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The Dirichlet problem for the equation of prescribed mean curvature

by

Guofang WANG

Institute of Mathematics, Academia Sinica, Beijing, 100080 China

ABSTRACT. — We prove that there exist at least two distinct solutions to the Dirichlet problem for the equation of prescribed mean curvature $\Delta X = 2H(X) X_u \wedge X_v$, the curvature function H being in a full neighborhood of a suitable constant.

Key words : Equation of prescribed mean curvature, Dirichlet problem, relative minimizer.

1. INTRODUCTION

Let $B = \{ \omega = (u, v) \in \mathbb{R}^2 / |\omega| < 1 \}$ be the unit disc in $\mathbb{R}^2$ with boundary ∂B. We consider the Dirichlet problem for the equation of prescribed mean curvature

$$\Delta X = 2H(X) X_u \wedge X_v, \quad \text{in } B, \tag{1.1}$$
$$X = X_D, \quad \text{on } \partial B. \tag{1.2}$$

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Here, $X_u = \frac{\partial}{\partial u} X$ and $X_v = \frac{\partial}{\partial v} X$ denote partial derivative, $\wedge$ and . are the exterior and inner product in $\mathbb{R}^3$ and $H : \mathbb{R}^3 \to \mathbb{R}$ is a given function, and X_D is a given function of class $C^2(\bar{B}, \mathbb{R}^3)$.

If $H \equiv H_0 = \text{Const.}$, solutions to (1.1), (1.2) can be characterized as critical points of the functional

$$E_{H_0}(X) = D(X) + 2 H_0 V(X), \tag{1.3}$$

in a space of admissible functions satisfying the boundary condition (1.2), where

$$D(X) = \frac{1}{2} \int_B |\nabla X|^2 \, d\omega \tag{1.4}$$

is the Dirichlet integral and

$$V(X) = \frac{1}{3} \int_B X . X_u \wedge X_v \, d\omega \tag{1.5}$$

is the algebraic volume of surface X.

THEOREM 1.1 ([Hi2], [Wet1], [Wet2] and [Stf1]). — *Suppose* $H \equiv H_0 \in \mathbb{R}$ *and let* $X_D \in H^{1,2}(B, \mathbb{R}^3)$ *be given. Assume that either*

(i) X_D *is bounded and*

$$|H_0| . \|X_D\|_{L^\infty} < 1, \tag{1.6}$$

or

(ii) *the condition*

$$H_0^2 D(X_D) < \frac{2}{3} \pi \tag{1.7}$$

is satisfied. Then there is a solution $\underline{X} \in \{X_D\} + H_0^{1,2}(B, \mathbb{R}^3)$ *to* (1.1), (1.2) *which is a strict relative minimizer of* E_{H_0} *in this space.*

Remark 1.2. — The observation that the solutions of Hildebrandt, Steffen and Wente are strict relative minima is due to Brezis-Coron [BC].

The existence of a second solution was proved independently by Brezis-Coron [BC] and Struwe [St2] with an important contribution by Steffen [Stf2] as follows

THEOREM 1.3 [Str3]. — *Let* $X_D \in H^{1,2} \cap L^\infty(B, \mathbb{R}^3)$ *be a non-constant vector,* H_0 *any real number different from zero. Suppose* E_{H_0} *admits a local minimum* $\underline{X}$ *in the class* $\{X_D\} + H_0^{1,2}(B, \mathbb{R}^3)$. *Then there exists a solution* $\bar{X} \in \{X_D\} + H_0^{1,2}(B, \mathbb{R}^3)$ *of* (1.1) *and* (1.2) *different from* $\underline{X}$ *and satisfying the condition*

$$E_{H_0}(\underline{X}) < E_{H_0}(\bar{X}) = \inf_{p \in P} \sup_{X \in \operatorname{im}(P)} E_H(X) < E_{H_0}(\underline{X}) + \frac{4\pi}{3|H_0|^2}, \tag{1.8}$$

where

$$P=\{p\in C^0([0,1],\{X_D\}+H_0^{1,2}(B,\mathbb{R}^3))\,|\,p(0)=\underline{X},\ E_{H_0}(p(1))<E_{H_0}(\underline{X})\}. \quad (1.9)$$

For variable curvature functions H results comparable to Theorem 1.1 have been obtain by Hildebrandt [Hi1] and Steffen [Stf1].

THEOREM 1.4 [Hi1]. — *Suppose* H *is of class* C^1 *and let* $X_D\in H^{1,2}\cap L^\infty(B,\mathbb{R}^3)$ *be given with* $\|X_D\|_{L^\infty}<1$. *Then if*

$$h\overset{\Delta}{=}\operatorname*{ess\,sup}_{|X|\leqq 1} H(X)<1$$

there exists a solution $\underline{X}\in\{X_D\}+H_0^{1,2}(B,\mathbb{R}^3)$ *to* (1.1), (1.2) *such that* $E_H(\underline{X})=\inf\{E_H(X);\ X\in M\}$, *where* M *is given by* (2.9) *below.*

If variable curvature function H is sufficiently close to a suitable constant, Struwe obtained [Str4].

