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On nonviscous compressible fluids in a time-dependent domain

by

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ABSTRACT. — We study the Euler equations for a nonviscous compressible barotropic fluid in a time-dependent domain of the three-dimensional space. We prove the existence of a unique local in time classical solution.

Key words : Euler equations, first order nonlinear hyperbolic systems, compressible fluids.

RÉSUMÉ. — On considère les équations d'Euler pour un fluide non visqueux, compressible et barotropique dans un domaine de l'espace dépendant du temps. On démontre l'existence locale d'une unique solution classique.

1. INTRODUCTION

In this paper we study the Euler equations for a non-viscous compressible barotropic fluid in a time-dependent domain Ω_t of the three-dimensional space. We assume that it is given an open bounded subset Ω of $\mathbf{R}^3$

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with smooth boundary Γ and a smooth map $\eta:[0,T_0]\times\bar\Omega\to\mathbf{R}^3$, for some $T_0>0$, such that $\eta(0,.)=\mathrm{Id}$ (identity map on $\bar\Omega$). For each t, η defines time-dependent domains $\Omega_t=\eta(t,\Omega)$, $\Omega_0\equiv\Omega$. We assume that $\eta(t,.):\Omega\to\Omega_t$ is a diffeomorphism for each $t\in(0,T_0]$; it follows that $\partial\Omega_t=\partial\eta(t,\Omega)=\eta(t,\Gamma)$. Let us denote by $n^t(y)$ the unit outward normal vector to $\partial\Omega_t$ at the point $y\in\partial\Omega_t$. Set $w(t,x)\equiv\dot\eta(t,x)$, $(t,x)\in Q_{T_0}\equiv[0,T_0]\times\bar\Omega$, where $\dot\eta$ denotes the time derivative, and define $\bar w(t,y)=\bar w(t,\eta(t,x))\equiv w(t,x)$ for $y=\eta(t,x)\in\Omega_t$, $x\in\Omega$.

Let us denote by $\bar u(t,y)=(\bar u_1,\bar u_2,\bar u_3)^*$, $\bar\rho(t,y)$ the unknown velocity and density of the fluid at time t at the point $y\in\Omega_t$. Then the equations of motion are

$$(\mathrm{E})\quad\begin{cases}\bar\rho[\dot{\bar u}+(\bar u.\nabla)\bar u-\bar b]+\nabla p(\bar\rho)=0 \quad \text{in } D_T\equiv\{(t,y)\in[0,T]\times\mathbf{R}^3/y\in\Omega_t\},\\ \dot{\bar\rho}+\operatorname{div}(\bar\rho\bar u)=0 \quad \text{in } D_T,\\ \bar u.n^t=\bar w.n^t \quad \text{on } S_T\equiv\{(t,y)\in[0,T]\times\mathbf{R}^3/y\in\partial\Omega_t\},\\ \bar u(0,y)=u_0(y), \qquad y\in\Omega,\\ \bar\rho(0,y)=\rho_0(y), \qquad y\in\Omega.\end{cases}$$

$\bar b(t,y)$ denotes the external force field per unit mass; the initial velocity $u_0(y)$ and the initial density $\rho_0(y)$ are given. We assume that the fluid is barotropic, *i.e.* the pressure p is a function of the density only: $p=p(\bar\rho)$. The known function $\xi\to p(\xi)$ verifies the physical hypothesis $p'(\xi)>0$ for $\xi>0$.

Because of the boundary condition $(\mathrm{E})_3$, (E) is a typical characteristic initial-boundary value problem for quasilinear symmetric hyperbolic systems. In a fixed domain, the compressible Euler equations (E) were studied by D. Ebin [3] in the case of subsonic flow and by H. Beirão da Veiga [2] and R. Agemi [1] in the general case; *see* also S. Schochet [7]. The equations of Ideal Magneto-Hydrodynamics were studied by T. Yanagisawa [8]. The aim of the present paper is to prove the existence of a local in time classical solution to the initial boundary value problem (E).

Before stating our result, let us introduce the functional space $Y_k(T_0)=\bigcap_{j=0}^{k} H^{j+1}(0,T_0;H^{k-j}(\Omega))$. Our result reads as follows:

THEOREM A. — *Let Ω be an open bounded subset of $\mathbf{R}^3$ with boundary Γ of class C^6. Let $\eta\in Y_5(T_0)$ such that $\eta(t,.)$ is a diffeomorphism for each $t\in[0,T_0]$ and $\eta(0,.)=\mathrm{Id}$. Assume that $\bar b\in H^3_{\mathrm{loc}}([0,T_0]\times\mathbf{R}^3)$, $p\in C^5$. Suppose that u_0, $\rho_0\in H^3(\Omega)$ with $\min_{x\in\bar\Omega}\rho_0(x)>0$ and that the (necessary) compatibility conditions through order 2 are satisfied:*

$$[\partial/\partial t+(\bar w.\nabla)]^k(\bar u-\bar w).n^t_{|t=0}=0 \quad on\ \Gamma, \tag{1.1}$$

for $k=0,1,2$. The above time derivatives are calculated from $(\mathrm{E})_{1,2}$ and then expressed in terms of the initial data and their derivatives.

Then there exists $T' \in (0, T_0]$ *such that* (E) *is uniquely solvable on* $[0, T']$ *by a classical solution* $(\bar{u}, \bar{\rho}) \in C^1(D_{T'})$.

If the data are smoother and more compatibility conditions are assumed, in the same way we can prove further regularity of the solution. We remark that this result can be extended to the nonisentropic case without any essential modification of the proof.

In order to solve (E), it is convenient to reduce it to a problem in a cylindrical domain with variable coefficients. Let us denote by $a_{ki}(t,x)$ the entries (k,i) of the Jacobian matrix $[D\eta]^{-1}$ (where $D\eta$ has the term $D_k \eta_i$ in the i-th row, k-th column); denote by $\mathcal{A} = \mathcal{A}(t,x)$ the transpose matrix of (a_{ki}), $\mathcal{A} = (a_{ki})^*$. Given (t,y), $y \in \Omega_t$, we have $y = \eta(t,x)$ for a suitable $x \in \Omega$. Performing the change of variables $(t,y) \to (t,x)$ we obtain

$$(D_k \equiv \partial/\partial x_k, \nabla = \nabla_x = (D_1, D_2, D_3))$$
$$\partial/\partial t \to \partial/\partial t - (w \,.\, \mathcal{A} \nabla) = \partial/\partial t - w_i a_{ki} D_k,$$
$$\partial/\partial y_i = a_{ki} D_k = (\mathcal{A} \nabla)_i, \qquad \nabla_y = \mathcal{A} \nabla.$$

If we set $u(t,x) = \bar{u}(t, \eta(t,x))$, $\rho(t,x) = \bar{\rho}(t, \eta(t,x))$, $b(t,x) = \bar{b}(t, \eta(t,x))$, problem (E) reduces to

$$(E') \quad \begin{cases} \rho[\dot{u} + ((u-w) \,.\, \mathcal{A} \nabla) u - b] + \mathcal{A} \nabla p(\rho) = 0 & \text{in } Q_T \equiv [0,T] \times \Omega, \\ \dot{\rho} + (u-w) \,.\, \mathcal{A} \nabla \rho + \rho \mathcal{A} \nabla \,.\, u = 0 & \text{in } Q_T, \\ u \,.\, N = w \,.\, N & \text{on } \Sigma_T \equiv [0,T] \times \Gamma, \\ u(0) = u_0 & \text{in } \Omega, \\ \rho(0) = \rho_0 & \text{in } \Omega, \end{cases}$$

where $N(t,x)$ is the normal to $\partial\eta(t,\Omega)$ calculated in $\eta(t,x)$, *i.e.* $N(t,x) = n^t(\eta(t,x))$. We want to write (E′) as a quasilinear symmetric hyperbolic system with a homogeneous boundary condition. Hence we introduce the new unknown $v \equiv u - w$ and consider the pressure p as an independent variable so that $\rho = \rho(p)$, $\rho \in C^5$. We obtain

$$(P) \quad \begin{cases} \rho(p)[\dot{v} + (v \,.\, \mathcal{A} \nabla) v + \dot{w} + (v \,.\, \mathcal{A} \nabla) w - b] + \mathcal{A} \nabla p = 0 & \text{in } Q_T, \\ (\rho'/\rho)(p)[\dot{p} + v \,.\, \mathcal{A} \nabla p] + \mathcal{A} \nabla \,.\, v + \mathcal{A} \nabla \,.\, w = 0 & \text{in } Q_T, \\ v \,.\, N = 0 & \text{on } \Sigma_T, \\ v(0) = v_0 \equiv u_0 - w(0) & \text{in } \Omega, \\ p(0) = p_0 \equiv p(\rho_0) & \text{in } \Omega. \end{cases}$$

Before stating our result concerning the solvability of (P) let us introduce the functional space $X_k(T) = \bigcap_{j=0}^{k} C^j([0,T]; H^{k-j}(\Omega))$.

THEOREM B. — *Let* Ω *be an open bounded subset of* $\mathbf{R}^3$ *with boundary* Γ *of class* C^6. *Let* $\eta \in Y_5(T_0)$ *such that* $\eta(t,.)$ *is a diffeomorphism for each* $t \in [0, T_0]$ *and* $\eta(0,.) = \mathrm{Id}$. *Assume that* $b \in H^3(Q_{T_0})$, $\rho \in C^5$. *Let* v_0, $p_0 \in H^3(\Omega)$ *such that* $\min_{x \in \bar{\Omega}} [\rho(p_0(x)), \rho'(p_0(x))] > 0$. *Assume that the* *(necessary) compatibility conditions through order* 2 *are satisfied:*

$$\partial^k/\partial t^k (v.\mathrm{N})_{|t=0}=0 \quad on\ \Gamma, \qquad k=0,1,2, \tag{1.2}$$

where the above time derivatives are calculated from $(\mathrm{P})_{1,2}$ *and then expressed in terms of the initial data and their derivatives.*

Then there exists $\mathrm{T}'\in(0,\mathrm{T}_0]$ *such that* (P) *is uniquely solvable on* $[0,\mathrm{T}']$ *by* $(v,p)\in\mathrm{X}_3(\mathrm{T}')$.

