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Global strong solutions in Sobolev or Lebesgue spaces to the incompressible Navier-Stokes equations in $\mathbb{R}^3$

by

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ABSTRACT. – We construct global strong solutions of the Navier-Stokes equations with sufficiently oscillating initial data. We will show that the condition is for the norm in some Besov space to be small enough.

RÉSUMÉ. – Nous construisons des solutions fortes globales des équations de Navier-Stokes, pour des données initiales suffisamment oscillantes. Cette condition se traduit en terme de norme petite dans un certain espace de Besov.

INTRODUCTION

We are interested in the following system, for $x \in \mathbb{R}^3$ and $t > 0$,

$$(1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)\, u = \nu \Delta u - \nabla p \\ \nabla \cdot u = 0, \end{cases}$$

with initial data $u(x,0) = u_0(x)$.For the sake of simplicity, we suppose that $\nu = 1$; a simple rescaling allows us to obtain any other value. Local existence and uniqueness in the Sobolev space $H^s(\mathbb{R}^3)$ and the Lebesgue space $L^p(\mathbb{R}^3)$ are known, if $s > 1/2$ and $p > 3$ (*see* [4]). We have global solutions for small initial data in $L^3(\mathbb{R}^3)$ (*see* [9] or [4]) and $H^{\frac{1}{2}}(\mathbb{R}^3)$ (*see* [4] and [5]), or in $L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$, with $p > 3$ (*see* [1]). We shall extend the results of [4], for $s > 1/2$ and $p > 3$. By adapting the auxiliary spaces used in [4], we shall prove the existence and uniqueness of global solutions in $H^s(\mathbb{R}^3)$ provided the initial data are small in a sense which will be made precise later, and in $L^p(\mathbb{R}^3)$ up to additional conditions on u_0. Let us define the homogeneous Besov spaces $\dot{B}^{\alpha}_{p,q}$:

DEFINITION 1. – *Let us choose $\phi \in \mathcal{S}(\mathbb{R}^n)$ a radial function so that Supp $\hat{\phi} \subset \{|\xi| < 1+\varepsilon\}$, and $\hat{\phi}(\xi) = 1$ for $|\xi| < 1$. Define $\phi_j(x) = 2^{nj}\phi(2^j x)$, S_j the convolution operator with ϕ_j, and $\Delta_j = S_{j+1} - S_j$. Let $f \in \mathcal{S}'(\mathbb{R}^n)$, $\alpha \in \mathbb{R}$, $1 < p, q \leq +\infty$, $f \in \dot{B}^{\alpha}_{p,q}$ if and only if*

$$\left[\sum_{-\infty}^{+\infty} (2^{j\alpha} \|\Delta_j f\|_{L^p})^q\right]^{\frac{1}{q}} < +\infty.$$

The reader should consult [12], [2], or [16] where the properties of Besov spaces are exposed in detail. Let us see how homogeneous Besov spaces arise. If we want to construct a global solution, it is useful to control a norm remaining invariant by the rescaling $f(x) \rightarrow \lambda f(\lambda x)$. If this can be achieved in a Besov space with $\alpha < 0$ and therefore bigger than the usual space where we want to obtain a solution, we will have weaker assumptions on u_0.

Let us give the results in the case of Sobolev spaces. BC denotes the class of bounded continuous functions.

THEOREM 1. – *There exists an universal constant $\beta > 0$ such that, if $s > \frac{1}{2}$, $u_0 \in H^s(\mathbb{R}^3)$, $\nabla \cdot u_0 = 0$ and*

$$\| u \|_{\dot{B}^{-1/4}_{4,\infty}} < \beta, \tag{2}$$

then there exists a unique solution u of (1) *such that*

$$u \in BC([0,\infty), H^s(\mathbb{R}^3)). \tag{3}$$

Moreover, the following properties hold for u:

- $\| u(\cdot,t) \|_{L^2}$ *is decreasing, and for every $t > 1$,*

$$\| u(\cdot,t) - e^{t\Delta}u_0 \|_{L^2} \leq \frac{C(\beta, u_0)}{t^{1/4}}. \tag{4}$$

• *For every* $t > 1$,

$$\| (-\Delta)^{s/2} u(\cdot, t) \|_{L^2} \leq \frac{C(\beta, u_0, s)}{t^{s/2}}. \tag{5}$$

• *For every* $t > 0$,

$$\| u(\cdot, t) \|_\infty \leq \frac{C(\beta, u_0)}{\sqrt{t}}. \tag{6}$$

• *If* $s \in (1, 3/2]$, *for every* $t < 1$,

$$\| u(\cdot, t) - e^{t\Delta} u_0 \|_\infty \leq C(\beta, u_0). \tag{7}$$

Note that the space $\dot{B}^{-1/4}_{4,\infty}$ is invariant under the scaling $f(x) \to \lambda f(\lambda x)$, and $\dot{H}^{\frac{1}{2}} \subset \dot{B}^{-1/4}_{4,\infty}$. It is very interesting that we do not need a small $H^{\frac{1}{2}}$-norm to obtain a global solution (*see* [4]). On the other hand, if we want to include the case 1/2, u is unique in the space

$$\begin{cases} u \in BC([0, +\infty), H^{\frac{1}{2}}) \\ t^{1/8} u(\cdot, t) \in BC([0, +\infty), L^4) \\ \lim_{t \to 0} t^{1/8} \|u\|_{L^4} = 0. \end{cases}$$

which was used in [4], the starting point of the present work. The weak condition (2) is the only remaining obstacle to the problem of existence of global smooth solutions to the Navier-Stokes equations, and we remark that β does not depend on s. The decay estimates (4) can be found in [8], in a slightly different context. We recall it here as a natural consequence of the construction of u.

