Ann. Inst. Henri Poincaré,

Vol. 13, n° 3, 1996, p. 337-379 *Analyse non linéaire*

Properties of pseudoholomorphic curves in symplectisations I: Asymptotics

by

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ABSTRACT. – Given an oriented, compact, 3-dimenional contact manifold (M, λ) we study maps $\tilde{u} = (a, u) : \mathbb{C} \to \mathbb{R} \times M$ satisfying the Cauch-Riemann type equation $\tilde{u}_s + \tilde{J}(\tilde{u})\tilde{u}_t = 0$, with a very special almost complex structure $\tilde{J}$ related to the contact form λ on M. If the energy is positive and bounded, $0 < E(\tilde{(u)} < \infty$, then the asymptotic behavior of $u : \mathbb{C} \to M$ as $|z| \to \infty$ is intimately related to the dynamics of the Reeb vector field X_λ on M. Assuming the periodic solutions of X_λ to be non degenerate, we shall show that $\lim_{R\to\infty} u(Re^{2\pi it}) = x(Tt)$ for a T-periodic solution x with $E(\tilde{u}) = T$. The main result is an asymptotic formula which demonstrates the exponential nature of this limit. Some consequences for the geometry of the maps $u : \mathbb{C} \to M$ are deduced.

Key words: Finite energy planes, pseudoholomorphic curves, contact forms.

RÉSUMÉ. – Étant donnée (M, λ) une variété de type contact, compacte, orientée et de dimension trois, nous étudions les applications $\tilde{u} = (a, u) : \mathbb{C} \to \mathbb{R} \times M$ solutions de l'équation de type Cauchy-Riemann $\tilde{u}_s + \tilde{J}(\tilde{u})\tilde{u}_t = 0$ où $\tilde{J}$ est une structure presque complexe très particulière, reliée à la forme de contact λ définie sur M. Lorsque l'énergie est positive et bornée, $0 < E(\tilde{u}) < \infty$, le comportement asymptotique d'une solution $u : \mathbb{C} \to M$ quand $|z| \to \infty$, est intimement lié à la dynamique du champ de Reeb X_λ défini sur M. Supposant que les orbites périodiques

Mathematics Subject Classification. 58Gxx, 53C15.

Annales de l'Institut Henri Poincaré - Analyse non linéaire - 0294-1449

Vol. 13/96/03/$ 4.00/

de X_λ sont non dégénérées, nous allons montrer que lorsque $R \to \infty$, $\lim u(Re^{2\pi it}) = x(Tt)$, où x est une orbite T-périodique de X_λ vérifiant $T = E(\tilde{u})$. Le principal résultat de cet article est une formule asymptotique qui établit la nature exponentielle de cette limite. On en déduit certaines conséquences pour la géometrie des solutions $u : \mathbb{C} \to M$.

CONTENTS

1. INTRODUCTION, NOTATIONS, RESULTS

We consider a compact oriented 3-manifold M and choose a contact form λ. Its existence is guaranteed by J. Martinet [11]. We recall that a contact form λ is a 1-form on M such that $\lambda \wedge d\lambda$ defines a volume-form on M. In the following we shall always assume that the orientation of M agrees with the one induced by this volume-form. Since the functional $\lambda_m: T_mM \to \mathbb{R}$ does not vanish, with the contact-form λ there is associated a 2-dimensional vectorbundle $\xi \to M$ over M, whose fibre $\xi_m \subset T_mM$ is defined by

$$\xi_m = \ker(\lambda_m), \quad m \in M.$$

This vectorbundle is a so called contact structure of M. The skew symmetric form $\omega = \omega_\lambda = d\lambda \mid \xi \oplus \xi$ is nondegenerate on each fibre and hence defines a symplectic form on the vectorspaces $\xi_m \subset T_mM$. We denote by (ξ, ω) this symplectic vector-bundle. In addition, again in view of the fact that $\lambda \wedge d\lambda$ is a volume-form, the kernel $\ker d\lambda \subset TM$ is 1-dimensional and defines a line-bundle $l = l_\lambda$ transversal to ξ. Its fibre is defined by

$$l_m = \{h \in T_mM \mid d\lambda(h,k) = 0, \quad \text{for all} \quad k \in T_mM\}.$$

This line-bundle has a distinguished section: there is a unique nonvanishing vector-field $X = X_\lambda$ defined by

$$(1) \qquad i_X d\lambda = 0 \quad \text{and} \quad i_X \lambda = 1;$$

$X = X_\lambda$ is called the Reeb-vector-field of λ. Summarising we see that the tangent bundle TM of M splits into the line-bundle $l \to M$ having the preferred section X_λ and a symplectic bundle $\xi \to M$ having the preferred symplectic form $\omega = d\lambda$:

$$\begin{aligned} TM &= (l, X) \oplus (\xi, d\lambda) \\ &= \mathbb{R} X \oplus \xi. \end{aligned}$$

If φ_t denotes the flow of X satisfying $\frac{d}{dt}\varphi_t(m) = X(\varphi_t(m))$, $m \in M$, we conclude from (1) that $\frac{d}{dt}(\varphi_t^* \lambda) = 0$ and $\frac{d}{dt}(\varphi_t^* d\lambda) = 0$. Consequently $d\varphi_t$ leaves the vectorbundle ξ invariant:

$$d\varphi_t(\xi_m) = \xi_{\varphi_t(m)}, \quad m \in M.$$

Moreover, since X is time-independent we conclude from $\varphi_t \circ \varphi_s = \varphi_{t+s}$, as usual,

$$d\varphi_t X(m) = X(\varphi_t(m)),$$

and see that $d\varphi_t$ leaves the splitting $\mathbb{R}X \oplus \xi$ of TM invariant. With

$$\pi: TM = \mathbb{R}X \oplus \xi \to \xi$$

we shall denote in the following the projection along X. It satisfies:

$$(2) \qquad h = \lambda(h)X(m) + \pi(h)$$

for every $h \in T_m M = \mathbb{R}X(m) \oplus \xi_m$. The symplectic vectorbundle $(\xi, d\lambda) \to M$ has a distinguished class of almost complex structures $J: \xi \to \xi$ satisfying $J(m) \in \mathcal{L}(\xi_m, \xi_m)$ and $J(m)^2 = -I$, which are compatible with $d\lambda$ in the sense that

$$(3) \qquad g_J(a, b) = d\lambda(a, J(m)b)$$

defines a positive definite inner product on each fibre ξ_m. This space of complex structures is contractible, as is well known, *see* f.e. ([1], [3], [9]).

Fixing an almost complex structure J compatible with $d\lambda$ we are interested in the first order elliptic system for functions

$$\tilde{u} := (a, u) : \mathbb{C} \to \mathbb{R} \times M,$$

defined by

$$\begin{aligned} \pi\frac{\partial u}{\partial s} + J(u)\pi\frac{\partial u}{\partial t} &= 0 \\ (u^*\lambda) \circ i &= \mathrm{da}, \end{aligned} \tag{4}$$

where $z = s + it \in \mathbb{C}$. In order to reformulate this equation we introduce the almost complex structure $\tilde{J}$ on the 4-manifold $\mathbb{R} \times M$ as follows:

$$\tilde{J}(a, m)(h, k) = (-\lambda_m(k), J(m)\pi k + hX(m))$$

for $(h, k) \in T_{(a,m)}(\mathbb{R} \times M)$. One verifies immediately using (2) that $\tilde{J}^2(h, k) = (-h, -k)$. The equation (4) is equivalent to the equation

$$\begin{aligned} \tilde{u} &= (a, u) : \mathbb{C} \to \mathbb{R} \times M \\ \tilde{u}_s + \tilde{J}(\tilde{u})\tilde{u}_t &= 0. \end{aligned} \tag{5}$$

There are plenty of solutions of (5) which are not interesting to us: for example, if $x : \mathbb{R} \to M$ is a solution of the Reeb field $\dot{x} = X(x)$ on M, then

$$\tilde{u}(s + it) := (s, x(t)) \in \mathbb{R} \times M \tag{6}$$

is a solution, as is readily verified. As was shown in Hofer [4], there is, however, an interesting class of solutions singled out by an "energy requirement". Let

$$\Sigma = \{f \in C^\infty(\mathbb{R}, [0, 1]) \mid f' \geq 0\}$$

and define for $f \in \Sigma$ the 1-form λ_f on $\mathbb{R} \times M$ by:

$$\lambda_f(a, m)(h, k) = f(a)\lambda_m(k).$$

For a solution $\tilde{u} = (a, u)$ of (5) one computes

$$\begin{aligned} \tilde{u}^* d\lambda_f = \frac{1}{2}[&f'(a)(a_s^2 + a_t^2 + \lambda(u_s)^2 + \lambda(u_t)^2) \\ &+ f(a)\left(|\pi u_s|_J^2 + |\pi u_t|_J^2\right)]ds \wedge dt, \end{aligned}$$

which is a nonnegative integrand. Here, and also in the following, we used the norm $|h|_J^2 := g_J(h,h)$ for $h \in \xi$, where g_J is defined in (3). Therefore, if $\tilde{u}$ is a solution of (5), then

$$0 \leq \int_{\mathbb{C}} \tilde{u}^* d\lambda_f \leq \infty,$$

and we define its energy $E(\tilde{u}) \in [0,\infty]$ by

$$E(\tilde{u}) = \sup_{f\in\Sigma} \int_{\mathbb{C}} \tilde{u}^* d\lambda_f.$$

DEFINITION 1.1. – *A finite energy plane is a solution* $\tilde{u} = (a,u)$ *of* (5) *satisfying, in addition,*

$$0 < E(\tilde{u}) < \infty.$$

For the trivial solutions $\tilde{u}$ defined in (6) we have $E(\tilde{u}) = \infty$. Indeed, taking a function $f \in \Sigma$ with $f' \neq 0$ we compute

$$\int_{\mathbb{C}} \tilde{u}^* d\lambda_f = (f(\infty) - f(-\infty)) \int_{\mathbb{R}} dt = \infty.$$

The significance of the concept of "finite energy plane" lies in the following result relating finite energy planes to periodic orbits of the Reeb vectorfield X, *see* [4].

THEOREM 1.2. – *Assume* $\tilde{u} = (a,u)\colon \mathbb{C} \to \mathbb{R} \times M$ *is a finite energy plane. Then*

$$T := \int_{\mathbb{C}} u^* d\lambda > 0$$

and there exists a sequence $R_k \to \infty$ *such that* $\lim_{k\to\infty} u(R_k e^{2\pi i t}) = x(Tt)$ *in* $C^\infty(\mathbb{R})$ *for a* T*-periodic solution* $x(t)$ *of the Reeb vectorfield* $\dot{x}(t) = X(x(t))$. *If this solution is nondegenerate then*

$$\lim_{R\to\infty} u(Re^{2\pi i t}) = x(Tt),$$

with convergence in $C^\infty(\mathbb{R})$.

The first part of Theorem 1.2 has been proved in [4]. The strengthening for a non-degenerate asymptotic limit will be proved in the present paper. The limit x associated to a suitable sequence $R_k \to \infty$ will be called, in the following, an asymptotic limit. As stated, the asymptotic limit is unique provided there exist a non-degenerate one. In general this should not be case. However we do not know an explicit counter example.

As for the existence question, we recall that if M has a non-vanishing $\pi_2(M)$ then for every contact form λ and every compatible almost complex structure J there exists a finite energy plane. One can also show that for the three-sphere S^3 there exists a finite energy plane for every choice of contact form and compatible J. All these results, with the exception of the case where λ is a tight contact form on S^3 have been proved in [4]. Theorem 1.2 then shows that the associated Reeb vectorfields X possess periodic solutions.

We are not concerned in the following with the existence question. Rather we assume the existence of a finite energy plane and the aim is to describe precisely its asymptotic behaviour as $|z| \to \infty$. We shall assume, however, that the T-periodic solution $x(t)$ guaranteed by the first part of Theorem 1.2 is nondegenerate. This requires that it has only one Floquet multiplier equal to 1, and hence is isolated in the set of periodic solutions of the Reeb vectorfield having their periods close to T. We reformulate the second part of Theorem 1.2 as follows

THEOREM 1.3. – *Let $\tilde{u} = (a, u): \mathbb{C} \to \mathbb{R} \times M$ be a nonconstant, finite energy plane as in Theorem 1.2, with an asymptotic T-periodic orbit $x(t)$ which is nondegenerate, then*

$$\lim_{R \to \infty} u(Re^{2\pi it}) = x(T \cdot t),$$

moreover the convergence is in $C^\infty(\mathbb{R})$.

The theorem allows us to study, for R large, the finite energy plane in a tubular neighborhood of its limit $x(t)$. It is convenient to consider the holomorphic cylinder $\tilde{v} = \tilde{u} \circ \varphi = (a, v)$, with the biholomorphic map $\varphi: \mathbb{R} \times S^1 \to \mathbb{C} \setminus \{0\}$ defined by $\varphi(s, t) = e^{2\pi(s+it)}$. Then $v(s, t) \to x(Tt)$ as $s \to \infty$ in $C^\infty(S^1)$. We shall construct local coordinates $\mathbb{R} \times \mathbb{R}^2$ in a tubular neighborhood of $x(t)$. In these coordinates the map $\tilde{v}$ is represented by $(a, v) = (a(s, t), \vartheta(s, t), z(s, t)): [s_0, \infty) \times \mathbb{R} \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2$. If $T = k\tau$, $k \geq 1$, where τ is the minimal period of the periodic solution $x(t)$, then $\vartheta(s, t + 1) = \vartheta(s, t) + k$, while the other functions a, z are 1-periodic in t. The main contents of this paper is the proof of the following asymptotic description of a non degenerate finite energy plane.