The proof of theorem **B** consists of the following steps. First, by means of an iteration scheme, we construct successive approximation which are solutions of linearized equations obtained from (P) such that the boundary is non characteristic (*cf.* Schochet [7], Yanagisawa [8]). Second, we establish uniform *a priori* estimates for the approximating solutions. A standard approach gives interior estimates or estimates for the tangential derivatives near the boundary; the crucial point (here as in all characteristic hyperbolic mixed problems) is how to get estimates for the normal derivatives. These are achieved by combining an estimate for the transformed vorticity, which satisfies a transport equation, with the fact that the rank of the boundary matrix is two. A similar idea was first introduced by Beirão da Veiga [2]. Finally, by passing to the limit of the approximate solutions, we get the solution of the original initial boundary value problem (P).

2. PRELIMINARIES

Let us denote with $\mathrm{C}^0(\bar\Omega)$ the space of continuous (and bounded) functions on $\bar\Omega$, and with $\mathrm{C}^k(\bar\Omega)$ (k positive integer) the space of functions with derivatives up to order k in $\mathrm{C}^0(\bar\Omega)$. Given $k>0$, we denote by $\mathrm{H}^k(\Omega)$ the corresponding Sobolev spaces of exponent k on Ω, with norm $\|.\|_k$. By $(.,.)$ we denote the scalar product in $\mathrm{L}^2(\Omega)$. We don't introduce different notations for spaces of vector- and matrix-valued functions. If X is a Banach space, $\mathrm{L}^2(0,\mathrm{T};\mathrm{X})$, $\mathrm{L}^\infty(0,\mathrm{T};\mathrm{X})$, $\mathrm{H}^k(0,\mathrm{T};\mathrm{X})$, $\mathrm{C}^k([0,\mathrm{T}];\mathrm{X})$ are the spaces of X-valued functions in L^2, L^∞, H^k and C^k respectively. We denote the norm of $\mathrm{L}^\infty(0,\mathrm{T};\mathrm{H}^k(\Omega))$ by $\|\cdot\|_{k,\mathrm{T}}$; the norm of $\mathrm{X}_k(\mathrm{T})=\bigcap_{j=0}^{k}\mathrm{C}^j([0,\mathrm{T}];\mathrm{H}^{k-j}(\Omega))$ is $|||\mathrm{U}|||_{k,\mathrm{T}}\equiv\sum_{j=0}^{k}\|\partial^j\mathrm{U}/\partial t^j\|_{k-j,\mathrm{T}}$. The space $\mathrm{H}^k(\mathrm{Q_T})=\bigcap_{j=0}^{k}\mathrm{H}^j(0,\mathrm{T};\mathrm{H}^{k-j}(\Omega))$ is equipped with the norm $\|\mathrm{U}\|_{k,\mathrm{Q_T}}\equiv\sum_{j=0}^{k}\|\partial^j\mathrm{U}/\partial t^j\|_{2,k-j,\mathrm{T}}$, where $\|\cdot\|_{2,k-j,\mathrm{T}}$ is the norm of $\mathrm{L}^2(0,\mathrm{T};\mathrm{H}^{k-j}(\Omega))$. Moreover, let us set $|||\mathrm{U}(t)|||\equiv\sum_{j=0}^{k}\|(\partial^j\mathrm{U}/\partial t^j)(t)\|_{k-j}$

for each $U \in X_k(T)$. For $U_0=(v_0, p_0)^*$ we set $|||U_0|||_k \equiv \sum_{j=0}^{k} \|\partial^j U_0/\partial t^j\|_{k-j}$, where $\partial^j U_0/\partial t^j$ is obtained from $(P)_{1,2}$ and expressed in terms of U_0 and its derivatives. Finally, observe that $Y_k(T) \subset X_k(T) \subset H^k(Q_T)$, $k=3, 4, 5$, $H^k(Q_T) \subset Y_{k-1}(T)$, $k=4, 5$, where every imbedding is continuous. Everyone of these spaces is an algebra. We shall make use of the summation convention over repeated indices. We shall introduce several constants C, C_0, C'_0, C_i which will depend at most on the data of the problem Ω, η, v_0, p_0, ρ, unless explicitly indicated.

Now we study the regularity of functions depending on η, which will be useful in the sequel. For it we make use of well known tools as the Hölder inequality and Sobolev imbeddings. From $\eta \in Y_5(T_0)$ we have $w \in H^5(Q_{T_0})$. Since η is a diffeomorphism and because of its regularity, it follows that the coefficients a_{ki} of $[D\eta]^{-1}$ are in $Y_4(T_0)$. On the boundary Γ, in each local chart $\phi=\phi(\xi_1, \xi_2)$, since η is a diffeomorphism, the unit vector $N(t, x)$ can be written as

$$N(t,x)=\frac{D\eta(t,x)\tau_1(x) \wedge D\eta(t,x)\tau_2(x)}{|D\eta(t,x)\tau_1(x) \wedge D\eta(t,x)\tau_2(x)|}, \tag{2.1}$$

where

$$\tau_1(x) \equiv \frac{(\partial\phi/\partial\xi_1)(\phi^{-1}(x))}{|(\partial\phi/\partial\xi_1)(\phi^{-1}(x))|},$$

$$\tau_2(x) \equiv \frac{(\partial\phi/\partial\xi_2)(\phi^{-1}(x))-[(\partial\phi/\partial\xi_2)(\phi^{-1}(x))\tau_1(x)]\tau_1(x)}{|(\partial\phi/\partial\xi_2)(\phi^{-1}(x))-[(\partial\phi/\partial\xi_2)(\phi^{-1}(x))\tau_1(x)]\tau_1(x)|}.$$

Extend τ_1, τ_2 to the interior of Ω in such a way that $\tau_i \in C^5(\bar{\Omega})$. Given $T_0>0$, we can find $\delta>0$ such that in $\Omega_{2\delta}=\{x \in \Omega/\mathrm{dist}(x, \Gamma) \leq 2\delta\}$ and for $t \in [0, T_0]$

$$|D\eta(t,x)\tau_1(x) \wedge D\eta(t,x)\tau_2(x)|>0. \tag{2.2}$$

From (2.1) we can define N also in $[0, T_0] \times \Omega_{2\delta}$ and obtain $N \in Y_4(T_0, \Omega_{2\delta})$ (this last symbol is self-explanatory). Furthermore, a direct computation shows that on Γ we have

$$J(t,x)\mathscr{A}(t,x)n(x)=\psi_0(t,x)N(t,x), \qquad t \in [0, T_0], \quad x \in \Gamma, \tag{2.3}$$

where $J \equiv \det[D\eta]$, $n=n(x)$ denotes the unit outward normal vector to Γ and the scalar function $\psi_0=\psi_0(t, x)$, in each local chart $\phi=\phi(\xi_1, \xi_2)$ of Γ, is given by

$$\psi_0=\frac{|\partial(\eta\circ\varphi)/\partial\xi_1 \wedge \partial(\eta\circ\varphi)/\partial\xi_2|}{|\partial\varphi/\partial\xi_1 \wedge \partial\varphi/\partial\xi_2|}.$$

3. THE ITERATION SCHEME

First of all we write (P) in the following matrix form:

$$A_0 \frac{\partial U}{\partial t} + \sum_{j=1}^{3} A_j D_j U + DU = F, \tag{3.1}$$

where $U=(v_1, v_2, v_3, p)^*$,

$$A_0 = A_0(U) = \begin{pmatrix} \rho & \cdot & \cdot & \cdot \\ \cdot & \rho & \cdot & \cdot \\ \cdot & \cdot & \rho & \cdot \\ \cdot & \cdot & \cdot & \rho'/\rho \end{pmatrix},$$

$$A_j = A_j(U) = \begin{pmatrix} \rho\, v_i a_{ji} & \cdot & \cdot & a_{j1} \\ \cdot & \rho\, v_i a_{ji} & \cdot & a_{j2} \\ \cdot & \cdot & \rho\, v_i a_{ji} & a_{j3} \\ a_{j1} & a_{j2} & a_{j3} & (\rho'/\rho)\, v_i a_{ji} \end{pmatrix},$$

$$D = D(U) = \rho(p) \begin{pmatrix} a_{j1} D_j w_1 & a_{j2} D_j w_1 & a_{j3} D_j w_1 & \cdot \\ a_{j1} D_j w_2 & a_{j2} D_j w_2 & a_{j3} D_j w_2 & \cdot \\ a_{j1} D_j w_3 & a_{j2} D_j w_3 & a_{j3} D_j w_3 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

$$F = F(U) = (\rho(p)(b - \dot{w}), -\mathscr{A}\nabla . w)^*.$$

For the sake of brevity we introduce the differential operator

$$L(U) \equiv A_0(U) \frac{\partial}{\partial t} + \sum_{j=1}^{3} A_j(U) D_j + D(U)$$

so that we can write (3.1) as

$$L(U)\, U = F(U). \tag{3.2}$$

The boundary matrix $A_n(U)$ is defined by

$$A_n(U) \equiv \sum_{j=1}^{3} A_j(U) n_j = \begin{pmatrix} \rho v . \mathscr{A} n I_3 & \mathscr{A} n \\ (\mathscr{A} n)^* & (\rho'/\rho)\, v . \mathscr{A} n \end{pmatrix},$$

where I_3 is the 3×3 unit matrix. Because of (2.3), on the boundary $A_n(U)$ reduces to

$$A_n(U)_{|\Sigma_T} = \frac{\psi_0}{J} \begin{pmatrix} \rho v . N I_3 & N \\ N^* & (\rho'/\rho)\, v . N \end{pmatrix} = \frac{\psi_0}{J} \begin{pmatrix} 0_3 & N \\ N^* & 0 \end{pmatrix},$$

because of the boundary condition $v . N = 0$ on Σ_T (0_3 is the 3×3 null matrix). Since $\det A_n(U) = 0$ on Σ_T, the boundary matrix is singular, namely the boundary is characteristic for (P). To define our approximating solutions, we modify (P) in such a way that the boundary is no more characteristic (*cf.* Schochet [7]). Before of that, in order for the approximate solutions defined below to be sufficiently smooth for the computations leading to the *a priori* estimates, we regularize our data as follows. Take a sequence $\{w^{(m)}\}_{m=0}^{\infty}$ in $H^6(Q_{T_0})$ such that $w^{(m)} \to w$ in $H^5(Q_{T_0})$. Define $\eta^{(m)} \in Y_6(T_0)$ as $\eta^{(m)}(t, x) = x + \int_0^t w^{(m)}(s, x)\, ds$. For any m we have $\eta^{(m)}(0, .) = \mathrm{Id}$ and, for x, $y \in \bar{\Omega}$,