In the Lebesgue spaces, the analogue is

THEOREM 2. – *Let* $p > 3/2$, *there exists* $\delta(p) > 0$ *such that, if* $u_0 \in L^p \cap \dot{B}^{-(1-\frac{3}{2p})}_{2p,\infty}$, $\nabla \cdot u_0 = 0$ *and*

$$\| u_0 \|_{\dot{B}^{-(1-\frac{3}{2p})}_{2p,\infty}} < \delta(p), \tag{8}$$

then there exists a unique solution u such that

$$\begin{cases} u \in BC([0, +\infty), L^p) \\ t^{\frac{1}{2} - \frac{3}{4p}} u(\cdot, t) \in BC([0, +\infty), L^{2p}) \\ \lim_{t \to 0} t^{\frac{1}{2} - \frac{3}{4p}} \|u\|_{L^{2p}} = 0. \end{cases}$$

The restriction $p > 3/2$ is due to technical considerations, and we could probably obtain 1 instead of $3/2$, by sligthly modifying the Besov space involved.

PROPOSITION 1. – *The constant $\delta(p)$ satisfies:*

$$\lim_{p \to +\infty} \delta(p) = 0,$$
$$\lim_{p \to 3/2} \delta(p) = 0.$$

PROPOSITION 2. – *In Theorem 2, we can replace $u_0 \in L^p \cap \dot{B}_{2p,\infty}^{-(1-\frac{3}{2p})}$ by $u_0 \in L^p \cap L^3$, and if $p > 3$ by $L^2 \cap L^p$.*

If $u_0 \in H^s, s \geq 1/2$, then as $\dot{H}^{\frac{1}{2}} \subset \dot{B}_{4,\infty}^{-1/4}$, we have a natural candidate for the useful Besov space. On the contrary, if we take L^p, we may use two different Besov spaces: the first one is $\dot{B}_{2p,\infty}^{-\frac{3}{2p}}$, as $L^p \subset \dot{B}_{2p,\infty}^{-\frac{3}{2p}}$. But this space is not invariant by the rescaling. The "right" space is $\dot{B}_{2p,\infty}^{-(1-\frac{3}{2p})}$, but unfortunately $L^p \not\subset \dot{B}_{2p,\infty}^{-(1-\frac{3}{2p})}$. This explains the additional condition imposed on u_0 in Theorem 2. Both spaces coincide only when $1 - \frac{3}{2p} = \frac{3}{2p}$, which means $p = 3$. The reader should refer to [9] and [4] for details.

Proofs. – We first reformulate the problem in order to obtain an integral equation for u. This is standard practice, and was first employed by Kato and Fujita (*see* [10] [11]), and very often used since (*see* [7] [6] [15]). All these authors use semi-group theory, but in the present case, we do not need this formalism, for the exact expression of the heat kernel in $\mathbb{R}^3$ allows us to obtain directly the estimates we need (*see* [9]). Let $\mathbb{P}$ be the projection operator from $(L^2(\mathbb{R}^3))^3$ onto the subspace of divergence-free vectors, denoted by $\mathbb{P}L^2$, and R_j the Riesz transform with symbol $\frac{\xi_j}{|\xi|}$. We easily see that

$$\mathbb{P}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} - \begin{pmatrix} R_1\sigma \\ R_2\sigma \\ R_3\sigma \end{pmatrix} \tag{9}$$

where $\sigma = \sum_j R_j u_j$.It is well-known that $\mathbb{P}$ can be extended to a bounded operator from $(L^p)^3$ onto $\mathbb{P}L^p$, $1 < p < +\infty$, and from $(H^s)^3$onto $\mathbb{P}H^s$, $s \geq 0$. Note that $\mathbb{P}$ commutes with $S(t) = e^{t\Delta}$, whereas on an open set Ω, we need to introduce the Stokes operator $-\mathbb{P}\Delta$ and the associated semi-group. Note that

$$\text{Ker}\mathbb{P} = \{u \,|\, \exists \phi \text{ such that } u = \nabla \phi\}.$$

Using $\mathbb{P}$, (1) becomes an evolution equation

$$(10) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \mathbb{P}\nabla \cdot (u \otimes u), \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x). \end{cases}$$

We replace $(u \cdot \nabla)u$ by $\nabla \cdot (u \otimes u)$ to avoid problems of definition, and this is possible only because $\nabla \cdot u = 0$. It is then standard to study (10) via the corresponding integral equation

$$(11) \qquad u(x,t) = S(t)u_0(x) - \int_0^t \mathbb{P}S(t-s)\nabla \cdot (u \otimes u)(x,s)ds$$

in a space of divergence free vectors. The integral should be seen as a Bochner integral. In the general case of evolution equations, a solution of (11) might not be a solution of (10). However, in the case of the Navier-Stokes equations without external forces, it is true without any extra assumptions. Actually, the solutions of (11) are $C^\infty((0,+\infty) \times \mathbb{R}^3)$ and verify the equations (1) in the classical sense, as we recover easily the pressure up to a constant by

$$(12) \qquad -\Delta p = \sum_{i,j} \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}.$$

The reader should refer to [7] [10] or [13] for proofs.We remark that since a solution of (1) is necessarily a solution of (11), uniqueness for (11) guarantees uniqueness for (1). We aim to solve (11) by successive approximations, with the following lemma:

LEMMA 1. – *Let E and F be two Banach functional spaces, endowed with the norms $\| \cdot \| = \| \cdot \|_E$ and $| \cdot | = \| \cdot \|_F$, B a continuous bilinear operator from $F \times F \to E$ and $F \times F \to F$:*

$$\| B(u,v) \| \le \eta \mid u \mid\mid v \mid$$
$$\mid B(u,v) \mid \le \gamma \mid u \mid\mid v \mid,$$

and define the sequence $X_0 = 0$, $X_{n+1} = Y + B(X_n, X_n)$, where Y belongs to E and to F. If

$$(13) \qquad 4\gamma \mid Y \mid < 1,$$

then the sequence converges in both spaces E and F, and the limit X sastisfies

$$X = Y + B(X, X) \tag{14}$$

and

$$| X | < 2 | Y | . \tag{15}$$

The proof is left to the reader. Note that the value of η has no influence on the convergence. Now we have to study the following bilinear operator

$$B(u, v) = \int_0^t \mathbb{P}S(t-s)\nabla \cdot (u \otimes u) ds. \tag{16}$$

In order to simplify the notations, we limit ourselves to the following scalar operator

$$B(f, g) = \int_0^t \frac{1}{(t-s)^2}\theta\left(\frac{\cdot}{(t-s)^2}\right) * fg(s) ds. \tag{17}$$

As $\mathbb{P}S(t-s)\nabla\cdot$ is a matrix of convolution operators, the components are all operators like (17), with

$$e^{|\xi|^2}\hat{\theta}(\xi) = \frac{\xi_j \xi_k \xi_l}{| \xi |^2} \quad \text{(with a } -\xi_j \text{ on the diagonal)}. \tag{18}$$

LEMMA 2. – $\theta(x) \in C^\infty(\mathbb{R}^3)$ *and* $\theta \in L^1 \cap L^\infty$.

This can be easily seen on the Fourier transform of θ.

In what follows, C denotes a constant which may vary from one line to another.

Proof of Theorem 1

PROPOSITION 3. – *Let* $1/2 < s \le 3/4$, *then there exists a solution* u *of* (11) *such that*

$$\begin{cases} u \in BC([0, +\infty), \dot{H}^s) = E, \\ \omega(t)u(x, t) \in BC([0, +\infty), L^4) = F, \end{cases} \tag{19}$$

where $\omega(t) = t^{3/8 - s/2}$ *if* $0 < t < 1$ *and* $\omega(t) = t^{1/8}$ *if* $t \ge 1$.

We want to apply Lemma (1) where E and F are defined by the norms

$$\| u \| = \sup_t \| u \|_{\dot{H}^s}$$
$$| u | = \sup_t \omega(t) \| u \|_{L^4}.$$

If we use Hölder and Young inequalities for $B(f,g)$, Λ being the operator with symbol $| \xi |$,

$$\| \Lambda^s B(f,g)(t) \|_{L^2} \leq \| \theta \|_{L^1} | f || g | \int_0^t \frac{1}{(t-\tau)^{1/2+s/2}\omega^2(\tau)} d\tau, \tag{20}$$

$$\| B(f,g)(t) \|_{L^4} \leq \| \theta \|_{L^{4/3}} | f || g | \int_0^t \frac{1}{(t-\tau)^{7/8}\omega^2(\tau)} d\tau. \tag{21}$$

We shall then verify that, for all $t > 0$,

$$I_1 = \int_0^t \frac{1}{(t-\tau)^{1/2+s/2}\omega^2(\tau)} d\tau < +\infty,$$
$$I_2 = \int_0^t \frac{1}{(t-\tau)^{7/8}\omega^2(\tau)} d\tau < +\infty.$$

Easy calculations actually show that for $t < 1$,

$$I_i < Ct^{s/2-1/4}$$

and for $t > 1$

$$I_i < Ct^{1/4-s/2}. \tag{22}$$

The continuity at $t = 0$ comes from the estimate when $t < 1$. In order to include the case $s = 1/2$, we have to impose $\lim_{t\to 0} t^{1/8} \| u \|_{L^4} = 0$ (*see* [4]). Note that the constant γ of Lemma 1 is

$$\gamma = \int_0^1 \frac{\| \theta \|_{L^{4/3}}}{(1-\tau)^{7/8}\tau^{1/4}} d\tau. \tag{23}$$

Therefore, if $S(t)u_0$ satisfies condition (13), we obtain $u \in BC([0,+\infty), \dot{H}^s)$.

PROPOSITION 4. – *We have*

$$u \in BC([0,+\infty), L^2). \tag{24}$$

Let $G = BC([0,+\infty), L^2)$; B is bicontinuous from $G \times F$ to G:

$$(25)\quad \| B(f,g)(t) \|_{L^2} \leq | g | \sup_{[0,t]} \| f \|_{L^2} \| \theta \|_{L^{4/3}} \int_0^t \frac{1}{(t-\tau)^{7/8}\omega(\tau)} d\tau.$$

Let

$$I_3 = \int_0^t \frac{\| \theta \|_{L^{4/3}}}{(t-\tau)^{7/8}\omega(\tau)} d\tau,$$

for $t < 1$, $I_3 < Ct^{s/2-1/4}$, and for all t

$$I_3 \leq \int_0^1 \frac{\| \theta \|_{L^{4/3}}}{(1-\tau)^{7/8}\tau^{7/8}} d\tau = \rho.$$

G being a Banach space, we can use a contraction argument to show that the sequence defined previously converges in G. It is sufficient that $2 \mid u \mid \rho < 1$, which is true as $\rho \leq \gamma$ and u verifies (15). Therefore, we proved (24) and hence Proposition 3, and shown that $\| u(\cdot,t) \|_{L^2}$ is uniformly bounded.

