Ann. Inst. Henri Poincaré,

Vol. 9, n° 6, 1992, p. 657-681. *Analyse non linéaire*

Singular minimisers in the Calculus of Variations: a degenerate form of cavitation

by

J. SIVALOGANATHAN

School of Mathematical Sciences, University of Bath, Bath, Avon, U.K

ABSTRACT. — This paper demonstrates the existence of singular minimisers to a class of variational problems that correspond to a degenerate form of cavitation and studies the stability of these singular maps with respect to three dimensional variations.

INTRODUCTION

This paper presents an instructive example of a class of variational problems whose minimisers are closely linked with the mathematical phenomenon of cavitation. The first analytic study of this phenomenon was made by Ball [1], motivated by earlier work of Gent and Lindley [9], and has since been studied by a number of authors (*see*, for example, [12], [13], [18], [19], [21], [22], [26], [27], [28]). The physical interpretation of these results is that a ball of isotropic nonlinearly elastic material held in tension under prescribed boundary displacements can lower its bulk energy

Classification A.M.S. : 49 H (73 G).

Annales de l'Institut Henri Poincaré - Analyse non linéaire - 0294-1449
Vol. 9/92/06/657/25/$4.50/

by forming a hole at its centre, if the boundary displacement is sufficiently large.

This paper demonstrates the existence of singular minimisers to a simple integral functional that correspond to a degenerate form of cavitation. Surprisingly, this phenomenon can occur when the underlying equations are *linear*. We prove global stability of these singular mappings with respect to variations in the shape and position of the hole.

To fix ideas let $B=\{\mathbf{x}\in\mathbf{R}^3:|\mathbf{x}|<1\}$ be the region occupied by an elastic body in its reference state and let $\mathbf{u}:B\to\mathbf{R}^3$ be a *deformation* of B. Typically, in elasticity, one requires that $\mathbf{u}$ satisfy a local invertibility condition

$$\det\nabla\mathbf{u}(\mathbf{x})>0 \quad \text{for almost every } \mathbf{x}\in B \tag{1}$$

to avoid local interpretation of matter under the deformation.

For the purposes of this example, attention is restricted to the Dirichlet problem with radial boundary data so that the deformations $\mathbf{u}$ satisfy the condition

$$\mathbf{u}|_{\partial B}=\lambda\mathbf{x} \quad \text{for some } \lambda>0. \tag{2}$$

Define the energy associated with $\mathbf{u}$, denoted $E(\mathbf{u})$, by

$$E(\mathbf{u})=\frac{1}{2}\int_B|\nabla\mathbf{u}|^2\,dx, \tag{3}$$

the corresponding Euler-Lagrange equations are then

$$\Delta\mathbf{u}=0, \qquad \mathbf{x}\in B. \tag{4}$$

The Dirichlet integral is chosen for ease of exposition but the idea of the example extends easily to more general energy functions.

It is easily verified that the *homogeneous* deformation

$$\mathbf{u}_\lambda^{\mathrm{hom}}(\mathbf{x})\equiv\lambda\mathbf{x}, \qquad \mathbf{x}\in B, \tag{5}$$

which corresponds to a uniform dilation of B, is a solution of (4) satisfying (2). Thus by the convexity of E

$$\left.\begin{aligned} E(\mathbf{u}_\lambda^{\mathrm{hom}})=\int_B\frac{1}{2}|\nabla\mathbf{u}_\lambda^{\mathrm{hom}}|^2+\nabla\mathbf{u}_\lambda^{\mathrm{hom}}.(\nabla\mathbf{u}-\nabla\mathbf{u}_\lambda^{\mathrm{hom}})\leqq E(\mathbf{u})\\ \forall(\mathbf{u}-\lambda\mathbf{x})\in W_0^{1,2}(B;\mathbf{R}^3)\end{aligned}\right\} \tag{6}$$

and so the homogeneous deformation is the global minimiser for the Dirichlet problem. Now consider the related energy functional

$$E_\alpha(\mathbf{u})=\int_B\frac{1}{2}|\nabla\mathbf{u}|^2+\alpha\det\nabla\mathbf{u}\,dx, \qquad \alpha>0. \tag{7}$$

Formally, the Euler-Lagrange equations for E_α are identical with those for E (*i.e.* given by (4)) since $\det \nabla \mathbf{u}$ is a null lagrangian [1]. In fact for fixed λ, on sufficiently smooth spaces of functions, e. g. $W^{1,p}(B;\mathbf{R}^3)$, $p>3$, the two functionals E and E_α agree up to a constant since

$$\left.\begin{array}{c}\displaystyle\int_B \det \nabla \mathbf{u}(\mathbf{x})\,dx=\int_B \det \nabla \mathbf{u}_\lambda^{\text{hom}}=\frac{4}{3}\pi\lambda^3\\ \forall\,(\mathbf{u}-\lambda\,\mathbf{x})\in W_0^{1,p}(B),\end{array}\right\} \tag{8}$$

which is the deformed volume of B.

Thus by (6), (7) and (8)

$$E_\alpha(\mathbf{u}_\lambda^{\text{hom}})\leqq E_\alpha(\mathbf{u}),\qquad \forall\,(\mathbf{u}-\lambda\,\mathbf{x})\in W_0^{1,p}(B),\quad p>3 \tag{9}$$

and so again the homogeneous deformations are globally minimising for the Dirichlet problem. In this respect E and E_α behave similarly.

However, in classes containing discontinuous maps, for example in $W^{1,p}(B)$, $p<3$, the two functionals behave quite differently (in fact E_α loses its convexity for large α). The reason is that, even though (8) holds for $p>3$, for $p<3$ it is possible to effect a reduction in energy through the introduction of "holes". To see why singularities might form, let $\tilde{\mathbf{u}}$ be the radial deformation

$$\tilde{\mathbf{u}}(\mathbf{x})=r(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|} \tag{10}$$

where the absolutely continuous function $r:[0,1]\to\mathbf{R}$ is non decreasing, non-negative, with $r(0)>0$, $r(1)=1$, so that in particular

$$\tilde{\mathbf{u}}(\mathbf{x})\,|_{\partial B}=\mathbf{x}. \tag{11}$$

Then a straightforward calculation (*see* e. g. Ball [1], Sivaloganathan [21]) gives

$$\int_B \det \nabla \tilde{\mathbf{u}}=4\pi\int_0^1 R^2\left[r'\left(\frac{r}{R}\right)^2\right]dR=\frac{4}{3}\pi\,[1-r^3(0)]. \tag{12}$$

If $r(0)>0$, then $\tilde{\mathbf{u}}$ opens a hole of radius $r(0)$ at the centre of the ball and so

$$\frac{4}{3}\pi\,[1-r^3(0)]=\int_B \det \nabla \tilde{\mathbf{u}}<\int_B \det \nabla \mathbf{u}_1^{\text{hom}}=\int_B 1=\frac{4}{3}\pi, \tag{13}$$

[1] *i.e.* the Euler-Lagrange equations for $\int_B \det \nabla \mathbf{u}$ are identically satisfied by all smooth deformations—*see* Ball, Currie and Olver [3], Edelen [6] and Olver and Sivaloganathan [16] for details and further references.

i.e. the deformed volume of B under $\tilde{\mathbf{u}}$ is less than that under the identity map $\mathbf{u}_1^{\text{hom}}$, since $\tilde{\mathbf{u}}$ introduces a hole of volume $\frac{4}{3}\pi r^3(0)$.

Now let

$$\mathbf{u}_\lambda(\mathbf{x}) = \lambda \tilde{\mathbf{u}}(\mathbf{x}), \qquad \mathbf{x} \in \mathrm{B} \tag{14}$$

then by (11)

$$\mathbf{u}_\lambda(\mathbf{x})\big|_{\partial \mathrm{B}} = \lambda \mathbf{x}. \tag{15}$$

Notice now that

$$\mathrm{E}_\alpha(\mathbf{u}_\lambda) = \lambda^2 \frac{1}{2} \int_{\mathrm{B}} |\nabla \tilde{\mathbf{u}}|^2 + \alpha \lambda^3 \int_{\mathrm{B}} \det \nabla \tilde{\mathbf{u}}$$

and that

$$\mathrm{E}_\alpha(\mathbf{u}_\lambda^{\text{hom}}) = \lambda^2 \frac{1}{2} \int_{\mathrm{B}} |\, 1\, |^2 + \alpha \lambda^3 \int_{\mathrm{B}} 1 \tag{16}$$

hence by (13) and (16), for λ sufficiently large,

$$\mathrm{E}_\alpha(\mathbf{u}_\lambda) < \mathrm{E}_\alpha(\mathbf{u}_\lambda^{\text{hom}}),$$

so that it is energetically more favourable to introduce a hole [2].

We first study the minimisers of E_α in the class of radial maps, that is maps of the form (10). In order to obtain the existence of minimisers we are forced to weaken the constraint (1) and allow deformations satisfying

$$\det \nabla \mathbf{u}(\mathbf{x}) \geqq 0, \qquad \mathbf{x} \in \mathrm{B}. \tag{17}$$

In the class of radial maps satisfying (17), it is shown that there is a critical value of the boundary displacement

$$\lambda_{\text{crit}} = \frac{4}{3\alpha}$$

with the property that the homogeneous map is the radial minimiser for all $\lambda \leqq \lambda_{\text{crit}}$ and such that a radial map which produces a cavity is the radial minimiser for $\lambda > \lambda_{\text{crit}}$. The radial minimisers constructed for $\lambda > \lambda_{\text{crit}}$ satisfy (1) away from the origin but are of the form const. $\frac{\mathbf{x}}{|\mathbf{x}|}$ in a neighbourhood of the origin on which they are not invertible. The size of this neighbourhood and of the cavity increases monotonically with λ until, for $\lambda \geqq \frac{2}{\alpha}$, the radial minimiser is $\lambda \frac{\mathbf{x}}{|\mathbf{x}|}$, *i.e.* the hole has filled the ball.