$$\eta^{(m)}(t, x) - \eta^{(m)}(t, y) = x - y + \int_0^t [w^{(m)}(s, x) - w^{(m)}(s, y)]\, ds$$

which gives

$$|\eta^{(m)}(t, x) - \eta^{(m)}(t, y)| \geqq |x-y| - C\|w^{(m)}\|_{2.3.T_0} t^{1/2} |x-y| \geqq (1 - C t^{1/2}) |x-y|, \quad (3.3)$$

since $w^{(m)}$ is bounded in $L^2(0, T_0; H^3(\Omega)) \subset L^2(0, T_0; C^1(\bar{\Omega}))$. Take $T_1 \in (0, T_0]$ such that $1 - CT_1^{1/2} \geqq 1/2$; from (3.3) we obtain that $\eta^{(m)}$ is injective and, because of its regularity, is a diffeomorphism. Correspondingly, define $\mathscr{A}^{(m)}$, $N^{(m)}$, $J^{(m)}$, $\psi_0^{(m)}$; then we have

$$\left.\begin{aligned} N^{(m)}(0) = n \quad \text{on } \Gamma, \qquad \mathscr{A}^{(m)}(0) = I_3 \quad \text{in } \Omega, \\ J^{(m)} \mathscr{A}^{(m)} n = \psi_0^{(m)} N^{(m)} \quad \text{on} \quad \Sigma_{T_1}. \end{aligned}\right\} \quad (3.4)$$

Observe also that from $\eta^{(m)} \in Y_6(T_1)$ it follows that the coefficients $a_{ki}^{(m)}$ of $[D\eta^{(m)}]^{-1}$ are in $Y_5(T_1)$ and $N^{(m)} \in Y_5(T_1, \Omega_\delta)$; here we have extended $N^{(m)}$ to a neighborhood Ω_{δ_m} of Γ, as we did for N, where *a priori* Ω_{δ_m} depends on m. Since $\eta^{(m)} \to \eta$ in $Y_5(T_1)$ we can take $\delta_m \geqq \delta$. For $m = 0, 1, \ldots$ consider the operator

$$L^{(m)}(U) \equiv A_0(U) \frac{\partial}{\partial t} + \sum_{j=1}^{3} A_j^{(m)}(U) D_j + D^{(m)}(U) \quad (3.5)$$

where

$$A_j^{(m)}(U) \equiv \begin{pmatrix} \rho v_i a_{ji}^{(m-1)} & . & . & a_{j1}^{(m)} \\ . & \rho v_i a_{ji}^{(m-1)} & . & a_{j2}^{(m)} \\ . & . & \rho v_i a_{ji}^{(m-1)} & a_{j3}^{(m)} \\ a_{j1}^{(m)} & a_{j2}^{(m)} & a_{j3}^{(m)} & (\rho'/\rho) v_i a_{ji}^{(m-1)} \end{pmatrix},$$

$$D^{(m)}(U) \equiv \rho(p) \begin{pmatrix} a_{j1}^{(m)} D_j w_1^{(m)} & a_{j2}^{(m)} D_j w_1^{(m)} & a_{j3}^{(m)} D_j w_1^{(m)} & . \\ a_{j1}^{(m)} D_j w_2^{(m)} & a_{j2}^{(m)} D_j w_2^{(m)} & a_{j3}^{(m)} D_j w_2^{(m)} & . \\ a_{j1}^{(m)} D_j w_3^{(m)} & a_{j2}^{(m)} D_j w_3^{(m)} & a_{j3}^{(m)} D_j w_3^{(m)} & . \\ . & . & . & . \end{pmatrix}.$$

The corresponding boundary matrix is

$$A_n^{(m)}(U) \equiv \sum_{j=1}^{3} A_j^{(m)}(U) n_j = \begin{pmatrix} \rho v . \mathscr{A}^{(m-1)} n I_3 & \mathscr{A}^{(m)} n \\ (\mathscr{A}^{(m)} n)^* & (\rho'/\rho) v . \mathscr{A}^{(m-1)} n \end{pmatrix};$$

if $U=(v,p)^*$ is such that $v.N^{(m-1)}=0$ on Σ_T, on the boundary $A_n^{(m)}(U)$ reduces to

$$A_n^{(m)}(U)_{|\Sigma_T}=\frac{\psi_0^{(m)}}{J^{(m)}}\begin{pmatrix}0_3 & N^{(m)}\\ (N^{(m)})^* & 0\end{pmatrix}, \qquad (3.6)$$

because of $(3.4)_3$. Like A_n, $A_n^{(m)}$ is a singular matrix. Second, take a sequence $\{b^{(m)}\}_{m=0}^{\infty}$ in $H^4(Q_{T_1})$ such that

$$\|b-b^{(m)}\|_{3,Q_{T_1}}<\gamma 2^{-m}, \qquad (3.7)$$

where γ is a positive parameter. Define

$$F^{(m)}(U)\equiv(\rho(p)(b^{(m)}-\dot{w}^{(m)}), -\mathscr{A}^{(m)}\nabla . w^{(m)})^*.$$

Next, we approximate the initial data $U_0=(v_0,p_0)^*\in H^3(\Omega)$ by functions $U_0^{(m)}\in H^5(\Omega)$ that satisfy suitable compatibility conditions through order 3.

LEMMA 3.1. – *Given any* $\gamma^*<\frac{1}{2}|||U_0|||_2$, *there exists a sequence* $\{U_0^{(m)}\}_{m=0}^{\infty}$ *in* $H^5(\Omega)$ *such that*:

(i) $U_0^{(0)}$ *satisfies the compatibility conditions of order* 3 *for the equations* $L^{(0)}(U)U=F^{(0)}(U)$ *with respect to the boundary condition* $U.N^{(0)}=0$ *on* Σ_{T_1}.

(ii) $U_0^{(m)}$ $(m=1,2,\ldots)$ *satisfies the compatibility conditions of order* 3 *for the equations* $L^{(m)}(U^{(m-1)})U=F^{(m)}(U^{(m-1)})$ *with respect to the boundary condition* $U.N^{(m)}=0$ *on* Σ_{T_1}. *Here* $U^{(m-1)}$ *satisfies* $U^{(m-1)}.N^{(m-1)}=0$ *on* Σ_{T_1}, $(\partial^k U^{(m-1)}/\partial t^k)$ $(0)=(\partial^k/\partial t^k)U_0^{(m-1)}$ $(k=0,1,2)$, *where* $(\partial^k/\partial t^k)U_0^{(m-1)}$ *denotes the k-th time derivative of* $U_0^{(m-1)}$ *determined in the preceding steps by calculating the time derivatives from the corresponding equations and expressing them in terms of* $U_0^{(m-1)}$. *Moreover* $U_0^{(m)}$ *satisfies*

$$|||U_0-U_0^{(m)}|||_3<\gamma^* 2^{-m} \quad \textit{for} \quad m=0,1,\ldots \qquad (3.8)$$

Proof. – Such approximations are constructed in Rauch-Massey [6], Lemma 3.3, for linear equations with a nonsingular boundary matrix. The method can be extended to the present case following the arguments of [7], p. 52, 53, if we can check the relation

$$\text{Range } M=\text{Range } M(A_n^{(m)})^k \quad \text{on } \Gamma \qquad (3.9)$$

for $k=0$, 1, 2, 3, where M is the matrix giving the boundary condition $U.N^{(m)}=0$ on Σ_{T_1}:

$$M=((N^{(m)})^*,0).$$

The range of M is **R**. On the other hand, on Γ we have

$$(A_n^{(m)})^k = \left(\frac{\psi_0^{(m)}}{J^{(m)}}\right)^k \begin{pmatrix} 0_3 & N^{(m)} \\ (N^{(m)})^* & 0 \end{pmatrix} \quad \text{if} \quad k=1,3,$$

$$(A_n^{(m)})^k = \left(\frac{\psi_0^{(m)}}{J^{(m)}}\right)^k \begin{pmatrix} N^{(m)} \otimes N^{(m)} & 0 \\ 0 & 1 \end{pmatrix} \quad \text{if} \quad k=2,$$

so that

$$M\,(A_n^{(m)})^k = \left(\frac{\psi_0^{(m)}}{J^{(m)}}\right)^k (0,0,0,1) \quad \text{if} \quad k=1,3,$$

$$M\,(A_n^{(m)})^k = \left(\frac{\psi_0^{(m)}}{J^{(m)}}\right)^k M \quad \text{if} \quad k=2.$$

In both cases the range is **R**, so that (3.9) holds. The proof of Lemma 3.1 is complete. ■

Now we modify the operator $L^{(m)}\,(m=0,1,\ldots)$ given in (3.5) in order to have a nonsingular boundary matrix: we extend the normal vector n to be $C^5(\bar{\Omega})$ and consider

$$L^{(m,\varepsilon)}(U) \equiv A_0(U)\frac{\partial}{\partial t} + \sum_{j=1}^{3} A_j^{(m,\varepsilon)}(U)\,D_j + D^{(m)}(U)$$

where

$$A_j^{(m,\varepsilon)}(U) \equiv A_j^{(m)}(U) + \varepsilon \begin{pmatrix} n_j I_3 & 0 \\ 0 & 0 \end{pmatrix}.$$

Correspondingly we take

$$F^{(m,\varepsilon)}(U,\hat{U}) \equiv (\rho(p)(b^{(m)} - \dot{w}^{(m)}) + \varepsilon\,(n\,.\,\nabla)\,\hat{v},\ -\mathscr{A}^{(m)}\nabla\,.\,w^{(m)})^*,$$

where $\hat{U}=(\hat{v},\hat{p})^*$ is a suitable smooth vector-function which will be specified later. The boundary matrix is now

$$A_n^{(m,\varepsilon)}(U) = \begin{pmatrix} (\rho\, v\,.\,\mathscr{A}^{(m-1)} n + \varepsilon)\, I_3 & \mathscr{A}^{(m)} n \\ (\mathscr{A}^{(m)} n)^* & (\rho'/\rho)\, v\,.\,\mathscr{A}^{(m-1)} n \end{pmatrix}.$$

LEMMA 3.2. – *Let* $\bar{U}=(\bar{v},\bar{p})^*$ *such that* $\bar{v}\,.\,N^{(m-1)}=0$ *on* Γ. *For any* $\varepsilon>0$, *on* Γ $A_n^{(m,\varepsilon)}(\bar{U})$ *is non singular. Moreover, the linear space* $B^{(m)}(t,x)=\{U=(v,p)^*/v\,.\,N^{(m)}(t,x)=0 \text{ on } \Gamma\}$ *of* $\mathbf{R}^4$ *is maximally non negative with respect to* $A_n^{(m,\varepsilon)}(\bar{U})$.