We now show (6): the following estimation is verified by the heat kernel,

$$(26)\quad \sup_{[0,t]} \sqrt{t} \| S(t)u_0 \|_\infty \leq C.$$

We have

$$\| B(f,g)(t) \|_\infty \leq \| \theta \|_{L^{4/3}} \int_0^t \frac{\| f(s) \|_{L^4} \| g(s) \|_\infty}{(t-s)^{7/8}} ds.$$

Let us denote $W(f,t) = \sup_{[0,t]} \sqrt{t} \| f(.,s) \|_\infty$, then

$$(27)\quad W(B(f,g),t) \leq | f | W(g,t) \| \theta \|_{L^{4/3}} \int_0^t \frac{\sqrt{t}}{(t-s)^{7/8}s^{5/8}} ds.$$

Let

$$I_4 = \int_0^1 \frac{\| \theta \|_{L^{4/3}}}{(1-\mu)^{7/8}\mu^{5/8}} d\mu$$

then, as $I_4 \leq 2\gamma$, we have $2W(S(t)u_0,t)I_4 < 1$. Therefore,

$$(28)\quad \sup_{[0,t]} \sqrt{t} \| u(\cdot,t) \|_\infty \leq \frac{C}{1 - 2I_4 \| S(t)u_0 \|}.$$

Now we can prove (4) as follows:

$$\| B(f,g)(t) \|_{L^2} \leq \int_0^t \frac{C}{(t-s)^{2-\frac{3}{2q}}} \| g \|_{L^2} \| f \|_{L^\beta} ds,$$

where

$$\frac{1}{2} = \frac{1}{q} + \frac{1}{2} + \frac{1}{\beta} - 1.$$

If we take q such that $\frac{3}{2q} = 1 + \varepsilon$, $\varepsilon > 0$, using interpolation and (28) we get, for $t > 1$,

$$\|f\|_{L^p} \leq \|f\|_{L^2}{}^{\frac{2}{p}} \|f\|_\infty{}^{1-\frac{2}{p}} \tag{29}$$

and

$$\| B(f,g)(t) \|_{L^2} \leq \int_0^t \frac{C}{(t-s)^{1-\varepsilon} s^{\frac{1}{6}+\frac{2}{3}\varepsilon}} ds$$

$$\| B(f,g)(t) \|_{L^2} \leq \frac{C(\varepsilon)}{t^{\frac{1}{6}-\frac{\varepsilon}{3}}}. \tag{30}$$

On the other hand, we know by (26) that $\forall q \geq 2$,

$$\sup_{[0,t]} t^{\frac{3}{2}(\frac{1}{2}-\frac{1}{q})} \| S(t)u_0 \|_q \leq C. \tag{31}$$

Therefore, as u sastisfies (14), we will improve (30) in the following way: let

$$B_1(f,g) = \int_0^1 \frac{1}{(t-s)^2} \theta\left(\frac{\cdot}{\sqrt{t-s}}\right) * fg(s) ds \tag{32}$$

$$B_2(f,g) = \int_1^t \frac{1}{(t-s)^2} \theta\left(\frac{\cdot}{\sqrt{t-s}}\right) * fg(s) ds. \tag{33}$$

The term B_1 can be handled very easily, so that $\forall \eta > 0$,

$$\| B_1(f,g)(t) \|_{L^2} \leq \frac{C}{t^{1-\eta}}.$$

Now, we split $B_2(u,u)$ in three parts. By (31) we have

$$\begin{aligned} \|B_2(S(t)u_0, S(t)u_0)\|_{L^2} &\leq \int_1^t \frac{C}{(t-s)^{2-\frac{3}{2q}} s^{\frac{3}{2}(1-\frac{1}{\alpha}-\frac{1}{\beta})}} ds \\ &\leq \frac{C}{t^{1-\frac{3}{2q}+\frac{3}{2}(1-(\frac{3}{2}-\frac{1}{q}))}} \\ &\leq \frac{C}{t^{1/4}}, \end{aligned}$$

as

$$\frac{1}{2} = \frac{1}{\gamma} + \frac{1}{\alpha} + \frac{1}{\beta} - 1.$$

We remark that the exponent $1/4$ cannot be improved, as it does not depend on γ, α and β.

LEMMA 3. – *Suppose that for* $0 < \mu$

$$\| B(u,u)(t) \|_{L^2} \leq \frac{C}{t^\mu},$$

then

$$\|B_2(S(t)u_0, B(u,u))\|_{L^2} \leq \frac{C}{t^{1/4+\mu}},$$

and there exists $\nu > 0$ *such that*

$$\|B_2(B(u,u), B(u,u))\|_{L^2} \leq \frac{C}{t^{\mu+\nu}}.$$

By (31)

$$\begin{aligned}\|B_2(S(t)u_0, B(u,u))\|_{L^2} &\leq \int_1^t \frac{C}{(t-s)^{2-\frac{3}{2q}} s^{\frac{3}{2}(\frac{1}{2}-\frac{1}{\beta})} s^\mu} ds \\ &\leq \frac{C}{t^{-\varepsilon+\frac{3}{2}(\frac{1}{6}+\frac{2}{3}\varepsilon)+\mu}} \\ &\leq \frac{C}{t^{1/4+\mu}},\end{aligned}$$

and, by (28) and (29)

$$\begin{aligned}\|B_2(B(u,u), B(u,u))\|_{L^2} &\leq \int_1^t \frac{C}{(t-s)^{2-\frac{3}{2q}} s^\mu s^{\frac{1}{2}(1-\frac{2}{\beta})+\frac{2}{\beta}\mu}} ds \\ &\leq \frac{C}{t^{-\frac{1}{3}\varepsilon+\frac{1}{6}+(\frac{5}{3}-\frac{4}{3}\varepsilon)\mu}}.\end{aligned}$$

