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Global analysis for a two-dimensional elliptic eigenvalue problem with the exponential nonlinearity

by

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ABSTRACT. – For the Emden-Fowler equation $-\Delta u = \lambda e^u$ in $\Omega \subset \mathbf{R}^2$, the connectivity of the trivial solution and the one-point blow-up singular limit is studied with respect to the parameter $\lambda > 0$. The connectivity is assured when the domain Ω is simply connected and the total mass $\Sigma = \int_\Omega \lambda e^u \, dx$ tends to 8π from below, which is a generalization for the case that Ω is a ball.

Key words : Emden-Fowler equation, singular limit, global bifurcation, rearrangement.

1. INTRODUCTION

In the present paper, we shall study the global bifurcation problem for the nonlinear elliptic eigenvalue problem (P): find $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ and

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$\lambda \in R_+ \equiv (0, +\infty)$ satisfying

$$-\Delta u = \lambda e^u \qquad (\text{in } \Omega) \tag{1}$$
$$u = 0 \qquad (\text{on } \partial\Omega), \tag{2}$$

where $\Omega \subset R^n$ is a bounded domain with smooth boundary $\partial\Omega$ and λ is a positive constant.

We shall study the two-dimensional problem, but a lot of work has been done for (P) including higher dimensional cases, for instance, Keller-Cohen [9], Fujita [6], Laetsch [10], Keener-Keller [8], Crandall-Rabinowitz [5]. They can be summarized in the following way:

FACT 1. – *There exists* $\bar{\lambda} \in (0, +\infty)$ *such that* (P) *has no solution* $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ *for* $\lambda > \bar{\lambda}$, *while* (P) *has at least one solution for* $0 < \lambda < \bar{\lambda}$.

FACT 2. – *For each fixed* λ *the set of solutions* $\{u\}$ *for* (P), *which is denoted by* S_λ, *has a minimal element* $u = \underline{u}_\lambda$ *whenever* $S_\lambda \neq 0$. *That is,* $\underline{u}_\lambda \in S_\lambda$, *and* $\underline{u}_\lambda \leqq u$ *holds for any* $u \in S_\lambda$.

FACT 3. – *There exists no triple* $\{u_1, u_2, u_3\} \subset S_\lambda$ *satisfying* $u_1 \leqq u_2 \leqq u_3$ *and* $u_1 \neq u_2 \neq u_3$.

FACT 4. – *Minimal solutions* $\{(\lambda, \underline{u}) \mid 0 < \lambda < \bar{\lambda}\}$ *form a branch* $\underline{S}$, *that is, one-dimensional manifold, in* $\lambda - u$ *plane starting from* $(\lambda, u) = (0, 0)$.

FACT 5. – *When* $n \leqq 9$, $\underline{S}$ *continues up to* $\lambda = \bar{\lambda}$ *and then bends back.*

FACT 6. – *When* $n \leqq 2$ *and* $\lambda \in (0, \bar{\lambda})$ *we have* $S_\lambda \neq \{\underline{u}_\lambda\}$, *that is, there exists a nonminimal solution then.*

In the case of $n = 2$, the problem (P) has a complex structure found by Liouville [12]. Utilizing it, Weston [20] and Moseley [13] have constructed a branch S^* of nonminimal solutions via singular perturbation method for generic simply connected domains $\Omega \subset R^2$. Their solutions make one-point blow-up as $\lambda \downarrow 0$.

One the contrary, the asymptotic behavior of solutions $\{u\}$ as $\lambda \downarrow 0$ has been studied by [15]. Singular limits of (P) are classified in the following way for the general domain $\Omega \subset R^2$.

THEOREM 1. – *Let* $h = \begin{pmatrix} u \\ \lambda \end{pmatrix}$ *be the classical solution of* (P), *and set*

$$\Sigma = \int_\Omega \lambda e^u \, dx. \tag{3}$$

Then $\{\Sigma\}$ *accumulates to some* $8\pi m$ *as* $\lambda \downarrow 0$, *where* $m = 0, 1, 2, \ldots, +\infty$. *The solutions* $\{u\}$ *behave as follows:*

(*a*) *If* $m = 0$, *then* $|u|_{L^\infty(\Omega)} \to 0$, i.e., *uniform convergence to the trivial solution* $u = 0$ *for* $\lambda = 0$.

(*b*) *If* $0 < m < +\infty$, *then there exists a set* $S \subset \Omega$ *of m-points such that* $u|_S \to +\infty$ *and* $|u|_{L^\infty_{loc}(\bar{\Omega} \setminus S)} \in O(1)$, i.e., *m-point blow-up.*

(*c*) *If* $m = +\infty$, *then* $u(x) \to +\infty$ *for any* $x \in \Omega$, i.e., *entire blow-up.*

Furthermore, in the case (*b*) the singular limit $u_0 = u_0(x)$ and the location of blow-up points $S \subset \Omega$ are described in terms of the Green function $G = G(x, y)$ of $-\Delta$ under the Dirichlet boundary condition in Ω. For instance, if $m = 1$ then the singular limit $u_0 = u_0(x)$ must be

$$u_0(x) = 8\pi G(x, \kappa) \tag{4}$$

and the blow-up point $\kappa \in \Omega$ must satisfy

$$\nabla R(\kappa) = 0, \tag{5}$$

where $R(x) = \left[G(x, y) + \frac{1}{2\pi} \log|x - y| \right]_{y=x}$ denotes the Robin function.