Proof. – The first part of the Lemma holds since

$$\det A_n^{(m,\varepsilon)}(\bar{U}) = -\varepsilon^2 \left(\frac{\psi_0^{(m)}}{J^{(m)}}\right)^2.$$

If $U \in B^{(m)}$, on Γ we have

$$U^* A_n^{(m,\varepsilon)}(\bar{U})\, U = \varepsilon\, |v|^2 + 2\frac{\psi_0^{(m)}}{J^{(m)}} v \,.\, N^{(m)} p = \varepsilon\, |v|^2 \geqq 0,$$

i.e. $B^{(m)}$ is non negative. To show its maximality, we observe that $\dim B^{(m)} = 3$. Hence if $B^{(m)}$ is not maximal, the maximal subspace is necessarily $\mathbf{R}^4$. But for instance the vector $U' = \left(\frac{\psi_0^{(m)}}{J^{(m)}} N^{(m)}, -1-\varepsilon\right)^*$ satisfies

$$(U')^* A_n^{(m,\varepsilon)}(\bar{U})\, U' = \left|\frac{\psi_0^{(m)}}{J^{(m)}}\right|^2 (-2-\varepsilon) < 0$$

(recall that $|N^{(m)}(t,x)| = 1$) so that $\mathbf{R}^4$ cannot be maximally non negative with respect to $A_n^{(m,\varepsilon)}(\bar{U})$. This gives a contradiction, *i.e.* the thesis. ■

We proceed now with the construction of the first approximate solution of the iteration scheme.

LEMMA 3.3. — *There exists* $U^{(0)} = (v^{(0)}, p^{(0)})^* \in X_4(T_1)$ *such that*

$$\left.\begin{aligned} &v^{(0)} \,.\, N^{(0)} = 0 \quad \text{on } \Sigma_{T_1}, \\ &(\partial^k U^{(0)}/\partial t^k)(0) = (\partial^k/\partial t^k)\, U_0^{(0)} \quad \text{for } k = 0, 1, 2, 3, \end{aligned}\right\} \tag{3.10}$$

where $(\partial^k/\partial t^k)\, U_0^{(0)}$ *denotes the k-th time derivative at* $t=0$ *of a solution to problem* $L^{(0)}(U)\, U = F^{(0)}(U)$ *and initial data* $U_0^{(0)}$.

Proof. — Given $\varepsilon > 0$, we consider the following linear mixed problem:

$$\left.\begin{aligned} &L^{(0,\varepsilon)}(\hat{U}^{(0)})\, U = F^{(0,\varepsilon)}(\hat{U}^{(0)}, \hat{U}^{(0)}) \quad \text{in } Q_{T_1}, \\ &v^{(0)} \,.\, N^{(0)} = 0 \quad \text{on } \Sigma_{T_1}, \\ &U(0) = U_0^{(0)} \quad \text{in } \Omega, \end{aligned}\right\} \tag{3.11}$$

where $\hat{U}^{(0)} \in H^5(Q_{T_1})$ is such that

$$(\partial^k \hat{U}^{(0)}/\partial t^k)(0) = (\partial^k/\partial t^k)\, U_0^{(0)} \qquad \text{for } k = 0, 1, 2, 3, \tag{3.12}$$

and satisfies $\|\hat{U}^{(0)}\|_{5, Q_{T_1}} \leqq C \|U_0^{(0)}\|_5$. For such extension *see* [5]. Because of Lemma 3.1 (i) and (3.12), the compatibility conditions through order 3 hold. Because of the assumptions on p_0, $\rho(.)$, we can find a constant $C_0 > 1$ such that

$$\frac{4}{C_0} I_4 \leqq A_0(U_0) \leqq \frac{C_0}{4} I_4 \quad \text{in } \bar{\Omega}. \tag{3.13}$$

Take γ^* in (3.8) small enough in such a way that from (3.13) we obtain

$$\frac{2}{C_0} I_4 \leqq A_0(U_0^{(m)}) \leqq \frac{C_0}{2} I_4 \quad \text{in } \bar{\Omega}, \qquad m = 0, 1, \ldots \tag{3.14}$$

Since $\hat{U}^{(0)} \in H^5(Q_{T_1})$, $\mathscr{A}^{(0)} \in Y_5(T_1)$, $w^{(0)} \in H^6(Q_{T_1})$, $b^{(0)} \in H^4(Q_{T_1})$, we have

$$A_0(\hat{U}^{(0)}),\quad A_j^{(0,\varepsilon)}(\hat{U}^{(0)}),\quad D^{(0)}(\hat{U}^{(0)}) \in X_4(T_1),\quad F^{(0,\varepsilon)}(\hat{U}^{(0)}, \hat{U}^{(0)}) \in H^4(Q_{T_1}).$$

Then from Lemma 3.2 we can apply the theory for linear non characteristic hyperbolic mixed problems and show the existence of a unique solution $U = U^{(0)} \in X_4(T_1)$ (*cf.* Rauch and Massey [6] and Appendix A in Schochet [7]). $(3.10)_2$ follows from $(3.11)_1$, (3.12). The proof is complete. ■

Given $U^{(0)}$, we define the approximate solutions for $m = 1, 2, \ldots$ by means of the iteration scheme:

$$\left.\begin{aligned} L^{(m,\varepsilon_m)}(U^{(m-1)})\,U^{(m)} &= F^{(m,\varepsilon_m)}(U^{(m-1)}, \hat{U}^{(m)}) \quad \text{in } Q_T,\\ v^{(m)}.\,N^{(m)} &= 0 \quad \text{on } \Sigma_T,\\ U^{(m)}(0) &= U_0^{(m)} \quad \text{in } \Omega. \end{aligned}\right\} \qquad (3.15)_m$$

Here $U^{(m)} = (v^{(m)}, p^{(m)})^*$, $\hat{U}^{(m)} \in H^5(Q_{T_1})$ is such that

$$(\partial^k \hat{U}^{(m)}/\partial t^k)(0) = (\partial^k/\partial t^k)\,U_0^{(m)} \qquad \text{for} \quad k = 0, 1, 2, 3, \qquad (3.16)$$

and satisfies $\|\hat{U}^{(m)}\|_{5, Q_{T_1}} \leq C \|U_0^{(m)}\|_5$; $\varepsilon_m \downarrow 0$ is chosen in such a way that

$$\varepsilon_m \|(n.\nabla)\,\hat{U}^{(m)}\|_{3, Q_{T_1}} \leq \varepsilon_m C \|U_0^{(m)}\|_4 \leq 2^{-m} C_0' \|U_0\|_3. \qquad (3.17)$$

4. UNIFORM ESTIMATES AND PROOF OF THEOREM B

We define an invariant set for the iteration scheme. For, set $E_1 \equiv 3 |||U_0|||_2$, $E_2 \equiv 2\,C^*(E_1)\,(|||U_0|||_3 + R)$, where $C^* > 1$ is a nondecreasing function of E_1 which will be specified in (4.25),

$$R \equiv \sup_m \{ |||b^{(m)}|||_{2,T_1} + |||\dot{w}^{(m)}|||_{2,T_1} + |||\mathscr{A}^{(m)}\nabla.\,w^{(m)}|||_{2,T_1} + \varepsilon_m |||(n.\nabla)\,\hat{v}^{(m)}|||_{2,T_1} < +\infty$$

(interpolation inequalities yield the boundedness of R). Define

$$S(T) \equiv \{ U = (v, p)^* / U \in X_4(T),\ |||U|||_{2,T} \leq E_1,\ |||U|||_{3,T} \leq E_2 \}.$$

LEMMA 4.1. – *There exists* $T > 0$ *such that, if* $(3.15)_k$ *has a unique solution* $U^{(k)} \in S(T)$ *for* $k = 0, \ldots, m-1$, *then* $(3.15)_m$ *has a unique solution* $U^{(m)}$ *again in* $S(T)$.

Proof. – First of all, since $U^{(0)} \in X_4(T_1)$ we have

$$|||U^{(0)} - U_0|||_{2,T} \leq |||U^{(0)} - U_0|||_{3,T} \leq |||U^{(0)} - U_0^{(0)}|||_{3,T} + |||U_0^{(0)} - U_0|||_3$$
$$\leq |||\dot{U}^{(0)}|||_{3,T_1} T + \gamma^* \leq |||\dot{U}^{(0)}|||_{3,T_1} T + \frac{1}{2} |||U_0|||_2.$$

If $T \leqq T_1$ is such that $|||\dot{U}^{(0)}|||_{3, T_1} T \leqq \frac{1}{2} |||U_0|||_2$ we have

$$|||U^{(0)} - U_0|||_{2, T} \leqq |||U_0|||_2, \qquad |||U^{(0)} - U_0|||_{3, T} \leqq |||U_0|||_3$$

from which $U^{(0)} \in S(T)$.

For $U \in S(T)$, let us assume that $U(0)$ is such that

$$\frac{2}{C_0} I_4 \leqq A_0(U(0)) \leqq \frac{C_0}{2} I_4 \quad \text{in } \bar{\Omega}.$$

Since we have

$$\| U(t) - U(0) \|_{C^0(\bar{\Omega})} \leqq \| \dot{U} \|_{C^0(\bar{Q}_T)} t \leqq C \| \dot{U} \|_{2, T} t \leqq C E_2 t,$$

there exists $T_2 \in (0, T_1]$ small enough such that

$$\frac{1}{C_0} I_4 \leqq A_0(U(t)) \leqq C_0 I_4 \quad \text{in } \bar{Q}_{T_2}. \tag{4.1}$$

Take now $U^{(m-1)} \in S(T)$ with $T \leqq T_2$. Because of (3.14) applied to $U_0^{(m-1)}$, $U^{(m-1)}$ verifies (4.1). Since $U^{(m-1)} \in X_4(T)$, $\mathscr{A}^{(m)} \in Y_5(T)$, $w^{(m)} \in H^6(Q_T)$, $b^{(m)} \in H^4(Q_T)$, we have $A_0(U^{(m-1)})$, $A_j^{(m, \varepsilon_m)}(U^{(m-1)})$, $D^{(m)}(U^{(m-1)}) \in X_4(T)$, $F^{(m, \varepsilon_m)}(U^{(m-1)}, \hat{U}^{(m)}) \in H^4(Q_T)$. Lemma 3.2 and the results for linear non characteristic hyperbolic mixed problems yields the existence of a unique solution $U^{(m)} \in X_4(T)$ of $(3.15)_m$. Let us show now the key *a priori* estimates.