We can start with $\mu = 1/6 - \varepsilon$, and obtain any exponent $\eta > 1/4$. Thus,

$$\|B_2(u,u)\|_{L^2} \leq \frac{C}{t^{1/4}}. \tag{34}$$

We constructed u for $s \leq 3/4$. Now we will see that if $s > 3/4$, u as above is actually in H^s. We limit ourselves to the bilinear form (17), as the term $S(t)u_0$ satisfies at least the same estimates.

LEMMA 4. – *Let* $f, g \in H^s(\mathbb{R}^3), 3/4 \le s < 3/2,$

$$\| \Lambda^{2s-3/2}(fg) \|_{L^2} \le C \| \Lambda^s f \|_{L^2} \| \Lambda^s g \|_{L^2} . \tag{35}$$

For a proof *see* the Appendix. Suppose now that $s > 3/4$, and u is the solution of Proposition 3 for $s = 3/4$. Then, if $\eta \le 1/4$, we obtain for $t < 1$

$$\| \Lambda^{\frac{3}{4}+\eta} B(f,g)(t) \|_{L^2} \le \int_0^t \frac{C}{(t-\tau)^{\frac{1}{2}+\frac{3/4+\eta}{2}}} d\tau$$

using Lemma 4 and the boundedness of f in $H^{3/4}$, so that

$$\| \Lambda^{\frac{3}{4}+\eta} B(f,g)(t) \|_{L^2} \le C t^{\frac{1}{8}-\frac{\eta}{2}}, \tag{36}$$

which gives the continuity at zero. For $t > 1$, we have by (29)

$$\|B(f,g)\|_{L^4} \le \frac{C}{t^{1/4}}$$

which allows us to improve (22), for $s \le 3/4$

$$\begin{aligned} \| \Lambda^s B(f,g)(t) \|_{L^2} &\le \int_0^1 \frac{\| \theta \|_{L^1}}{(t-\tau)^{1/2+s/2} \omega^2(\tau)} d\tau \mid f \mid \mid g \mid \\ &+ \int_1^t \frac{C \| \theta \|_{L^1}}{(t-\tau)^{1/2+s/2} \tau^{1/2}} d\tau \\ &\le \frac{C}{t^{s/2}}. \end{aligned}$$

Then

$$\| \Lambda^{\frac{3}{4}} f(t) \|_{L^2} \le \frac{C}{t^{\frac{3}{8}}}$$

and,

$$\| \Lambda^{\frac{3}{4}+\eta} B_1(f,g) \|_{L^2} \le \frac{C}{(t-1)^{\frac{1}{2}+\frac{3/4+\eta}{2}}}$$

and

$$\| \Lambda^{\frac{3}{4}+\eta} B_2(f,g) \|_{L^2} \le \int_1^t \frac{C}{(t-\tau)^{\frac{1}{2}+\frac{3/4+\eta}{2}} \tau^{\frac{1}{2}}} d\tau$$

then, for all t

$$\| \Lambda^{\frac{3}{4}+\eta} B(f,g)(t) \|_{L^2} \le \frac{C}{1+t^{\frac{3}{8}+\frac{\eta}{2}}}.$$

We have thus obtained $u \in H^{\frac{3}{4}+\eta}$. By applying the same argument we can reach the value $s > 3/2$, as

$$\Lambda^{s+\eta}B(f,g)(t) = \int_0^t \frac{1}{(t-\tau)^2}\Lambda^{\eta+3/2-s}\theta * \Lambda^{2s-3/2}(fg)(\tau)d\tau,$$

with $\eta + 3/2 - s < 1$. Before dealing with the case $s > 3/2$, let us briefly show (7) . By Sobolev's injection theorem (*see* [14]), if $s < 3/2$ then

$$\| f \|_{L^p} \leq C \| \Lambda^s f \|_{L^2}, \tag{37}$$

with $1/p = 1/2 - s/3$. If $s = 1 + \alpha$, $\alpha < 1/2$, we obtain, for $t < 1$

$$\| B(f,g)(t) \|_\infty \leq \int_0^t \frac{C}{(t-\tau)^{1-\alpha}} d\tau.$$

For small t, $B(f,g)$ is bounded and tends to zero as t goes to zero. Now we treat the case where $s > 3/2$, using the following estimate

LEMMA 5. – *Let* $f, g \in H^s(\mathbb{R}^3), s > 3/2$,

$$\| \Lambda^s(fg) \|_{L^2} \leq C(s)(\| \Lambda^s f \|_{L^2} \| g \|_\infty + \| \Lambda^s g \|_{L^2} \| f \|_\infty). \tag{38}$$

For a proof, *see* the appendix. We will then show

LEMMA 6. – *Let* $s > 3/2$, *for all* $t > 0$

$$\| \Lambda^s u(\cdot,t) \|_{L^2} \leq \frac{C(s)}{1 + t^{\frac{s}{2}}}. \tag{39}$$

This can be achieved by successive iterations, starting from the previous estimate for $s = 2$, and applying Lemma 5. Let us see how it works at each step. Let $\eta < 1$, we first treat the case $t < 1$.