THEOREM 1.4. – *There exist constants $c \in \mathbb{R}$ and $d > 0$ such that*

$$\left|\partial^\beta [a(s, t) - Ts - c]\right| \leq Me^{-ds}$$
$$\left|\partial^\beta [\vartheta(s, t) - kt]\right| \leq Me^{-ds}$$

for all multi-indices β, with constants $M = M_\beta$. Moreover, we have the asymptotic formula for the transversal approach to $x(t)$:

$$z(s,t) = e^{\int_{s_0}^{s} \mu(\tau)d\tau}[e(t) + r(s,t)] \in \mathbb{R}^2$$

where $\partial^\beta r(s,t) \to 0$ as $s \to \infty$ uniformly in t for all derivatives. Here $\mu: [s_0, \infty) \to \mathbb{R}$ is a smooth function satisfying $\mu(s) \to \lambda < 0$ as $s \to \infty$. The number λ is an eigenvalue of a selfadjoint operator A in $L^2(S^1, \mathbb{R}^2)$ related to the linearized Reeb vectorfield X along the limit orbit $x(t)$. The operator is defined by $A = -J_0 \frac{d}{dt} - S_\infty(t)$, with $S_\infty(t) = S_\infty(t+1)$ a symmetric, 1-periodic, smooth 2×2 matrix function defined by $S_\infty(t) = -J_0 \pi_m dX(m) \pi_m$, where $m = (kt, 0) \in \mathbb{R} \times \mathbb{R}^2$. Moreover,

$$e(t) = e(t+1) \neq 0,$$

is an eigenvector of A belonging to the eigenvalue $\lambda < 0$.

From this asymptotic description of $\tilde{u}$ we shall deduce, using the similarity principle, the following global consequences

THEOREM 1.5. – *Let $\tilde{u} = (q, u): \mathbb{C} \to \mathbb{R} \times M$ be a nonconstant finite energy plane with nondegenerate asymptotic periodic orbit $x(t)$. Let $P = \{x(t) \mid t \in \mathbb{R}\} \subset M$. Then the sets*

$$\{z \in \mathbb{C} \mid u(z) \in P\}$$
$$\{z \in \mathbb{C} \mid \pi \circ Tu(z) = 0\}$$

consist of finitely many points.

This means that the map $u: \mathbb{C} \to M$ intersects its limit $x(t)$ in at most finitely many points. Moreover, the tangent map Tu has maximal rank except at finitely many points, using that $\pi \circ Tu(z): T_z\mathbb{C} \to \xi_{u(z)}$ is complex linear, in view of the identity $\pi \circ Tu \circ i = J \circ \pi \circ Tu$.

These results will be important in a series of applications of holomorphic curves methods to problems in low-dimensional topology and Hamiltonian dynamics, *see* ([5], [6], [7], [8]). There we will use holomorphic curve methods in symplectisations to construct open book decompositions for certain three-manifolds, [7], as well as global surfaces of sections for Hamiltonian flows on three-dimensional energy surfaces, [8]. In particular it turns out that a Hamiltonian flow on a strictly convex energy surface in $\mathbb{R}^4$ has either precisely 2 or infinitely many periodic orbits, *see* [8].

There are three technical ingredients to any application. The first is a complete description of the behavior of finite energy planes at infinity, which is the same as the behaviour of a finite energy surface near a non removable singularity. This is the contents of the present paper. The second ingredient is the study of embedding properties of finite energy surfaces and their projections into the contact manifold. Here methods from algebraic topology like intersection theory, Maslov indices and winding numbers combined with the asymptotic analysis from the present paper will play a crucial role, *see* [5]. The third ingredient is a Fredholm theory and implicit function type techniques in order to describe families of finite energy planes, *see* [6].

2. PERIODIC ORBITS OF X_λ AND LOCAL COORDINATES NEAR THE ENDS

We consider a T-periodic solution $x(t)$ for the Reeb vectorfield $\dot{x} = X_\lambda(x)$. Then $x(0) = x(T)$ and for the linearization of the flow φ_t we have

$$d\varphi_T X(x(0)) = X(\varphi_T(x(0))) = X(x(0)).$$

Hence 1 is an eigenvalue of $d\varphi_T(x(0)) \in \mathcal{L}(T_{x(0)}M, T_{x(0)}M)$. The periodic solution in called nondegenerate if this is the only eigenvalue equal to 1 of the linear map $d\varphi_T(x(0))$. Since $d\varphi_T$ leaves the splitting $X(x(0)) \oplus \xi_{x(0)}$ invariant this is equivalent to the requirement that

$$d\varphi_T(x(0)): \xi_{x(0)} \to \xi_{x(0)}$$

has no eigenvalue equal to 1. Dynamically a nondegenerate T-periodic solution is isolated on M in the set of periodic solutions having periods close to T. In order to study the asymptotic behaviour it is convenient in the following to consider a cylinder instead of a plane. Let $\varphi: \mathbb{R} \times S^1 \to \mathbb{C} \setminus \{0\}$ be the biholomorphic map defined by

$$\varphi(s,t) = e^{2\pi(s+it)},$$

where $S^1 = \mathbb{R}/\mathbb{Z}$. If $\tilde{u} = (a,u)$ is a finite energy plane, we define the cylinder

$$\tilde{v}: \mathbb{R} \times S^1 \to \mathbb{R} \times M$$

by

$$\tilde{v} = \tilde{u} \circ \varphi.$$

In what follows we will use a letter a to denote a map $\mathbb{C} \to \mathbb{R}$ and a map $\mathbb{R} \times S^1 \to \mathbb{R}$ obtained by composing a with φ. Then $\tilde{v} = (a, v)$ satisfies

$$(7) \qquad \begin{aligned} &\tilde{v}_s + \tilde{J}(\tilde{v})\tilde{v}_t = 0 \qquad \text{on } \mathbb{R} \times S^1 \\ &\int_{\mathbb{R}\times S^1} v^* d\lambda = \int_{\mathbb{C}} u^* d\lambda > 0 \\ &0 < E(\tilde{v}) = E(\tilde{u}) < \infty. \end{aligned}$$

A solution $\tilde{v}$ of (7) satisfies the estimate

$$\sup_{\mathbb{R}\times S^1} |\nabla \tilde{v}(s,t)| < \infty,$$

from which one derives estimates for all derivatives

$$(8) \qquad \sup_{\mathbb{R}\times S^1} |\partial^\alpha \tilde{v}(s,t)| < \infty, \quad |\alpha| \geq 1.$$

For a proof of these crucial estimates, based on a "bubbling off" analysis and elliptic estimates we refer to Hofer [4].

In order to prove Theorem 1.3 we start with

PROPOSITION 2.1. – *Let $\tilde{u}$ be a finite energy plane and assume there exists a sequence (R_k) such that $u(R_k e^{2\pi i t}) \to x(Tt)$ in $C^\infty(S^1, M)$. Assume further that x is a non-degenerate T-periodic solution of $\dot{x} = X(x)$. Then given any S^1-invariant C^∞-neighbourhood W of $x(T\cdot)$ in $C^\infty(S^1, M)$ there exists $R_0 > 0$ such that $u(Re^{2\pi i \cdot}) \in W$ for all $R \geq R_0$.*

The S^1-action on $C^\infty(S^1, M)$ is the one induced by the operation of S^1 on itself. Here we view M as being embedded in some R^n. The Frechet space $C^\infty(S^1, R^n)$ is equipped with a translation invariant and S^1-invariant metric d which we restrict to the subspace $C^\infty(S^1, M)$. The proof will follow from Theorem 1.2 and the estimates (8) using the assumption that the periodic solution x is isolated. We work with $\tilde{v}$. Let (s_k) be the sequence defined by (R_k) via $e^{2\pi s_k} = R_k$. We have, by assumption,

$$\lim_{k\to\infty} v(s_k, \cdot) = x(T\cdot).$$

Define the closed subset $\mathcal{S}$ of the loop space of M by

$$\mathcal{S} = \overline{\{v(s,\cdot) \mid s \in \mathbb{R}\}} \subset C^\infty(S^1, M).$$

In view of the estimates (8) $\mathcal{S}$ is a compact set of $C^\infty(S^1, M)$. We need the following

LEMMA 2.2. – *Let x be a T-periodic non-degenerate solution of $\dot{x} = X(x)$ and $\mathcal{S}$ a compact family of loops in $C^\infty(S^1, M)$. Denote by $x_T \in C^\infty(S^1, M)$ the associated loop $t \mapsto x(Tt)$. Then, for a given open S^1-invariant neighbourhood W of x_T there exist two open S^1-invariant neighbourhoods U and V and numbers $\delta, \varepsilon > 0$ satisfying*

$$B_\epsilon(V) \subset U \subset \bar{U} \subset W$$

and

$$\int_{S^1} |\pi \dot{y}|^2 dt \geq \delta$$

for all $y \in \mathcal{S} \cap (U \setminus V)$.

Proof. – We find an open S^1-invariant neighbourhood U of x_T, $U \subset W$, such that for every loop $y \in U$ satisfying $\int_{S^1} |\pi \dot{y}|^2 dt = 0$, we have that $\int_{S^1} y^* \lambda$ is near to T, and $y(S^1)$ is close to $x(S^1)$. Recall that $\int_{S^1} x^* \lambda = T$. Therefore, $y(S^1)$ belongs to a periodic orbit of the Reeb vectorfield X, which corresponds to a fixed point of the Poincaré map associated to the distinguished T-periodic solution x. Clearly, if U is small enough we have $y(S^1) = \{x(t) \mid t \in \mathbb{R}\}$, provided $y \in \bar{U}$ and $\int_{S^1} |\pi \dot{y}|^2 dt = 0$, by our non-degeneracy assumption. Next take an S^1-invariant neighbourhood V such that there exists a number $\varepsilon > 0$ with $B_\varepsilon(V) \subset U$. By the compactness of $\mathcal{S}$, the map $y \to \int_{S^1} |\pi \dot{y}|^2 dt$ is bounded away from 0 on $\mathcal{S} \cap (\bar{U} \setminus V)$. This completes the proof of the lemma. □

Proof of Proposition 2.1. – In view of Lemma 2.2 we have

$$\int_{S^1} |\pi \dot{y}|^2 dt \geq \delta$$

for all $y \in \mathcal{S} \cap (U \setminus V)$. We know that $v(s_k, \cdot) \to x(T\cdot)$ with x being a T-periodic non-degenerate orbit of X. Hence we may assume that $v(s_k, \cdot) \in U$ for all k. If our assertion is wrong we find a sequence $(s_k^{'})$ such that $v(s_k^{'}, \cdot) \notin V$. Eventually taking suitable subsequences we may assume that

$$s_k \leq s_k^{'} < s_{k+1}.$$

In view of the uniform gradient estimate of v in (8), the solution v needs a uniform time in order to travel through the set $U \setminus V$: there exists an $\varepsilon > 0$ and $[a_k, b_k] \subset [s_k, s_k']$ with $b_k - a_k \geq \varepsilon$ for all k such that

$$v([a_k, b_k], \cdot) \subset \bar{U} \setminus V.$$

Hence we compute

$$\int_{s_k}^{s'_k} \int_{S^1} |\pi v_t|^2 ds\, dt \geq \int_{a_k}^{b_k} \int_{S^1} |\pi v_t|^2 ds\, dt \geq \delta\varepsilon.$$

There is an infinite sequence of such $k's$ and we, therefore, find

$$\sum_k \int_{s_k}^{s_{k+1}} \int_{S^1} |\pi v_t|^2 ds\, dt = \infty.$$

This contradicts the assumption

$$\int_{\mathbb{R}\times S^1} v^* d\lambda = \int_{\mathbb{R}} \int_{S^1} |\pi v_t|^2 ds\, dt < \infty.$$

The proof of Proposition 2.1 is complete. □

We shall study the finite energy cylinder $\tilde{v} = (a, v): \mathbb{R} \times S^1 \to \mathbb{R} \times M$ as introduced in the previous paragraph. We know by the previous discussion that given any S^1-invariant neighbourhood W of $x(T\cdot)$ in $C^\infty(S^1, M)$ we have $v(s, \cdot) \in W$ for all s large enough. Hence we can study the solution $v: \mathbb{R} \times S^1 \to M$ for large s, say $s \geq s_0$ locally in a tubular neighborhood of the periodic solution x. For this purpose we shall first introduce convenient local coordinates in M near the periodic solution $x(t)$. The coordinates will be given by

$$S^1 \times \mathbb{R}^2, \quad f \cdot \lambda_0$$

where the periodic solution is on $S^1 \times \{0\}$, f is a positive function and $f \cdot \lambda_0$ is a contact form with

$$\lambda_0 = d\vartheta + x dy$$

being the standard contact form on $S^1 \times \mathbb{R}^2$. Since $S^1 = \mathbb{R}/\mathbb{Z}$ we work in the covering space and denote by $(\vartheta, x, y) \in \mathbb{R}^3$ the coordinates, ϑ mod 1. Recall first that if $\varphi: (N, \mu) \to (M, \lambda)$ is a diffeomorphism between two contact manifolds satisfying $\varphi^* \lambda = \mu$, then the corresponding Reeb vectorfields are transformed into each other by

$$X_\mu = (d\varphi)^{-1} \cdot X_\lambda \circ \varphi,$$

as is easily verified. Hence φ maps the solutions of X_μ onto the solutions of X_λ: for the flows we have $\varphi_t^\lambda \circ \varphi = \varphi \circ \varphi_t^\mu$, for all $t \in \mathbb{R}$. This will be used in the proof of

LEMMA 2.3. – *Let (M, λ) be a 3-dimensional contact manifold, and let $x(t)$ be a T-periodic solution of the corresponding Reeb vectorfield $\dot{x} = X_\lambda(x)$ on M. Let τ be the minimal period such that $T = k\tau$ for some positive integer k. Then there is an open neighborhood $U \subset S^1 \times \mathbb{R}^2$ of $S^1 \times \{0\}$ and an open neighborhood $V \subset M$ of $P = \{x(t) \mid t \in \mathbb{R}\}$ and a diffeomorphism $\varphi: U \to V$ mapping $S^1 \times \{0\}$ onto P such that*

$$\varphi^* \lambda = f \cdot \lambda_0, \tag{9}$$

with a positive smooth function $f: U \to \mathbb{R}$ satisfying

$$f(\vartheta, 0, 0) = \tau \quad \textit{and} \quad df(\vartheta, 0, 0) = 0 \tag{10}$$

for all $\vartheta \in S^1$.