THEOREM 1.5. — *Suppose* $X_D\in C^2(B,\mathbb{R}^2)$ *is non-constant and suppose that for* $H_0\in\mathbb{R}\setminus\{0\}$ *the functional* E_{H_0} *admits a relative minimizer in* $\{X_D\}+H_0^{1,2}(B,\mathbb{R}^3)$. *Then there exists a number* $\alpha>0$ *such that for a dense set* $\mathscr{A}$ *of curvature functions* H *in the* α*-neighborhood of* H_0, *the Dirichlet problem* (1.1), (1.2) *admits at least two distinct regular solutions in* $\{X_D\}+H_0^{1,2}(B,\mathbb{R}^3)$.

Here the α-neighborhood of H_0 is defined as

$$[H-H_0]\overset{\Delta}{=}\operatorname*{ess\,sup}_{X\in\mathbb{R}^3}\{(1+|X|)(|H(X)-H_0|+|\nabla H(X)|)+|Q(X)-H_0X|+|\nabla Q(X)-H_0\,\mathrm{id}|\}\leqq\alpha, \quad (1.10)$$

where Q is given by (2.3) below.

In this paper, we improve Theorem 1.5 and obtain that

THEOREM 1.6. — *Suppose* $X_D\in C^2(\mathbb{B},\mathbb{R}^3)$ *is non-constant, and suppose that for* $H_0\in\mathbb{R}\setminus\{0\}$ *the functional* E_{H_0} *admits a relative minimizer in* $\{X_D\}+H_0^{1,2}$. *Then there exists a number* $\alpha>0$ *such that if* $[H-H_0]<\alpha$, E_H *admits two solutions in* $\{X_D\}+H_0^{1,2}$.

From the proof of Theorem 1.5 [Str4], we have a relative minimizer of E_H for a full α-neighborhood of H_0 and another "large" critical point of E_H for $H\in\mathscr{A}$. We call the former S-solution and the latter L-solution.

First, we show that the S-solution is also a "strict" relative minimizer—its E_H—energy is less than that of the L-solution—provided that $[H-H_0]$ is small enough. Next, we give *a priori* estimates for solutions of the Dirichlet problem—which are of crucial importance to our result—though they are not given explicitly. Then, we can use the solutions obtained by Struwe in [Str4]—the L-solutions—for a dense set $\mathscr{A}$ in

$\mathscr{H}_\alpha=\{H|[H-H_0]<\alpha\}$ to approximate a solution of E_H for any $H\in\mathscr{H}_\alpha$ which is different from the S-solution.

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2. PRELIMINARIES

For variable curvature function H, solutions to (1.1), (1.2) can be characterized as critical points of the functional

$$E_H(X)=D(X)+2V_H(X) \tag{2.1}$$

in the space $\{X_D\}+H_0^{1,2}(B,\mathbb{R}^2)$. Here, the H-volume introduced by Hildebrandt is given by

$$V_H(X)=\frac{1}{3}\int_B Q(X).X_u\wedge X_v\,dw, \tag{2.2}$$

where

$$Q(x_1,x_2,x_3)=\left(\int_0^{x_1}H(s,x_2,x_3)\,ds,\right.$$
$$\left.\int_0^{x_2}H(x_1,s,x_3)\,ds,\int_0^{x_3}H(x_1,x_2,s)\,ds\right). \tag{2.3}$$

We list some useful lemmas.

LEMMA 2.1 (Isoperimetric inequality, *cf.* [Wet1]):

$$36\pi(V(X))^2\leqq D(X)^3, \tag{2.4}$$

for $X\in H_0^{1,2}(B,\mathbb{R}^3)$.

LEMMA 2.2 ([BC], [Str4], Prop. 3.1). – *Suppose* $X_D\in C^2(\bar{B},\mathbb{R}^3)$ *is non-constant, and suppose that for* $H\equiv H_0\neq 0$ *the functional* E_{H_0} *admits a relative minimizer* $\underline{X}_0\in\{X_D\}+C^2\cap H_0^{1,2}(B,\mathbb{R}^3)$. *Then there exists a radius* $R>0$, *a function* $X_1\in\{X_D\}+C^2\cap H_0^{1,2}(\bar{B},\mathbb{R}^3)$ *with* $D(X_1-\underline{X}_0)\geqq R$, *and a continuous path* $p\in C^0([0,1];\{X_D\}+C^2\cap H_0^{1,2}(\bar{B},\mathbb{R}^3))$ *connecting* $\underline{X}_0=p(0)$ *with* $X_1=p(1)$ *such that the estimates*

$$E_{H_0}(X_1)<\inf\{E_{H_0}(X);X-\underline{X}_0\in H_0^{1,2},D(X-\underline{X}_0)\leqq R\}\leqq E_{H_0}(\underline{X}_0) \tag{2.5}$$
$$<\inf\{E_{H_0}(X);X-\underline{X}_0\in H_0^{1,2},D(X-\underline{X}_0)=R\} \tag{2.6}$$
$$\leqq\sup\{E_{H_0}(X);X\in p([0,1])\} \tag{2.7}$$
$$<E_{H_0}(\underline{X}_0)+4\pi/3H_0^2 \tag{2.8}$$

hold.