---

[2] For conditions on r ensuring that $\tilde{\mathbf{u}} \in \mathrm{W}^{1,2}(\mathrm{B})$ *see* Ball [1].

These results are qualitatively in agreement with those for the fully nonlinear case studied by Ball [1], Stuart [26] and Sivaloganathan [21] (but there is no analogue in these works of the hole filling the ball. Also, our minimisers do not satisfy the natural boundary condition imposed in these works that the cavity surface be traction free).

It should be stressed that (7) is not necessarily intended to model the stored energy of an elastic material: in particular, any material characterized by (7) does not have a natural state. Rather, the interest in this functional is that it gives a lower bound for the energy and subsequently on the properties of a class of polyconvex stored energy functions (*see* concluding remarks). Also, the transparent nature of our example yields insight into the behaviour of more complex nonlinear energy functionals.

Theorem 2.4 studies the stability of the radial minimisers with respect to smooth variations (which allows movement of the position of the hole once formed). It is proved in particular that, if $\mathbf{u}_0$ is the radial minimiser, then amongst all admissible variations [*see* (2.4)], $\varphi \in W_0^{1,p}(B)$, $p>3$,

$$E_\alpha(\mathbf{u}_0+\varphi) \geqq E_\alpha(\mathbf{u}_0) + \int_B |\nabla \varphi|^2.$$

Thus, in this class, it is energetically more favourable to produce a hole at the centre of the ball.

Theorem 3.3 examines the stability of the radial cavitating maps with respect to variations in the hole shape. It is shown that, in the class $C^1(\bar{B}\setminus\{\mathbf{0}\})$, the radial minimiser is the global minimiser amongst deformations satisfying an appropriate invertibility hypothesis (3.1). The proof of this result relies on the classical isoperimetric inequality.

I remark finally that I know of no existence or regularity theory for minimisers of integral functionals of the form (7) on maps satisfying (17). (*See* Ball and Murat [2] for some of the problems encountered in trying to obtain existence theories in $W^{1,p}$, $p<3$; *see* Evans [7], Fusco and Hutchinson [8] for regularity results for minimisers of certain polyconvex energy functionals; and *see* Muller [15] for interesting properties of mappings satisfying (17).)

Further results relating to stability of cavitating equilibria can be found in James and Spector [15] Sivaloganathan [23], [24], and Spector [25].

RADIAL MAPS

Let $\mathbf{u}: B \to \mathbf{R}^3$ be a radial deformation so that

$$\mathbf{u}(\mathbf{x}) = \frac{r(R)}{R}\mathbf{x}, \qquad R = |\mathbf{x}|, \qquad \forall\, \mathbf{x} \in B, \tag{1.1}$$

where $r:[0,1]\to\mathbf{R}$, then a straightforward calculation gives

$$E_\alpha(\mathbf{u})=4\pi I_\alpha(r)\stackrel{\text{def}}{=}4\pi\int_0^1 R^2\left\{\frac{1}{2}\left[(r')^2+2\left(\frac{r}{R}\right)^2+\alpha r'\left(\frac{r}{R}\right)^2\right]\right\}dR$$

$$:=4\pi\int_0^1\frac{1}{2}R^2(r')^2+r^2+\alpha r' r^2\,dR \quad (1.2)$$

(*see* e. g. [1]).

In order that $\mathbf{u}$ satisfy (5) and (1), we require that r lies in the set

$$\mathscr{A}_\lambda=\{r\in W^{1,1}((0,1)):r(1)=\lambda,\ r(0)\geqq 0,\ r'(R)>0 \text{ a.e. } R\in(0,1)\}$$

(*see* e. g. [1]).

This set, although convex, is not strongly closed and hence not weakly closed $\Big($the constraint $r'>0$ a.e. can be lost in the strong limit: just consider $(r_n)\in\mathscr{A}_\lambda$, $r_n(R)=\frac{\lambda R}{n}+\lambda\left(1-\frac{1}{n}\right)$ $R\in[0,1]$, $n\in\mathbf{N}$, then (r_n) converges in $W^{1,1}((0,1))$ to the constant function with value λ which does not lie in $\mathscr{A}_\lambda\Big)$. Thus in general one does not expect I_α to attain a minimum on $\mathscr{A}_\lambda$ (we shall see later that for large λ it does not). If however we define I_α on

$$\bar{\mathscr{A}}_\lambda=\{r\in W^{1,1}((0,1)):r(1)=\lambda,\quad r(0)\geqq 0,\quad r'\geqq 0 \text{ a.e. } R\in(0,1)\} \quad (1.3)$$

then I_α does attain a minimum as demonstrated by Proposition 1.1.

PROPOSITION 1.1. — *For each* $\alpha,\lambda>0$ *the functional* I_α *attains a minimum on* $\bar{\mathscr{A}}_\lambda$.

Proof. — We give a sketch of the proof: let (r_n) be a minimising sequence for I_α then,

$$\int_{1/m}^1\frac{1}{m^2}(r_n')^2\,dR\leqq I_\alpha(r_n)\leqq\text{const.},\qquad \forall n\in\mathbf{N} \text{ for each } m\in\mathbf{N}. \quad (1.4)$$

Thus given $m\in\mathbf{N}$ there is a subsequence (r_{n_k}') of (r_n') converging weakly in $L^2\left(\frac{1}{m},1\right)$ to some $\chi\in L^2\left(\frac{1}{m},1\right)$. Repeat this process inductively choosing subsequences for a monotone increasing sequence of integers (m_i) and take a diagonal sequence which we again denote by (r_n') then $r_n'\xrightarrow{L^2(1/m,1)}\chi$ as $n\to\infty$ for each $m\in\mathbf{N}$ (χ is well defined by uniqueness of weak limits and non-negative for example by Mazur's Theorem). Now define

$$\tilde{r}(R)=\lambda-\int_R^1\chi(s)\,ds \quad (1.5)$$

then it is easily verified that

$$r_n \xrightarrow{W^{1,1}(\delta,1)} \tilde{r} \quad \text{as } n \to \infty \text{ for each } \delta \in (0,1).$$

Hence $r_n \xrightarrow{C([\delta,1])} \tilde{r}$ for each $\delta \in (0,1)$ and so $\tilde{r}(R) \geqq 0$ on $[0,1]$, it then follows from the non-negativity of χ that $\tilde{r} \in \bar{\mathscr{A}}_\lambda$. Now, standard lower semi-continuity results imply that

$$\int_\delta^1 \frac{1}{2} R^2 (\tilde{r}')^2 + (\tilde{r})^2 + \alpha \tilde{r}' \tilde{r}^2 \leqq \varliminf_{n \to \infty} \int_\delta^1 \frac{1}{2} R^2 (r_n')^2 + r_n^2 + \alpha r_n' r_n^2 \quad \text{for each } \delta \in (0,1) \tag{1.6}$$

so that

$$\int_\delta^1 \frac{1}{2} R^2 (\tilde{r}')^2 + (\tilde{r})^2 + \alpha \tilde{r}' \tilde{r}^2 \leqq \operatorname*{Inf}_{\bar{\mathscr{A}}_\lambda} I_\alpha \quad \text{for each } \delta \in (0,1). \tag{1.7}$$

Hence by the monotone convergence theorem

$$I_\alpha(\tilde{r}) \leqq \operatorname*{Inf}_{\bar{\mathscr{A}}_\lambda} I_\alpha$$

and so $\tilde{r}$ is a minimiser of I_α on $\bar{\mathscr{A}}_\lambda$.

CANDIDATES FOR THE MINIMISER

It is easily verified that, formally, the Euler equation for I_α is

$$\frac{d}{dR}[R^2 r'] = 2r. \tag{1.8}$$

A minimiser of I_α on $\bar{\mathscr{A}}_\lambda$ does not necessarily satisfy this equation since there may be sets of non-zero measure on which its derivative is zero and where a corresponding differential *inequality* would then be satisfied. The only solutions of equation (1.8) satisfying $r(1) = \lambda$ are of the form

$$r(R) = AR + \frac{(\lambda - A)}{R^2}, \tag{1.9}$$

where $A \in \mathbf{R}$ is a constant.

We now construct candidates for the minimiser of I_α by smoothly piecing together a solution of the form (1.9) on a subinterval $[R_0, 1]$ of $[0, 1]$ together with a constant function on $[0, R_0]$, where R_0 is suitably chosen. More precisely let

$$\bar{r}(R) = \begin{cases} r(R) & \text{on } [R_0, \lambda] \\ r(R_0) & \text{on } [0, R_0] \end{cases} \tag{1.10}$$

where r is given by (1.9) and R_0 is chosen so that $r'(R_0)=0$. A straightforward calculation then gives

$$R_0=\left[2\left(\frac{\lambda}{A}-1\right)\right]^{1/3}, \tag{1.11}$$

$$\bar{r}(R_0)=\frac{3}{2}A\left[2\left(\frac{\lambda}{A}-1\right)\right]^{1/3} \tag{1.12}$$

and $\bar{r}\in\mathscr{A}_\lambda$ if and only if

$$\frac{2}{3}\lambda\leqq A\leqq\lambda. \tag{1.13}$$

The first inequality in (1.13) ensures that $\bar{r}'(1)\geqq 0$ and the second ensures that $\bar{r}$ given by (1.9) is convex so that there exists a point $R_0\in[0, 1]$ with $r'(R_0)=0$. Notice that the choice $A=\lambda$ gives $R_0=0$, and the corresponding map $\bar{r}$ is then the homogeneous map λR which corresponds to (5).