Proof. – Let $\varphi_0: U \to V$ be a local diffeomorphism mapping $S^1 \times \{0\}$ onto P such that the contact structure $\ker(\varphi_0^* \lambda)$ is transversal to $S^1 \times \{0\}$. By J. Martinet [11], we find a local diffeomorphism $\varphi_1: U \to U'$ in the local coordinates, where U and U' are open neighborhoods of $S^1 \times \{0\} \subset S^1 \times \mathbb{R}^2$, satisfying $\varphi_1(S^1 \times \{0\}) = S^1 \times \{0\}$ and

$$\varphi_1^*(\varphi_0^* \lambda) = g\lambda_0$$

with a nonvanishing smooth function $g: U \to \mathbb{R}$. Denoting in the covering space $(\vartheta, x, y) \in \mathbb{R}^3$ the coordinates, the function g is periodic in ϑ of period 1. The Reeb vectorfield $X_{g\lambda_0}$ associated with the contact form $g\lambda_0$ on $S^1 \times \mathbb{R}^2$ is computed to be

$$X_{g\lambda_0}(\vartheta, x, y) = \left(\frac{1}{g} + \frac{x}{g^2} g_x\right) \frac{\partial}{\partial \vartheta} + \frac{1}{g^2}(g_y - x g_\vartheta) \frac{\partial}{\partial x} - \left(\frac{1}{g^2} g_x\right) \frac{\partial}{\partial y}.$$

By construction, in view of the remark previous to the Lemma this Reeb vectorfield is tangential to the periodic solution $(\alpha(t), 0, 0) = (\varphi_0 \circ \varphi_1)^{-1}(x(t))$, where $\alpha(t + \tau) = \alpha(t) + 1$. Recall that τ is the minimal period of $x(t)$. As usual, we work in the covering space $\mathbb{R}$ of $S^1 = \mathbb{R}/\mathbb{Z}$. Therefore

$$g_x(\vartheta, 0, 0) = g_y(\vartheta, 0, 0) = 0$$
$$X_{g\lambda_0}(\vartheta, 0, 0) = \frac{1}{g(\vartheta, 0, 0)} \frac{\partial}{\partial \vartheta}$$

and

$$\dot{\alpha}(t) = \frac{1}{g(\alpha(t), 0, 0)}. \tag{11}$$

Finally, we define a diffeomorphism $\varphi_2 : S^1 \times \mathbb{R}^2 \to S^1 \times \mathbb{R}^2$ leaving $S^1 \times \{0\}$ invariant, by

$$\varphi_2(\vartheta, x, y) = (a(\vartheta), \dot{a}(\vartheta)x, y),$$

where $a(\vartheta) = \alpha(\tau\vartheta)$, so that $a(\vartheta + 1) = a(\vartheta) + 1$. Then the composition $\varphi = \varphi_0 \circ \varphi_1 \circ \varphi_2$ is a local diffeomorphism $S^1 \times \mathbb{R}^2 \to M$ mapping the periodic solution $S^1 \times \{0\}$ onto $x(t)$. It satisfies $\varphi^*\lambda = f\lambda_0$, with the function f defined by

$$f(\vartheta, x, y) = g(a(\vartheta), \dot{a}(\vartheta)x, y)\dot{a}(\vartheta).$$

The function f satisfies $f(\vartheta + 1, x, y) = f(\vartheta, x, y)$ and a computation, using (11) shows that $f = \tau$ and $f_\vartheta = f_x = f_y = 0$ at every point $(\vartheta, 0, 0) \in S^1 \times \{0\}$ as desired. This finishes the proof of the Lemma. □

From now on we shall work in the local coordinates $(\vartheta, x, y) \in \mathbb{R} \times \mathbb{R}^2$, ϑ mod 1 with the contact structure

$$\lambda = f \cdot \lambda_0, \quad \lambda_0 = d\vartheta + x dy$$

with a smooth and positive function $f : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$, defined near $\mathbb{R} \times \{0\}$ and periodic in ϑ: $f(\vartheta, x, y) = f(\vartheta + 1, x, y)$ and satisfying (10). The Reeb-vectorfield $X = X_\lambda(\vartheta, x, y) \in \mathbb{R}^3$ is periodic in ϑ, satisfies

$$X(\vartheta, 0, 0) = \frac{1}{\tau}(1, 0, 0),$$

and is given by

$$X(\vartheta, x, y) = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \frac{1}{f^2} \begin{pmatrix} f + x f_x \\ f_y - x f_\vartheta \\ -f_x \end{pmatrix}.$$

The contact plane ξ_m at $m = (\vartheta, x, y) \in \mathbb{R}^3$, defined by $\xi_m = \{k \in \mathbb{R}^3 \mid \lambda_m(k) = 0\}$, is the two dimensional plane

$$\xi_m = \operatorname{span}\langle e_1, e_2 \rangle,$$

where

$$e_1 = (0, 1, 0)$$
$$e_2 = (-x, 0, 1)$$

at the point $m = (\vartheta, x, y)$. Since

$$d\lambda(e_1, e_2) = f d\lambda_0(e_1, e_2) = f$$

at $m = (\vartheta, x, y)$ we find that the symplectic structure $d\lambda \mid \xi_m \oplus \xi_m$ is, in the basis e_1, e_2 of ξ_m given by the skew symmetric 2×2-matrix $\Omega = \Omega(\vartheta, x, y)$:

$$\Omega = fJ_0, \qquad \text{where} \quad J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is the standard symplectic structure of $\mathbb{R}^2$.

Given to us is an almost complex structure $j_m: \xi_m \to \xi_m$ compatible with $d\lambda \mid \xi_m$ and induced by the diffeomorphism $\varphi: \mathbb{R}^3 \to M$ of Lemma 2.3:

$$j_m: (d\varphi_m)^{-1} \circ J_{\varphi(m)} \circ d\varphi_m: \xi_m \to \xi_m,$$

where J is the almost complex structure chosen in Theorem 1.2. Since j_m is compatible with $d\lambda \mid \xi_m$ it is, in the basis e_1, e_2 of ξ_m, represented by a 2×2-matrix $M = M(m)$ depending smoothly on m and satisfying

$$M^2 = -1, \quad M^T \Omega M = \Omega, \quad M^T \Omega > 0. \tag{12}$$

The second condition is, in view of $f > 0$, equivalent to $M^T J_0 M = J_0$, hence equivalent to $\det M = 1$. The last condition requires the inner product $g(z, z') = \langle \Omega z, Mz' \rangle = \langle M^T \Omega z, z' \rangle$, for the coordinates $z, z' \in \mathbb{R}^2$ of ξ_m, to be positive definite. It is equivalent to $\Omega^T M > 0$ and hence, since $f > 0$, equivalent to

$$-J_0 M > 0.$$

Finally, as in the introduction, we shall denote by $\pi: \mathbb{R}^3 \to \xi$ the projection

$$\pi_m: \mathbb{R}^3 = \mathbb{R}X(m) \oplus \xi_m \to \xi_m$$

along the Reeb vectorfield X onto the contact planes. It is given by the formula

$$\pi_m(k) = k - \lambda_m(k) X(m) \in \xi_m \tag{13}$$

for $k \in \mathbb{R}^3$.

If now $\tilde{v} = (a, v): \mathbb{R} \times S^1 \to M$ is the positive energy cylinder of Theorem 1.2, we write in our local coordinates

$$\tilde{u} = (a, u) = (a, \varphi^{-1} \circ v): [s_0, \infty) \times S^1 \to \mathbb{R}^4$$

for some $s_0 > 0$ large. We shall use the notations

$$\begin{aligned} u(s,t) &= (u^1(s,t), u^2(s,t), u^3(s,t)) \\ &= (\vartheta(s,t), x(s,t), y(s,t)). \end{aligned}$$

Working, as usual, in the covering space $\mathbb{R}$ of $S^1 = \mathbb{R}/\mathbb{Z}$, the functions $a(s,t), x(s,t)$ and $y(s,t)$ are 1-periodic in the t variable. The function $\vartheta(s,t)$, however, represents a map from S^1 onto S^1 and satisfies $\vartheta(s,t+1) = \vartheta(s,t) + k$. Indeed, this follows from the fact that for any sequence $s_n \to \infty$ there is a constant $c \in [0,1)$ such that $u(s_n,t) \to \xi(Tt+c)$ as $n \to \infty$. Here $\xi(t)$ is the T-periodic solution of $\dot{\xi} = X(\xi)$ and $T = k\tau$ with the minimal period τ. In view of $X(\vartheta,0,0) = \frac{1}{\tau}(1,0,0)$ we find that $\xi(Tt) = (Tt/\tau,0,0) = (kt,0,0)$ and the claim follows. By construction, these functions solve the equation

$$\begin{aligned} \tilde{u} = (a,u)&: [s_0,\infty) \times \mathbb{R} \to \mathbb{R}^4 \\ \tilde{u}_s + \tilde{J}(\tilde{u})\tilde{u}_t &= 0. \end{aligned} \tag{14}$$

At the point $(a,m) \in \mathbb{R}^4$, the $\tilde{J}$ is given by

$$\tilde{J}(a,m)(h,k) = (-\lambda_m(k), j_m(\pi k) + hX(m)),$$

where $\pi = \pi_m$ is the projection as in (13). More explicitely we therefore can write the equation (14) as follows:

$$a_s - \lambda(u_t) = 0 \tag{15}$$

$$(\lambda(u_s) + a_t)X(u) = 0 \tag{16}$$

$$\pi u_s + j(\pi u_t) = 0 \tag{17}$$

with $\lambda = \lambda_u$ and $\pi = \pi_u$. Note that $X(u) \neq 0$. Abbreviating the partial derivatives

$$u_s = (\vartheta_s, x_s, y_s), \quad u_t = (\vartheta_t, x_t, y_t)$$

we next express the equation (17) in the basis e_1, e_2 of the contact plane ξ_u. In view of (15) and (16) and using the formula (13) we obtain for (15)-(17) the equations

$$\begin{aligned} a_s - \lambda(u_t) &= 0 \\ a_t + \lambda(u_s) &= 0 \end{aligned} \tag{18}$$

$$\begin{pmatrix} x_s \\ y_s \end{pmatrix} + M\begin{pmatrix} x_t \\ y_t \end{pmatrix} + a_t\begin{pmatrix} X_2(u) \\ X_3(u) \end{pmatrix} - a_s M\begin{pmatrix} X_2(u) \\ X_3(u) \end{pmatrix} = 0. \tag{19}$$

Here $M = M(u)$ is the 2×2 matrix which represents the almost complex structure j_u in the basis (e_1, e_2) of ξ_u. For the derivatives of ϑ we find the additional equations

$$\begin{aligned}\vartheta_s &= -a_t X_1(u) - x(y_s + a_t) X_3(u) \\ \vartheta_t &= a_s X_1(u) - x(y_t - a_s) X_3(u)\end{aligned}$$

where $x = x(s,t)$. In view of the definition $\lambda = f \cdot \lambda_0$ we can rewrite (18) and find

$$\begin{aligned}a_s &= (\vartheta_t + x y_t) f(u) \\ a_t &= -(\vartheta_s + x y_s) f(u),\end{aligned} \tag{20}$$

with $x = x(s,t)$.

We need the following lemma which will be a consequence of proposition 2.1 and a standard bubbling-off argument as given in [4].

LEMMA 2.4. – *As* $s \to \infty$

$$\begin{aligned}\partial^\alpha x(s,t) &\to 0 \\ \partial^\alpha y(s,t) &\to 0,\end{aligned}$$

uniformly in t, for all derivatives $\alpha = (\alpha_1, \alpha_2)$. *Moreover,*

$$\begin{aligned}\partial^\alpha[\vartheta(s,t) - kt] &\to 0 \\ \partial^\alpha[a(s,t) - Ts] &\to 0,\end{aligned}$$

uniformly in t, provided $|\alpha| \geq 1$.

Proof. – First we recall from [4] that for a finite energy plane $\tilde{v} = (a, v)$ all the partial derivatives of v, and the partial derivatives of a satisfying $|\alpha| \geq 1$ are uniformly bounded. (Here we view M as beeing embedded in some $\mathbb{R}^n$). In addition, we recall from [4] that every sequence $v(s_k, t)$, with $s_k \to \infty$, possesses a subsequence converging with all its t derivatives uniformly to a T-periodic solution of X. By Proposition 2.1 we, therefore, conclude that the statement for the functions $x(s,t)$ and $y(s,t)$ hold true for all the derivatives in time, *i.e.* for $\alpha = (0,k)$, $k \geq 0$. That it holds true for all derivatives follows from the equations (19) and (20) together with the second statement.