DEFINITION 2.3:

$$M=\{X\in\{X_D\}+H_0^{1,2}(B,\mathbb{R}^3);\ D(X-\underline{X}_0)\leqq R\},\qquad (2.9)$$

where $\underline{X}_0$ and R are as in Lemma 2.2.

LEMMA 2.4 [BC]. – *For* $H_0\in\mathbb{R}\setminus\{0\}$. *Suppose that* E_{H_0} *admits a relative minimizer* $\underline{X}_0\in\{X_D\}+C^2\cap H_0^{1,2}(\bar{B},\mathbb{R}^3)$. *Then there is* $\delta>0$ *such that*

$$\int|\nabla\varphi|^2+4H_0\int\underline{X}_0\cdot\varphi_u\wedge\varphi_v\geqq\delta\int|\nabla\varphi|^2,\quad \textit{for all}\ \ \varphi\in H_0^{1,2}.\qquad (2.10)$$

Let

$$P=\{p\in C^0([0,1];\{X_D\}+H_0^{1,2}(B,\mathbb{R}^3)),\ p(0)=\underline{X}_0,p(1)=X_1\}$$

and set

$$\gamma_{H,\rho}=\inf_{p\in P}\ \sup_{X\in \mathrm{im}(p)} E_{H,\rho}(X)$$

where $E_{H,\rho}(X)=(1+\rho)\,E_{H/(1+\rho)}(X)$. Using Lemma 2.2 we have (*see* [Str4], (3.5))

$$\begin{aligned}
E_{H,\rho}(X_1)&<\inf\{E_H(X);\ X-\underline{X}_0\in H_0^{1,2},\ D(X-\underline{X}_0)\leqq R\}\leqq E_{H,\rho}(\underline{X}_0) && (2.5)_\rho\\
&<\inf\{E_H(X);\ X-\underline{X}_0\in H_0^{1,2},\ D(X-\underline{X}_0)=R\}\\
&\qquad\leqq\inf\{E_{H,\rho}(X);\ \ldots\} && (2.6)_\rho\\
&\leqq\sup\{E_{H,\rho}(X);\ X\in p([0,1])\} && (2.7)_\rho\\
&<E_H(\underline{X}_0)+\beta\leqq E_{H,\rho}(\underline{X}_0)+\beta && (2.8)_\rho
\end{aligned}$$

and

$$E_{H,\rho}(\underline{X}_0)<\gamma_{H,0}\leqq\gamma_{H,\rho}\leqq\gamma_{H,\alpha}<E_H(\underline{X}_0)+\beta,\qquad (2.11)$$

for $\rho\in[0,\alpha]$. Here X_1, $\underline{X}_0$ and P are as in Lemma 2.2, α is small enough and fixed and $\beta<4\pi/3H_0^2$ is independent of H. Moreover, we have

LEMMA 2.5. – *There exists a constant number* ε_0 *independent of* α *such that*

$$\begin{aligned}
E_{H_0}(\underline{X}_0)+\varepsilon_0&<\inf\{E_H(X);\ X-\underline{X}_0\in H_0^{1,2},\ D(X-X_0)=R\}\\
&\leqq\inf\{E_{H,\rho}(X);\ X-\underline{X}_0\in H_0^{1,2},D(X-X_0)=R\}\\
&\leqq\sup\{E_{H,\rho}(X);\ X\in p([0,1])\}\\
&<E_H(\underline{X}_0)+\beta-\varepsilon_0\leqq E_{H,\rho}(\underline{X}_0)+\beta-\varepsilon_0
\end{aligned}$$

provided that α *is small enough, where* $\underline{X}_0$, R *and* p *are as in Lemma* 2.2.

Proof. – Set

$$\varepsilon_0=\frac{1}{4}\min\{(E_{H_0}(\underline{X}_0)+4\pi/3H_0^2-\sup\{E_H(X);\ X\in p([0,1])\}),\\ (\inf\{E_{H_0}(X);\ X-\underline{X}_0\in H_0^{1,2},\ D(X-\underline{X}_0)=R\}-E_{H_0}(\underline{X}_0))\}.$$

It is easy to see that Lemma 2.5 follows from Lemma 2.2 for α small enough.

Q.E.D.

LEMMA 2.6. — *There exists a constant c independent of* α *such that if* $H \in \mathscr{A}$ (see *Theorem* 1.5),

$$D(\underline{X}-\bar{X})>c,$$

where $\underline{X}$ *(resp.* $\bar{X}$*) is the* S*-solution (resp.* L*-solution) to* (1.1), (1.2).

Proof. — It follows the proof of Theorem 1.5 ([Str4], Theorem 3.1) and Lemma 2.5.

Q.E.D.