When Ω is a ball $B \equiv \{ |x| < 1 \}$ only the cases (*a*) and (*b*) with $m = 1$ occur in Theorem 1, and for the latter case

$$\kappa = 0 \qquad \text{and} \qquad u_0(x) = 8\pi G(x, \kappa) = 4 \log \frac{1}{|x|}.$$

On the other hand all possibilities $m = 0, 1, 2, \ldots, +\infty$ are expected when Ω is an annulus $A \equiv \{ a < |x| < 1 \}$, where $0 < a < 1$, *See* [11] and [14].

A natural question is how these singular limits are globally related to each other in $\lambda - u$ plane. In fact if $\Omega = B$, the singular limit $u_0(x) = 4 \log \frac{1}{|x|}$ is connected to the trivial solution $u = 0$ (*Fig.* 1).

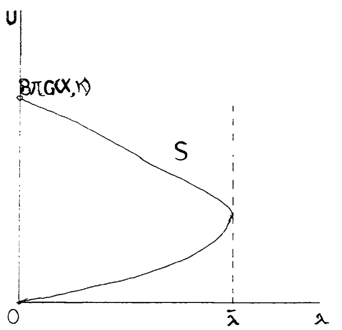

FIG. 1

Our purpose is to show that this phenomenon holds in more general situations. We can prove the following theorem, which is a refinement of our previous work [18]:

THEOREM 2. — *Let Ω be simply connected. Suppose that there exists a family of classical solutions $\{u\}$ of* (P) *satisfying $\Sigma = \int_\Omega \lambda e^u \, dx \uparrow 8\pi$ with*

$\lambda \downarrow 0$. *Then the singular limit in Theorem* 1, $u_0(x)=8\pi G(x, \kappa)$, *is connected to the trivial solution* $u=0$ *in* $\lambda-u$ *plane through a branch* S *bending just once.*

As for Weston-Moseley's branch S* of nonminimal solutions, we have a quantitive criterior for Σ to tend to 8π from below ([18], Proposition 1). Namely, given simply connected domain $\Omega \subset \mathbf{R}^2$, we take a $\kappa \in \Omega$ satisfying (5) and a univalent holomorphic function $g_\kappa : B=\{|x|<1\} \to \Omega$ with $g_\kappa(0)=\kappa$. It follows from (5) that $g''_\kappa(0)=0$. Under some generic assumption for κ other than $|g'''_\kappa(0)/g'_\kappa(0)| \neq 2$, Weston-Moseley's branch S* can be constructed, of which solutions $\{u^*\}$ make one-point blow-up at $\kappa \in \Omega$ as $\lambda \downarrow 0$. Then the relation

$$\int_\Omega \lambda e^{u^*} dx = 8\pi + C\lambda + o(\lambda) \qquad (as\ \lambda \downarrow 0) \tag{6}$$

holds with

$$\frac{C}{\pi} = -|a_1|^2 + \sum_{k=3}^{\infty} \frac{k^2}{k-2} |a_k|^2, \tag{7}$$

where

$$g_\kappa(z) = \kappa + a_1 z + \sum_{k=3}^{\infty} a_k z^k. \tag{8}$$

Therefore, if $C<0$ then S* is connected to $\underline{S}$, the branch of minimal solutions.

(Weston-Moseley's branch is constructed by a modified Newton method. The generic assumption on κ stated above is related to the degree of degeneracy of the linearized operator, and is rather implicit and complicated. However, in the case that Ω is convex with two axile symmetries, say, a rectangle, that condition holds. Furthermore, we have $|g'''_\kappa(0)/g'_\kappa(0)|<2$ in this case. *See* Moseley [13] and Wente [19].)

In the previous work [18, Theorem 3] we actually showed that phenomenon of connectivity when Ω is close to a ball. But we could not give a quantitative criterion about how Ω should be close to a ball to assure us of such a connectivity of S* and $\underline{S}$. In fact we have $a_k=0\,(k \geqq 3)$ when Ω is a ball in (8) and hence $C/\pi = -|a_1|^2 < 0$.

As Bandle [1] reveals, the problem (P) with $n=2$ has a geometric structure other than complex one. This structure will be fully utilized in proving our Theorem 2. Namely, employing the technique of rearrangement, we can reduce the theorem to the radial case $\Omega=B$. The assumption of simply connectedness of Ω is necessary in developping those procedures of rearrangement (Proposition 9).