In order to prove the second statement, *i.e.* the statement for the functions a and ϑ we argue by contradiction. If the assertion is wrong, we find a sequence (s_k, t_k) with $s_k \to \infty$ and $t_k \to t_0 \in [0,1]$ satisfying

$$|\partial^\alpha(a - Ts, \vartheta - kt)(s_k, t_k)| \geq \varepsilon, \tag{21}$$

for some $\varepsilon > 0$ and some multi-index α of order at least 1. We can always add a real constant to a and an integer to ϑ so that still the equations (18) and (19) hold. This will also not affect our assertion. Define a sequence of functions $((a_k, b_k))$ by

$$(a_k(s,t), b_k(s,t)) = (a(s+s_k,t) - a(s_k,t_k), \vartheta(s+s_k,t) - \vartheta(s_k,t_k)).$$

Eventually taking a subsequence the above sequence has a C^∞_{loc}-convergent subsequence, whose limit we denote by $(\hat{a}, \hat{b})$. The map $(\hat{a}, \hat{b})$ is then defined on $\mathbb{R} \times \mathbb{R}$, and $\hat{a}$ is 1-periodic in t while $\hat{b}$ satisfies $\hat{b}(s,t+1) = \hat{b}(s,t) + k$. Moreover, it solves the equation

$$\begin{aligned} \hat{a}_s &= \tau \hat{b}_t \\ \hat{a}_t &= -\tau \hat{b}_s \end{aligned}$$

on $\mathbb{R} \times \mathbb{R}$. Indeed, this follows from (19) and (20) taking into account that $f(\vartheta,0,0) = \tau$, $X_2(\vartheta,0,0) = X_3(\vartheta,0,0) = 0$, and that $(x(s,t), y(s,t)) \to (0,0)$ as $s \to \infty$. Hence, the function $f(s+it) = \hat{a}(s,t) + i\tau\hat{b}(s,t)$ is holomorphic on $\mathbb{C}$. Since its first derivative is bounded, the function f is linear, and hence

$$(\hat{a}, \hat{b})(s,t) = (Ts + c,\ kt + d),$$

with real constants c and d. Recall that $T = k\tau$. Consequently, since $|\alpha| \geq 1$, we deduce that

$$\partial^\alpha(a - Ts, \vartheta - kt)(s_k,t_k) \to \partial^\alpha(\hat{a} - Ts, \hat{b} - kt)(0,t_0) = (0,0). \tag{22}$$

Clearly (22) contradicts (21). This completes the proof of the lemma. □

If $X = (X_1, X_2, X_3)$ is the Reeb vectorfield, we next introduce

$$Y(t,x,y) = \begin{pmatrix} X_2(t,x,y) \\ X_3(t,x,y) \end{pmatrix} \in \mathbb{R}^2.$$

Since $X(t,0,0) = (1/\tau, 0, 0)$ we have $Y(t,0) = 0$ and, therefore, by the mean value theorem

$$Y(t,x,y) = D(t,x,y)\begin{pmatrix} x \\ y \end{pmatrix},$$

with the matrix function

$$D(t,x,y) = \int_0^1 dY(t,\tau x,\tau y)d\tau.$$

In particular,

$$D(t,0,0) = dY(t,0,0) = \begin{pmatrix} f_{xy} & f_{yy} \\ -f_{xx} & -f_{xy} \end{pmatrix},$$

where the righthand side is evaluated at $(t,0,0)$. Introducing

$$z = \begin{pmatrix} x \\ y \end{pmatrix}$$

and the matrix functions

$$\begin{aligned} J(s,t) &= M(u(s,t)) \\ S(s,t) &= [a_t - a_s J(s,t)]D(u(s,t)), \end{aligned} \tag{23}$$

we can represent the equation (19) in the form

$$z_s + J(s,t)z_t + S(s,t)z = 0 \tag{24}$$

for $z = z(s,t)$. In order to simplify the presentation we assume now that the almost complex structure M is given by

$$M = J_0,$$

i.e. M agrees with the standard symplectic structure, which is a constant matrix. We shall reduce the case of a general M to this special case later on.

Next we will show that $\|z(s)\|_{L^2(S^1)}$ converges exponentially to 0 as $s \to \infty$. Define the operators $A(s)$ in $L^2(S^1, \mathbb{R}^2)$ by

$$A(s) = -J_0 \frac{d}{dt} - S(s,t) : W^{1,2}(S^1, \mathbb{R}^2) \subset L^2(S^1, \mathbb{R}^2) \to L^2(S^1, \mathbb{R}^2).$$

By Lemma 2.4, for every sequence $s_n \to \infty$, there is a sequence $c_n \in [0,1]$ such that

$$[\vartheta(s_n, t) - c_n] \to kt \bmod \mathbb{Z}$$

in $C^\infty_{loc}(\mathbb{R})$. Introducing the matrix

$$S^c_\infty(t) = -J_0 dY(kt + c, 0, 0)$$

we conclude, using Lemma 2.4, that

$$S(s_n, t) - S^{c_n}_\infty(t) \to 0, \tag{25}$$

as $n \to \infty$ in $C^\infty(\mathbb{R})$. For any $c \in \mathbb{R}$ the matrix $S^c_\infty(t)$ is symmetric. Indeed

$$\begin{aligned} S^c_\infty(t) &= -J_0 dY(kt+c,0,0) \\ &= \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}, \end{aligned}$$

where the last matrix is evaluated at $(kt+c,0,0)$. This allows to introduce the selfadjoint operators A^c_∞ in $L^2(S^1, \mathbb{R}^2)$ by

$$A^c_\infty = -J_0 \frac{d}{dt} - S^c_\infty(t): D_{A^c_\infty} \subset L^2(S^1) \to L^2(S^1),$$

with domain of definition $D_{A^c_\infty} = W^{1,2}(S^1)$. Since the inclusion $W^{1,2}(S^1) \to L^2(S^1)$ is compact, the resolvent of A^c_∞ is compact. Hence the spectrum $\sigma(A^c_\infty)$ consists of isolated eigenvalues of multiplicity at most 2, which accumulate at $+\infty$ and $-\infty$. The spectrum $\sigma(A^c_\infty)$ does not depend on the value of c. Observe that A^c_∞ is a relatively compact perturbation of the selfadjoint operator $-J_0 \frac{d}{dt}$ whose spectrum is the set $2\pi\mathbb{Z}$.

LEMMA 2.5. – *The periodic solution $x(t) = (kt, 0, 0) \in \mathbb{R}^3$ of the Reeb vectorfield X is nondegenerate if and only if*

$$0 \notin \sigma(A^c_\infty).$$

Proof. – Let φ_t denote the flow of $X = (X_1, Y)$. Then the derivative $d\varphi_t$ along the solution $x(t) = (kt,0,0)$ leaves the splitting $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R}^2 = \mathbb{R}X(m) \oplus \xi_m$, $m = (kt,0,0)$, invariant. Hence it is of the form:

$$d\varphi_t(t,0,0) = \begin{bmatrix} 1 & 0 \\ 0 & R(t) \end{bmatrix}$$

where $R(t)$ is the resolvent solution satisfying $\frac{d}{dt}R(t) = dY(kt,0,0)R(t)$ and $R(0) = I$. If the solution $x(t)$ is degenerate, then 1 is an eigenvalue of $R(1)$. Let $e \in \mathbb{R}^2$ be a corresponding eigenvector. Then $w(t) = R(t)e$ is 1-periodic and solves the equation

$$\dot{w}(t) = dY(kt+c,0)w(t), \tag{26}$$

or, equivalently,

$$-J_0 \frac{d}{dt} w(t) - S^c_\infty(t) w(t) = 0 \tag{27}$$

and hence $w \in \ker(A^c_\infty)$. Conversely if $0 \neq w \in \ker(A_\infty)$ then $w(t+1) = w(t)$ is a solution of (27) and (26). Hence $R(t)\ w(0)$ and

$w(t)$ are solutions of (26) having the same initial condition $w(0)$ and therefore $w(t) = R(t)w(0)$. Consequently $R(1)w(0) = w(1) = w(0)$ and hence 1 is an eigenvalue of $R(1)$ so that $x(t)$ is degenerate. $\square$

We shall use Lemma 2.5 in order to prove

LEMMA 2.6. – *Assume the periodic solution $x(t) = (kt, 0, 0)$ of the Reeb vector field X is non degenerate. Then the function $z(s,t) = (x(s,t), y(s,t))$ converges exponentially to zero in L^2: there exist $s_1 > 0$ and $r > 0$ such that*

$$\|z(s)\|_{L^2(S^1)} \leq \|z(s_1)\|_{L^2(S^1)} e^{-r(s-s_1)}$$

for $s \geq s_1$.

Proof. – We shall denote the $L^2(S^1)$-norm by $\| \ \|$. In view of Lemma 2.5 we have the estimate $\|A^c_\infty \eta\| \geq \delta_0 \|\eta\|$ for all $\eta \in D_{A^c_\infty}$ and all $c \in [0,1]$. The constant $\delta_0 > 0$ is independent of c. Consequently, in view of (25), we find constants $\delta_1 > 0$ and $s_1 \geq 0$ such that

$$\|A(s)\eta\| \geq \delta_1 \|\eta\|,$$

for all $\eta \in W^{1,2}(S^1, \mathbb{R}^2)$ and for all $s \geq s_1$. We introduce the smooth function $g(s)$ by

$$g(s) = \frac{1}{2}\|z(s)\|^2.$$

We differentiate g twice. In view of (24), we have $z_s = A(s)z(s)$. Abbreviating $z = z(s)$, we obtain the formula

$$g''(s) = 2\|A(s)z\|^2 + \langle (S - S^*)z, A(s)z\rangle - \langle S_s z, z\rangle.$$

Since $S^c_\infty(t)$ is symmetric, we conclude from (25) that $|S(s,t) - S^*(s,t)| \to 0$ as $s \to \infty$ uniformly in t. Using Lemma 2.4 we conclude from (23) that $|S_s(s,t)| \to 0$ as $s \to \infty$, uniformly in t. Consequently, denoting by $0(s)$ a positive function converging to 0 for $s \to \infty$ we can estimate

$$g''(s) \geq 2\|A(s)z\|(\|A(s)z\| - 0(s)\|z\|) - 0(s)\|z\|^2.$$

Choosing s_1 sufficiently large we find a constant $\delta > 0$, such that

$$g''(s) \geq \delta g(s), \qquad \text{if } s \geq s_1.$$

Since $g(s)$ is bounded, we deduce

$$g(s) \leq g(s_1)e^{-\sqrt{\delta}(s-s_1)}, \quad s \geq s_1,$$

and the Lemma is proved. $\square$

We will now use the exponential estimate of z in order to finish the proof of Theorem 1.3. Recall that a and ϑ: $[s_1, \infty) \times \mathbb{R} \to \mathbb{R}$ are smooth solutions of

$$\begin{aligned} a_s &= (\vartheta_t + xy_t) f(\vartheta, z) \\ a_t &= -(\vartheta_s + xy_s) f(\vartheta, z). \end{aligned} \tag{28}$$

Moreover $f(\vartheta, 0) \equiv \tau$ and we can write

$$f(\vartheta, z) = \tau + \int_0^1 f_z(\vartheta, \mu z) d\mu z \equiv \tau + k(s,t) z.$$

Define the 1-periodic functions $\tilde{a}, \tilde{b}$: $[s_1, \infty) \times S^1 \to \mathbb{R}$ by

$$\begin{aligned} \tilde{a}(s,t) &= a(s,t) - Ts \\ \tilde{\vartheta}(s,t) &= \vartheta(s,t) - kt. \end{aligned}$$

We have that

$$\begin{aligned} \partial^\beta \tilde{a}(s,t) &\to 0 \\ \partial^\beta \tilde{\vartheta}(s,t) &\to 0, \end{aligned}$$

for $|\beta| \geq 1$ as $s \to \infty$, uniformly in t. The equation (28) becomes

$$\begin{aligned} \tilde{a}_s &= \tilde{\vartheta}_t + xy_t + (\vartheta_t + xy_t) kz \\ \tilde{\vartheta}_s &= -\tilde{a}_t - xy_s - (\vartheta_s + xy_s) kz. \end{aligned} \tag{29}$$

Abbreviating

$$w(s,t) = \begin{pmatrix} \tilde{a} \\ \tilde{\vartheta} \end{pmatrix},$$

we can write the equation (29) in the form

$$w_s + J_0 w_t = h, \qquad J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{30}$$

with a smooth function h: $[s_1, \infty) \times S^1 \to \mathbb{R}$ satisfying, in view of the exponential estimate of $\|z(s)\|$, the exponential estimate

$$\|h(s)\| \leq M e^{-rs},$$

for some constants M and $r > 0$. Integrating the equation (30) over the interval $[0, 1]$ we obtain

$$\left| \int_0^1 w_s(s,t) dt \right| = \left| \int_0^1 h(s,t) dt \right| \leq \|h(s)\|_{L^2(S^1)} \leq M e^{-rs},$$

for $s \geq s_1$. In particular, we have

$$\left|\int_0^1 \vartheta_s(s,t)dt\right| \leq Me^{-rs}. \tag{31}$$

LEMMA 2.7. – *If the periodic solution $x(t) = (kt, 0, 0)$ of X is non degenerate, then there exists a constant $\vartheta_0 \in \mathbb{R}$ such that*

$$\partial^\alpha\left[\vartheta(s,t) - kt - \vartheta_0\right] \to 0$$

as $s \to \infty$, uniformly in t, for all derivatives α.

Without loss of generality we shall put later on $\vartheta_0 = 0$.

Proof. – In view of (31), the meanvalues of the (in the variable t) periodic functions $\vartheta(s,t) - kt$ over $t \in [0,1]$ is a Cauchy sequence. By Lemma 2.4, $\partial^\alpha\left[\vartheta(s,t) - kt\right] \to 0$ as $s \to \infty$, if $|\alpha| \geq 1$, uniformly in t. It follows that $\vartheta(s,\cdot) - kt \to \vartheta_0$ in $C^\infty(\mathbb{R})$ as $s \to \infty$, as claimed in the Lemma. □

With Lemma 2.4 and Lemma 2.7 the proof of Theorem 1.3 is complete.

We also observe that with these Lemmata, putting $\vartheta_0 = 0$, we have

$$S(s,t) \to J_0 dY(kt,0,0)$$

as $s \to \infty$. Abbreviating in the following

$$\begin{aligned} S_\infty(t) &= -J_0 dY(kt,0,0)\\ A_\infty &= -J_0\frac{d}{dt} - S_\infty(t)\end{aligned}$$

we have

$$S(s,t) \to S_\infty(t), \quad s \to \infty$$

in $C^\infty(\mathbb{R})$. Summarizing we consider a finite energy cylinder $\tilde{v} = (a,v)\colon \mathbb{R} \times S^1 \to \mathbb{R} \times M$ and assume that it is nondegenerate in the sense of Theorem 1.2: there exists a nondegenerate T-periodic solution $x(t)$ of the Reeb vectorfield $\dot{x} = X(x)$ such that $v(s,\cdot) \to x$ as $s \to \infty$ in C^∞. The period is $T = k\tau$, with the minimal period τ. Then there are local coordinates in a tubular neighborhood of $x \subset M$ in which the cylinder is represented by the functions

$$\tilde{u} = (a,u)\colon [s_0,\infty) \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}^3$$

for some large $s_0 > 0$.