First we observe that, since $w^{(m)} \to w$ in $H^5(Q_{T_0})$, then $w^{(m)}$, $\eta^{(m)}$, $\mathscr{A}^{(m)}$, $N^{(m)}$ are bounded respectively in $H^5(Q_{T_0})$, $Y_5(T_0)$, $Y_4(T_0)$, $Y_4(T_0, \Omega_\delta)$. Since $U^{(m-1)} \in S(T)$, we have that $A_0(U^{(m-1)})$, $A_j^{(m, \varepsilon_m)}(U^{(m-1)})$, $D^{(m)}(U^{(m-1)})$ are bounded in $X_3(T)$, $F^{(m, \varepsilon_m)}(U^{(m-1)}, \hat{U}^{(m)})$ is bounded in $H^3(Q_T)$. An explicit boundedness in terms of $|||U^{(m-1)}|||_{2, T}$, $|||U^{(m-1)}|||_{3, T}$ will be necessary and will be pointed out later. Given $\delta > 0$ such that (2.2) holds [we shall definitely choose it in (4.19)], consider a partition of unity $\{\varphi_j\}_{j=0}^h$ in Ω, namely $\varphi_j \in C_0^\infty(\mathbf{R}^3)$ with $\sum_{j=0}^h \varphi_j(x) = 1$ in Ω; the compact support Ω_j of φ_j, $j = 1, \ldots, h$, is such that $\Omega_j \cap \Gamma \neq \varnothing$, $\Omega_j \cap \Omega \subseteqq \Omega_\delta$; the support Ω_0 of φ_0 does not intersect Γ. Assume that $\Omega_j (j \neq 0)$ is contained in a sufficiently small neighborhood of a point $(x_1^j, x_2^j, x_3^j) \in \Omega_j \cap \Gamma$ such that in it $\Omega_j \cap \Gamma$ can be represented by the equation

$$x_3 = \psi_j(x_1, x_2).$$

In $\Omega_j (j \neq 0)$ we introduce the standard change of independent variables $z = \Psi_j(x)$

$$z_1 = x_1 - x_1^j, \qquad z_2 = x_2 - x_2^j, \qquad z_3 = x_3 - \psi_j(x_1, x_2)$$

and assume that

$$\Psi_j(\Omega_j) \subseteqq \{ z \in \mathbf{R}^3 / |z| \leqq 1 \}, \qquad \Psi_j(\Omega_j \cap \Omega) = \{ z \in \mathbf{R}^3 / |z| \leqq 1,\ z_3 > 0 \}.$$

We change the dependent variables $\varphi_j U^{(m)}$ by an orthogonal matrix-valued function $H_j^{(m)}(t, x)$ which transforms the boundary space $B^{(m)}(t, x)$ (*see* Lemma 3.2) into a new boundary space B constant on $\{ z \in \mathbf{R}^3 / |z| \leqq 1,\ z_3 = 0 \}$. We construct $H_j^{(m)}(t,x)$ as in Lemma 3.1 of Ikawa [4]. Since $B^{(m)}(t,x)$ is the linear space of vectors U in $\mathbf{R}^4$ orthogonal to $(N^{(m)}(t,x), 0)^* \in Y_5(T_1, \Omega_\delta)$, we can find an orthogonal base $e_i^{(m)}(t,x) = \{ e_{i1}^{(m)}(t,x), \ldots, e_{i4}^{(m)}(t,x) \} \in Y_5(T_1)$ $(i = 1, 2, 3)$ of $B^{(m)}(t, x)$ in Ω_j. This is possible when Ω_j is sufficiently small. This base of $B^{(m)}(t,x)$ and the vector $e_4^{(m)}(t,x) = (N^{(m)}(t,x)^*, 0)$ form an orthonormal base of $\mathbf{R}^4$. Define $H_j^{(m)}(t,x)$ by

$$H_j^{(m)}(t, x) = [e_{ik}^{(m)}(t, x)]_{i,\, k = 1, \ldots, 4}.$$

$H_j^{(m)}(t, x)$ is a unitary matrix with coefficients in $Y_5(T_1, \Omega_j)$, bounded uniformly in $Y_4(T_1, \Omega_j)$ with respect to m. If we set $V = H_j^{(m)}(t, x)\, U$, then $U \in B^{(m)}(t, x)$ is equivalent to $V_4 = 0$. In every Ω_j we consider as a new dependent variable the function $\tilde{U}_j^{(m)} = (H_j^{(m)} \varphi_j U^{(m)})^{\tilde{}}(t, z)$, where for a function $w(x)$ defined in $\Omega_j \cap \Omega$ we denote by $\tilde{w}(z)$ the function $\tilde{w}(z) = \tilde{w}(\Psi_j(x)) = w(x)$. The new unknown $\tilde{U}_j^{(m)}$ satisfies the system (for the sake of simplicity we write $\tilde{U}$ instead of $\tilde{U}_j^{(m)}$, H instead of $H_j^{(m)}$)

$$A'_0 \frac{\partial \tilde{U}}{\partial t} + \sum_{j=1}^{3} A'_j \frac{\partial \tilde{U}}{\partial z_j} + D' \tilde{U} = F' \quad \text{in } [0, T] \times \mathbf{R}_+^3 \tag{4.2}$$

where

$$A'_0 = [H A_0 (U^{(m-1)}) H^*]^{\tilde{}},$$

$$A'_j = \left[H A_k^{(m, \varepsilon_m)} (U^{(m-1)}) \frac{\partial z_j}{\partial x_k} H^* \right]^{\tilde{}},$$

$$D' = \left[H D H^* - H A_0 (U^{(m-1)}) H^* \frac{\partial H}{\partial t} H^* - H A_k^{(m, \varepsilon_m)} (U^{(m-1)}) H^* \frac{\partial H}{\partial x_k} H^* \right]^{\tilde{}},$$

$$F' = \left[H A_k^{(m, \varepsilon_m)} (U^{(m-1)}) U^{(m)} \frac{\partial \varphi_j}{\partial x_k} + H \varphi_j F^{(m, \varepsilon_m)} \right]^{\tilde{}}.$$

The new boundary space is now $B = \{ V \in \mathbf{R}^4 / V_4 = 0 \}$ so that the new boundary condition is

$$\tilde{U}(t, z) \in B \quad \text{if} \quad z_3 = 0. \tag{4.3}$$

Applying time and tangential derivative operators $D_{\tan}^{\alpha} = \partial_t^{\alpha_0} \partial_{z_1}^{\alpha_1} \partial_{z_2}^{\alpha_2}$ with $|\alpha| = \alpha_0 + \alpha_1 + \alpha_2 \leqq 3$, $\alpha_0 \leqq 2$, to the equations (4.2), multiplying by $D_{\tan}^{\alpha} \tilde{U}$

and integrating in space we obtain

$$\frac{d}{dt}(A'_0\, D^\alpha_{\tan}\tilde{U},\, D^\alpha_{\tan}\tilde{U})$$
$$=\left(\frac{\partial A'_0}{\partial t} D^\alpha_{\tan}\tilde{U},\, D^\alpha_{\tan}\tilde{U}\right)-2\left(A'_j\frac{\partial}{\partial z_j} D^\alpha_{\tan}\tilde{U},\, D^\alpha_{\tan}\tilde{U}\right)$$
$$-2\sum_{|\beta|+|\gamma|=|\alpha|,\,|\beta|\geqq 1}\left(A'_0\, D^\beta_{\tan}[(A'_0)^{-1}A'_j]\frac{\partial}{\partial z_j} D^\gamma_{\tan}\tilde{U},\, D^\alpha_{\tan}\tilde{U}\right)$$
$$-2\,(A'_0\, D^\alpha_{\tan}(A'_0)^{-1}(D'\,\tilde{U}-F'),\, D^\alpha_{\tan}\tilde{U}). \quad (4.4)$$

We integrate now by parts the second term in the right-hand side of (4.4). We obtain

$$2\left(A'_j\frac{\partial}{\partial z_j} D^\alpha_{\tan}\tilde{U},\, D^\alpha_{\tan}\tilde{U}\right)$$
$$=\int_{z_3=0} A'_3\, D^\alpha_{\tan}\tilde{U}\,.\,D^\alpha_{\tan}\tilde{U}-\left(\frac{\partial A'_j}{\partial z_j} D^\alpha_{\tan}\tilde{U},\, D^\alpha_{\tan}\tilde{U}\right) \quad (4.5)$$

since A'_j are symmetric. Since

$$\nabla z_3=\left(-\frac{\partial\psi_j}{\partial x_1},\, -\frac{\partial\psi_j}{\partial x_2},\, 1\right)=(1+|\nabla\psi_j|^2)^{1/2}\, n,$$

we have

$$A'_3=[(1+|\nabla\psi_j|^2)^{1/2}\, HA^{(m,\,\varepsilon_m)}_n\, H^*]^\sim.$$

Observing now that $D^\alpha_{\tan}\tilde{U}\in B$, since the boundary space $B^{(m)}(t,x)$ is non negative with respect to $A^{(m,\,\varepsilon_m)}_n$ (Lemma 3.2) we get

$$A'_3\, D^\alpha_{\tan}\tilde{U}\,.\,D^\alpha_{\tan}\tilde{U}\geqq 0. \quad (4.6)$$