3. THE "STRICT" RELATIVE MINIMA

In this section, we will prove that the S-solution to the Dirichlet problem for the equation of prescribed mean curvature H is a "strict" relative minimum in the space $\{X_D\}+H_0^{1,2}(B, \mathbb{R}^3)$, provided that $[H-H_0]$ is small enough. Here $H_0 \neq 0$ is a constant with the property that E_{H_0} admits a relative minimizer $\underline{X}_0 \in \{X_D\}+C^2 \cap H_0^{1,2}(B, \mathbb{R}^3)$.

LEMMA 3.1. — *There exists a constant number* $\alpha>0$ *with the property that if* $[H-H_0]<\alpha$ *there is a constant* $\delta>0$ *depending only on* α *and* X_D *such that*

$$\int|\nabla \varphi|^2+4\int Q(\underline{X})\,\varphi_u \wedge \varphi_v \geqq \delta \int|\nabla \varphi|^2, \quad \textit{for any } \varphi \in H_0^{1,2}. \quad (3.1)$$

Here $\underline{X}=\underline{X}_H$ *is the* S*-solution to* (1.1), (1.2).

Proof. — Let $\underline{X}_0$ be the small solution of E_{H_0} in the space $\{X_D\}+H_0^{1,2}$. By Brezis-Coron [BC]—*see* Lemma 2.4—there exists a constant $\delta_1>0$ such that

$$\int|\nabla \varphi|^2+4H_0\int \underline{X}_0\,\varphi_u \wedge \varphi_v \geqq \delta_1 \int|\nabla \varphi|^2, \qquad \varphi \in H_0^{1,2}. \qquad (3.2)$$

Thus for any $\varphi \in H_0^{1,2}$

$$\begin{aligned}\int |\nabla \varphi|^2 + 4 \int Q(\underline{X})\, \varphi_u \wedge \varphi_v &= \int |\nabla \varphi|^2 + 4 H_0 \int \underline{X}_0\, \varphi_u \wedge \varphi_v \\ &+ 4 \int (Q(\underline{X}) - H_0 \underline{X})\, \varphi_u \wedge \varphi_v + 4 H_0 \int (\underline{X} - X_0)\, \varphi_u \wedge \varphi_v \\ &\geqq (\delta_1 - 2\alpha) \int |\nabla \varphi|^2 + 4 H_0 \int (\underline{X} - X_0)\, \varphi_u \wedge \varphi_v.\end{aligned}$$

Therefore, Lemma 3.1 follows from the following

LEMMA 3.2. – *For any* $\varepsilon > 0$, *there exists a constant* $\alpha > 0$ *with the property that for any curvature function* H *with* $[H - H_0] < \alpha$, *if* $\underline{X}_H$ *is the* S-*solution to* (1.1), (1.2), *then*

$$\| \underline{X}_H - \underline{X}_0 \|_{L^\infty} < \varepsilon. \tag{3.3}$$

Proof. – If the Lemma is false, we may assume that there exist $\varepsilon_0 > 0$ and a sequence $\{\underline{X}_i\}$ of the S-solutions of E_H with $H = H_i$ and $[H_i - H_0] \to 0$ as $i \to 0$ such that $\| \underline{X}_{H_i} - \underline{X}_0 \|_{L^\infty} \geqq \varepsilon_0$. Noticing that $X_i \in M$, we know that $D(\underline{X}_i)$ are bounded uniformly in i. Thus we may assume that $\{X_i\}$ converges to $\underline{X}$ weakly in $\{X_D\} + H_0^{1,2}$ for some $\underline{X} \in \{X_D\} + H_0^{1,2}$. It is easy to see that $\underline{X} \in M$ (*see* [Str4]). Recall

$$M = \{ X \in \{X_D\} + H_0^{1,2}(B, \mathbb{R}^3);\ D(X - \underline{X}_0) \leqq R \}.$$

But then

$$\left.\begin{aligned} E_{H_0}(\underline{X}) &\geqq \inf_{X \in M} E_{H_0}(X) \\ &= \inf_{X \in M} \lim_{i \to \infty} E_{H_i}(X) \\ &\geqq \lim_{i \to \infty} \inf_{X \in M} E_{H_i}(X) \\ &= \lim_{i \to \infty} E_{H_i}(\underline{X}_i). \end{aligned}\right\} \tag{3.4}$$

Hence, by Theorem 4.5 below, $\underline{X}_i \to \underline{X}$ strongly in $\{X_D\} + H_0^{1,2}$ and uniformly in $\bar{B}$ and $E_{H_0}(\underline{X}) = \lim_{i \to \infty} E_{H_i}(\underline{X}_i)$ by (3.4). We have

$$E_{H_0}(\underline{X}) = \inf_{X \in M} E_{H_0}(X) = E_{H_0}(\underline{X}_0).$$

Hence the uniqueness of the small solution of E_{H_0} in $\{X_D\} + H_0^{1,2}$ [BC] shows that $\underline{X} = \underline{X}_0$. Therefore, $\underline{X}_i \to \underline{X}_0$ uniformly in $\bar{B}$ which contradicts the above assumption. This completes the proof of Lemma 3.2.