$$\| \Lambda^{s+\eta}B(f,g)(t) \|_{L^2} \leq \int_0^t \frac{1}{(t-\tau)^2} \| \Lambda^\eta \theta\left(\frac{\cdot}{\sqrt{t-\tau}}\right) * \Lambda^s(fg) \|_{L^2} d\tau,$$

and as f and g are bounded in L^∞ and in H^s,

$$\| \Lambda^{s+\eta}B(f,g)(t) \|_{L^2} \leq C(s)t^{\frac{1}{2}-\frac{\eta}{2}}. \tag{40}$$

For $t > 1$,

$$\| \Lambda^{s+\eta}B_1(f,g) \|_{L^2}$$
$$\leq \int_0^1 \frac{1}{(t-\tau)^{\frac{1}{2}+\frac{s+\eta-1/2}{2}}} \| \Lambda^{s+\eta-\frac{1}{2}}\theta \|_{L^1} \| \Lambda f \|_{L^2} \| \Lambda g \|_{L^2} d\tau,$$

by using Lemma 4. Then

$$\| \Lambda^{s+\eta} B_1(f,g) \|_{L^2} \leq \frac{C(s)}{(t-1)^{\frac{s+\eta+1/2}{2}}}.$$

For $t > 1$,

$$\| \Lambda^{s+\eta} B_2(f,g) \|_{L^2} \leq \int_1^t \frac{1}{(t-\tau)^2} \| \Lambda^\eta \theta\left(\frac{\cdot}{\sqrt{t-\tau}}\right) * \Lambda^s(fg) \|_{L^2} \, d\tau,$$

and using Lemma 5 we deduce the estimate for $s+\eta$ from the estimate for s:

$$\| \Lambda^{s+\eta} B_2(f,g) \|_{L^2} \leq \frac{C(s)}{t^{\frac{s+\eta}{2}}}. \tag{41}$$

This achieves the proof of the existence of $u \in BC([0,+\infty), H^s)$. Now, we observe that, as we have local existence and uniqueness for $s > 1/2$ (*see* [4]), our solution is unique by applying this theorem on intervals covering $[0,\infty)$. In the case $s = 1/2$, it is necessary to establish uniqueness directly, (*see* [4] or [11]). The reader should refer to [11] or [7], in order to see why a solution of (11) is actually a solution in the classical sense. We can nevertheless make a few remarks. By the same process we use to gain the regularity $s - 3/4$, we can establish, independently of s, estimates in H^r, $r > s$: for all $t > 0$, there exists $\pi(r) > 0$

$$\| \Lambda^r u(\cdot,t) \|_{L^2} \leq \frac{C(r)}{t^{\pi(r)}},$$

and $\Lambda^r u$ is holderian on every interval $[t_0, t_1]$, provided $t_1 > t_0 > 0$.This provides the regularity in the space variables. As for regularity in time, it suffices to use the relation, which can be established without knowing (10)

$$u(t) = S(t-\varepsilon)u(\varepsilon) + \int_\varepsilon^t \mathbb{P} S(t-\tau)\nabla \cdot (u \otimes u) d\tau, \tag{42}$$

and the following lemma, (*see* [11] or [7] for a proof).

LEMMA 7. – *Let* $u(t) = \int_0^t e^{-(t-s)\Delta} f(s) ds$, $t \in [0,T]$, $f \in C^\eta([0,T], B)$, $\eta < 1$, *B a Banach space. Then* $u \in C^{1+\nu}((0,T], B)$, $Au \in C^\nu((0,T], B)$, *and*

$$\partial_t u = -\Delta u + f,$$

for all $\nu < \eta$.

We then obtain the C^∞ regularity of u, for $t > 0$, with a bootstrap argument. Let us see how condition (13) can be expressed on u_0 in terms of Besov spaces. We set $|S(t)u_0| < \beta$, where β has been chosen so that our scheme converges in F. Remember that

$$|S(t)u_0| = \sup_t \omega(t) \parallel S(t)u_0 \parallel_{L^4} .$$

Therefore, as $3/8 - \inf(s, 3/4)/2 < 1/8$,

$$\sup_t t^{3/8-\inf(s,3/4)/2} \parallel S(t)u_0 \parallel_{L^4} < \beta \tag{43}$$

and

$$\sup_t t^{1/8} \parallel S(t)u_0 \parallel_{L^4} < \beta. \tag{44}$$

LEMMA 8. – *Let* $u_0 \in \mathcal{S}(\mathbb{R}^3)$, $\alpha > 0$, *and* $\gamma > 1$; $\sup_t t^{\alpha/2} \parallel S(t)u_0 \parallel_{L^\gamma}$ *is a norm on* $\dot{B}^{-\alpha}_{\beta,\infty}$ *which is equivalent to the classical dyadic norm.*

We refer to [4] or [12] for a proof. In our case, except for $s = 3/4$, the condition on u_0 is equivalent to

$$\begin{cases} \parallel u_0 \parallel_{\dot{B}^{-1/4}_{4,\infty}} \leq \beta, \\ \parallel u_0 \parallel_{\dot{B}^{-(3/4-\inf(s,3/4))}_{4,\infty}} \leq \beta. \end{cases}$$

Thus, as $\dot{H}^{\frac{1}{2}} \subset \dot{B}^{-1/4}_{4,\infty}$ and $\dot{H}^{\inf(s,3/4)} \subset \dot{B}^{-(3/4-\inf(s,3/4))}_{4,\infty}$, u_0 belongs to both Besov spaces. If u is a solution with initial condition u_0, $\lambda u(\lambda x, \lambda^2 t)$ is a solution with $\lambda u_0(\lambda x)$ as initial data. The condition (44) is independent of λ for the norm is invariant by scaling. And (43) can be forced by a suitable choice of λ. For $s = 3/4$, we know that $H^{\frac{3}{4}} \subset L^4$, and we conclude in the same way. This ends the proof.