We next examine the minimising properties of the maps $\bar{r}$ which we have constructed. In particular, we show that *given any cavity size* there is a unique member of the family (1.10) which minimises I_α.

PROPOSITION 1.2. — *Let* $\beta\in[0,\lambda]$, $\lambda\geqq 0$, *and let*

$$\bar{r}=\begin{cases} AR+\dfrac{\lambda-A}{R^2} & \text{on } [R_0, 1] \\ \beta & \text{on } [0, R_0], \end{cases} \tag{1.14}$$

where $\frac{2}{3}\lambda\leqq A\leqq\lambda$ *is chosen such that*

$$\beta=\frac{3}{2}A\left[2\left(\frac{\lambda}{A}-1\right)\right]^{1/3} \tag{1.15}$$

and

$$R_0=\left[2\left(\frac{\lambda}{A}-1\right)\right]^{1/3}, \tag{1.16}$$

then

$$I_\alpha(\bar{r})\leqq I_\alpha(r), \qquad \forall\, r\in\mathscr{A}_\lambda \quad \text{with } r(0)=\beta.$$

Proof. — First observe that for A in the range $\left[\frac{2}{3}\lambda, \lambda\right]$ for each $\beta\in[0,\lambda]$ there is a unique value of A satisfying (1.15).

Since $r(0)=\bar{r}(0)$ it follows that

$$\int_0^1 r^2(R)\, r'(R)\, dR=\int_0^1 \bar{r}^2(R)\, \bar{r}'(R)\, dR,$$

hence it is sufficient to prove that

$$\int_0^1 \frac{1}{2} R^2 (\bar{r}')^2 + \bar{r}^2 \leqq \int_0^1 \frac{1}{2} R^2 (r')^2 + r^2. \tag{1.17}$$

To see this let

$$r = \bar{r} + \varphi, \tag{1.18}$$

then the righthand side of (1.17) equals

$$\int_0^1 \frac{1}{2} R^2 (\bar{r}' + \varphi')^2 + (\bar{r} + \varphi)^2$$
$$= \int_0^1 \frac{1}{2} R^2 (\bar{r}')^2 + \bar{r}^2 + R^2 \bar{r}' \varphi' + 2 \bar{r} \varphi + \frac{1}{2} R^2 (\varphi')^2 + \varphi^2$$

on using (1.14) and the fact that $\bar{r}$ satisfies (1.8) on $[R_0, 1]$ this becomes

$$\int_0^1 \frac{1}{2} R^2 (\bar{r}')^2 + \bar{r}^2 + \int_{R_0}^1 \frac{d}{dR} [(R^2 \bar{r}') \varphi] + \int_0^{R_0} 2 \beta \varphi + \int_0^1 \frac{1}{2} R^2 (\varphi')^2 + \varphi^2.$$

Since $\varphi(1) = 0$ and $\bar{r}'(R_0) = 0$ it follows that the above expression equals

$$\int_0^1 \frac{1}{2} R^2 (\bar{r}')^2 + \bar{r}^2 + \int_0^{R_0} 2 \beta \varphi + \int_0^1 \frac{1}{2} R^2 (\varphi')^2 + \varphi^2. \tag{1.19}$$

By (1.18), since $r \in \bar{\mathscr{A}}_\lambda$ by assumption, it follows that

$$r'(R) = \bar{r}'(R) + \varphi'(R) = \varphi'(R) \geqq 0 \quad \text{a.e. } R \in [0, R_0],$$

thus since $\varphi(0) = 0$ [as $r(0) = \bar{r}(0) = \beta$] it follows that

$$\varphi(R) \geqq 0, \qquad R \in [0, R_0]. \tag{1.20}$$

Thus by (1.19) and (1.20), (1.17) then follows and hence the proposition.

The last proposition implies that, in searching for a minimiser of I_α on $\bar{\mathscr{A}}_\lambda$, it is sufficient to look in the class of maps defined by (1.14), we study this in the next Theorem.

THEOREM 1.3. — *Let* $\lambda_{\text{crit}}=\dfrac{4}{3\alpha}$, *then:*

(i) *If* $\lambda\leqq\lambda_{\text{crit}}$, $\tilde{r}(\mathrm{R})\equiv\lambda\mathrm{R}$ *is the global minimiser of* I_α *on* $\bar{\mathscr{A}}_\lambda$.

(ii) *If* $\dfrac{2}{\alpha}>\lambda>\lambda_{\text{crit}}$ *then*

$$\tilde{r}(\mathrm{R})=\begin{cases}\dfrac{4}{3\alpha}\mathrm{R}+\dfrac{(\lambda-4/3\alpha)}{\mathrm{R}^2}, & \mathrm{R}\in[\mathrm{R}_0,1]\\ \dfrac{2}{\alpha}\left[2\left(\dfrac{3\alpha\lambda}{4}-1\right)\right]^{1/3}, & \mathrm{R}\in[0,\mathrm{R}_0]\end{cases}$$

with $\mathrm{R}_0=\left[2\left(\dfrac{3\alpha\lambda}{4}-1\right)\right]^{1/3}$, *is the global minimiser of* I_α *on* $\bar{\mathscr{A}}_\lambda$.

(iii) *If* $\lambda\geqq\dfrac{2}{\alpha}$ *then*

$$\tilde{r}(\mathrm{R})\equiv\lambda,\qquad \mathrm{R}\in[0,1]$$

is the global minimiser of I_α *on* $\bar{\mathscr{A}}_\lambda$.

Proof. — By proposition 1.2, in order to determine the minimiser of I_α on $\bar{\mathscr{A}}_\lambda$ it is sufficient, for each $\lambda>0$, to determine the minimiser of I_α in the class of mappings of the form (1.14). To this end we first calculate the value of I_α on a mapping of the form (1.14) and then minimise over admissible values of A.

$$\begin{aligned}\int_0^1 \frac{1}{2}\mathrm{R}^2\left[(\tilde{r}')^2+2\left(\frac{\tilde{r}}{\mathrm{R}}\right)^2\right]+\alpha r' r^2 &= \int_0^{\mathrm{R}_0}\frac{1}{2}\mathrm{R}^2\left[(\tilde{r}')^2+2\left(\frac{\tilde{r}}{\mathrm{R}}\right)^2\right]+\int_{\mathrm{R}_0}^1\frac{1}{2}\mathrm{R}^2\left[(\tilde{r}')^2+2\left(\frac{\tilde{r}}{\mathrm{R}}\right)^2\right]+\frac{\alpha}{3}(\lambda^3-\tilde{r}^3(0))\\ &=\int_0^{\mathrm{R}_0}\beta^2+\int_{\mathrm{R}_0}^1\frac{1}{2}\mathrm{R}^2\left[\left(\mathrm{A}-2\frac{(\lambda-\mathrm{A})}{\mathrm{R}^3}\right)^2+2\left(\mathrm{A}+\frac{(\lambda-\mathrm{A})}{\mathrm{R}^3}\right)^2\right]+\frac{\alpha}{3}(\lambda^3-\beta^3)\\ &=\beta^2\mathrm{R}_0+\int_{\mathrm{R}_0}^1\frac{1}{2}\mathrm{R}^2\left[3\mathrm{A}^2+6\frac{(\lambda-\mathrm{A})^2}{\mathrm{R}^6}\right]d\mathrm{R}+\frac{\alpha}{3}(\lambda^3-\beta^3)\\ &=\frac{3^2}{2^2}\mathrm{A}^2\left[2\left(\frac{\lambda}{\mathrm{A}}-1\right)\right]+\frac{1}{2}\mathrm{A}^2\left[1-2\left(\frac{\lambda}{\mathrm{A}}-1\right)\right]\\ &\quad+(\lambda-\mathrm{A})^2\left[\frac{1}{2((\lambda/\mathrm{A})-1)}-1\right]+\frac{\alpha}{3}\left(\lambda^3-\frac{3^3}{2^2}\mathrm{A}^2(\lambda-\mathrm{A})\right)\\ &=\frac{9}{2}\mathrm{A}(\lambda-\mathrm{A})+\frac{1}{2}[3\mathrm{A}^2-2\lambda\mathrm{A}]\\ &\quad+\frac{1}{2}[(\lambda-\mathrm{A})\mathrm{A}-2(\lambda-\mathrm{A})^2]+\frac{\alpha}{3}\left(\lambda^3-\frac{3^3}{2^2}\mathrm{A}^2(\lambda-\mathrm{A})\right)\end{aligned}$$

$$= \frac{9}{2} A (\lambda - A) + \frac{1}{2} \lambda (3A - 2\lambda) + \frac{\alpha}{3} \left(\lambda^3 - \frac{3^3}{2^2} A^2 (\lambda - A) \right)$$
$$= \frac{9}{4} \alpha A^3 - \frac{9}{2} \left(1 + \frac{\alpha\lambda}{2} \right) A^2 + 6\lambda A$$
$$+ \lambda^2 \left(\frac{\alpha\lambda}{3} - 1 \right) \stackrel{\text{def}}{=} : F(A, \alpha, \lambda). \tag{1.21}$$

For α and λ fixed we now examine the behaviour of F as a function of A. Firstly its turning points are solutions of

$$\frac{\partial F}{\partial A} = \frac{27}{4} \alpha A^2 - 9 \left(1 + \frac{\alpha\lambda}{2} \right) A + 6\lambda = 0$$
$$\Leftrightarrow \quad \alpha \left(\frac{3}{2} A \right)^2 - (2 + \alpha\lambda) \left(\frac{3}{2} A \right) + 2\lambda = 0$$
$$\Leftrightarrow \quad \frac{3}{2} A = \frac{(2 + \alpha\lambda) \pm \sqrt{(2 + \alpha\lambda)^2 - 8\alpha\lambda}}{2\alpha}$$
$$= \frac{(2 + \alpha\lambda) \pm |2 - \alpha\lambda|}{2\alpha} = \frac{2}{\alpha} \quad \text{or} \quad \lambda$$

i.e.