Q.E.D.

If H is sufficiently close to H_0, we have

PROPOSITION 3.3. – *If* $H \in \mathscr{A}$ *and* X *is the* L-*solution to* (1.1), (1.2), *then* $E_H(X) > E_H(\underline{X})$, *where* $\underline{X}$ *is the* S-*solution to* (1.1), (1.2).

Proof. – Let $\varphi = X - \underline{X} \in H_0^{1,2}(B, \mathbb{R}^3)$. Noting that $X = \underline{X} + \varphi$ and $\underline{X}$ satisfy the equation (1.1), we have

$$\begin{aligned}
E_H(X) &= E_H(\underline{X}+\varphi)\\
&= \frac{1}{2}\int |\nabla(\underline{X}+\varphi)|^2 + \frac{2}{3}\int Q(\underline{X}+\varphi)(\underline{X}+\varphi)_u \wedge (\underline{X}+\varphi)_v\\
&= E_H(\underline{X}) + \frac{1}{2}\int |\nabla\varphi|^2 + \frac{2}{3}\int Q(\varphi)(\underline{X}_u \wedge \varphi_v + \varphi_u \wedge \underline{X}_v)\\
&\quad + \frac{2}{3}\int (Q(\underline{X}+\varphi) - Q(\varphi))\,\varphi_u \wedge \varphi_v\\
&\quad + \frac{2}{3}\int Q(\varphi)\,\varphi_u \wedge \varphi_v + O(\alpha)\left(\int |\nabla\varphi|^2\right)^{1/2} \qquad (3.5)
\end{aligned}$$

by (1.10). Testing (1.1) with φ we get

$$\begin{aligned}
0 &= \int \nabla\varphi\,\nabla(\underline{X}+\varphi) + 2\int H(\underline{X}+\varphi)\,\varphi\,(\underline{X}+\varphi)_u \wedge (X+\varphi)_v\\
&= \int |\nabla\varphi|^2 + 4\int Q(\underline{X})\,\varphi_u \wedge \varphi_v + 2\int Q(\varphi)\,\varphi_u \wedge \varphi_v\\
&\quad + O(\alpha)\left(\left(\int |\nabla\varphi|^2\right)^{1/2} + \int |\nabla\varphi|^2\right) \qquad (3.6)
\end{aligned}$$

by (1.10). From (3.5), (3.6) it is clear

$$\begin{aligned}
E_H(X) &= E_H(\underline{X}) + \frac{1}{2}\int |\nabla\varphi|^2 + 2\int Q(\underline{X})\,\varphi_u \wedge \varphi_v\\
&\quad + \frac{2}{3}\int Q(\varphi)\,\varphi_u \wedge \varphi_v + c\,\alpha\left(\left(\int |\nabla\varphi|^2\right)^{1/2} + \int |\nabla\varphi|^2\right)\\
&= E_H(\underline{X}) + \frac{1}{6}\left(\int |\nabla\varphi|^2 + 4\int Q(\underline{X})\,\varphi_u \wedge \varphi_v\right)\\
&\quad + O(\alpha)\left(\left(\int |\nabla\varphi|^2\right)^{1/2} + \int |\nabla\varphi|^2\right).
\end{aligned}$$

By Lemma 3.1, we get

$$E_H(X) - E_H(\underline{X}) \geqq \delta\int |\nabla\varphi|^2 - c\,\alpha\left(\left(\int |\nabla\varphi|^2\right)^{1/2} + \int |\nabla\varphi|^2\right). \qquad (3.7)$$

Therefore, from Lemma 2.6 we have

$$E_H(X) > E_H(\underline{X})$$

provided that α is small enough.

Q.E.D.

PROPOSITION 3.4. — *If $\alpha > 0$ is small enough, for $H \in \mathscr{H}_\alpha$ there exist a $\rho_0 > 0$ and a dense set A in $[0, \rho_0]$ such that if $\rho \in A$, then $E_{H/(1+\rho)}$ admits two distinct regular solutions in $\{X_D\} + H_0^{1,2}$, one is the S-solution $\underline{X}_H$ and the other is the L-solution $\bar{X}$ with*

$$E_H(\underline{X}_0) < \gamma_{H,0} \leqq (1+\rho) E_{H/(1+\rho)}(\bar{X}) \leqq \gamma_{H,\alpha} < E_H(\underline{X}_0) + \beta,$$

where $\gamma_{H,0}$, $\gamma_{H,\alpha}$ and β are given in section 2 and $\underline{X}_0$ is the small solution of E_{H_0}.

Proof. — Proposition 3.4 follows from the proof of Theorem 1.5 (*see* [Str4]) and Proposition 3.3.