The functions $\tilde{u}(s,t) = (a(s,t), \vartheta(s,t), x(s,t), y(s,t))$ are periodic in t except ϑ which satisfies $\vartheta(s,t+1) = \vartheta(s,t) + k$. The convergence to the periodic solution becomes

$$u(s,t) \to (kt, 0, 0) \in \mathbb{R}^3 \quad (s \to \infty)$$

and (a,u) satisfy as $s \to \infty$ the properties of Lemma 2.4. The functions solve the equations

$$\begin{aligned} a_s &= (\vartheta_t + xy_t) f(u) \\ a_t &= -(\vartheta_s + xy_s) f(u) \end{aligned} \tag{32}$$

$$z_s + J_0 z_t + S(s,t) z = 0, \tag{33}$$

where $z = (x,y)$. Here $f = f(t,z)\colon \mathbb{R}^3 \to \mathbb{R}$ is smooth, 1-periodic in t and satisfies $f(t,0) = \tau$ and $df(t,0) = 0$. Moreover $S(s,t) \to S_\infty(t)$ as $s \to \infty$ in C^∞. The matrix S_∞ is periodic in t, symmetric and satisfies $0 \notin \sigma(A_\infty)$.

Our aim is to prove the following result about the asymptotic behaviour of nondegenerate finite energy planes locally near the limit periodic solution.

THEOREM 2.8 (Asymptotic behaviour of nondegenerate finite energy planes). – *Assume the functions* $(a,u)\colon [s_0,\infty) \times \mathbb{R} \to \mathbb{R}^4$ *meet the above conditions. Then*
either

(i) *There exists* $c \in \mathbb{R}$, *such that*

$$(a(s,t), \vartheta(s,t), z(s,t)) = (Ts + c, kt, 0).$$

or

(ii) *There are constants* $c \in \mathbb{R}$, $d > 0$ *and* $M_\alpha > 0$ *for all* $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N} \times \mathbb{N}$ *such that*

$$\begin{aligned} |\partial^\alpha [a(s,t) - Ts - c]| &\le M_\alpha e^{-d \cdot s} \\ |\partial^\alpha [\vartheta(s,t) - kt]| &\le M_\alpha e^{-d \cdot s} \end{aligned}$$

for all $s \ge s_0$, $t \in \mathbb{R}$. *Moreover*

$$z(s,t) = e^{\int_{s_0}^{s} \gamma(\tau) d\tau} [e(t) + r(s,t)].$$

Here $e \neq 0$ *is an eigenvector of the selfadjoint operator* A_∞ *corresponding to a negative eigenvalue* $\lambda < 0$ *and* $\gamma\colon [s_0,\infty) \to \mathbb{R}$ *is a smooth function* *satisfying* $\gamma(s) \to \lambda$ *as* $s \to \infty$. *In particular* $e(t) \neq 0$ *pointwise and the remainder* $r(s,t)$ *satisfies*

$$\partial^\alpha r(s,t) \to 0$$

for all derivatives $\alpha = (\alpha_1, \alpha_2)$, *uniformly in* $t \in \mathbb{R}$.

We point out, that the first alternative does not occur if the data (a,u) is the restriction of a finite energy plane in view of its global properties. Indeed, if (i) holds we have $\tilde{v}(s,t) = (Ts + c, x(t))$ for all $s \geq s_0$, where x is a solution of the Reeb vectorfield X. Since the original equation is translation invariant (in s), $\tilde{v}(s,t) = (Ts + c, x(t))$ for all $(s,t) \in \mathbb{R} \times S^1$, and hence is a trivial solution. We conclude that $v^* d\lambda = 0$ (introduction), contradicting

$$\int_{\mathbb{R}\times S^1} v^* d\lambda = T > 0$$

in Theorem 1.2.

Remark 2.9. – We have assumed above, that the representation of the given almost complex structure in ξ_m, namely the matrix $M(m)$ agrees with the standard structure J_0 (in the bases (e_1, e_2) of ξ). This is no loss of generality as we shall demonstrate next by a change of coordintes in ξ_m. Recall by (12), that $-J_0 M(m) > 0$, *i.e.* is a positive definite matrix, which, in addition, is symplectic. Define, in the coordinates $z \in \mathbb{R}^2$ of ξ_m at the point $m = (\vartheta, x, y)$, the new coordinates $z' \in \mathbb{R}^2$ by

$$z = T(m)z'$$
$$T(m) = (-J_0 M(m))^{-1/2}.$$

Then $T = T(m)$ is symmetric and symplectic, so that $TJ_0T = J_0$, and we claim that

$$T(m)^{-1} M(m) T(m) = J_0, \qquad \text{all } m. \tag{34}$$

Indeed with T also T^{-1} is symmetric and symplectic: $T^{-1}J_0T^{-1} = J_0$. Since, by definition, $T^{-2} = -J_0 M$ and $J_0^{-1} = -J_0 = J_0^T$, we have $T^{-1} = J_0 T(-J_0)$ and hence $T^{-1}M = J_0 T(-J_0 M) = J_0 T T^{-2} = JT^{-1}$. We have proved

$$T^{-1}(m) M(m) = J_0 T(m)^{-1}, \tag{35}$$

which is equivalent to (34). Introduce now

$$T(s,t) \equiv T(u(s,t)), \quad m = u(s,t)$$

and define ξ by

(36) $$z(s,t) = T(s,t)\xi(s,t).$$

Using (34) and (35) one derives from the equation (24) for $z(s,t)$ that ξ is a solution of the equation:

$$\xi_s + J_0\xi_t + \hat{S}(t,s)\xi = 0,$$

where

$$\hat{S}(t,s) = J_0 T^{-1} T_t + T^{-1} T_s + T^{-1} S T.$$

The equation has the same form as (24). Moreover, from $T(s,t) \to T_\infty(t)$ as $s \to \infty$ in $C^\infty(\mathbb{R})$, where

$$T_\infty(t) = (-J_0 M(kt,0,0))^{-1/2},$$

we find that $\hat{S}(t,s) \to \hat{S}_\infty(t)$ as $s \to \infty$ in $C^\infty(\mathbb{R})$, where

$$\hat{S}_\infty(t) = -J_0[T_\infty^{-1} dY T_\infty - T_\infty^{-1}\dot{T}_\infty].$$

The dot denotes the derivative of T_∞ in t, and $dY(m)$ is the restriction of the linearized Reeb-vector-field $\pi_m dX(m)\pi_m$ along $m = (kt,0,0)$, as introduced above. Since JdY is symmetric, as we have seen, one verifies that $\hat{S}_\infty(t)$ is symmetric, using that T_∞ is symmetric and symplectic. Hence the operator

$$\hat{A}_\infty = -J_0\frac{d}{dt} - \hat{S}_\infty(t)$$

is selfadjoint, and, as before, $0 \notin \sigma(\hat{A}_\infty)$ if and only if the periodic solution of the Reeb vectorfield X is non degenerate. This follows immediately from Lemma 2.5, observing that the linear and periodic vectorfield in $z \in \mathbb{R}^2$:

$$\dot{z} = dY(kt,0)z$$

is, by the periodic transformation $z = T_\infty(t)\zeta$, transformed into the vector field

$$\dot{\zeta} = [T_\infty^{-1} dY T_\infty - T_\infty^{-1}\dot{T}_\infty]\zeta.$$

It follows that Theorem 2.8 holds true for $\xi(s,t)$ replacing $z(s,t)$ in the statement, and hence, by formula (36), also for the original $z(s,t)$, for every almost symplectic structure compatible with λ.

3. PROOF OF THE ASYMPTOTIC FORMULA (THEOREM 2.8)

We shall consider the smooth function $(a, \vartheta, z)\colon [s_0, \infty) \times \mathbb{R} \to \mathbb{R}^4$ which meets the assumptions of Theorem 2.8, *i.e.* solves the equations (32) and (33) and has the asymptotic properties as described in Lemma 2.4. In the following we shall denote by $z(s)$ the periodic function $z(s, t)$ and abbreviate $L^2 \equiv L^2(S^1)$. We start investigating the alternative (i) of Theorem 2.8. Recall that the period of the limiting periodic solution $x(t)$ of the Reeb vector field satisfies $T = k\tau$ with the minimal period τ. In order to simplify the notation in the proof we assume

$$T = \tau = k = 1;$$

furthermore we often drop the index in $J = J_0$. We begin with a proposition concerning the L^2-convergence of (x, y). It is of course related to our previous discussion. However the conclusion is now somewhat stronger, since we have convergence of $u(s, \cdot)$ as $s \to \infty$.

PROPOSITION 3.1. – *Assume* $(a, \vartheta, z)\colon [s_0, \infty) \times \mathbb{R} \to \mathbb{R}^4$ *solves the equations* (32) *and* (33) *and has the asymptotic properties of Lemma* 2.4. *Then* $\|z(s)\|_{L^2} \to 0$ *as* $s \to \infty$. *If*

$$\|z(s^*)\|_{L^2} = 0$$

for some $s^* \geq s_0$, *then*

$$(a(s,t), \vartheta(s,t), z(s,t)) \equiv (s + c, t, 0),$$

for all $s \geq s_0$ *and* $t \in \mathbb{R}$ *with some constant* $c \in \mathbb{R}$.

Proof. – The first claim is an immediate consequence of Lemma 2.4. Assume that $\|z(s^*)\| = 0$, then $z(s^*, t) = 0$ for all $t \in \mathbb{R}$ and we pick a $t^* \in \mathbb{R}$ such that $z(s^*, t^*) = 0$. Since $z(s, t)$ solves the partial differential equation (33) there is an open neighborhood $D \subset \mathbb{R}^2$ of the zero (s^*, t^*) of z, on which z can be represented as

$$z(s,t) = \Phi(\zeta) h(\zeta), \quad \zeta = s + it \in \mathbb{C}.$$

Here $\Phi\colon D \to GL(\mathbb{C})$ is continuous and $h\colon D \subset \mathbb{C} \to \mathbb{C}$ is a holomorphic function. (This is the generalized similarity principle for which we refer to H. Hofer and E. Zehnder [9]). Since (s^*, t^*) is a cluster point of zeroes of z we conclude that $h \equiv 0$ on D and hence $z(s, t) \equiv 0$ on D. Consequently

$z(s,t) = 0$ for all $s \geq s_0$ and $t \in \mathbb{R}$. We conclude from the equation (32), using that

$$f(t,0) = 1,$$

$$a_s = \vartheta_t \qquad \text{and} \qquad a_t = -\vartheta_s.$$

By Lemma 2.4, we can expand into Fourier series in t:

$$a(s,t) = s + \sum_{k \in \mathbb{Z}} a_k(s) e^{2\pi i k t}$$
$$\vartheta(s,t) = t + \sum_{k \in \mathbb{Z}} b_k(s) e^{2\pi i k t}.$$

The coefficients solve the equations

$$\frac{d}{ds}\begin{pmatrix} a_k \\ b_k \end{pmatrix} = \begin{pmatrix} 0 & 2\pi i k \\ -2\pi i k & 0 \end{pmatrix}\begin{pmatrix} a_k \\ b_k \end{pmatrix}, \qquad k \in \mathbb{Z}.$$

whose solutions are all periodic. In view of the asymptotic behaviour *i.e.* Lemma 2.4 we, therefore, conclude $a_k(s) = b_k(s) \equiv 0$, if $k \neq 0$, $a_0(s) = \text{const}$ and $b_0(s) = 0$, as claimed in the proposition. □

For the remainder of this section we shall assume that $\|z(s)\|_{L^2} \neq 0$ for all $s \geq s_0$. We know that $\|z(s)\| \to 0$ as $s \to \infty$. Using the assumption $0 \notin \sigma(A_\infty)$, we shall derive an exponential formula.

LEMMA 3.2

$$\|z(s)\|_{L^2} = e^{\int_{s_0}^{s} \alpha(\tau) d\tau} \cdot \|z(s_0)\|_{L^2},$$

for a smooth function $\alpha: [s_0, \infty) \to \mathbb{R}$ satisfying $\lim_{s \to \infty} \alpha(s) = \lambda < 0$ and $\lambda \in \sigma(A_\infty)$.

Proof. – We assume $\|z(s)\| \neq 0$ and abbreviate $\| \; \| \equiv \| \; \|_{L^2}$. Introduce the smooth function

$$\xi(s,t) = \frac{z(s,t)}{\|z(s)\|}, \qquad \text{then} \qquad \|\xi(s)\| = 1.$$

Differentiating in s, using that z solves the equation (33) we obtain

$$\xi_s = -J\xi_t - S\xi - \frac{1}{2}\frac{\frac{d}{ds}\|z\|^2}{\|z\|^2}\xi, \tag{37}$$

abbreviating $J \equiv J_0$. Denoting by $\langle \cdot, \cdot \rangle$ the $L^2(S^1)$ scalarproduct we conclude from $\langle \xi, \xi \rangle = 1$ that $\langle \xi_s, \xi \rangle = 0$, and inserting the above equation we find

$$\frac{1}{2} \frac{\frac{d}{ds}||z||^2}{||z||^2} = \langle -J\xi_t - S\xi, \xi \rangle \equiv \alpha(s), \tag{38}$$

so that

$$||z(s)|| = e^{\int_{s_0}^{s} \alpha(\tau) d\tau} ||z(s_0)||. \tag{39}$$

We have to show that the smooth function α converges as $s \to \infty$ to a negative eigenvalue of A_∞. Dropping the subscript we recall that $A = A_\infty = -J\frac{d}{dt} - S_\infty(t)$ and write

$$-J\frac{d}{dt} - S(s,t) = A + \varepsilon \qquad \text{and} \qquad \varepsilon \equiv (S_\infty - S).$$

Then, by the previous section, $\varepsilon(s,t) \to 0$ as $s \to \infty$ in $C^\infty(\mathbb{R})$. Denoting by the prime the derivative in the s-variable we can write the equation (37) for ξ and the smooth function α in (38) as

$$\xi' = (A + \varepsilon)\xi - \alpha\xi \tag{40}$$

$$\alpha(s) = \langle (A + \varepsilon)\xi, \xi \rangle. \tag{41}$$

We differentiate this function. Using the selfadjointnes of A and the identity $\langle \xi, (A + \varepsilon)\xi - \alpha\xi \rangle = 0$, we obtain

$$\alpha' = 2||\xi'||^2 - \langle \varepsilon\xi, \xi' \rangle + \langle \varepsilon'\xi, \xi \rangle. \tag{42}$$