$$A = \frac{4}{3\alpha} \quad \text{or} \quad \frac{2}{3}\lambda. \tag{1.22}$$

We next evaluate F at these points:

$$F\left(\frac{4}{3\alpha}, \alpha, \lambda \right) = \frac{9\alpha}{4} \left(\frac{4}{3\alpha} \right)^3 - \frac{9}{2} \left(1 + \frac{\alpha\lambda}{2} \right)$$
$$\times \left(\frac{4}{3\alpha} \right)^2 + 6\lambda \left(\frac{4}{3\alpha} \right) + \lambda^2 \left(\frac{\alpha\lambda}{3} - 1 \right)$$
$$= \frac{16}{3\alpha^2} - \frac{8}{\alpha^2} \left(1 + \frac{\alpha\lambda}{2} \right) + \frac{8\lambda}{\alpha} + \lambda^2 \left(\frac{\alpha\lambda}{3} - 1 \right) \tag{1.23}$$

in order to facilitate comparison with the energy of the homogeneous map $r_\lambda^{\text{hom}}(R) = \lambda R$ note that

$$I_\alpha (r_\lambda^{\text{hom}}) = \frac{\lambda^2}{2} + \frac{\alpha\lambda^3}{3} \tag{1.24}$$

and write (1.23) as

$$F\left(\frac{4}{3\alpha}, \alpha, \lambda \right) = \left[\frac{-8}{3\alpha^2} + \frac{4\lambda}{\alpha} - \frac{3}{2}\lambda^2 \right] + \frac{\lambda^2}{2} + \frac{\alpha\lambda^3}{3}$$
$$= \frac{-3}{2} \left(\lambda - \frac{4}{3\alpha} \right)^2 + I_\alpha (r_\lambda^{\text{hom}}). \tag{1.25}$$

We next calculate

$$\begin{aligned}F\left(\frac{2}{3}\lambda,\alpha,\lambda\right)&=\frac{9\alpha}{4}\left(\frac{2}{3}\lambda\right)^3-\frac{9}{2}\left(\frac{2}{3}\lambda\right)^2\\&\quad\times\left(1+\frac{\alpha\lambda}{2}\right)+6\lambda\left(\frac{2}{3}\lambda\right)+\lambda^2\left(\frac{\alpha\lambda}{3}-1\right)\\&=\frac{2}{3}\alpha\lambda^3-2\lambda^2\left(1+\frac{\alpha\lambda}{2}\right)+4\lambda^2+\lambda^2\left(\frac{\alpha\lambda}{3}-1\right)\\&=\frac{\lambda^2}{2}\left(1-2\frac{\alpha\lambda}{3}\right)+I_\alpha(r_\lambda^{\text{hom}}).\end{aligned}\tag{1.26}$$

From the definition of F (1.21) and the above calculations (1.24)-(1.26) we obtain the following schematic graphs for the various regimes of λ.

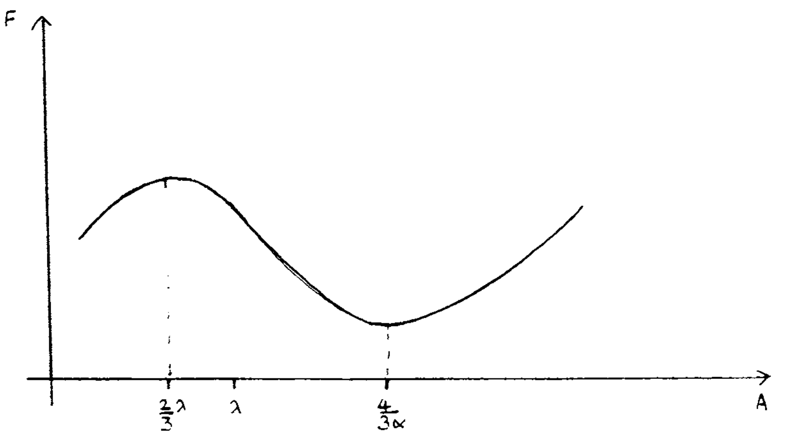
In interpreting these pictures, one should note the following points:

(i) The value $A=\lambda$ corresponds to $\bar{r}$ being the homogeneous map λR.

(ii) As discussed at the beginning of this section, the admissible values of A are restricted to lie between $\frac{2}{3}\lambda$ and λ [*see* (1.13)].

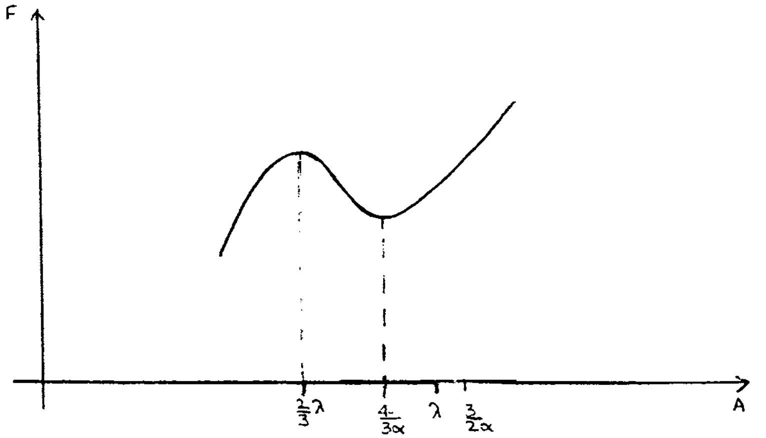
(iii) The value $A=\frac{2}{3}\lambda$ corresponds via (1.16) to $R_0=1$ and hence by (1.14) the corresponding $\bar{r}$ is the constant function with value λ.

(i) $\lambda\leqq\frac{4}{3\alpha}=\lambda_{\text{crit}}$



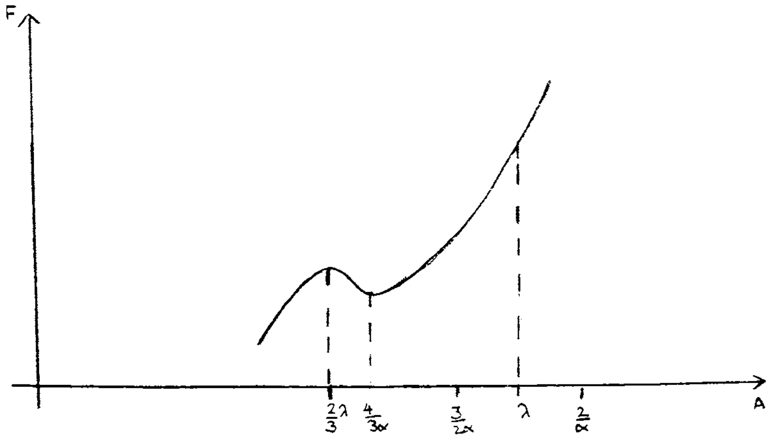
In this case the only admissible minimum at $A=\lambda$ *i.e.* ar $\bar{r}\equiv\lambda R$.

(ii) $\frac{4}{3\alpha} < \lambda \leqq \frac{3}{2\alpha}$



In this case the only admissible minimum is at $A=\frac{4}{3\alpha}$, *i.e.* to $\bar{r}$ given by (ii) in the statement of the theorem.

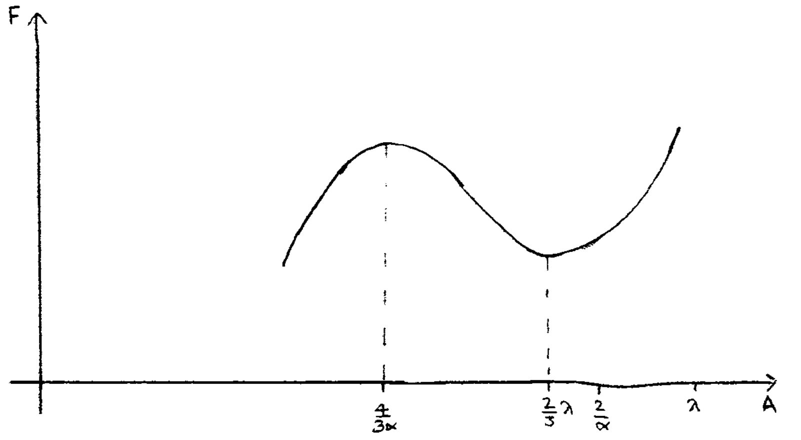
(iii) $\frac{3}{2\alpha} < \lambda \leqq \frac{2}{\alpha}$



In this case the only admissible minimum is again at $A=\frac{4}{3\alpha}$.

Notice, however, that in this regime the constant map $\bar{r}\equiv\lambda$ $\left(\text{corresponding to } A=\frac{2}{3}\lambda\right)$ has less energy than the homogeneous map $\bar{r}\equiv\lambda R$ (corresponding to $A=\lambda$).

(iv) $\frac{2}{\alpha} < \lambda$



In this regime the only admissible minimum is at $A = \frac{2}{3}\lambda$ which corresponds to the constant map $\bar{r} \equiv \lambda$.

Remark 1.4. – Notice than in case (iii) in the above Theorem the map $\bar{r} \equiv \lambda$ corresponds via (10) to the map $\tilde{\mathbf{u}}(\mathbf{x}) = \lambda \frac{\mathbf{x}}{|\mathbf{x}|}$, *i.e.* the hole has completely filled the ball.