4. CONVERGENCE OF SURFACES OF PRESCRIBED MEAN CURVATURE

As in [Pc], we can also establish a convergence theorem of surfaces of prescribed mean curvature with the Dirichlet boundary condition. Let $H: \mathbb{R}^3 \to \mathbb{R}$ satisfy

$$\left.\begin{aligned} &H \in C^1(\mathbb{R}^3, \mathbb{R}) \\ &\|H\|_{L^\infty(\mathbb{R}^3)} + \|(1+|X|)|\nabla H(X)|\|_{L^\infty(\mathbb{R}^3)} < +\infty. \end{aligned}\right\} \tag{4.1}$$

THEOREM 4.1. — *Let H_i satisfy (4.1) and $\|H_i\|_{L^\infty} \leqq K$ uniformly, and $H_i \to H$ a.e. on $\mathbb{R}^3$. Suppose $X_i \in \{X_D\} + H_0^{1,2}(B, \mathbb{R}^3) \cap C^2(\bar{B}, \mathbb{R}^3)$ is a sequence of solutions to (1.1), (1.2) with $H = H_i$ and $\int_B |\nabla X_i|^2 d\omega \leqq c$ uniformly. Assume that $X_i \to X$ weakly in $H^{1,2}(B, \mathbb{R}^3)$ for some function $X \in \{X_D\} + H_0^{1,2}(B, \mathbb{R}^3)$. Then $X_i \to X$ strongly in $\{X_D\} + H_{0,\,loc}^{1,2}(B \setminus S, \mathbb{R}^3)$ where S is a finite subset of $\bar{B}$. Moreover, X satisfies*

$$\left.\begin{aligned} \Delta X &= 2H(X) X_u \wedge X_v, \quad \text{in } B, \\ X &= X_D, \quad \text{on } \partial B. \end{aligned}\right\} \tag{4.2}$$

Proof. — The proof is similar to that of Proposition 2.6 in [Str4], thus we only sketch it. Set

$$S = \bigcap_{r>0} \left\{ w \in \bar{B} / \lim_{i \to \infty} \inf \int_{B(w,r) \cap B} |\nabla X_i|^2 \geqq \mu_0 \right\} \tag{4.3}$$

where μ_0 is a constant like μ_0 in [Str4]. By the same argument of [Str4] or [Pc], we have (by taking subsequence)

$$X_i \to X \quad \text{strongly in } C^1(\bar{B}\setminus S, \mathbb{R}^3),$$

and S is a finite subset of $\bar{B}$. Moreover, X satisfies (4.2).

Q.E.D.

LEMMA 4.2. — *If* $X \in C^2(\mathbb{R}^2_+, \mathbb{R}^3)$ *satisfies*

$$\left.\begin{aligned} \Delta X &= 2H(X) X_u \wedge X_v, && \text{in } \mathbb{R}^2_+ \\ X &= \text{const.}, && \text{on } \partial\mathbb{R}^2_+ \end{aligned}\right\} \tag{4.4}$$

then $X \equiv \text{const.}$

The Lemma easily follows from [Wet2]. For the convenience of the reader we give a complete proof.

Proof of Lemma 4.2. — Note that $X_u.(X_u \wedge X_v) \equiv 0$ in $\mathbb{R}^2_+$, from (4.4) we have

$$X_u . \Delta X = 0, \quad \text{in } \mathbb{R}^2_+$$

It's easy to see that

$$\begin{aligned} 0 = X_u . \Delta X &= X_u . \operatorname{div} \nabla X \\ &= \operatorname{div}\left\{((1,0)\nabla X)\nabla X - \frac{1}{2}(1,0)|\nabla X|^2\right\}. \end{aligned}$$

By Stokes' formula, we have

$$\int_{\partial\mathbb{R}^2_+} n.((1,0)\nabla X)\nabla X - \frac{1}{2} n.(1,0)|\nabla X|^2 \, d\omega = 0 \tag{4.5}$$

where $n = (-1, 0)$ is the outer normal to $\mathbb{R}^2_+$ at $\partial\mathbb{R}^2_+$. Since $X \equiv \text{const.}$ on $\partial\mathbb{R}^2_+$, $\nabla X = (\nabla X . n) n$ on $\partial\mathbb{R}^2_+$. Hence, from (4.5) we get

$$\int_{\partial\mathbb{R}^2_+} |\nabla X|^2 \, d\omega = 0.$$

Therefore $X_n \equiv 0$ on $\partial\mathbb{R}^2_+$. By the argument of Wente [Wet3], $X \equiv \text{const.}$ in $\mathbb{R}^2_+$.

Q.E.D.

LEMMA 4.3. — *Let* $[H - H_0] < \infty$. *If* $X \in H^{1,2}_0(\mathbb{R}^2, \mathbb{R}^3)$ *satisfies* $\Delta X = 2H(X) X_u \wedge X_v$ *in* $\mathbb{R}^2$, *and is non-constant then*

$$E_H(X) \geq \frac{4\pi}{3H_0^2} - c_0[H - H_0], \tag{4.6}$$

where c_0 *is independent of* H *and* $[H - H_0]$.

Proof. — It is easy to prove this lemma, we omit it.