In view of $||\xi|| = 1$, we have the estimates:

$$|\langle \varepsilon\xi, \xi' \rangle| \leq 0(s)||\xi'(s)||$$
$$|\langle \varepsilon'\xi, \xi \rangle| \leq 0(s),$$

where $0(s) \to 0$, as $s \to \infty$. Consequently

$$\alpha'(s) \geq 2||\xi'|| \left[||\xi'|| - 0(s)\right] - 0(s). \tag{43}$$

Next we claim that α is bounded

$$|\alpha(s)| \leq C, \quad s \geq s_0 \tag{44}$$

for some $C > 0$. Arguing by contradiction we assume that α is not bounded from above. Then there is a sequence $s_n \to \infty$ such that $\alpha(s_n) \to \infty$. On the other hand if $\alpha(s) \geq \delta > 0$ for all s large, then $\|z(s)\| \to +\infty$ in view of (39) which contradicts $\|z(s)\| \to 0$, as $s \to \infty$. Hence there exists another sequence $s'_n \to \infty$ such that $\alpha(s'_n) < \delta$ and so the function α has an "oscillatory" behaviour as $s \to \infty$. Since $\sigma\left(-J\frac{d}{dt}\right) = 2\pi\mathbb{Z}$ it follows from Kato's perturbation theory for isolated eigenvalues of selfadjoint operators [10], that there is an $L > 0$ and an integer m, so that every interval of $\mathbb{R}$ having length equal to L contains at most m points of the spectrum $\sigma(A = -J\frac{d}{dt} - S_\infty)$ belonging to the perturbed operator A. Consequently there are spectral gaps of fixed size: there is a sequence $r_n \to \infty$ and a constant $d > 0$ satisfying

$$[r_n - d, r_n + d] \cap \sigma(A) = \emptyset. \tag{45}$$

Hence by the oscillatory behaviour of α we find a sequence $\tau_n \to \infty$ satisfying $\alpha(\tau_n) = r_n$ and $\alpha'(\tau_n) \leq 0$. It then follows from (43) that $\|\xi'(\tau_n)\| \to 0$ as $n \to \infty$. Since, by (40), $\xi' = A\xi - \alpha(s)\xi + \varepsilon$, we can estimate

$$\begin{aligned} \|\xi'(s)\| &\geq \|[A - \alpha(s)]\xi\| - 0(s) \\ &\geq \operatorname{dist}(\alpha(s), \sigma(A)) - 0(s). \end{aligned} \tag{46}$$

Here we have used, that $\|\xi\| = 1$ and that for the resolvent of a selfadjoint operator $\|(A - \mu)^{-1}\| = [\operatorname{dist}(\mu, \sigma(A))]^{-1}$. Using (46) we conclude from (45) that $\|\xi'(\tau_n)\| \geq \frac{d}{2} > 0$ contradicting $\|\xi'(\tau_n\| \to 0$ as $m \to \infty$. This contradiction shows that α is indeed bounded from above. The same argument shows that α is also bounded from below, proving the claim (44).

There exists a sequence $s_n \to \infty$ such that $\|\xi'(s_n)\| \to 0$. Indeed, otherwise, for all large s, $\|\xi'\| \geq \delta > 0$, hence $\alpha' \geq \delta^2$ in view of (43), and $\alpha(s) \geq \delta^2(s - s_0) + \alpha(s_0)$, so that $\|z(s)\| \to \infty$, in view of (39). This contradicts $\|z(s)\| \to 0$. Since α is bounded, the sequence $\alpha(s_n)$ has a convergent subsequence, $\lim_{n\to\infty} \alpha(s_n) = \lambda$ and we conclude from (46) that $\lambda \in \sigma(A)$. Since α is bounded, every sequence $\alpha(\tau_n)$, $\tau_n \to \infty$ possesses a convergent subsequence, $\lim_{n\to\alpha} \alpha(\tau_n) = \mu$ and we claim that $\mu = \lambda$. Indeed, if f.e. $\mu < \lambda$, then α has again an oscillatory behaviour and we can pick $\mu < \nu < \lambda$ satisfying $\nu \notin \sigma(A)$, and a sequence $s_n \to \infty$ satisfying $\alpha(s_n) = \nu$ and $\alpha'(s_n) \leq 0$. Consequently, in view of (43), $\|\xi'(s_n)\| \to 0$ and hence, in view of (46), $\nu \in \sigma(A)$, contradicting

$\nu \notin \sigma(A)$. We have proved that $\lim_{s\to\infty} \alpha(s) = \lambda$ and $\lambda \in \sigma(A)$. Clearly $\lambda \leq 0$, since otherwise $\|z(s)\| \to +\infty$. But then $\lambda < 0$ in view of our assumption $0 \notin \sigma(A)$. This finishes the proof of Lemma 3.2. □

For the solution $z = z(s,t)$ of the equation (33) we have, in view of Lemma 3.2, the formula

$$z(s,t) = \|z(s_0)\| e^{\int_{s_0}^{s} \alpha(\tau) d\tau} \xi(s,t), \qquad \|\xi(s)\| = 1. \tag{47}$$

The exponential decay of z as $s \to \infty$ will be concluded from the C^∞ bounds of ξ and α in the next Lemma.

LEMMA 3.3. – *Define, as in Lemma* 3.2,

$$\xi(s,t) = \frac{z(s,t)}{\|z(s)\|}$$

$$\alpha(s) = \langle A_\infty \xi, \xi \rangle + \langle (S_\infty - S)\xi, \xi \rangle.$$

Then, for every $\beta = (\beta_1, \beta_2) \in \mathbb{N} \times \mathbb{N}$ *and* $j \in \mathbb{N}$:

$$\sup_{s,t} |\partial^\beta \xi(s,t)| < \infty$$

$$\sup_{s} |\partial^i \alpha(s)| < \infty.$$

Proof. – By (40) the 1-periodic smooth function $\xi: [s_0, \infty) \times \mathbb{R} \to \mathbb{R}^2$ solves the partial differential equation

$$\bar\partial \xi = -S(s,t)\xi - \alpha(s)\xi, \qquad \bar\partial = \frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t}. \tag{48}$$

The smooth functions S and α satisfy

$$\sup |\partial^\beta S| < \infty \qquad \text{and} \qquad \sup |\alpha| < \infty,$$

for all multi-indices β. The supremum is taken over all $s \geq s_0$ and $t \in \mathbb{R}$. Recall also that the constant matrix J_0 satisfies $J_0^2 = -1$. In order to establish at first $W^{k,2}_{\text{loc}}$-estimates for ξ we make use of the following well-known a-priori estimate for the elliptic operator $\bar\partial$. If $B \subset \mathbb{R}^2$ is a disc of radius $R > 0$, $k \geq 1$ an integer and $p > 1$, then there exists a constant $M = M(k,p,R)$ such that

$$\|\varphi\|_{W^{k,p}(B)} \leq M \|\bar\partial \varphi\|_{W^{k-1,p}(B)} \tag{49}$$

for every $\varphi \in C^\infty(\mathbb{R}^2)$ having its support in the interior of B. For a proof we refer to E. Stein [12] or to C. Abbas and H. Hofer [1].

We fix in the following $p = 2$ and an integer $R \geq 2$ and define for every $x^* \in \mathbb{R}^2$ a monotone decreasing sequence of closed discs centered at x^* and having radii R_j, $j \geq 0$:

$$D_j = B(x^*, R_j),\ R_j = \frac{R}{2}(1 + 2^{-j}).$$

Then $D_j \supset D_{j+1} \supset \ldots \supset B\left(x^*, \frac{R}{2}\right)$. With ζ_j, $j \geq 1$ we denote a sequence of bump-functions $\zeta_j \in C^\infty(\mathbb{R}^2, \mathbb{R})$ satisfying $\zeta_j \equiv 1$ on D_j and having supports in the interior of the larger discs D_{j-1}, moreover, $0 \leq \zeta_j \leq 1$. Abbreviating $\zeta_j = \zeta$ and $D_0 = D$ we conclude from (49)

$$\|\xi\|_{W^{1,2}(D)} \leq \|\zeta\xi\|_{W^{1,2}(D)} \leq M\|\bar{\partial}(\zeta\xi)\|_{L^2(D)}.$$

Using $\bar{\partial}(\zeta\xi) = (\bar{\partial}\zeta)\xi + \zeta(\bar{\partial}\xi)$ and the equation (48) for ξ, namely $\bar{\partial}\xi = -S\xi - \alpha\xi$, we can estimate

$$\|\xi\|_{W^{1,2}(D_1)} \leq M \cdot C\|\xi\|_{L^2(D)},$$

where the constant C depends on $\|\zeta\|_{C^1(\mathbb{R}^2)}$, $\sup|S|$ and $\sup|\alpha|$. Now, by definition of ξ, the L^2-norm over the time $\|\xi(s)\|_{L^2(S)} = 1$ for all s, and hence $\|\xi\|_{L^2(D)} \leq (2R)^2$ independently of the choice of the center x^* of the disc D. Consequently there exists a constant $C_1 > 0$ independent of x^*, such that for all discs

$$\|\xi\|_{W^{1,2}(D_1)} \leq C_1.$$

From (42) we conclude that $|\frac{\partial}{\partial s}\alpha(s)| \leq c_1$ for all $s \geq s_0$. Using the *a-priori* estimates

$$\|\xi\|_{W^{k,2}(D_k)} \leq M\|\bar{\partial}(\zeta_k\xi)\|_{W^{k-1,2}(D_{k-1})}$$

for all $k \geq 1$, one finds, proceeding inductively, constants $C_k > 0$ such that

$$\|\xi\|_{W^{k,2}(D_k)} \leq C_k,\ k \geq 1. \tag{50}$$

These constants do not depend on x^*. Here one uses crucially that the derivatives $\partial^\beta S$ are bounded on the whole space. The estimate $|\partial^k\alpha(s)| \leq c_k$ for all $s \geq s_0$ follows at each iteration step using the equation (42) for α. Finally, by means of the well-known Sobolev-embedding theorems we deduce from the uniform $W^{k,2}_{\text{loc}}$-estimate (50) the pointwise estimates for $j \geq 0$:

$$|\partial^\beta\xi(s,t)| \leq c'_j,$$
$$|\partial^j\alpha(s)| \leq c'_j$$

for $|\beta| \leq j$, and for all $s \geq s_0$ and $t \in \mathbb{R}$. This finishes the proof of Lemma 3.3. □

Recall that $\alpha(s) \to \lambda$ as $s \to \infty$ and $\lambda < 0$. We deduce from Lemma 3.3 and the representation (47) for z:

COROLLARY 3.4. – *Let $0 < r < |\lambda|$, then*

$$|\partial^\beta z(s,t)| \leq Me^{-rs},$$

for all derivatives β, with constants $M = M_\beta$.

PROPOSITION 3.5. – *Recall that $\alpha(s) \to \lambda$ as $s \to \infty$ with $\lambda < 0$ and $\lambda \in \sigma(A_\infty)$. There exists an eigenvector $e(t+1) = e(t)$ of $A_\infty e = \lambda e$ satisfying $\|e\|_{L^2(S^1)} = 1$ and*

$$\xi(s,t) \to e(t) \quad \text{in} \quad C^\infty(\mathbb{R}) \quad \text{as} \quad s \to \infty.$$

Proof. – In view of the previous Lemma it is sufficient to prove the convergence in $W^{1,2}(S^1)$. We start with

LEMMA 3.6. – *Let $E \subset L^2(S^1)$ be the eigenspace of A_∞ belonging to $\lambda \in \sigma(A_\infty)$. Then*

$$\text{dist}(\xi(s), E) \to 0 \quad \text{as} \quad s \to \infty,$$

where the distance is taken in the $W^{1,2}(S^1)$-norm.

Proof of Lemma 3.6. – Arguing by contradiction we assume that $\text{dist}(\xi(s_n), E) \geq \varepsilon$ for some $\varepsilon > 0$ and for a sequence $s_n \to \infty$. Since, by Lemma 3.3, the derivatives of $\xi(s,t)$ are uniformly bounded, there is a constant $c > 0$, such that

$$\|\xi(s) - \xi(s')\|_{W^{1,2}} \leq c|s - s'|.$$

Therefore we find intervals around s_n, $I_n = [s_n - d, s_n + d]$, with some $d > 0$, such that

$$\text{dist}(\xi(s), E) \geq \frac{\varepsilon}{2}, \qquad s \in I_n. \tag{51}$$

We claim that there exists a sequence $\tau_k \in I_{n_k}$ such that

$$\|A_\infty \xi(\tau_k) - \lambda \xi(\tau_k)\|_{L^2(S^1)} \to 0 \tag{52}$$

as $k \to \infty$. Indeed, we recall, using (46) and $\|\xi\| = 1$, that

$$\|\xi'(s)\| \geq \|[A - \alpha(s)]\xi\| - 0(s) \geq \|[A - \lambda]\xi\| - |\lambda - \alpha(s)| - 0(s).$$

Therefore it is sufficient to prove that $\|\xi'(\tau_k)\| \to 0$. If not, then we have $\|\xi'(s)\| \geq \delta$ for $s \in I_n$, all n, with some $\delta > 0$. Hence $\alpha'(s) \geq \delta^2$ in view of the estimate (43), $s \in I_n$ and hence, by the mean value theorem $|\alpha(s_n + d) - \alpha(s_n - d)| \geq 2d\delta^2 > 0$ for all n, which contradicts the convergence $\alpha(s) \to \lambda$ as $s \to \infty$ and proves the claim (52).