For the remainder of this paper, given $\lambda > \lambda_{\text{crit}}$, we will refer to the radial map given by (1.1) and (ii) [or (iii)] of Theorem 1.3 as the *radial cavitating map*.

Though the results of Theorem 1.3 are qualitatively in agreement with those for the fully nonlinear case studied in Ball [1], Stuart [26], Sivaloganathan [21], the following differences should be noted: (i) the radial cavitating maps are not invertible in a neighbourhood of the origin where they do not satisfy the radial equilibrium equation (1.8) (rather, a corresponding differential inequality holds in this neighbourhood); (ii) if one were to view (7) as the stored energy functional for an elastic material, which may be inappropriate since materials characterised by (7) do not have a natural state, then the radial cavitating maps give rise to a radial Cauchy stress of magnitude α on the cavity surface. Notice also, that by the scaling arguments used in the introduction, changing the boundary displacement λ whilst keeping α fixed is equivalent to fixing λ and varying α.

The example treated in the theorem also appears to have connections with the earlier studies of point defects in elasticity: the region $B_{R_0}(\mathbf{0})$ could be associated with the classical notion of a defect core.

Finally, it is interesting to note that the ideas of this section can be extended to prove the existence of cavitating equilibria for stored energy functions with slow growth (*see* [24]).

THREE DIMENSIONAL STABILITY OF RADIAL MAPS

In this section we study the stability, with respect to smooth three dimensional variations, of the radial maps given by (1.10) studied in the last section, that is we take

$$\mathbf{u}_0(\mathbf{x}) = \frac{r(\mathrm{R})}{\mathrm{R}}\mathbf{x}, \qquad \mathrm{R} = |\mathbf{x}|, \qquad \mathbf{x} \in \mathrm{B}, \tag{2.1}$$

where

$$r(\mathrm{R}) = \begin{cases} \mathrm{AR} + \dfrac{\lambda - \mathrm{A}}{\mathrm{R}^2}, & \mathrm{R} \in [\mathrm{R}_0, 1] \\ \mathrm{AR}_0 + \dfrac{\lambda - \mathrm{A}}{\mathrm{R}_0^2}, & \mathrm{R}_0 \in [0, \mathrm{R}_0] \end{cases} \tag{2.2}$$

and

$$\mathrm{R}_0 = \left[2\left(\frac{\lambda}{\mathrm{A}} - 1\right)\right]^{1/3}. \tag{2.3}$$

DEFINITION 2.1. — Given any deformation $\mathbf{u}$ of B, we will say that $\varphi \in W_0^{1,p}(\mathrm{B})$, $p > 3$ is an *admissible variation* of $\mathbf{u}$ if for almost all $\mathbf{x} \in \mathrm{B}$

$$\det(\nabla \mathbf{u}(\mathbf{x}) + \varepsilon \nabla \varphi(\mathbf{x})) \geqq 0 \tag{2.4}$$

for all non negative ε sufficiently small (depending on $\mathbf{x}$).

Remark 2.2. — If $\mathbf{u}$ is given by (2.1)-(2.3), then this condition is only a restriction on $\boldsymbol{\varphi}$ on $\overline{\mathrm{B}_{\mathrm{R}_0}(0)}$ since $\det \nabla \mathbf{u}_0(\mathbf{x}) > 0$, $\forall \mathbf{x} \in \mathrm{B} \setminus \overline{\mathrm{B}}_{\mathrm{R}_0}$ and (2.4) is then automatically satisfied on this set for any choice of $\boldsymbol{\varphi}$. The notion of admissible variation reflects the fact that on sets where $\mathbf{u}_0$ is not invertible, only variations which do not reverse the local orientation are admissible.

Remark 2.3. — Notice that $\mathbf{u}_0$ given by (2.1)-(2.3) satisfies $\mathbf{u}_0|_{\partial \mathrm{B}} = \lambda \mathbf{x}$, $\mathbf{u}_0 \in W^{1,2}(\mathrm{B}) \cap C^1(\overline{\mathrm{B}} \setminus \{\mathbf{0}\})$ and that

$$\nabla \mathbf{u}_0(\mathbf{x}) = \frac{r(\mathrm{R})}{\mathrm{R}}\mathbf{1} + \left(r' - \frac{r}{\mathrm{R}}\right)\frac{\mathbf{x} \otimes \mathbf{x}}{\mathrm{R}^2}, \tag{2.5}$$

where r is given by (2.2). (The fact that $\mathbf{u}_0 \in W^{1,2}(\mathrm{B})$ and that the pointwise derivative coincides with the distributional derivative follows, for example, from Ball [1]).

The next theorem concerns the stability of cavitating maps with respect to smooth variations.

THEOREM 2.4. — *Let* $\mathbf{u}_0$ *be defined by* (2.1)-(2.3) *then*

$$E_\alpha(\mathbf{u}_0+\boldsymbol{\varphi}) \geqq E_\alpha(\mathbf{u}_0) + \frac{1}{2}\int_B |\nabla \boldsymbol{\varphi}|^2 \tag{2.6}$$

for all admissible variations $\boldsymbol{\varphi} \in W_0^{1,p}(B)$, $p>3$.

The proof of this result requires the next Lemma and is given following it.

LEMMA 2.5. — *Let* $\mathbf{u}_0$ *be defined by* (2.1)-(2.3) *then, for any* $p>3$,

$$\int_B \det \nabla(\mathbf{u}_0+\boldsymbol{\varphi}) = \int_B \det \nabla \mathbf{u}_0, \qquad \forall\, \boldsymbol{\varphi} \in W_0^{1,p}(B). \tag{2.7}$$

Proof. — We prove the result for $\boldsymbol{\varphi} \in C_0^\infty(B)$, the claim of the lemma then follows by density.

First notice that

$$\det \nabla(\mathbf{u}_0+\boldsymbol{\varphi}) = \det \nabla \mathbf{u}_0 + (\operatorname{Adj} \nabla \mathbf{u}_0)_i^\alpha \varphi^i_{,\alpha} + u^i_{0,\alpha}(\operatorname{Adj}\nabla \boldsymbol{\varphi})_i^\alpha + \det \nabla \boldsymbol{\varphi}, \tag{2.8}$$

where $\operatorname{Adj}\nabla \mathbf{u}_0$ denotes the adjugate matrix of $\nabla \mathbf{u}_0$. Since $\boldsymbol{\varphi}$ is smooth and vanishes on ∂B it follows that

$$\int_B \det \nabla \boldsymbol{\varphi} = \int_B \frac{1}{6}(\varepsilon^{ijk}\varepsilon_{\alpha\beta\gamma}\varphi^i \varphi^j_{,\beta}\varphi^k_{,\gamma})_{,\alpha} = 0. \tag{2.9}$$

We next prove that

$$\int_B (\operatorname{Adj}\nabla \mathbf{u}_0)_i^\alpha \varphi^i_{,\alpha} = 0. \tag{2.10}$$

We again use the divergence structure of subdeterminants:

by remark 2.3, $\mathbf{u}_0$ is in $C^1(B\setminus\{\mathbf{0}\})$ and $\operatorname{Adj}\nabla \mathbf{u}_0 \in L^1(B)$ so that

$$\begin{aligned}\int_B (\operatorname{Adj}\nabla \mathbf{u}_0)_i^\alpha \varphi^i_{,\alpha} &= \lim_{\varepsilon\to 0}\int_{B\setminus B_\varepsilon} \frac{1}{2}(\varepsilon^{ijk}\varepsilon_{\alpha\beta\gamma}u_0^j u^k_{0,\gamma}\varphi^i_{,\alpha})_{,\beta} \\ &= \int_{\partial B}\frac{1}{2}\varepsilon^{ijk}\varepsilon_{\alpha\beta\gamma}\lambda x^j u^k_{0,\gamma}\varphi^i_{,\alpha} n^\beta\, dS \\ &\quad - \lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon}\frac{1}{2}\varepsilon^{ijk}\varepsilon_{\alpha\beta\gamma} r(R_0)\frac{x^j}{|\mathbf{x}|}u^k_{0,\gamma}\varphi^i_{,\alpha} n^\beta\, dS, \end{aligned} \tag{2.11}$$

where we have used the form $\mathbf{u}_0$, and r is given by (2.2). From (2.5) and (2.2) we see that $|\nabla \mathbf{u}_0|$ is $O\left(\frac{1}{|\mathbf{x}|}\right)$ as $\mathbf{x}\to\mathbf{0}$ and hence the second integral on the right-hand side of (2.11) converges to zero as $\varepsilon\to 0$. Notice also that the first integral in (2.11) is also zero since φ has compact support in B. Combining these two facts then yields (2.10). An analogous argument gives

$$\int_{B} u_{0,\alpha}^{i} (\operatorname{Adj} \nabla \varphi)_{i}^{\alpha} = 0. \tag{2.12}$$

The claim of the Lemma (2.7), now follows from (2.8)-(2.10) and (2.12).

Remark 2.6. — The claim of the lemma is intuitively clear: if the variation φ is smooth then it can introduce no further "holes" hence the deformed volume of B under $\mathbf{u}_0$ is the same as that under $\mathbf{u}_0 + \varphi$.

We now give the proof of Theorem 2.4.