Hence from (4.4)-(4.6) we obtain

$$\frac{d}{dt}(A'_0\, D^\alpha_{\tan}\tilde{U},\, D^\alpha_{\tan}\tilde{U})\leqq\left(\frac{\partial A'_0}{\partial t} D^\alpha_{\tan}\tilde{U},\, D^\alpha_{\tan}\tilde{U}\right)+\left(\frac{\partial A'_j}{\partial z_j} D^\alpha_{\tan}\tilde{U},\, D^\alpha_{\tan}\tilde{U}\right)$$
$$-2\sum_{|\beta|+|\gamma|=|\alpha|,\,|\beta|\geqq 1}\left(A'_0\, D^\beta_{\tan}[(A'_0)^{-1}A'_j]\frac{\partial}{\partial z_j} D^\gamma_{\tan}\tilde{U},\, D^\alpha_{\tan}\tilde{U}\right)$$
$$-2\,(A'_0\, D^\alpha_{\tan}(A'_0)^{-1}(D'\,\tilde{U}-F'),\, D^\alpha_{\tan}\tilde{U}).$$

Summing over all α, $|\alpha|\leqq 3$, we obtain

$$\frac{d}{dt}\sum_{|\alpha|\leqq 3}(A'_0\, D^\alpha_{\tan}\tilde{U},\, D^\alpha_{\tan}\tilde{U})\leqq[a_1(t)\,\|U\|_3+a'_1(t)\, R^{(m)}_3]\,|||\tilde{U}|||_{3,\tan} \quad (4.7)$$

where $|||\tilde{U}|||_{3,\tan} \equiv \sum_{|\alpha| \leqq 3} \|\tilde{U}\|_{|\alpha|}$, $a_1(t) \in L^2(0,T)$, $a'_1(t) \in L^\infty(0,T)$, a_1 and a'_1 are non decreasing functions of E_2,

$$R_k^{(m)}(t) \equiv |||b^{(m)}(t)|||_k + |||\dot{w}^{(m)}(t)|||_k + |||\mathscr{A}^{(m)}\nabla . w^{(m)}(t)|||_k + \varepsilon_m |||(n.\nabla)\hat{v}^{(m)}(t)|||_k.$$

Integrating (4.7) in time between 0 and t and using (4.1) we obtain

$$|||\tilde{U}(t)|||_{3,\tan} \leqq C_0 |||\tilde{U}(0)|||_{3,\tan} + \int_0^t [a_2(s)|||U^{(m)}(s)|||_3 + a'_2(s) R_3^{(m)}(s)]\, ds \quad (4.8)$$

where $a_2(t) \in L^2(0,T)$, $a'_2(t) \in L^\infty(0,T)$ are non decreasing functions of E_2. In the same way we obtain the interior estimate

$$|||\varphi_0 U^{(m)}(t)|||_3 \leqq C_0 |||\varphi_0 U^{(m)}(0)|||_3 + \int_0^t [a_3(s)|||U^{(m)}(s)|||_3 + a'_3(s) R_3^{(m)}(s)]\, ds \quad (4.9)$$

where $a_3(t) \in L^2(0,T)$, $a'_3(t) \in L^\infty(0,T)$ are non decreasing functions of E_2.

The next step is to find *a priori* estimates for the normal derivatives. If the boundary matrix were non singular, this would be obtained by inverting it in a neighborhood of the boundary and representing the normal derivatives in terms of tangential derivatives. In our case the boundary is characteristic so that this is not possible. In any way, we observe that the rank of the original boundary matrix A_n is 2. This suggests that we could express two of the four first order normal derivatives of U by means of the other two first-order normal derivatives and of tangential derivatives. On the other hand, we know that the vorticity satisfies a first order symmetric system. Combining in a suitable way the information coming from these two properties we obtain the estimates for the normal derivatives. More precisely we proceed as follows. In $(3.15)_m$ the equations for the velocity have the form $(\rho^{(m-1)} \equiv \rho(p^{(m-1)}))$

$$\rho^{(m-1)}[\dot{v}^{(m)} + (v^{(m-1)} . \mathscr{A}^{(m-1)}\nabla) v^{(m)} + (v^{(m)} . \mathscr{A}^{(m)}\nabla) w^{(m)} + \dot{w}^{(m)} - b^{(m)}] + \mathscr{A}^{(m)}\nabla p^{(m)} + \varepsilon_m (n.\nabla) v^{(m)} = \varepsilon_m (n.\nabla)\hat{v}^{(m)}.$$

Dividing by $\rho^{(m-1)}$ and introducing the Eulerian coordinates $(t,y) \equiv (t, \eta^{(m)}(t,x))$ gives

$$\dot{\bar{v}}^{(m)} + (\bar{w}^{(m)} . \nabla)\bar{v}^{(m)} + (\bar{v}^{(m-1)} . \bar{\mathscr{A}}^{(m-1)}(\bar{\mathscr{A}}^{(m)})^{-1}\nabla)\bar{v}^{(m)} + (\bar{v}^{(m)} . \nabla)\bar{w}^{(m)} + \dot{\bar{w}}^{(m)} - \bar{b}^{(m)} + (1/(\bar{\rho}^{(m-1)})\nabla\bar{p}^{(m)} + (\varepsilon_m/(\bar{\rho}^{(m-1)})(\bar{n} . (\bar{\mathscr{A}}^{(m)})^{-1}\nabla)\bar{v}^{(m)} = (\varepsilon_m/(\bar{\rho}^{(m-1)})(\bar{n} . (\bar{\mathscr{A}}^{(m)})^{-1}\nabla)\bar{\hat{v}}^{(m)} \quad (4.10)$$

where $\bar{v}(t,y) = \bar{v}(t, \eta^{(m)}(t,x)) \equiv v(t,x)$, etc., since $\partial/\partial t \to \partial/\partial t + (\bar{w}^{(m)} . \nabla_y)$, $\mathscr{A}^{(m)}\nabla_x \to \nabla_y$. Taking the curl of (4.10) gives $(\bar{\zeta}^{(m)}(t,y) \equiv \mathrm{rot}\,\bar{v}^{(m)}(t,y))$

$$\dot{\bar{\zeta}}^{(m)} + [\bar{w}^{(m)} . \nabla + \bar{v}^{(m-1)} . \bar{\mathscr{A}}^{(m-1)}(\bar{\mathscr{A}}^{(m)})^{-1}\nabla + (\varepsilon_m/(\bar{\rho}^{(m-1)})\bar{n} . (\bar{\mathscr{A}}^{(m)})^{-1}\nabla]\bar{\zeta}^{(m)} = \mathrm{rot}\,[\bar{b}^{(m)} - \dot{\bar{w}}^{(m)} + (\varepsilon_m/(\bar{\rho}^{(m-1)})(\bar{n} . (\bar{\mathscr{A}}^{(m)})^{-1}\nabla)\bar{\hat{v}}^{(m)}] + \text{l. o. t.}$$

where l. o. t. means terms with zero and first order derivatives of $(\bar{v}^{(m)}, \bar{p}^{(m)})$ [observe that rot $(1/(\bar{\rho}^{(m-1)})\ \nabla \bar{p}^{(m)} = \nabla (1/(\bar{\rho}^{(m-1)}) \wedge \nabla \bar{p}^{(m)}$ because of rot $\nabla = 0$]. Changing again the independent variables from (t, y) to (t, x) gives [we set $\zeta^{(m)}(t, x) \equiv \bar{\zeta}^{(m)}(t, \eta^{(m)}(t, x)) = \bar{\zeta}^{(m)}(t, y)$, $\zeta^{(m)} = \mathscr{A}^{(m)} \nabla \wedge v^{(m)}$]

$$\begin{aligned}\dot{\zeta}^{(m)} + [v^{(m-1)} . \mathscr{A}^{(m-1)} \nabla + (\varepsilon_m/\rho^{(m-1)})\, n . \nabla]\, \zeta^{(m)} \\ = \mathscr{A}^{(m)} \nabla \wedge [b^{(m)} - \dot{w}^{(m)} + (\varepsilon_m/(\rho^{(m-1)})\,(n . \nabla)\, \hat{v}^{(m)}] + \text{l. o. t.} \equiv \mathrm{K} \quad (4.11)\end{aligned}$$

where now l. o. t. means terms with zero and first order derivatives of $(v^{(m)}, p^{(m)})$.

Applying a derivative operator D^α (in space-time) with $|\alpha| \leqq 2$ to (4.11), multiplying by $\mathrm{D}^\alpha \zeta^{(m)}$ and integrating over Ω gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt} \| \mathrm{D}^\alpha \zeta^{(m)} \|^2 \\ &\quad + ((v^{(m-1)} . \mathscr{A}^{(m-1)} \nabla + (\varepsilon_m/\rho^{(m-1)})\, n . \nabla)\, \mathrm{D}^\alpha \zeta^{(m)}, \mathrm{D}^\alpha \zeta^{(m)}) \\ &= -(\mathrm{D}^\alpha (v^{(m-1)} . \mathscr{A}^{(m-1)} \nabla)\, \zeta^{(m)} - (v^{(m-1)} . \mathscr{A}^{(m-1)} \nabla)\, \mathrm{D}^\alpha \zeta^{(m)}, \mathrm{D}^\alpha \zeta^{(m)}) \\ &\quad - (\mathrm{D}^\alpha (\varepsilon_m/\rho^{(m-1)})\,(n . \nabla)\, \zeta^{(m)} \\ &\quad - (\varepsilon_m/\rho^{(m-1)})\,(n . \nabla)\, \mathrm{D}^\alpha \zeta^{(m)}, \mathrm{D}^\alpha \zeta^{(m)}) + (\mathrm{D}^\alpha \mathrm{K}, \mathrm{D}^\alpha \zeta^{(m)}). \quad (4.12)\end{aligned}$$