Now, by Lemma 3.3 we know that $\|\xi(\tau_k)\|_{W^{2,2}(S^1)} \leq C$ and hence, since $W^{2,2}(S^1)$ is compactly embedded in $W^{1,2}(S^1)$, we find a subsequence which converges in $W^{1,2}$ such that $\xi(\tau_k) \to e \in W^{1,2}(S^1)$. It follows from $\|\xi(s)\| = 1$ that $\|e\|_{L^2} = 1$ and, in view of (50), that $A_\infty e = \lambda e$, so that $e \in E$. This contradicts (51) and hence the Lemma 3.6 is proved. □

LEMMA 3.7. – *There exists* $e \in E$, i.e. $A_\infty e = \lambda e$ *such that* $\|e\|_{L^2(S^1)} = 1$ *and*

$$\xi(s) \to e \quad \text{in} \quad W^{1,2}(S^1) \quad \text{as} \quad s \to \infty.$$

Proof. – Let P denote the orthogonal projection of $L^2(S^1)$ onto the eigenspace E of A_∞, and define

$$\zeta(s) = P\xi(s).$$

Recall that, by (40), ξ solves the equation $\xi' = A_\infty\xi + \varepsilon\xi - \alpha\xi$. Using $A_\infty(P\xi) = PA_\infty\xi = \lambda(P\xi)$ we find for $\zeta(s)$ the equation

$$\zeta' = [\lambda - \alpha(s)]\zeta + P\varepsilon\xi. \tag{53}$$

From Lemma 3.6 we conclude

$$\|\zeta(s) - \xi(s)\|_{L^2} \to 0 \quad \text{and} \quad \|\zeta(s)\|_{L^2} \to 1 \tag{54}$$

as $s \to \infty$. Therefore $\|\zeta(s)\| \geq \frac{1}{2}$ for large s and we define the smooth function η by

$$\eta(s,t) = \frac{\zeta(s,t)}{\|\zeta(s)\|}, \qquad \text{then} \qquad \|\eta(s)\| = 1.$$

This function satisfies the differential equation

$$\eta' = \frac{\zeta'}{\|\zeta\|} - \frac{1}{2}\frac{\frac{d}{ds}\|\zeta\|^2}{\|\zeta\|^2} \cdot \frac{\zeta}{\|\zeta\|}.$$

Using that $\langle \eta', \eta\rangle = 0$ we find, inserting the equation (53) for ζ' that

$$\eta' = \frac{P\varepsilon\xi}{\|\zeta\|} - \frac{\langle P\varepsilon\xi, \eta\rangle}{\|\zeta\|} \cdot \eta.$$

Since, in the L^2-norms, $||\xi|| = ||\eta|| = 1$ and $||\zeta|| \geq \frac{1}{2}$ we find the estimate

$$||\eta'(s)||_{L^2(S^1)} \leq 4||\varepsilon(s)||_{L^2(S^1)},$$

where $\varepsilon(s,t) = S(s,t) - S_\infty(t)$. By definition, $S(s,t) = N(\vartheta(s,t), z(s,t))$ and $S_\infty(t) = N(t,0)$ for a smooth matrix function N. Therefore we can estimate:

$$||\varepsilon(s)||_{L^2} \leq C(||z(s)||_{L^2} + ||\tilde{\vartheta}(s)||_{L^2}),$$

with the t-periodic function $\tilde{\vartheta}(s,t) = \vartheta(s,t) - t$. We shall prove below that $||\tilde{\vartheta}(s)||_{L^2} \leq Ce^{-rs}$ for some $r > 0$. Consequently we find together with the exponential estimate (47) for z, that

$$||\eta'(s)||_{L^2} \leq Ce^{-rs} \tag{55}$$

for some $r > 0$. Take any sequence $s_n \to \infty$, since $\xi(s_n)$ is, by Lemma 3.3, bounded in $W^{2,2}(S^1)$ it possesses a subsequence converging in $W^{1,2}(S^1)$, such that $\xi(s_n) \to e \in W^{1,2}(S^1)$. From Lemma 3.6 we conclude that $e \in E$ and it remains to prove the uniqueness of this limit. Assume $\xi(s_n) \to e$ and $\xi(\tau_n) \to e'$ in $W^{1,2}(S^1)$, then, by (54), $\eta(s_n) \to e$ and $\eta(\tau_n) \to e'$ in $L^2(S^1)$. Using (55) we can estimate in L^2

$$||\eta(s_n) - \eta(\tau_n)|| \leq \left| \int_{s_n}^{\tau_n} ||\eta'(s)|| ds \right| \leq C \left| \int_{s_n}^{\tau_n} e^{-rs} ds \right| \to 0$$

for $n \to \infty$. Hence $e = e'$ and the proof of Lemma 3.7 and, therefore, also the proof of Proposition 3.5 is finished. □

As a consequence of Lemma 3.7 and the C^∞ bounds of Lemma 3.6 we have $\xi(s,t) \to e(t)$ as $s \to \infty$ in $C^\infty(\mathbb{R})$. Now define $r(s,t) = \xi(s,t) - e(t)$. Using the equation (40) for the derivative of ξ in the s-variable, the convergence $\alpha(s) \to \lambda$, and $(A - \lambda)e = e$, we deduce inductively that $\partial^\beta r(s,t) \to 0$ as $s \to \infty$, uniformly in t, for all derivatives. Recalling formula (47), we have established the asymptotic formula for the function z in theorem 2.8. It remains to demonstrate the exponential decay of the functions $a - Ts$ and $\vartheta - kt$. Again, for simplicity of the notation, we assume $T = k = \tau = 1$.

4. END OF THE PROOF

We shall now use the exponential estimate of z (Corollary to Lemma 3.3) in order to derive the desired exponential estimate for the functions a and ϑ

by means of the maximum principle. Recall that a and ϑ: $[s_0, \infty) \times \mathbb{R} \to \mathbb{R}$ are smooth solutions of

$$\begin{aligned} a_s &= (\vartheta_t + xy_t) f(\vartheta, z) \\ a_t &= -(\vartheta_s + xy_s) f(\vartheta, z). \end{aligned} \tag{56}$$

Moreover $f(t, 0) \equiv 1$ and we can write

$$f(\vartheta, z) = 1 + \int_0^1 f_z(\vartheta, \tau z) d\tau z \equiv 1 + k(s, t) z.$$

Introduce the 1-periodic functions $\tilde{a}, \tilde{b}$: $[s_0, \infty) \times S^1 \to \mathbb{R}$ by

$$\begin{aligned} \tilde{a}(s, t) &= a(s, t) - s \\ \tilde{\vartheta}(s, t) &= \vartheta(s, t) - t. \end{aligned}$$

By Lemma 2.4,

$$\begin{aligned} \partial^\beta \tilde{a}(s, t) &\to 0 \qquad |\beta| \geq 1 \\ \partial^\beta \tilde{\vartheta}(s, t) &\to 0 \qquad |\beta| \geq 1. \end{aligned}$$

as $s \to \infty$, uniformly in t. Moreover we have by our preceeding discussion that

$$\tilde{\vartheta}(s, t) \to 0$$

as $s \to \infty$ uniformly in t. The equation (56) becomes

$$\begin{aligned} \tilde{a}_s &= \tilde{\vartheta}_t + xy_t + (\vartheta_t + xy_t) kz \\ \tilde{\vartheta}_s &= -\tilde{a}_t - xy_s - (\vartheta_s + xy_s) kz. \end{aligned} \tag{57}$$

Hence, abbreviating

$$w(s, t) = \begin{pmatrix} \tilde{a} \\ \tilde{\vartheta} \end{pmatrix},$$

we can write the equation (57) in the form

$$w_s + J_0 w_t = h, \qquad J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{58}$$

with a smooth function h: $[s_0, \infty) \times S^1 \to \mathbb{R}$ satisfying, in view of the corollary to Lemma 3.3, the exponential estimates

$$|\partial^\beta h(s, t)| \leq M e^{-rs}, \qquad |\beta| \geq 0$$

for constants $M = M_\beta$, and $0 < r < |\lambda|$. Our aim is to deduce similar estimates for w. We start with a simple

LEMMA 4.1. – *Assume* $v: [s_0, \infty) \times S^1 \to \mathbb{R}^2$ *is smooth, bounded and solves*

$$v_s + J_0 v_t = g, \qquad \text{where} \qquad \|g(s)\| \leq Me^{-rs},$$

for some $r > 0$, *where the norm is the* $L^2(S^1)$*-norm. If* v *satisfies* $v_t(s,t) \to 0$ *as* $s \to \infty$ *uniformly in t, and moreover has vanishing mean values,*

$$\int_0^1 v(s,t)dt \equiv 0,$$

then:

$$\int_{s_0}^{s} e^{2\rho s} \|v(s)\|^2 ds < \infty,$$

for every $0 \leq \rho < r$ *and* $\rho < \frac{1}{2}$.

Proof. – We first show that $\|g(s)\| \in L^2$ implies $\|v(s)\| \in L^2$, the norm denoting the $L^2(S^1)$-norm. We make use of the following pointwise identities for a function $w = w(s,t)$:

$$\begin{aligned} 2\langle w_s, J_0 w_t\rangle &= \frac{d}{ds}\langle w, J_0 w_t\rangle - \frac{d}{dt}\langle w, J_0 w_s\rangle \\ |w_s|^2 + |w_t|^2 &= |w_s + J_0 w_t|^2 + 2\langle w_s, J_0 w_t\rangle. \end{aligned} \tag{59}$$

Since v has mean values zero we can estimate $\|v(s)\| \leq \|v_t(s)\|$. Using (59), integrating by parts, and observing that the integral of the derivative of a periodic function over a period vanishes, and v solves the equation $v_s + J_0 v_t = g$, we obtain

$$\begin{aligned} \int_{s_0}^{s} \|v(s)\|^2 &\leq \int_{s_0}^{s} (\|v_s(s)\|^2 + \|v_t(s)\|^2) ds \\ &= \int_{s_0}^{s} \|g(s)\|^2 ds + \langle v(s), J_0 v_t(s)\rangle - \langle v(s_0), J_0 v_t(s_0)\rangle, \end{aligned}$$

where $\langle\ ,\ \rangle$, denotes the inner product in $L^2(S^1)$. Since $\|g(s)\| \in L^2$ we conclude for the limits $s \to \infty$:

$$\int_{s_0}^{\infty} \|v(s)\|^2 \leq \int_{s_0}^{\infty} \|g(s)\|^2 - \langle v(s_0), J_0 v_t(s_0)\rangle.$$

Take now an increasing sequence of monotone increasing functions $\gamma_n: \mathbb{R} \to \mathbb{R}$ satisfying $\gamma_n(s) = s$ for $0 \leq s \leq n$, $0 \leq \gamma_n'(s) \leq 1$ for

$s \in \mathbb{R}$, and $\gamma_n(s) = \text{const}$ for $s \geq n+1$. Let $\rho > 0$ and define the sequence $\hat{v}_n = \hat{v}$ as

$$\hat{v}(s,t) = e^{\rho\gamma_n(s)} v(s,t).$$

Then $\hat{v}$ is smooth, bounded, satisfies $\hat{v}_t(s,t) \to 0$ as $s \to \infty$, has mean values zero and $\|\hat{v}(s)\| \in L^2$. Differentiating we obtain

$$\hat{v}_s + J_0\hat{v}_t = e^{\rho\gamma_n(s)} g + \rho\gamma_n'(s)\hat{v}.$$

If $0 < \rho < r$ we conclude, in view of the exponential decay of g, for $n \geq s_0$

$$\begin{aligned}\int_{s_0}^{\infty} \|\hat{v}\|^2 ds &\leq \int_{s_0}^{\infty} \|\hat{v}_s + J_0\hat{v}_t\|^2 - \langle \hat{v}(s_0), J_0\hat{v}_t(s_0)\rangle \\ &\leq 2\int_{s_0}^{\infty} e^{2\rho s}\|g(s)\|^2 + 2\rho \int_{s_0}^{\infty} \|\hat{v}\|^2 - e^{2\rho s_0}\langle v(s_0), J_0 v_t(s_0)\rangle.\end{aligned}$$

Hence

$$(1-2\rho)\int_{s_0}^{\infty} e^{2\rho\gamma_n(s)}\|v(s)\|^2 ds \leq M$$

with a constant M independent of n. Let $\rho < 1/2$; taking the limit as $n \to \infty$ we conclude that $e^{\rho s}\|v(s)\| \in L^2$ as claimed. $\square$

By means of the maximum principle we shall deduce

LEMMA 4.2. – *If $0 < \rho < r$ and $\rho < \frac{1}{2}$, then a solution w of* (58) *satisfies*

$$|\partial^\beta w(s,t)| \leq M e^{-\rho s},$$

with constants $M = M_\beta$, for all derivatives $\partial^\beta = \partial_t^{\beta_1}\partial_s^{\beta_2}$ satisfying $\beta_1 \geq 1$, $\beta_2 \geq 0$.

Proof. – Let w be a solution of (58) and abbreviate

$$v = \partial^\beta w,\ \partial^\beta w = \partial_t^{\beta_1}\partial_s^{\beta_2} w.$$

If $\beta_1 \geq 1$, the mean values of v over a period vanish, moreover v solves the equation

$$v_s + J_0 v_t = g,$$

with $g = \partial^\beta h$ and hence $|g(s,t)| \leq M e^{-rs}$, for some $M > 0$. We can apply Lemma 4.1 and conclude, in view of $\|v(s)\| \leq \|v_t(s)\|$ that

$$\int_{s_0}^{\infty} e^{2\rho s}\|v(s)\| ds \leq \int_{s_0}^{\infty} e^{2\rho s}\|v_t(s)\| ds < \infty \tag{60}$$

for every $\rho < r$ and $\rho < 1/2$. Define

$$\varphi(s,t) = e^{\rho s} v(s,t).$$

Due to (60) there exists, for given $\delta > 0$, an increasing sequence $s_n \to \infty$, satisfying $0 < s_{n+1} - s_n \leq \delta$ such that $\|\varphi(s_n)\| \to 0$ and $\|\varphi_t(s_n)\| \to 0$ and, therefore

$$\varphi(s_n, t) \to 0 \qquad (n \to \infty) \tag{61}$$

uniformly in t. We shall conclude that $|\varphi(s,t)| \leq C$, for some $C > 0$. For this purpose we compute the Laplacian

$$\Delta\varphi = \rho\varphi_s + e^{\rho s}(v_{ss} + v_{tt}).$$

From $v_s + J_0 v_t = g$ we derive $v_{ss} + v_{tt} = g_s - J_0 g_t$, so that

$$\Delta\varphi = \rho\varphi_s + R(s,t), \tag{62}$$

where $R(s,t) = e^{\rho s}(g_s - J_0 g_t)$ is, in the sup norm, arbitrarily small for s large, in view of the exponential decay of g. Set now

$$\psi(s,t) = |\varphi(s,t)|^2,$$

then

$$\Delta\psi = 2\langle\Delta\varphi, \varphi\rangle + 2|\varphi_s|^2 + 2|\varphi_t|^2.$$

By (62) we estimate

$$2|\langle\Delta\varphi, \varphi\rangle| \leq \rho(|\varphi_s|^2 + |\varphi|^2) + \frac{1}{\mu}|R|^2 + \mu|\varphi|^2,$$

for every $\mu > 0$. Hence

$$\begin{aligned} -\Delta\psi - \psi &= -2\langle\Delta\varphi, \varphi\rangle - 2|\varphi_s|^2 - 2|\varphi_t|^2 - |\varphi|^2 \\ &\leq (\rho - 2)|\varphi_s|^2 - 2|\varphi_t|^2 + (\rho + \mu - 1)|\varphi|^2 + \frac{1}{\mu}|R|^2. \end{aligned}$$

Choosing $\mu = \frac{1}{2}$ we conclude, using $\rho < \frac{1}{2}$, that

$$-\Delta\psi - \psi \leq 2|R|^2 \leq a$$

for some constant $a \geq 0$. In view of (61), we know, in addition, that $\psi(s_n, t) \leq 1$ for all t and n large. Choosing $\delta < \pi$ we have verified the assumptions of a version of the maximum principle for the cylinder due to A. Floer and H. Hofer in ([2], Proposition 8). We conclude that $\psi(s,t) \leq C$ for some constant C and for all (s,t). Consequently $e^{\rho s}|v(s,t)| \leq M$ as claimed in the Lemma.