Proof of Theorem 2. – We introduce as before two Banach spaces $E = BC([0, +\infty), L^p)$ with the natural norm

$$\parallel f \parallel = \sup_t \parallel f(\cdot, t) \parallel_{L^p},$$

and $F = \{f | t^{\frac{1}{2}-\frac{3}{4p}} f \in BC([0, +\infty), L^{2p})\}$ with the norm

$$| f | = \sup_t t^{\frac{1}{2}-\frac{3}{4p}} \parallel f(\cdot, t) \parallel_{L^{2p}} .$$

Then, we see that

$$\parallel B(f,g)(t) \parallel_{L^p} \leq \parallel f \parallel \, | g | \parallel \theta \parallel_{L^q} \int_0^t \frac{1}{(t-s)^{2-\frac{3}{2q}} s^{\frac{1}{2}-\frac{3}{4p}}} ds, \tag{45}$$

$$\parallel B(f,g)(t) \parallel_{L^{2p}} \leq | f | \, | g | \parallel \theta \parallel_{L^q} \int_0^t \frac{1}{(t-s)^{2-\frac{3}{2q}} s^{1-\frac{3}{2p}}} ds, \tag{46}$$

$$\text{where} \quad \frac{1}{p} = \frac{1}{q} + \frac{1}{p} + \frac{1}{2p} - 1. \tag{47}$$

which gives the continuity of B from $F \times F \to F$ and $F \times E \to E$, with constants $\gamma(p)$ and $\eta(p)$.

$$\gamma(p) = \| \theta \|_{L^q} t^{\frac{1}{2}-\frac{3}{4p}} \int_0^t \frac{1}{(t-s)^{2-\frac{3}{2q}} s^{1-\frac{3}{2p}}} ds,$$

$$\eta(p) = \| \theta \|_{L^q} \int_0^t \frac{1}{(t-s)^{2-\frac{3}{2q}} s^{\frac{1}{2}-\frac{3}{4p}}} ds,$$

and a simple rescaling shows both quantities are bounded. Then if we use the same sequence as before, Lemma 1 gives us the convergence in F, and we obtain the convergence in E by acontraction argument, as $\eta(p) \leq \gamma(p)$, we obtain $2|u|\eta(p) < 1$. The continuity at $t = 0$ comes from a slight modification of (45), as we can replace $| f |$ by $\sup_{[0,t]} \tau^{\frac{1}{2}-\frac{3}{4p}} \| f(\cdot,\tau) \|_{L^{2p}}$, which tends to zero with t. Actually, the value of $t^{\frac{1}{2}-\frac{3}{4p}} \| f(\cdot,t) \|_{L^{2p}}$ could only be zero: the first term $u_1 = S(t)u_0$ tends to zero, for if we consider a sequence of C_0^∞ functions $(v_j)_j$ which approximate u_0,

$$t^{\frac{1}{2}-\frac{3}{4p}} \| S(t)u_0 \|_{L^{2p}} = t^{\frac{1}{2}-\frac{3}{4p}} \| S(t) \| \| u_0 - v_j \|_{L^p} + \| S(t) \| \, t^{\frac{1}{2}-\frac{3}{4p}} \| v_j \|_{L^2 p}.$$

By Lemma 8 the condition on u_0 becomes,

$$\| u_0 \|_{\dot{B}_{2p,\infty}^{-(1-\frac{3}{2p})}} \leq \delta(p),$$

where $\delta(p) \approx 1/\gamma(p)$. This proves Proposition 1. Proposition 2 results from the inclusion of L^3 in $\dot{B}_{2p,\infty}^{-(1-\frac{3}{2p})}$. Note that for $p = 2$, we impose the condition

$$t^{\frac{1}{4}} \| S(t)u_0 \|_{L^4} < \delta,$$

which is equivalent to the condition (2). For a general $u_0 \in L^2$, we only know

$$t^{\frac{3}{8}} \| S(t)u_0 \|_{L^4} < +\infty.$$

In other words, we do not know enough on low frequencies, and a sufficient condition is (2), of which $u_0 \in L^3$ or $u_0 \in H^{\frac{1}{2}}$ with small norms are particular cases. We obtained existence and uniqueness in a ball of F with Lemma 1 and uniqueness in the whole space can be obtained directly as in [11] or [4]. As in the Sobolev case, it is possible to obtain estimates on L^q norms of $u(\cdot,t)$, $q > p$, in order to show the C^∞ regularity for $t > 0$.