Proof of Theorem 2.4. — Let $\varphi \in C_0^1(B)$ be an admissible variation (the result for an admissible variation $\varphi \in W_0^{1,p}(B)$, $p > 3$, will then follow by a density argument). By Lemma 2.5 and the definition of E_α, (7), it is sufficient to prove that

$$\frac{1}{2}\int_{B} |\nabla \mathbf{u}_0 + \nabla \varphi|^2 \geqq \frac{1}{2}\int_{B} |\nabla \mathbf{u}_0|^2 + \frac{1}{2}\int_{B} |\nabla \varphi|^2. \tag{2.13}$$

To see this notice that

$$\frac{1}{2}\int_{B} |\nabla \mathbf{u}_0 + \nabla \varphi|^2 = \frac{1}{2}\int_{B} |\nabla \mathbf{u}_0|^2 + \frac{1}{2}\int_{B} |\nabla \varphi|^2 + \int_{B} \nabla \mathbf{u}_0 . \nabla \varphi \tag{2.14}$$

and that

$$\begin{aligned} \int_{B} \nabla \mathbf{u}_0 . \nabla \varphi &= \int_{B \setminus B_{R_0}} u_{0,\alpha}^{i} \varphi_{,\alpha}^{i} + \lim_{\varepsilon \to 0} \int_{B_{R_0} \setminus B_\varepsilon} u_{0,\alpha}^{i} \varphi_{,\alpha}^{i} \\ &= \int_{B \setminus B_{R_0}} (u_{0,\alpha}^{i} \varphi^{i})_{,\alpha} + \lim_{\varepsilon \to 0} \left[\int_{B_{R_0} \setminus B_\varepsilon} (u_{0,\alpha}^{i} \varphi^{i})_{,\alpha} - \Delta u_0^{i} \varphi^{i} \right] \end{aligned} \tag{2.15}$$

(since $\Delta u_0^i = 0$ on $B \setminus B_{R_0}$). An easy calculation gives

$$\Delta u_0^i = -2 \frac{x^i}{|\mathbf{x}|^3}, \qquad \mathbf{x} \in B_{R_0} \setminus \{\mathbf{0}\}, \tag{2.16}$$

substituting this into (2.15) and using the divergence theorem gives

$$\begin{aligned} &\int_{\partial B} \varphi^i u_{0,\alpha}^{i} n^\alpha \, dS - \int_{\partial B_{R_0}} \varphi^i u_{0,\alpha}^{i} n^\alpha \, dS + \int_{\partial B_{R_0}} \varphi^i u_{0,\alpha}^{i} n^\alpha \, dS \\ &\quad + \lim_{\varepsilon \to 0} \left[- \int_{\partial B_\varepsilon} u_{0,\alpha}^{i} n^\alpha \varphi^i \, dS + \int_{B_{R_0} \setminus B_\varepsilon} 2 \frac{x^i}{|\mathbf{x}|^3} \varphi^i(\mathbf{x}) \, dx \right] && (2.17) \\ &= \lim_{\varepsilon \to 0} \left[\int_{B_{R_0} \setminus B_\varepsilon} 2 \frac{x^i \varphi^i(x)}{|x|^3} - \int_{\partial B_\varepsilon} \varphi^i u_{0,\alpha}^{i} n^\alpha \, dS \right], && (2.18) \end{aligned}$$

where we have used the fact that $\mathbf{u}_0$ is C^1 across ∂B_{R_0} by construction. Analogous arguments to those used in the proof of Lemma 2.5 show that the second term in (2.18) converges to zero as $\varepsilon \to 0$. In order to prove (2.13), it is therefore sufficient, by (2.14), to prove that the first term in (2.18) is non-negative. This follows on observing that for almost all $\mathbf{x}$ we can write

$$\varphi^i(\mathbf{x}) = \varphi^i(\mathbf{0}) + \int_0^1 \varphi^i_{,\alpha}(t\,\mathbf{x})\,x^\alpha\,dt. \tag{2.19}$$

Hence

$$\int_{B_{R_0}} \frac{x^i \varphi^i(\mathbf{x})}{|\mathbf{x}|^3} = \int_{B_{R_0}} \frac{x^i \varphi^i(\mathbf{0})}{|\mathbf{x}|^3} + \frac{x^i x^\alpha}{|\mathbf{x}|^3}\left(\int_0^1 \varphi^i_{,\alpha}(t\,\mathbf{x})\,dt\right)dx. \tag{2.20}$$

But

$$\int_{B_{R_0}} \frac{\varphi^i(\mathbf{0})\,x^i}{|\mathbf{x}|^3} = \int_{B_{R_0}} -\left(\frac{\varphi^i(\mathbf{0})}{|\mathbf{x}|}\right)_{,i} = \int_{\partial B_{R_0}} -\frac{\varphi^i(\mathbf{0})}{|\mathbf{x}|}n^i = \int_{\partial B_{R_0}} -\frac{\varphi^i(\mathbf{0})}{R_0}n^i = 0.$$

Hence

$$\int_{B_{R_0}} \frac{x^i \varphi^i(x)}{|\mathbf{x}|^3} = \int_{B_{R_0}} \frac{x^i x^\alpha}{|\mathbf{x}|^3}\left(\int_0^1 \varphi^i_{,\alpha}(t\,\mathbf{x})\,dt\right)dx. \tag{2.21}$$

Finally observe that, since $\boldsymbol{\varphi}$ is on admissible variation,

$$\det \nabla(\mathbf{u}_0 + \varepsilon\varphi) \geqq 0$$

for almost every $\mathbf{x} \in B_{R_0}$ for $\varepsilon \geqq 0$ sufficiently small, hence, differentiating with respect to ε and noting that $\det \nabla \mathbf{u}_0 = 0$ on B_{R_0}, we obtain

$$(\operatorname{Adj} \nabla \mathbf{u}_0)^\alpha_i\, \varphi^i_{,\alpha} \geqq 0 \quad \text{almost everywhere on } B_{R_0}. \tag{2.22}$$

A straightforward calculation gives

$$(\operatorname{Adj} \nabla \mathbf{u}_0)^\alpha_i = \frac{x^i x^\alpha}{|\mathbf{x}|^4} \quad \text{almost everywhere on } B_{R_0} \tag{2.23}$$

and (2.22) then becomes

$$\frac{x^i x^\alpha}{|\mathbf{x}|^4}\varphi^i_{,\alpha}(\mathbf{x}) \geqq 0 \quad \text{for almost every } \mathbf{x} \in B_{R_0}. \tag{2.24}$$

It now follows from (2.24) that (2.21) is non-negative, hence (2.13) and the Theorem follow.

Remark 2.7. — The use of the invertibility condition in the proof of Theorem 2.4 is the three dimensional analogue of the argument used in the proof of the corresponding radial result Proposition 1.2.

Notice that in Theorem 2.4 the maps $\mathbf{u}_0 + \boldsymbol{\varphi}$ allow movement of the hole once it has formed at the origin; thus every radial map of this form (no matter the hole size) is stable in this respect. However, by Theorem 1.3, the radial cavitating map has the least energy out of all the radial maps.

THREE DIMENSIONAL STABILITY OF RADIAL MAPS: VARIATIONS IN HOLE SHAPE

The class of variations considered in Theorem 2.4 does not allow different shapes of holes, we next examine stability of the radial cavitating maps with respect to variations in hole shape. By Remark 2.3 the radial maps (2.1)-(2.3) are in $C^1(\bar{B}\setminus\{\mathbf{0}\})$, we will prove stability of these radial minimisers in an appropriate subclass of this space.

Definition 3.1. — We say that $\mathbf{u}\in C^1(B\setminus\{\mathbf{0}\})$ satisfies the invertibility condition (I) if, for each $R\in(0,1)$, the restriction of $\mathbf{u}$ to the sphere of radius R, S_R, can be extended as a C^1 diffeomorphism of $\bar{B}_R$.

Remark 3.2. — In particular mappings satisfying the above condition have the property that they are invertible on S_R for each R. We will require also the following observation: if $\mathbf{u}$ satisfies (I) then given $R\in(0,1)$ there exists a diffeomorphism $\bar{\mathbf{u}}$ of $\bar{B}_R$ agreeing with $\mathbf{u}$ on ∂B_R. Hence

$$\begin{aligned}\int_{B\setminus B_R}\det\nabla\mathbf{u}&=\int_{\partial B}\frac{1}{3}u^i(\operatorname{Adj}\nabla\mathbf{u})_i^\alpha n^\alpha\,dS-\int_{\partial B_R}\frac{1}{3}u^i(\operatorname{Adj}\nabla\mathbf{u})_i^\alpha n^\alpha\,dS\\&=\int_{\partial B}\frac{1}{3}u^i(\operatorname{Adj}\nabla\mathbf{u})_i^\alpha n^\alpha\,dS-\int_{\partial B_R}\frac{1}{3}\bar{u}^i(\operatorname{Adj}\nabla\bar{\mathbf{u}})_i^\alpha n^\alpha\,dS\\&=\int_{\partial B}\frac{1}{3}u^i(\operatorname{Adj}\nabla\mathbf{u})_i^\alpha n^\alpha\,dS-\int_{B_R}\det\nabla\bar{\mathbf{u}}.\end{aligned}$$

Notice that the cavitating equilibria given by (2.1)-(2.3) clearly satisfy (I). Moreover, if $\mathbf{u}|_{\partial B}=\lambda\,\mathbf{x}$ then

$$\int_{B\setminus B_R}\det\nabla\mathbf{u}=\frac{4}{3}\pi\lambda^3-\int_{B_R}\det\nabla\bar{\mathbf{u}}=\frac{4}{3}\pi\lambda^3-\operatorname{vol}(\bar{\mathbf{u}}(B_R)).$$