PROPOSITION 4.4. — *Let* $[H-H_0]<\infty$

$$\beta_H \geqq \frac{4\pi}{3H_0^2} - c_0[H-H_0], \tag{4.7}$$

where

$$\beta_H = \inf\{\liminf_{i\to\infty}(E_{H_i}(X_i)-E_H(X));\ X_i \textit{ are critical points of } E_{H_i} \textit{ and } X_i \to X \text{ in } H^{1,2} \textit{ weakly but not strongly}\}.$$

Proof. — For any such sequence $\{X_i\}$, using Theorem 4.1, we see that $X_i \to X$ strongly in $\{X_D\}+H^{1,2}_{0,\,\mathrm{loc}}(B\setminus S, \mathbb{R}^3)$ where S is a finite non-empty subset of $\bar{B}$ and is defined by (4.3). There are two possibilities: either, (i) $S\cap\partial B=\varphi$; or, (ii) $S\cap\partial B\neq\varphi$. In case (i) from Section 3 of [Pc], we have a function $X_0\in H^{1,2}_0(\mathbb{R}^2,\mathbb{R}^3)$ satisfying $\Delta X_0 = 2H(X_0)X_{0u}\wedge X_{0v}$ in $\mathbb{R}^2$ and

$$\liminf_{i\to\infty}(E_{H_i}(X_i)-E_H(X)) \geqq E_H(X_0) \geqq \frac{4\pi}{3H_0^2} - c_0[H-H_0]$$

by Lemma 4.3. Therefore, $\beta_H \geqq \frac{4\pi}{3H_0^2} - c_0[H-H_0]$. In case (ii), using the same argument in [BC2] and Lemma 4.2, we also have a "blow up" function X satisfying $\Delta X = 2H(X)X_u\wedge X_v$ in $\mathbb{R}^2$ and

$$\liminf_{i\to\infty}(E_{H_i}(X_i)-E_H(X)) \geqq E_H(X_0) \geqq \frac{4\pi}{3H_0^2} - c_0[H-H_0]$$

Q.E.D.

THEOREM 4.5. — *Let* α *be fixed as in section* 3 *and* $[H_i-H]<\alpha$ *and* $H_i\to H$ *a.e. in* B. *Suppose* $X_i\in\{X_D\}+H^{1,2}_0$ *is a sequence of solutions to* (1.1), (1.2) *with* $H=H_i$ *and*

$$|E_{H_i}(X_i)| \leqq c < \infty$$

uniformly in i. Then

$$D(X_i) \leqq c_1$$

uniformly for another constant number c_1. *Moreover, assume* $X_i\to X$ *weakly in* $H^{1,2}(B,\mathbb{R}^3)$ *for some* $X\in\{X_D\}+H^{1,2}_0$, *then* X *is a critical point of* E_H *in* $\{X_D\}+H^{1,2}_0$ *and either*

(i) $X_i\to X$ *strongly in* $H^{1,2}\cup L^\infty(B,\mathbb{R}^3)$ *with*

$$E_H(X) = \liminf_{i\to\infty} E_{H_i}(X_i),$$

or

(ii) $E_H(X) \leqq \liminf_{i\to\infty} E_{H_i}(X_i) - \frac{4\pi}{3H_0^2} + c_0[H-H_0].$

Proof. — Let $\underline{X}_i$ be the S-solution of (1.1)-(1.2) with $H=H_i$ in $\{X_D\}+H_0^{1,2}(B,\mathbb{R}^3)$ (*see* § 3) and $\varphi_i=X_i-\underline{X}_i\in H_0^{1,2}$. By (2.7), we have

$$-c+\delta\int|\nabla\varphi_i|^2\leqq E(X_i)-E(\underline{X}_i),$$

where δ depends only on α and X_D. Note that $E_{H_i}(\underline{X}_i)$, $E_{H_i}(X_i)$ and $D(X_i)$ are bounded uniformly. Hence, $D(X_i)\leqq c_1$ uniformly for some constant c_1.

Assume that $X_i\to X$ weakly in $H^{1,2}(B,\mathbb{R}^3)$. Now there are two possibilities either, (i) $S=\varnothing$, or, (ii) $S\neq\varnothing$ by Theorem 4.1.

In case (i) $X_i\to X$ strongly in $H^{1,2}\cap L^\infty$ (*see* [Str4] or [Pc]). In case (ii)

$$\liminf_{i\to\infty} E_{H_i}(X_i)-E_H(X)\geqq\beta_H\geqq\frac{4\pi}{3H_0}-c_0[H-H_0],$$

by Proposition 4.5. This completes the proof.

Q.E.D.

Remark 4.6. — For the Dirichlet problem Theorem 4.5 gives *a priori* bounds which are of crucial importance to our results.