Integrating by parts in the second term in the left-hand side of (4.12) we obtain

$$\begin{aligned}&\frac{1}{2}\int_\Gamma [v^{(m-1)} . \mathscr{A}^{(m-1)} n + (\varepsilon_m/\rho^{(m-1)})\, |n|^2]\, |\mathrm{D}^\alpha \zeta^{(m)}|^2 \\ &\quad - \frac{1}{2}(\mathrm{D}_i (v_j^{(m-1)} a_{ij}^{(m-1)} + (\varepsilon_m n_i/\rho^{(m-1)}))\, \mathrm{D}^\alpha \zeta^{(m)}, \mathrm{D}^\alpha \zeta^{(m)}) \\ &\quad \geqq -\frac{1}{2}(\mathrm{D}_i (v_j^{(m-1)} a_{ij}^{(m-1)} + (\varepsilon_m n_i/\rho^{(m-1)}))\, \mathrm{D}^\alpha \zeta^{(m)}, \mathrm{D}^\alpha \zeta^{(m)}) \quad (4.13)\end{aligned}$$

since $v^{(m-1)} . \mathscr{A}^{(m-1)} n = \dfrac{\psi_0^{(m-1)}}{\mathrm{J}^{(m-1)}} v^{(m-1)} . \mathrm{N}^{(m-1)} = 0$ on Γ and $\varepsilon_m/\rho^{(m-1)} > 0$. Hence from (4.12), (4.13), adding over all α with $|\alpha| \leqq 2$ and integrating in time between 0 and t, we can obtain

$$\begin{aligned}&|||\zeta^{(m)}(t)|||_2 \leqq \mathrm{C}_0 |||\zeta^{(m)}(0)|||_2 \\ &\qquad + \int_0^t [a_4(s)\, |||\mathrm{U}^{(m)}(s)|||_3 + a'_4(s)\, \mathrm{R}_3^{(m)}(s)]\, ds \quad (4.14)\end{aligned}$$

where $a_4(t) \in \mathrm{L}^2(0, \mathrm{T})$, $a'_4(t) \in \mathrm{L}^\infty(0, \mathrm{T})$ are non decreasing functions of E_2. (4.14) gives an estimate for $\zeta^{(m)} = \mathscr{A}^{(m)} \nabla \wedge v^{(m)}$ which in each Ω_j can be written, introducing again the independent variables $z = \Psi_j(x)$, as

$(\mathscr{A}^{(m)}[\mathrm{D}z]^*)^\sim \nabla_z \wedge (\mathrm{H}_{ij}\mathrm{U}_i^{(m)})^\sim$ for $j=1, 2, 3$. Considering the first two components we have more explicitly ($\tilde{\mathrm{U}}$ stands for $\tilde{\mathrm{U}}^{(m)}$, H for $\mathrm{H}^{(m)}$)

$$\left.\begin{aligned}
\tilde{\zeta}_1^{(m)} &= \left(\frac{\partial z_j}{\partial x_k}\right)^\sim \\
&\quad \times \left\{ (a_{k2}^{(m)}\mathrm{H}_{i3} - a_{k3}^{(m)}\mathrm{H}_{i2})^\sim \frac{\partial \tilde{\mathrm{U}}_i}{\partial z_j} + \left(a_{k2}^{(m)} \frac{\partial \mathrm{H}_{i3}}{\partial z_j} - a_{k3}^{(m)} \frac{\partial \mathrm{H}_{i2}}{\partial z_j} \right)^\sim \tilde{\mathrm{U}}_i \right\}, \\
\tilde{\zeta}_2^{(m)} &= \left(\frac{\partial z_j}{\partial x_k}\right)^\sim \\
&\quad \times \left\{ (a_{k3}^{(m)}\mathrm{H}_{i1} - a_{k1}^{(m)}\mathrm{H}_{i3})^\sim \frac{\partial \tilde{\mathrm{U}}_i}{\partial z_j} + \left(a_{k3}^{(m)} \frac{\partial \mathrm{H}_{i1}}{\partial z_j} - a_{k1}^{(m)} \frac{\partial \mathrm{H}_{i3}}{\partial z_j} \right)^\sim \tilde{\mathrm{U}}_i \right\}.
\end{aligned}\right\} \quad (4.15)$$

Isolating the normal derivatives $\dfrac{\partial \tilde{\mathrm{U}}}{\partial z_3}$ gives

$$\left.\begin{aligned}
\left(\frac{\partial z_3}{\partial x_k}\right)^\sim (a_{k2}^{(m)}\mathrm{H}_{i3} - a_{k3}^{(m)}\mathrm{H}_{i2})^\sim \frac{\partial \tilde{\mathrm{U}}_i}{\partial z_3} &= \tilde{\zeta}_1^{(m)} + \text{“tan”} \\
\left(\frac{\partial z_3}{\partial x_k}\right)^\sim (a_{k3}^{(m)}\mathrm{H}_{i1} - a_{k1}^{(m)}\mathrm{H}_{i3})^\sim \frac{\partial \tilde{\mathrm{U}}_i}{\partial z_3} &= \tilde{\zeta}_2^{(m)} + \text{“tan”}
\end{aligned}\right\} \quad (4.16)$$

where “tan” contains the tangential derivatives of (4.15) and lower order terms. Consider (4.2). We multiply it by $\tilde{\mathrm{H}}^*$ and isolate the normal derivatives obtaining

$$\left(\mathrm{A}_k^{(m,\,\varepsilon_m)} \frac{\partial z_3}{\partial x_k} \mathrm{H}^*\right)^\sim \frac{\partial \tilde{\mathrm{U}}}{\partial z_3} = -\tilde{\mathrm{H}}^* \left(\mathrm{A}'_0 \frac{\partial \tilde{\mathrm{U}}}{\partial t} + \sum_{j=1}^{2} \mathrm{A}'_j \frac{\partial \tilde{\mathrm{U}}}{\partial z_j} + \mathrm{D}'\tilde{\mathrm{U}} - \mathrm{F}' \right). \quad (4.17)$$

Consider now the linear system with unknown $\tilde{\mathrm{H}}^* \dfrac{\partial \tilde{\mathrm{U}}}{\partial z_3}$ formed by (4.16) and the third and fourth row of (4.17). We write it in the form

$$\tilde{\Lambda}\tilde{\mathrm{H}}^* \frac{\partial \tilde{\mathrm{U}}}{\partial z_3} = (\tilde{\zeta}_1^{(m)}, \tilde{\zeta}_2^{(m)}, 0, 0)^* + \text{“tan”}, \quad (4.18)$$

where the matrix Λ of the coefficients is given by

$$\Lambda \equiv \frac{\partial z_3}{\partial x_k} \begin{pmatrix}
0 & -a_{k3}^{(m)} & a_{k2}^{(m)} & 0 \\
a_{k3}^{(m)} & 0 & -a_{k1}^{(m)} & 0 \\
0 & 0 & \rho^{(m-1)} v_i^{(m-1)} a_{ki}^{(m-1)} + \varepsilon_m n_k & a_{k3}^{(m)} \\
a_{k1}^{(m)} & a_{k2}^{(m)} & a_{k3}^{(m)} & (\rho'/\rho)^{(m-1)} v_i^{(m-1)} a_{ki}^{(m-1)}
\end{pmatrix}.$$

We compute Λ for $t=0$, $x \in \Gamma$. Since

$$\nabla z_3 = \left(-\frac{\partial \psi_j}{\partial x_1}, -\frac{\partial \psi_j}{\partial x_2}, 1 \right) = (1 + |\nabla \psi_j|^2)^{1/2}\, n,$$
$$a_{ki}^{(m)}(0) = \delta_{ki},$$

we obtain

$$\Lambda_0 \equiv \Lambda_{|\, t=0,\, x\in\Gamma} = \begin{pmatrix} 0 & -1 & -\dfrac{\partial \psi_j}{\partial x_2} & 0 \\ 1 & 0 & \dfrac{\partial \psi_j}{\partial x_1} & 0 \\ 0 & 0 & \varepsilon_m (1+|\nabla \psi_j|^2)^{1/2} & 1 \\ -\dfrac{\partial \psi_j}{\partial x_1} & -\dfrac{\partial \psi_j}{\partial x_2} & 1 & 0 \end{pmatrix}$$

whose determinant is $\det \Lambda_0 = -[1+|\nabla \psi_j|^2]$. Consider $x\in\Omega_j \cap \Omega_\delta$ and take $\bar{x}\in\Gamma$ such that $\operatorname{dist}(x,\bar{x})\leqq\delta$ and such that the segment between x and $\bar{x}$ is all contained in Ω_δ. Applying the mean value theorem we can obtain

$$\begin{aligned} &|\det \Lambda(t,x)| \\ &\quad \geqq |\det \Lambda(0,\bar{x})| - \left[\left\| \frac{\partial}{\partial t} \det \Lambda \right\|_{C^0(\bar{Q}_T)} + \| \nabla \det \Lambda \|_{C^0(\bar{Q}_T)} \right] [|t| + |x-\bar{x}|]. \end{aligned}$$

Since the coefficients of Λ are bounded in

$$C^0([0,T]; H^3(\Omega)) \cap C^1([0,T]; H^2(\Omega))$$

uniformly in m we can find a constant C_1 such that

$$\left\| \frac{\partial}{\partial t} \det \Lambda \right\|_{C^0(\bar{Q}_T)} + \| \nabla \det \Lambda \|_{C^0(\bar{Q}_T)} \leqq C_1$$

uniformly in m. Then there exist $T_3\in(0,T_2]$ and $\delta>0$ such that, if $T\leqq T_3$,

$$|\det \Lambda(t,x)| \geqq 1 - C_1(T_2+\delta) \geqq \frac{1}{2} \quad \text{in } [0,T]\times\bar{\Omega}_\delta. \tag{4.19}$$

Hence we can invert Λ in (4.18) and express $\dfrac{\partial \tilde{U}}{\partial z_3}$ in terms of the right-hand side in $[0,T]\times\bar{\Omega}_\delta$. Hence we obtain

$$\frac{\partial \tilde{U}}{\partial z_3} = \tilde{H}\tilde{\Lambda}^{-1}[(\tilde{\zeta}_1^{(m)}, \tilde{\zeta}_2^{(m)}, 0, 0)^* + \text{"tan"}] \tag{4.20}$$

for each $t\in[0,T]$ and $z=\Psi_j(x)$, $x\in\Omega_j\cap\Omega_\delta$. We use (4.20) to estimate $\dfrac{\partial \tilde{U}}{\partial z_3}$ by means of the right-hand side. First we take the L^2-norm, secondly the L^2-norm of tangential derivatives of first order, then the L^2-norm of one normal derivative and so on. We obtain for each $t\in[0,T]$

$$\left|\left|\left| \frac{\partial \tilde{U}}{\partial z_3} \right|\right|\right|_2 \leqq C_2(E_1)[|||\zeta^{(m)}|||_2 + |||\tilde{U}^{(m)}|||_{3,\tan} + R_2^{(m)}], \tag{4.21}$$

where C_2 is a non decreasing function of E_1, since no derivative of $U^{(m-1)}$ is contained in Λ. At this step an essential tool is the estimate (*see* (4.6) at p. 56 of [7])