APPENDIX

We recall that if $\phi \in \mathcal{S}(\mathbb{R}^n)$ is a radial function so that Supp $\hat{\phi} \subset \{|\xi| < 1+\varepsilon\}$, and $\hat{\phi}(\xi) = 1$ for $|\xi| < 1$, we define $\phi_j(x) = 2^{nj}\phi(2^j x)$, S_j the convolution operator with ϕ_j, and $\Delta_j = S_{j+1} - S_j$. Then

$$I = \sum_{-\infty}^{+\infty} \Delta_j$$

and $f(x) \in \dot{H}^s(\mathbb{R}^n)$ if and only if, $\forall j$,

$$\|\Delta_j(f)\|_{L^2} \leq 2^{-js}\|f\|_{\dot{H}^s}\varepsilon_j \tag{48}$$

where $\sum \varepsilon_j^2 \leq 1$. We will show the two following inequalities, which are homogeneous variants of well-known inequalities:

for $s < \frac{n}{2}$,

$$\|\Lambda^{2s-\frac{n}{2}}(fg)\|_{L^2} \leq C\|\Lambda^s f\|_{L^2}\|\Lambda^s g\|_{L^2}, \tag{49}$$

for $s > \frac{n}{2}$,

$$\|\Lambda^s(fg)\|_{L^2} \leq C(s)(\|\Lambda^s f\|_{L^2}\|g\|_\infty + \|\Lambda^s g\|_{L^2}\|f\|_\infty). \tag{50}$$

Let us start with the first case: we will use a paraproduct decomposition (*see* [3]): for $f, g \in \mathcal{S}$,

$$\begin{aligned} f(x)g(x) &= \sum_j \Delta_j(f)\sum_l \Delta(g) \\ &= \sum_{|j-l|\leq 1} \Delta_j(f)\Delta_l(g) + \sum_{|j-l|\geq 1} \Delta_j(f)\Delta_l(g). \end{aligned} \tag{51}$$

The second sum is, by reordering the terms, a finite sum of terms like $S_2 = \sum_j S_{j-1}\Delta_j(g)$. We will treat only S_2, as the other ones are of the same kind. The Fourier transform of S_2 is supported in an annulus $[2^{j-1}(1-2\varepsilon), 2^{j+1}(1+2\varepsilon)]$. Using Bernstein's lemma,

$$\begin{aligned} \|\Delta_j(f)\|_\infty &\leq C2^{j\frac{n}{2}}\|\Delta_j f\|_{L^2} \\ &\leq C2^{j\frac{n}{2}-s}\|f\|_{\dot{H}^s}\varepsilon_j. \end{aligned}$$

Then,

$$\|S_j(f)\|_\infty \leq C\sum_{-\infty}^{j} 2^{q(\frac{n}{2}-s)}\varepsilon_q\|f\|_{\dot{H}^s}.$$

If $j < 0$,

$$\sum_{-\infty}^{j} 2^{q(\frac{n}{2}-s)}\varepsilon_q = 2^{j(\frac{n}{2}-s)}\tilde{\varepsilon_j},$$

and

$$\tilde{\varepsilon}_j = \sum_{-\infty}^{0} 2^{q(\frac{n}{2}-s)}\varepsilon_{j+q}$$

is a convolution product between l^1 and l^2, therefore in l^2. For $j \geq 0$,

$$\sum_{-\infty}^{j} 2^{q(\frac{n}{2}-s)}\varepsilon_q \leq C(1 + \ldots + 2^{j(\frac{n}{2})}\varepsilon_j).$$

if

$$2^{j(\frac{n}{2}-s)}\check{\varepsilon_j} = 1 + \ldots + 2^{j(\frac{n}{2}-s)}\varepsilon_j, \tag{52}$$

$(\check{\varepsilon_j})$ is in l^2 for the same reason as $\tilde{\varepsilon_j}$. This gives

$$\|S_j(f)\|_\infty \leq C2^{j(\frac{n}{2}-s)}\|f\|_{\dot{H}^s}\eta_j.$$

where $(\eta_j)_j \in l^2$. Then, if $(\mu_j)_j$ is associated to g,

$$\|S_{j-1}(f)\Delta_j(g)\|_{L^2} \leq 2^{j(\frac{n}{2}-s)}\|f\|_{\dot{H}^s}\|g\|_{\dot{H}^s}\eta_j\mu_j,$$

and as $(\eta_j\mu_j)_j \in l^1 \subset l^2$, $S_1 \in \dot{H}^{2s-\frac{n}{2}}$. The terms of the first sum in (51) are like $S_1 = \sum_j \Delta_j(f)\Delta_j(g)$, and in this case we only know that the support of the Fourier transform of $\Delta_j(f)\Delta_j(g)$ is in $\{|\xi| \leq C2^j\}$, and

$$\|\Delta_j(f)\Delta_j(g)\|_{L^1} \leq \varepsilon_j\mu_j 2^{-2js}\|f\|_{\dot{H}^s}\|g\|_{\dot{H}^s}.$$

Lemma 9. – *If* $u \in L^1$, supp $\hat{u} \subset B(0,R)$, *and* $\|\hat{u}(\xi)\|_\infty \leq R^{-2s}$, *then*

$$\|\Lambda^{2s-\frac{n}{2}}u\|_{L^2} \leq \int_{S^2} dS.$$

This comes from

$$\int_{|\xi|\leq R} (|\xi|^2)^{2s-\frac{n}{2}}|\hat{u}(\xi)|^2 d\xi \leq R^{-4s}\int_{|\xi|\leq R} |\xi|^{4s-n}d\xi$$
$$\leq R^{-4s}\int_{S^2}\int_0^R r^{4s-1}drdS.$$

then, applying Lemma 9 to $\Delta_j(f)\Delta_j(g)$,

$$\|S_{j-1}(f)\Delta_j(g)\|_{\dot{H}^{2s-\frac{n}{2}}} \leq C\varepsilon_j\mu_j\|f\|_{\dot{H}^s}\|g\|_{\dot{H}^s}.$$

As $(\eta_j\mu_j)_j \in l^1$, this ends the proof. The second inequality can be proved by the same estimates, except that we have a better estimate for $\|S_j(f)\|_\infty$ and $\|\Delta_j(f)\|_\infty$, both bounded by $\|f\|_\infty$.

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