Having proved the estimates for the derivatives we finally turn to the estimates of the functions and first claim, that there exists a constant $c \in \mathbb{R}$ such that

$$\tilde{a}(s,t) \to c, \qquad s \to \infty \tag{63}$$

uniformly in t. Indeed, from $|\tilde{\vartheta}_t(s,t)| \leq Me^{-\rho s}$, and the equation (57), we find $|\tilde{a}_s(s,t)| \leq Me^{-\rho s}$, so that

$$|\tilde{a}(s,t) - \tilde{a}(\sigma,t)| \leq \left| \int_s^\sigma |\tilde{a}_s(s,t)| ds \right| \to 0$$

for $s, \sigma \to \infty$, uniformly in t. Hence, $\tilde{a}(s,t) \to c(t)$ as $s \to \infty$, with a continuous and periodic function $c(t)$. Similarly for fixed t, τ we have $|\tilde{a}(s,t) - \tilde{a}(s,\tau)| \to 0$, as $s \to \infty$. Hence $c(t) \equiv c$ must be a constant as claimed in (63). Define

$$\tilde{w} = w - \begin{pmatrix} c \\ 0 \end{pmatrix} \qquad \text{and} \qquad u(s) = \int_0^1 \tilde{w}(s,t) dt.$$

Then $w - u$ has vanishing mean values over a period and hence

$$|\tilde{w}(s,t) - u(s)| \leq \|\tilde{w}_t(s)\| = \|w_t(s)\| \leq Me^{-\rho s}. \tag{64}$$

Using the equation $\tilde{w}_t + J_0 \tilde{w}_s = h$ we obtain, since the mean value of a derivative in t vanishes,

$$u'(s) = \int_0^1 \tilde{w}_s(s,t) dt = \int_0^1 h(s,t) dt.$$

Integrating we find, using $u(s) \to 0$, as $s \to \infty$,

$$0 = u(s_0) + \int_{s_0}^\infty \int_0^1 h(s,t) dt\, ds.$$

Consequently,

$$u(s) = -\int_s^\infty \int_0^1 h(s,t) dt,$$

and hence, $|u(s)| \leq Me^{-\rho s}$. Together with (64), we see that $|\tilde{w}(s,t)| \leq Me^{-\rho s}$, as desired. To sum up, we have demonstrated:

PROPOSITION 4.3. – *Assume* (a, ϑ, z) *meets the assumptions of Theorem* 1.2. *Then there exists a constant* $c \in \mathbb{R}$ *such that for* $0 < \rho < |\lambda|$ *and* $\rho < 1/2$,

$$|\partial^\beta(a(s,t) - s - c)| \le Me^{-\rho s}$$
$$|\partial^\beta(\vartheta(s,t) - t)| \le Me^{-\rho s},$$

for all derivatives β, *with constants* $M = M_\beta$.

This completes the proof of Theorem 2.8 about the asymptotics of nondegenerate finite energy planes. We shall use now the asymptotic formula in order to derive some global properties of nondegenerate finite energy planes.

5. INTERSECTIONS OF THE FINITE ENERGY PLANE WITH THE "LIMIT CYCLE"

If $(a, u): \mathbb{C} \to \mathbb{R} \times M$ is a finite energy plane, which is nondegenerate as in Theorem 1.2, then $u(Re^{2\pi i t}) \to p(Tt)$ as $R \to \infty$. Here $p(t)$ is a periodic solution of the Reeb vectorfield $\dot{x} = X(x)$ associated to the contact structure λ on M. The period T is positive and we assume that $T = 1$ (for notational convenience). It turns out that outside of a large disc the energy plane does not hit the "limit" periodic solution p. We shall abbreviate $P = \{p(t) \mid t \in \mathbb{R}\} \subset M$.

THEOREM 5.1. – *If* $(a, u): \mathbb{C} \to \mathbb{R} \times M$ *is a nondegenerate finite energy plane as described in Theorem* 1.2, *then there exists an* $R > 0$, *such that*
(i) $u(z) \notin P$ *if* $|z| \ge R$
(ii) $\pi u_s(z) \ne 0$ *if* $|z| \ge R$,
where $\pi: T_m M = \mathbb{R} X(m) \oplus \xi_m \to \xi_m$ *is the projection onto the contact plane.*

Proof. – The proof is an immediate application of the asymptotic formula in Theorem 2.8. We argue by contradiction and assume, in the cylinder variables $(s, t) \in \mathbb{R} \times S^1$, that $u(s_n, t_n) \in P$ for a sequence $s_n \to \infty$. We may assume that $t_n \to t^* \in S^1$. In the local coordinates near P, we then have $u = (\vartheta(s_n, t_n), z(s_n, t_n)) \in \mathbb{R}^3$ and $z(s_n, t_n) = 0$. By the formula

$$z(s,t) = e^{\int_{s_0}^s \alpha(\tau)d\tau}[e(t) + r(s,t)]$$

we have

$$e(t_n) + r(s_n, t_n) = 0.$$

Since $r(s,t) \to 0$ as $s \to \infty$ we conclude that $e(t^*) = 0$. This contradicts the fact, that the eigenfunction $e(t)$ does not vanish and proves statement (i).

Similarly one proves the second statement. We assume that $\pi(u_s(s_n,t_n)) = 0$ for a sequence $s_n \to \infty$. Hence for s_n large $u(s_n,t_n)$ is in our local coordinate neighborhood of the periodic solution. We can write $u(t,s) = (\vartheta, z)$ and $u_s(s,t) = (\vartheta_s, z_s)$. Since $\pi u_s = u_s - \lambda(u_s)X(u)$, with the Reeb vectorfield X, we have

$$u_s(s_n,t_n) = \lambda(u_s)X(u),$$

and hence, at (s_n,t_n),

$$\begin{pmatrix} \vartheta_s \\ z_s \end{pmatrix} = \lambda(u_s)\begin{pmatrix} X_1(u) \\ X_2(u) \end{pmatrix} \in \mathbb{R}^3$$

Since $X_2(\vartheta, 0) = 0$, we can write $X_2(\vartheta, z) = Rz$, with a matrix function $R = R(\vartheta, z)$, so that, at (s_n,t_n)

$$z_s = f(u)(\vartheta_s + xy_s)Rz.$$

Inserting the asymptotic formula in Theorem 2.8 the exponential terms cancel, and we find

$$\alpha(s_n)[e(t_n) + r(s_n,t_n)] + r_s(s_n,t_n) = f(u)(\vartheta_s + xy_s)R[e(t_n) + r(s_n,t_n)].$$

Recall now that $\alpha(s) \to \lambda < 0$, and $r(s,t)$, $r_s(s,t)$, $\vartheta_s(s,t)$, $x(s,t) \to 0$ as $s \to \infty$. We conclude $\lambda e(t^*) = 0$, contradicting again $e(t) \neq 0$. This finishes the proof of Theorem 5.1. □

Using the generalized similarity principle we shall deduce from Theorem 5.1 the

THEOREM 5.2. – *The sets*

$$\{z \in \mathbb{C} \mid u(z) \in P\}$$
$$\{z \in \mathbb{C} \mid \pi_{u(z)}(u_s(z)) = 0\}$$

consist of finitely many points.

Proof. – In order to prove the first statement we argue by contradiction and assume that there is an infinite sequence $z_n \in \mathbb{C}$ such that $u(z_n) \in P$. By Theorem 5.1 we can assume that $z_n \to z^* \in \mathbb{C}$ and $u(z^*) \in P$. By Darboux's theorem there is an open neighborhood of $u(z^*) \in M$ on which we find coordinates $(\vartheta, x, y) = (\vartheta, z) \in \mathbb{R}^3$, in which the contact form λ is represented as

$$\lambda = d\vartheta + xdy,$$

and in which $u(z^*)$ corresponds to the origin 0 in $\mathbb{R}^3$. Using the cylinder coordinates $(s,t) \in \mathbb{R} \times S^1$, the map $u(s,t) = (\vartheta(s,t), z(s,t)) \in \mathbb{R} \times \mathbb{R}^2$ satisfies, in our local coordinates, the equations

$$z_s + J(s,t)z_t = 0, \tag{65}$$

where $J(s,t)^2 = -1$. This is proved as in Section 3; this time $f = 1$ and $X(\vartheta, z) = (1,0,0)$. By assumption, we know that

$$z(s_n, t_n) = z(s^*, t^*) = 0$$

for a sequence $(s_n, t_n) \to (s^*, t^*)$. Consequently, by the generalized similarity principle [9], there is an open neighborhood D of (s^*, t^*) on which the solution z of the equation is represented by $z(s,t) = \Phi(z)h(z)$, where $z = s + it$, $\Phi: D \to GL(\mathbb{R}^2)$ is continuous, and $h: D \to \mathbb{R}^2 \cong \mathbb{C}$ is holomorphic. By assumption, $z^* = s^* + it^*$ is a cluster point of zeroes of the holomorphic function h. Therefore, $h \equiv 0$ and hence $z \equiv 0$ on D. Consequently $u(s,t) \in P$ for all (s,t) in the open set D. We have proved, in particular, that the set of points $z = (s,t)$ which are cluster points of z_j satisfying $u(z_j) \in P$ is an open set in $\mathbb{R} \times S^1$. It is clearly also a closed set and hence agrees with $\mathbb{R} \times S^1$ so that $u(s,t) \in P$ for all $(s,t) \in \mathbb{R} \times S^1$. This contradicts Theorem 5.1.

The second statement is proved similarly. Note that in the above local coordinates the Reeb vectorfield X is constant, $X(\vartheta, z) = (1,0,0)$. Hence, the condition $0 = \pi u_s = u_s - \lambda(u_s)X(u)$ becomes, in our local coordinates $(u = (\vartheta, z))$, $z_s = 0$. Introducing $\zeta = z_s$ we find, by differentiating (65) in the s-variable, that ζ solves the equations

$$\zeta_s + J(s,t)\zeta_t + A(s,t)\zeta = 0.$$

Moreover, $\zeta(s_n, t_n) = \zeta(s^*, t^*) = 0$ for a sequence $(s_n, t_n) \to (s^*, t^*)$. Hence, by the generalized similarity principle [9], $\zeta \equiv 0$ in an open neighborhood of $(s^*, t^*) = 0$. Consequently $\pi u_s(z) = 0$ in an open neighborhood of $z^* \in \mathbb{C}$.

Arguing as before this leads to a contradiction to the second statement in Theorem 5.1. The proof of the Theorem 5.2 is complete. □

REFERENCES

[1] C. ABBAS and H. HOFER, Holomorphic Curves and Global Questions in Contact Geometry, *Nachdiplomvorlesung E.T.H.Z.*, to appear in Birkhäuser.

[2] A. FLOER and H. HOFER, Symplectic homology I, open sets in $\mathbb{C}^n$, *Math. Z.*, Vol. **215**, 1994, pp. 37-88.

[3] M. GROMOV, Pseudoholomorphic curves in symplectic manifolds, *Invent. Math.*, Vol. **82**, 1985, pp. 307-347.

[4] H. HOFER, Pseudoholomorphic curves in symplectization with applications to the Weinstein conjecture in dimension three, *Invent. Math.*, Vol. **114**, 1993, pp. 515-563.

[5] H. HOFER, K. WYSOCKI and E. ZEHNDER, Properties of pseudoholomorphic curves in symplectisations II: Embedding controls and algebraic invariants, *Geometrical and Functional Analysis*, Vol. **5**, No. 2, 1995, pp. 270-328.

[6] H. HOFER, K. WYSOCKI and E. ZEHNDER, Properties of pseudoholomorphic curves in symplectisations III: Fredholm theory, preprint.

[7] H. HOFER, K. WYSOCKI and E. ZEHNDER, A characterisation of the tight three-sphere, to appear *Duke Math. Journal.*

[8] H. HOFER, K. WYSOCKI and E. ZEHNDER, The dynamics on a strictly convex energy surface in $\mathbf{R}^4$, preprint.

[9] H. HOFER and E. ZEHNDER, Symplectic Invariants and Hamiltonian Dynamics, Birkhäuser Advanced Texts, 1994..

[10] T. KATO, Perturbation theory for linear operators, Springer 1966.

[11] J. MARTINET, Formes de contact sur les variétés de dimension 3, *Springer Lecture Notes in Math.*, Vol. **209**, 1971, pp. 142-163.

[12] E. STEIN, Singular integrals and differentiability properties of functions, Princeton University Press, 1970.

(Manuscript received October 31, 1995.)