$$\left|\left|\left|\frac{\partial^{k_1+k_2+k_3}\tilde{U}}{\partial z_1^{k_1}\partial z_2^{k_2}\partial z_3^{k_3}}\right|\right|\right| \leq \varepsilon_2 \left\|\frac{\partial^{k_1+k_2+k_3}\tilde{U}}{\partial z_1^{k_1}\partial z_2^{k_2}\partial z_3^{k_3}}\right\| + C(\varepsilon_2)[|||\tilde{U}|||_{k_1+k_2+k_3,\,\tan} + |||\tilde{U}|||_{k_1+k_2+k_3-1}],$$

where $k_1+k_2>0$. Taking account of (4.21), we obtain for each $t\in[0,T]$ the estimate

$$\begin{aligned}|||U^{(m)}|||_3 &\leq |||\varphi_0 U^{(m)}|||_3 + \sum_{j=1}^{h}\left(|||\tilde{U}_j^{(m)}|||_{3,\,\tan} + \left|\left|\left|\frac{\partial \tilde{U}_j^{(m)}}{\partial z_3}\right|\right|\right|_2\right)\\ &\leq |||\varphi_0 U^{(m)}|||_3 + C_3(E_1)\left(|||\zeta^{(m)}|||_2 + \sum_{j=1}^{h}|||\tilde{U}_j^{(m)}|||_{3,\,\tan} + R_2^{(m)}\right), \end{aligned} \quad (4.22)$$

where $C_3>1$ is a non decreasing function of E_1. From (4.8) for $j=1,\ldots,h$, (4.9) and (4.14) we have

$$\begin{aligned}|||U^{(m)}(t)|||_3 \leq c_0 |||\varphi_0 U^{(m)}(0)|||_3 &+ C_3(E_1)(|||\zeta^{(m)}(0)|||_2 \\ &+ c_0\sum_{j=1}^{h}|||\tilde{U}_j^{(m)}(0)|||_{3,\,\tan} + R) \\ &+ \int_0^t [a_5(s)|||U^{(m)}(s)|||_3 + a_5'(s)R_3^{(m)}(s)]\,ds, \end{aligned}\quad (4.23)$$

where

$$R \equiv \sup_m \{|||b^{(m)}|||_{2,T_1} + |||\dot{w}^{(m)}|||_{2,T_1} + |||\mathscr{A}^{(m)}\nabla . w^{(m)}|||_{2,T_1} + \varepsilon_m|||(n.\nabla)\hat{v}^{(m)}|||_{2,T_1}\}$$

and $a_5(t)\in L^2(0,T)$, $a_5'(t)\in L^\infty(0,T)$ are non decreasing functions of E_2. The Gronwall's lemma yields

$$|||U^{(m)}|||_{3,T} \leq e^{C_4 T^{1/2}}[C_5^{(m)} + C_6 T^{1/2}], \quad (4.24)$$

where

$$C_4 \equiv \|a_5\|_{L^2(0,T)},$$

$$C_5^{(m)} \equiv c_0|||\varphi_0 U^{(m)}(0)|||_3 + C_3(E_1)\left(|||\zeta^{(m)}(0)|||_2 + c_0\sum_{j=1}^{h}|||\tilde{U}_j^{(m)}(0)|||_{3,\,\tan} + R\right),$$

$$C_6 \equiv \|a_5'\|_{L^\infty(0,T)}\sup_m\{\|b^{(m)}\|_{3,Q_{T_1}} + \|\dot{w}^{(m)}\|_{3,Q_{T_1}} + \|\mathscr{A}^{(m)}\nabla . w^{(m)}\|_{3,Q_{T_1}} + C_0'\|U_0\|_3\}$$

[*see* (3.17)]. Since from (3.8) $|||U_0^{(m)}|||_3 \leq |||U_0|||_3 + \gamma^* \leq (3/2)|||U_0|||_3$, we can find a non decreasing function $C^*(E_1)$ such that

$$C_5^{(m)} \leq C^*(E_1)[|||U_0|||_3 + R]. \quad (4.25)$$

The function $C^*(E_1)$ is used for the definition of the set $S(T)$. Then from (4.24) we can find $T_4 \in (0, T_3]$ such that for $T \leqq T_4$

$$|||U^{(m)}|||_{3,T} \leqq 2C^*(E_1)[|||U_0|||_3 + R] \equiv E_2. \tag{4.26}$$

Now, directly from the equations $(3.15)_m$, not integrating by parts, we have

$$\frac{d}{dt}|||U^{(m)}|||_2 \leqq a_6(t)[|||U^{(m)}|||_{3,T} + R]$$

where $a_6(t) \in L^\infty(0, T)$; using (3.8), (4.26) gives

$$\begin{aligned}|||U^{(m)}|||_{2,T} &\leqq |||U_0^{(m)}|||_2 + \|a_6\|_{L^\infty(0,T)}[E_2 + R]T \\ &\leqq (3/2)|||U_0|||_2 + \|a_6\|_{L^\infty(0,T)}[E_2 + R]T.\end{aligned}$$

Finally we take $T_5 \in (0, T_4]$ such that, for $T \leqq T_5$,

$$|||U^{(m)}|||_{2,T} \leqq 3|||U_0|||_2 \equiv E_1,$$

namely $U^{(m)} \in S(T)$. We have shown that $S(T)$ is an invariant set, for any $T \leqq T_5$. The proof of Lemma 4.1 is complete. ■

Next we show that the sequence $\{U^{(m)}\}_{m=1}^\infty$ converges in $X_0(T)$ for a sufficiently small T.

LEMMA 4.2. – *There exists* $T' \in (0, T_5]$ *such that*

$$\sum_{m=1}^\infty |||U^{(m)} - U^{(m-1)}|||_{0,T'} < +\infty. \tag{4.27}$$

Proof. – We consider the difference $U^{(m)} - U^{(m-1)}$ which satisfies the equations

$$\left.\begin{aligned}&L^{(m,\varepsilon_m)}(U^{(m-1)})(U^{(m)} - U^{(m-1)}) \\ &\qquad = F^{(m,\varepsilon_m)}(U^{(m-1)}, \hat{U}^{(m)}) - L^{(m,\varepsilon_m)}(U^{(m-1)})U^{(m-1)} \\ &\qquad + L^{(m-1,\varepsilon_{m-1})}(U^{(m-2)})U^{(m-1)} - F^{(m-1,\varepsilon_{m-1})}(U^{(m-2)}, \hat{U}^{(m-1)}), \\ &\qquad (v^{(m)} - v^{(m-1)}).N^{(m)} = 0 \quad \text{on } \Sigma_T.\end{aligned}\right\} \tag{4.28}$$

where $T \leqq T_5$. We multiply (4.28) by $U^{(m)} - U^{(m-1)}$ and integrate over Ω. Since $U^{(m)}, U^{(m-1)} \in S(T)$, taking account of (3.7), (3.8), (3.17) we obtain by means of a standard energy estimate

$$\begin{aligned}|||U^{(m)} - U^{(m-1)}|||_{0,T} &\leqq \|U_0^{(m)} - U_0^{(m-1)}\|_0 \\ &+ C_7 e^{C_8 T}(2^{-m} + T|||U^{(m-1)} - U^{(m-2)}|||_{0,T}).\end{aligned}$$

Taking now $T' \leqq T_5$ such that $C_7 e^{C_8 T'} T' \leqq 1/2$ we obtain

$$|||U^{(m)} - U^{(m-1)}|||_{0,T'} \leqq (1/2)\|U^{(m-1)} - U^{(m-2)}|||_{0,T'} + \alpha_m$$

where $\sum_{m=1}^\infty \alpha_m < +\infty$ because of (3.8). This estimate gives (4.27). The proof is complete. ■

From Lemma 4.1 and Lemma 4.2 we can deduce that there exists a function U such that $U^{(m)}$ converges to U in

$$X_{3-\delta}(T') \subset C^0([0,T']; H^{3-\delta}(\Omega)) \cap C^1([0,T']; H^{2-\delta}(\Omega))$$

for any small $\delta>0$. Accordingly, $U \in C^1(Q_{T'})$ because of Sobolev imbeddings. Passing to the limit in $(3.15)_m$ for $m\to\infty$ we see that U is a classical solution of (P). The uniqueness follows from standard energy estimates. In order to remove $\delta>0$ and obtain $U\in X_3(T')$ we proceed as in [8]. Since $U^{(m)}$ converges to U in $C^0([0,T']; H^{3-\delta}(\Omega))$ and $|||U^{(m)}|||_{3,T'} \leqq E_2$, $m=0, 1, \ldots$, since $(H^{3-\delta}(\Omega))'$ is dense in $(H^3(\Omega))'$ we can show that

$$U \in C_w([0,T']; H^3(\Omega)) \tag{4.29}$$

where C_w means weak continuity. Applying the tangential mollifier for the principal part of the equations for U corresponding to (4.2), carrying out estimates for $(H_j\varphi_j U)^\sim$ similar to (4.8) $(j=1, \ldots, h)$, taking account of $|||U|||_{3,T'} \leqq E_2$, we obtain

$$\|(H_j\varphi_j U)^\sim(t) - (H_j\varphi_j U)^\sim(s)\|_{3,\tan} \leqq C\,|t-s|$$

for t, $s\in[0,T']$, $j=1, \ldots, h$. Accordingly we have from (4.29) $D^3_{\tan}(H_j\varphi_j U)\in C^0([0,T']; L^2(\Omega))$, for $j=1, \ldots, h$. Furthermore, proceeding as for (4.14), (4.21) we show that $\varphi_j U \in C^0([0,T']; H^3(\Omega))$. Since $\varphi_0 U\in C^0([0,T']; H^3(\Omega))$, we have $U\in C^0([0,T']; H^3(\Omega))$. Directly from the equations of (P) we get $U\in X_3(T')$. The proof of Theorem B is complete.

4. PROOF OF THEOREM A

First we observe that the data $u_0 - w(0)$, $p(\rho_0)$, $b(t,x)=\bar{b}(t,\eta(t,x))$ and η satisfy the assumptions of Theorem B. The compatibility conditions (1.2) follow directly from (1.1). Hence there exists a unique classical solution (v,p) of (P) on the time interval $[0,T']$. Performing the change of variables $(t,x)\to(t,y)$ where $y=\eta(t,x)$ it is easily checked that the functions $\bar{u}(t,y)\equiv v(t,x)+w(t,x)$, $\bar{\rho}(t,y)\equiv\rho(p(t,x))$ are a classical solution of (E) on the same time interval $[0,T']$.

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