quad \forall 1 < \alpha < 2 \\ \lim_{\alpha \rightarrow 2} \xi_3\left(\frac{b}{a}, \alpha_2, \alpha, \alpha_1\right) = 0 \quad \text{if } \frac{(2-\alpha_1)b}{(2-\alpha_2)a} \text{ remains bounded.} \end{array} \right.$$

Finally obtain

$$(6.36) \quad \rho(t) \geq \sigma_1 \rho_* = \sigma_1 \left(\frac{(2-\alpha_2)a}{-2E} \right)^{1/\alpha}, \quad \forall t \in [\lambda_1, \lambda_2],$$

and

$$(6.37) \quad \rho(t) \leq \sigma_2 \rho_* = \sigma_1 \left(\frac{(2-\alpha_2)a}{-2E} \right)^{1/\alpha}, \quad \forall t \in [\lambda_1, \lambda_2],$$

where

$$\left(\sigma_1 \left(\frac{b}{a}, \alpha_2, \alpha, \alpha_1 \right) \right)^{(2+\alpha)/2} = 1 - \xi_3 \left(\frac{b}{a}, \alpha_2, \alpha, \alpha_1 \right),$$

and

$$\left(\sigma_2 \left(\frac{b}{a}, \alpha_2, \alpha, \alpha_1 \right) \right)^{(2+\alpha)2} = 1 + \xi_3 \left(\frac{b}{a}, \alpha_2, \alpha, \alpha_1 \right).$$

From (6.35), both σ_1 and σ_2 satisfy

$$(6.38) \quad \left\{ \begin{array}{l} \lim_{((2-\alpha_1)b/(2-\alpha_2)a) \rightarrow 1} \sigma_i \left(\frac{b}{a}, \alpha_2, \alpha, \alpha_1 \right) = 1, \quad \forall 1 < \alpha < 2 \\ \lim_{\alpha \rightarrow 2} \sigma_i \left(\frac{b}{a}, \alpha_2, \alpha, \alpha_1 \right) = 1 \quad \text{if } \frac{(2-\alpha_1)b}{(2-\alpha_2)a} \text{ remains bounded.} \end{array} \right.$$

Hence Proposition 6.3 is proved.

Proof of Proposition 5.6. – We assume that (5.41), (5.42) and (5.43) hold with Ξ , σ_1 and σ_2 respectively defined in (6.22), (6.36), (6.37). Assume first that $y \in F_0$. Then Proposition 6.1 says that there is $\bar{t} \in [\lambda_1, \lambda_2]$ such that $\dot{y}(\bar{t}) = 0$. We then deduce from (6.3), (H1) and Proposition 6.3 that $\sigma_2 \left(\frac{b}{a}, \alpha_2, \alpha, \alpha_1 \right)^\alpha < \frac{2}{2-\alpha_1}$, that is a contradiction.

Now we assume that the minimal period of y is $\frac{T}{k}$, for some integer $k \geq 2$. Then Wirtinger inequality leads to

$$(6.39) \quad \left(\frac{2\pi k}{\lambda} \right)^2 \int_0^T |\dot{y}|^2 \leq \int_0^\lambda |\ddot{y}|^2.$$

On the other hand, from (H5) we have

$$(6.40) \quad \int_0^\lambda |\dot{y}|^2 \geq -\alpha_1 \int_0^\lambda F(y)$$

and, from (H4), (H6) and Proposition 6.3 we obtain

$$(6.41) \quad \int_0^\lambda |\ddot{y}|^2 = \int_0^\lambda |\nabla F(y)|^2 \leq a^2 \alpha_2^2 \int_0^\lambda \frac{1}{|y|^{2\alpha+2}} \leq a \alpha_2^2 \left(\frac{-2E}{(2-\alpha_2)a} \right)^{(2+\alpha)/\alpha} \frac{1}{\sigma_1^{2+\alpha}} \int_0^\lambda -F(y).$$

From (6.40) and (6.41) we deduce

$$\left(\frac{2\pi k}{\lambda}\right)^2 \leq a \frac{\alpha_2^2}{\alpha_1} \left(\frac{-2E}{(2-\alpha_2)a}\right)^{(2+\alpha)/\alpha} \frac{1}{\sigma_1^{2+\alpha}}.$$

Therefore (6.1) implies

$$k^2 \leq \frac{b^{2/\alpha}}{a^{2/\alpha}} \frac{\alpha \alpha_2^2}{\alpha_1^3} \left(\frac{2-\alpha_1}{2-\alpha}\right)^2 \left(\frac{2-\alpha}{2-\alpha_2}\right)^{(2+\alpha)/\alpha} \frac{1}{\sigma_1^{2+\alpha}} < 4,$$

that is (k is an integer) $k=1$.

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