5. PROOF OF THEOREM 1.6

For any curvature function H with $[H-H_0]<\alpha$, there exists the S-solution $\underline{X}_H$ to (1.1), (1.2). On the other hand, by the results of Struwe [Str4] and proposition 3.3 there exists a sequence of $H_i=H/(1+\rho_i)$ tending to H such that E_{H_i} admits the L-solution $X_i\in\{X_H\}+H_0^{1,2}\cap C^2(\bar{B},\mathbb{R}^3)$ with

$$E_H(X_0)<\gamma_{H,0}\leqq(1+\rho_i)E_{H_i}(X_i)\leqq\gamma_{H,\alpha}<E_H(\underline{X}_D)+\beta$$

(*see* [Str4] or Prop. 3.4), where $\rho_i>0$ tends to 0 and $\gamma_{H,0}$, $\gamma_{H,\alpha}$, β and X_0 are as in section 3.

Now from Theorem 4.5, $X_i\to X$ weakly in $H^{1,2}(B,\mathbb{R}^3)$ (by taking subsequence) and X is a critical point of E_H in $\{X_D\}+H_0^{1,2}$ with the property that either,

(i) $X_i\to X$ strongly in $H^{1,2}$, or,

(ii) $X_i\to X$ weakly but not strongly in $H^{1,2}$.

In case (i) $E_H(X)=\liminf_{i\to\infty}(1+\rho_i)E_{H_i}(X_i)\geqq\gamma_{H,0}$. In case (ii),

$$E_H(X)\leqq\liminf_{i\to\infty}(1+\rho_i)E_{H_i}(X_i)-\beta_H$$
$$\leqq\gamma_{H,\alpha}-\beta_H.$$

Therefore, from (2.11), Lemma 3.2 and Proposition 4.4 it is easy to see that in any case

$$E_H(X) \neq E_H(\underline{X}_H).$$

This completes the proof of our theorem.

Q.E.D.

Remark 5.1. – From (3.7) and Lemma 3.2 case (ii) in the proof of Theorem 1.6 cannot in fact happen for small α.

Remark 5.2. – We expect that for small α if $[H-H_0]<\alpha$, E_H satisfies the Palais-Smale condition in $(-\infty, E_H(\underline{X}_0)+\beta_H)$. Here $\underline{X}_0$ is the S-solution of E_H in $\{X_D\}+H_0^{1,2}(B, \mathbb{R}^3)$.

REFERENCES

[BC] H. Brezis and J.-M. Coron, Multiple Solutions of H-Systems and Rellich's Conjecture, *Comm. Pure Appl. Math.*, Vol. **37**, 1984, pp. 149-187.

[BC2] H. Brezis and J.-M. Coron, Convergence of Solutions of H-Systems or How to Blow Bubbles, *Arch. Rational Mech. Anal.*, Vol. **89**, 1985, pp. 21-56.

[He] E. Heinz, Uber die Regularität schwacher Lösungen nicht linearer elliptischer Systeme, *Nachr. Akad. Wiss. Göttingen*, Vol. **II**, No. 1, 1986.

[Hi1] S. Hildebrandt, On the Plateau Problem for Surfaces of Constant Mean Curvature, *Comm. Pure Appl. Math.*, Vol. **23**, 1970, pp. 97-114.

[Hi2] S. Hildebrandt, Randwertprobleme für Flächen mit vorgeschriebener mittlerer Krümmung und Anwendungen auf die Kapillaritätstheorie, Teil I. Fest vorgegebener, *Rand. Math. Z.*, Vol. **112**, 1969, pp. 205-213.

[Pc] F. Pacard, Convergence of Surfaces of Prescribed Mean Curvature, *Nonlinear Analysis. Theory, Methods & Applications*, Vol. **13**, 1989, pp. 1269-1281.

[Stf1] K. Steffen, On the Existence of Surfaces with Prescribed Mean Curvature and Boundary, *Math. Z.*, Vol. **146**, 1976, pp. 113-135.

[Stf2] K. Steffen, On the Non-Uniqueness of Surfaces with Prescribed Constant Mean Curvature Spanning a Given Contour, *Arch. Rat. Mech. Anal.*, Vol. **94**, 1986, pp. 101-122.

[Str1] M. Struwe, Plateau's Problem and the Calculus of Variations, *Lecture Notes*, Princeton University Press (in press).

[Str2] M. Struwe, Non-Uniqueness in the Plateau Problem for Surfaces of Constant Mean Curvature, *Arch. Rat. Mech. Anal.*, Vol. **93**, 1986, pp. 135-157.

[Str3] M. Struwe, Large H-Surfaces via the Mountain-Pass-Lemma, *Math. Ann.*, Vol. **270**, 1985, pp. 441-459.

[Str4] M. Struwe, *Multiple Solutions to the Dirichlet Problem for the Equation of Prescribed Mean Curvature*, preprint.

[Wet1] H. C. Wente, An Existence Theorem for Surfaces of Constant Mean Curvature, *J. Math. Anal. Appl.*, Vol. **26**, 1969, pp. 318-344.

[Wet2] H. C. Wente, A General Existence Theorem for Surfaces of Constant Mean Curvature, *Math. Z.*, Vol. **120**, 1971, pp. 277-288.

[Wet3] H. C. Wente, The Differential Equation $\Delta x = 2H x_u \wedge x_v$ with Vanishing Boundary Values, *Proc. A.M.S.*, Vol. **50**, 1975, pp. 59-77.

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