Theorem 3.3. — *Let* $\lambda>0$ *and let* $\tilde{\mathbf{u}}$ *be the radial map*

$$\tilde{\mathbf{u}}(\mathbf{x})=\frac{\tilde{r}(R)}{R}\mathbf{x},\qquad R=|\mathbf{x}|,$$

where $\tilde{r}$ is the radial minimiser given by Theorem 1.3. *Then* $\tilde{\mathbf{u}}$ *is the global minimiser of* E_α *on*

$$\mathscr{A}_\lambda^{\text{smooth}}=\Big\{\mathbf{u}\in \mathrm{C}^1(\mathrm{B}\setminus\{\mathbf{0}\});\ \mathbf{u}|_{\partial \mathrm{B}}=\lambda\,\mathbf{x},\ \det\nabla\mathbf{u}\geqq 0 \textit{ in } \mathrm{B}\setminus\{\mathbf{0}\},$$
$$\mathbf{u} \textit{ satisfies either } (\mathbf{I}) \textit{ or } \int_{\mathrm{B}}\det\nabla\mathbf{u}\geqq\frac{4}{3}\pi\lambda^3\Big\}. \quad (3.1)$$

Proof. — Let S_R, $\mathrm{R}\in(0,1]$, denote the sphere of radius R, let $\mathbf{u}\in\mathscr{A}_\lambda^{\text{smooth}}$ and define

$$a=\mathrm{Inf}\{\mathrm{Area}(\mathbf{u}(\mathrm{S}_\mathrm{R}));\ 0<\mathrm{R}\leqq 1\}. \quad (3.2)$$

Let (R_n) in $(0, 1]$ be a sequence satisfying

$$\mathrm{Area}(\mathbf{u}(\mathrm{S}_{\mathrm{R}_n}))\to a \quad \textit{as } n\to\infty, \quad (3.3)$$

without loss of generality, assume that $\mathrm{R}_n\to\tilde{\mathrm{R}}$ as $n\to\infty$.

Notice that

$$a\leqq \mathrm{Area}(\mathbf{u}(\mathrm{S}_1))=4\pi\lambda^2, \quad (3.4)$$

so we can choose $\mathrm{A}\in\left[\frac{2}{3}\lambda,\lambda\right]$ such that the radial deformation $\mathbf{u}_0$, given by (2.1)-(2.3), produces a cavity of area a.

We will first assume that $\tilde{\mathrm{R}}>0$ so that

$$a=\mathrm{Area}(\mathbf{u}(\mathrm{S}_{\tilde{\mathrm{R}}}))$$

by continuity. The proof now proceeds in three stages:

Step 1. — We first prove that

$$\int_{\mathrm{B}}\frac{1}{2}|\nabla\mathbf{u}|^2\geqq\int_{\mathrm{B}}\frac{1}{2}|\nabla\mathbf{u}_0|^2. \quad (3.5)$$

It follows from the convexity of the Dirichlet integral, on using the facts that $\mathbf{u}|_{\partial \mathrm{B}}=\mathbf{u}_0|_{\partial \mathrm{B}}=\lambda\,\mathbf{x}$ and $\frac{\partial\mathbf{u}_0}{\partial\mathbf{n}}=\mathbf{0}$ on $\partial\mathrm{B}_{\mathrm{R}_0}$, that

$$\int_{\mathrm{B}\setminus\mathrm{B}_{\mathrm{R}_0}}\frac{1}{2}|\nabla\mathbf{u}|^2\geqq\int_{\mathrm{B}\setminus\mathrm{B}_{\mathrm{R}_0}}\frac{1}{2}|\nabla\mathbf{u}_0|^2+\int_{\mathrm{B}\setminus\mathrm{B}_{\mathrm{R}_0}}\nabla\mathbf{u}_0.(\nabla\mathbf{u}-\nabla\mathbf{u}_0)$$
$$=\int_{\mathrm{B}\setminus\mathrm{B}_{\mathrm{R}_0}}\frac{1}{2}|\nabla\mathbf{u}_0|^2. \quad (3.6)$$

We next obtain a lower bound on

$$\int_{\mathrm{B}_{\mathrm{R}_0}}\frac{1}{2}|\nabla\mathbf{u}|^2.$$

Since the integrand is invariant under orthogonal changes of coordinate, choose $\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{n}$, to be an orthonormal basis on the sphere of radius R (with $\mathbf{t}_1$, $\mathbf{t}_2$ tangent to S_R and $\mathbf{n}$ normal to S_R). Then

$$\frac{1}{2}|\nabla \mathbf{u}|^2 = \frac{1}{2}\left(\left|\frac{\partial \mathbf{u}}{\partial \mathbf{t}_1}\right|^2 + \left|\frac{\partial \mathbf{u}}{\partial \mathbf{t}_2}\right|^2 + \left|\frac{\partial \mathbf{u}}{\partial \mathbf{n}}\right|^2\right). \tag{3.7}$$

Hence using (3.7)

$$\int_{B_{R_0}} \frac{1}{2}|\nabla \mathbf{u}|^2 = \int_0^{R_0}\int_{S_R} \frac{1}{2}|\nabla \mathbf{u}|^2 \, dS\, dR \geqq \int_0^{R_0}\int_{S_R} \left|\frac{\partial \mathbf{u}}{\partial \mathbf{t}_1}\right| \cdot \left|\frac{\partial \mathbf{u}}{\partial \mathbf{t}_2}\right| dS\, dR$$

$$\geqq \int_0^{R_0}\int_{S_R} \left|\frac{\partial \mathbf{u}}{\partial \mathbf{t}_1} \wedge \frac{\partial \mathbf{u}}{\partial \mathbf{t}_2}\right| dS\, dR \geqq R_0\, a = \int_{B_{R_0}} \frac{1}{2}|\nabla \mathbf{u}_0|^2. \tag{3.8}$$

The last inequality follows from the definition of a since

$$\int_{S_R} \left|\frac{\partial \mathbf{u}}{\partial \mathbf{t}_1} \wedge \frac{\partial \mathbf{u}}{\partial \mathbf{t}_2}\right| dS$$

is the deformed surface area of the sphere S_R. Expression (3.5) now follows by (3.6) and (3.8).

Step 2. — It is a consequence of (3.5) that

$$\int_B \frac{1}{2}|\nabla \mathbf{u}|^2 + \alpha \det \nabla \mathbf{u} \geqq \int_B \frac{1}{2}|\nabla \mathbf{u}_0|^2 + \alpha \det \nabla \mathbf{u}. \tag{3.9}$$

We next prove that

$$\int_B \det \nabla \mathbf{u} \geqq \int_B \det \nabla \mathbf{u}_0. \tag{3.10}$$

We may suppose, without loss of generality, that $\mathbf{u}$ satisfies (I), otherwise (3.10) follows trivially. To see this first recall that, by the classical isoperimetric inequality, for a given surface area the sphere encloses the maximum volume (*see* e.g. Osserman [17] and the references within). Thus by Remark 3.2, and since $\mathbf{u}_0$ was chosen to have the same cavity area as $\mathbf{u}(\partial B_{\tilde{R}})$, it follows that

$$\int_B \det \nabla \mathbf{u} = \int_{B \setminus B_{\tilde{R}}} \det \nabla \mathbf{u} \geqq \int_{B \setminus B_{R_0}} \det \nabla \mathbf{u}_0 = \int_B \det \nabla \mathbf{u}_0. \tag{3.11}$$

In obtaining (3.11) we have also used the facts that $\det \nabla \mathbf{u} \geqq 0$, by assumption, and that $\det \nabla \mathbf{u}_0 = 0$ on B_{R_0}.

Step 3. — Steps 1 and 2 imply that, for each $\mathbf{u} \in \mathscr{A}_\lambda^{\text{smooth}}$ there exists a radial map $\mathbf{u}_0 \in \mathscr{A}_\lambda^{\text{smooth}}$ given by (2.1)-(2.3) (with an appropriate choice of A) that satisfies

$$E_\alpha(\mathbf{u}) \geqq E_\alpha(\mathbf{u}_0).$$

Since Theorem 1.3 gives the global minimiser in the class of radial maps this completes the proof of the Theorem in the case $\tilde{R}>0$.

If $\tilde{R}=0$ then proceed as in the case $\tilde{R}>0$, defining a by (3.2), the only difference being that one requires an approximation argument to obtain (3.10) in step 2.

Remark 3.3. — The main point of the restrictions on $\mathscr{A}_\lambda^{\text{smooth}}$ is to ensure that we can apply the isoperimetric inequality in step 2. An instructive example of the situations that can occur is given by

$$\mathbf{u}(\mathbf{x})=r(\mathrm{R})\frac{\mathbf{x}}{\mathrm{R}}, \qquad \mathbf{x}\in \mathrm{B}\setminus\{\mathbf{0}\}$$

where $r(\mathrm{R})=(\lambda-c)\mathrm{R}+c$ and c is a constant. Then

$$\det\nabla\mathbf{u}=(\lambda-c)\left(\lambda-c+\frac{c}{\mathrm{R}}\right)^2>0, \qquad \forall\,\mathrm{R}>0.$$

If $c\geqq 0$ then $\mathbf{u}$ produces a hole of radius c, $\mathbf{u}\subseteqq\lambda\,\mathrm{B}$ and

$$\int_{\mathrm{B}}\det\nabla\mathbf{u}=\frac{4}{3}\pi\lambda^3-\frac{4}{3}\pi c^3\leqq\frac{4}{3}\pi\lambda^3$$

which is the deformed volume of B. However, if $-\lambda<c<0$ then again $\mathbf{u}\subseteqq\lambda\,\mathrm{B}$ but this time

$$\int_{\mathrm{B}}\det\nabla\mathbf{u}=\frac{4}{3}\pi\lambda^3-\frac{4}{3}\pi c^3>\frac{4}{3}\pi\lambda^3.$$

If we write $\mathbf{u}=\mathbf{u}_0+\boldsymbol{\varphi}$ in the above theorem and if $\boldsymbol{\varphi}$ satisfies (2.4) on B then, as in the proof of Theorem 2.4, it can be demonstrated that the stronger result

$$\mathrm{E}_\alpha(\mathbf{u})\geqq\mathrm{E}_\alpha(\mathbf{u}_0)+\int_{\mathrm{B}}\frac{1}{2}|\nabla\boldsymbol{\varphi}|^2$$

in fact holds. These results easily extend by density to the Sobolev spaces $\mathrm{W}^{1,p}(\mathrm{B}\setminus\{\mathbf{0}\})$.

In extending this work further it would be interesting to isolate minimal measure theoretic conditions under which the claims of Theorem 3.3 are valid to a class of maps containing $\mathscr{A}_\lambda^{\text{smooth}}$: in this respect the work of Sverak [27] may be useful. It should be noted however that some restriction on the number of singular points is necessary: otherwise, given any cavitating deformation of B, it is possible to construct deformations with the same energy, in which an infinite number of holes are formed, by exploiting the scaling properties of the energy functional (this is implicit in the work of Ball and Murat [2], *see* also [24]). It would be natural if the results of Theorem 3.3 could be obtained through the use of symmetrisation arguements such as those of Polya and Szego [20] but I have so far been unable to do so.

Concluding Remarks. — It is interesting to note the apparent connection between the problem studied in this paper and the interesting work of Brezis, Coron and Lieb [5] on liquid crystals (*see* also Hardt, Kinderlerer and Lin [10]). In these works the authors study minimisers of $\int_B \frac{1}{2}|\nabla \mathbf{u}|^2$ amongst maps $\mathbf{u}: \mathrm{B} \to \mathrm{S}^2$ (in the liquid crystal problem the unit vector $\mathbf{u}(\mathbf{x})$ represents the orientation of the crystal molecule at the point $\mathbf{x}$). In particular, under appropriate boundary conditions, $\frac{\mathbf{x}}{|\mathbf{x}|}$ is the global minimiser for this problem in $\mathrm{H}^1(\mathrm{B}; \mathrm{S}^2)$ (*see* [4], [5], [11] and [14]). It appears from Theorem 3.3 that minimisers to our problem may behave like minimisers to this constrained problem for large boundary displacements (or equivalently for large values of α).

Finally we remark that, as pointed out in the introduction, the energy function (7) is not necessarily to be interpreted as the stored energy of an elastic material. Rather it may be useful in giving lower bound properties of certain nonlinear materials (and in studying the qualitative properties of minimisers). To demonstrate this consider, for example, the following class of polyconvex stored energy functions studied by Ball [1] in his work on cavitation:

$$\bar{\mathrm{E}}(\mathbf{u}) = \int_B \frac{1}{2}|\nabla \mathbf{u}|^2 + h(\det \nabla \mathbf{u})$$

where the superlinear function $h: (0, \infty) \to \mathbf{R}^-$ is C^1 and convex with $h(\delta) \to \infty$ as $\delta \to 0, \infty$. Then for the Dirichlet boundary value problem with boundary condition (5), for each λ

$$\bar{\mathrm{E}}(\mathbf{u}) \geqq \int_B \frac{1}{2}|\nabla \mathbf{u}|^2 + h(\lambda^3) + h'(\lambda^3)(\det \nabla \mathbf{u} - \lambda^3)\,dx \qquad \forall (\mathbf{u} - \lambda\,\mathbf{x}) \in \mathrm{W}_0^{1,2}(\mathrm{B}).$$

The functional on the righthand side is (up to a constant depending on λ) of the form (7). By the results of Ball [1], $\bar{\mathrm{E}}$ exhibits cavitation for large boundary displacements. The results of Theorem 3.2 then imply that this cannot occur if $\lambda\, h'(\lambda^3) \leqq \frac{4}{3}$ [3].

[3] *See* Stuart [27] for an alternative approach to estimating the critical displacement for cavitation.

This approach to studying the properties of nonlinear stored energy functions is further extended in [24].

ACKNOWLEDGEMENTS

I would like to thank Geoffrey Burton, Irene Fonseca, Maureen McIver, Nic Owen, Vladimir Sverak and Luc Tartar for their interest and comments on this work. Finally my thanks to John Toland for his advice, enthusiasm and encouragement.

REFERENCES

[1] J. M. BALL, Discontinuous Equilibrium Solutions and Cavitation in Nonlinear Elasticity, *Phil. Trans. Roy. Soc. London.*, Vol. **A306**, 1982, pp. 557-611.

[2] J. M. BALL and F. MURAT, $W^{1,p}$-Quasiconvexity and Variational Problems for Multiple Integrals, *J. Funct. Anal.*, Vol. **58**, 1984, pp. 225-253.

[3] J. M. BALL, J. C. CURRIE and P. J. OLVER, Null Lagrangians, Weak Continuity and Variational Problems of Arbitrary Order, *J. Funct. Anal.*, Vol. **41**, 1981, pp. 135-174.

[4] H. BREZIS, J. CORON and E. LIEB, Harmonic Maps with Defects, *Comm. Math. Phys.*, Vol. **107**, 1986, pp. 649-705.

[5] H. BREZIS, S^k-valued maps with singularities, *Montecatini Lecture Notes*, Springer, 1988.

[6] D. G. EDELEN, The null set of the Euler-Lagrange Operator, *Arch. Ration. Mech. Anal.*, Vol. **11**, 1962, pp. 117-121.

[7] L. C. EVANS, Quasiconvexity and Partial Regularity in the Calculus of Variations, *Arch. Ration. Mech. Anal.*, Vol. **95**, 1986, pp. 227-252.

[8] N. FUSCO and J. HUTCHINSON, Partial Regularity of Functions Minimising Quasiconvex Integrals, *Manuscripta Math.*, Vol. **54**, 1985, pp. 121-143.

[9] A. N. GENT and P. B. LINDLEY, Internal Rupture of Bonded Rubber Cylinders in Tension, *Proc. Roy. Soc. London*, Vol. **A249**, 1958, pp. 195-205.

[10] R. HARDT, D. KINDERLEHRER and F. H. LIN, Existence and Partial Regularity of Static Liquid Crystal Configurations, *Commun. Math. Phys.*, Vol. **105**, 1986, pp. 541-570.

[11] F. HELEIN, Applications harmoniques et applications minimisantes entre variétés Riemanniennes, *Thèse de Doctorat*, École Polytechnique, 1989.

[12] C. O. HORGAN and R. ABEYARATNE, A Bifurcation Problem for a Compressible Nonlinearly Elastic Medium: Growth of a Microvoid, *J. Elasticity*, Vol. **16**, 1986, pp. 189-200.

[13] R. D. JAMES and S. J. SPECTOR, The Formation of Filamentary Voids in Solids, *J. Mech. Phys. Sol.*, Vol. **39**, 1991, pp. 783-813.

[14] F. H. LIN, Une remarque sur l'application $\mathbf{x}/|\mathbf{x}|$, *C.R. Acad. Sci. Paris*, T. **305**, Series I, 1987, pp. 529-531.

[15] S. MULLER, Higher Integrability of Determinants and Weak Convergence in L^1, *J. reine angew. Math.*, Vol. **412**, 1990, pp. 20-34.

[16] P. J. OLVER and J. SIVALOGANATHAN, The Structure of Null Lagrangians, *Nonlinearity*, Vol. **1**, 1988, pp. 389-398.

[17] R. OSSERMAN, The Isoperimetric Inequality, *Bull. Amer. Math. Soc.*, Vol. **84**, 1978, pp. 1182-1238.

[18] K. A. PERICAK-SPECTOR and S. J. SPECTOR, Nonuniqueness for a Hyperbolic System: Cavitation in Nonlinear Elastodynamics, *Arch. Ration. Mech. Anal.*, Vol. **101**, 1988, pp. 293-317.

[19] P. Podio-Guidugli, G. Vergara Caffarelli and E. G. Virga, Discontinuous Energy Minimisers in Nonlinear Elastostatics: an Example of J. Ball Revisited, *J. Elasticity*, Vol. **16**, 1986, pp. 75-96.

[20] Polya and Szego, *Isoperimetric Inequalities in Mathematical Physics*, Princeton University Press, 1951.

[21] J. Sivaloganathan, Uniqueness of Regular and Singular Equilibria for Spherically Symmetric Problems of Nonlinear Elasticity, *Arch. Ration. Mech. Anal.*, Vol. **96**, 1986, pp. 97-136.

[22] J. Sivaloganathan, A Field Theory Approach to Stability of Equilibria in Radial Elasticity, *Math. Proc. Camb. Phil. Soc.*, Vol. **99**, 1986, pp. 589-604.

[23] J. Sivaloganathan, The Generalised Hamilton-Jacobi Inequality and the Stability of Equilibria in Nonlinear Elasticity, *Arch. Ration. Mech. Anal.*, Vol. **107**, No. 4, 1989, pp. 347-369.

[24] J. Sivaloganathan, in preparation.

[25] S. Spector, *Linear Deformations as Minimisers of the Energy*, preprint.

[26] C. A. Stuart, Radially Symmetric Cavitation for Hyperelastic Materials, *Ann. Inst. Henri Poincaré : Analyse non linéaire*, Vol. **2**, 1985, pp. 33-66.

[27] C. A. Stuart, *Estimating the Critical Radius for Radially Symmetric Cavitation*, preprint E.P.F.L. Lausanne, 1991.

[28] F. Meynard, Cavitation dans un milieu hyperélastique, *Thèse de Doctorat*, École Polytechnique Fédérale de Lausanne, 1990.

(*Manuscript received December* 20, 1990;
revised July 4, 1991.)