2. STRATEGY FOR THE PROOF

We recall the problem (P):

$$-\Delta u=\lambda e^u \qquad (in \quad \Omega\subset \mathrm{R}^2) \tag{9}$$
$$u=0 \qquad (on \quad \partial\Omega). \tag{10}$$

The basic idea is to parametrize the solutions $\left\{h=\begin{pmatrix}u\\ \lambda\end{pmatrix}\right\}$ in terms of

$$\Sigma=\int_\Omega \lambda e^u\,dx. \tag{11}$$

This is nothing but to introduce the nonlinear operator

$$\Psi=\Psi(.,\Sigma): \quad \begin{matrix} C_0^{2+\alpha}(\bar\Omega) \\ \times \\ \mathrm{R}\end{matrix} \to \begin{matrix} C^\alpha(\bar\Omega) \\ \times \\ \mathrm{R}\end{matrix}$$

for given Σ through

$$\Psi(h,\Sigma)=\begin{pmatrix}\Delta u+\lambda e^u \\ \int_\Omega \lambda e^u\,dx-\Sigma\end{pmatrix}, \qquad where \quad h=\begin{pmatrix}u\\ \lambda\end{pmatrix}. \tag{12}$$

Here, $C^\alpha(\bar\Omega)$ denotes the usual Schauder space for $0<\alpha<1$ and $C_0^{2+\alpha}(\bar\Omega)=\{v\in C^{2+\alpha}(\bar\Omega)\,|\,v=0\ on\,\partial\Omega\}$. Then, each zero point of $\Psi(.,\Sigma)$ represents a solution $h=\begin{pmatrix}u\\ \lambda\end{pmatrix}$ of (P) satisfying (11).

This formulation has a geometric meaning. The solution $h=\begin{pmatrix}u\\ \lambda\end{pmatrix}$ of (P) is associated with a conformal mapping $\bar g$ from $\Omega\subset \mathrm{R}^2$ into a two-dimensional round sphere of diameter 1. Then, $\Sigma'=\frac{1}{8}\int_\Omega \lambda e^u\,dx$ indicates the area of $\bar g(\Omega)$ as an immersion. Therefore, we are trying to parametrize those surfaces by their area. This idea was also taken up in [14] in classifying radial solutions on annului. *See* [14] for details.

Later we shall show the following lemmas.

LEMMA 1. — *For each* $\delta>0$ *the set* $\left\{h=\begin{pmatrix}u\\ \lambda\end{pmatrix}\,|\,\Psi(h,\Sigma)=0 \textit{ for some } \Sigma\in[0,8\pi-\delta]\right\}$ *is compact.*

LEMMA 2. – *If* $\Psi(h, \Sigma)=0$ *with some* $\Sigma \in [0, 8\pi)$, *then the linearized operator*

$$d_h \Psi(.,\Sigma) : \begin{array}{c} C_0^{2+\alpha}(\bar{\Omega}) \\ \times \\ \mathbf{R} \end{array} \rightarrow \begin{array}{c} C^{\alpha}(\bar{\Omega}) \\ \times \\ \mathbf{R} \end{array}$$

is an isomorphism.

LEMMA 3. – *If* $\Psi(h, \Sigma)=0$ *for some* $\Sigma \in (0, 8\pi)$ *and* $h=\begin{pmatrix} u \\ \lambda \end{pmatrix}$, *then* $\mu_2(p, \Omega)>0$, *where* $p=\lambda e^u$.

Here and henceforth, $\mu_j(p, \Omega)$ ($j=1, 2, \ldots$) denotes the j-th eifenvalue of the differential operator $-\Delta-p$ in Ω under the Dirichlet boundary condition. That operator will be denoted by $-\Delta_D(\Omega)-p$. Thus, Lemma 3 indicates that the second eigenvalue of the linearized operator for (P) with respect to u is positive whenever $\Sigma=\int_\Omega \lambda e^u \, dx<8\pi$.

Those lammas imply our Theorem 2 in the following way. First, consider the set of zero points of Ψ in $\Sigma-h$ plane. Every zero point (h, Σ) of Ψ generates a branch of its zero points whenever $0 \leqq \Sigma < 8\pi$ by the implicit function theorem and Lemma 2. That branch continues up to $\Sigma=0$ by the compactness in Lemma 1. However, only the trivial solution $h=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is admitted for the problem (P) satisfying (11) with $\Sigma=0$. This implies the unique existence of a non-bending and non-bifurcating branch C of zeros of $\Psi : \{(\Sigma, h) \mid \Psi(h, \Sigma)=0\}$ in $\Sigma-h$ plane starting from

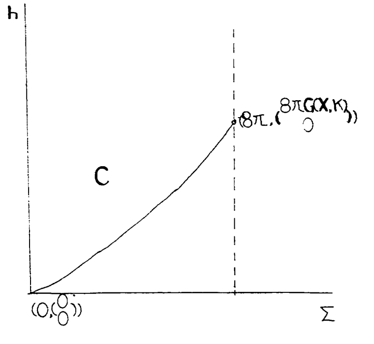

FIG. 2

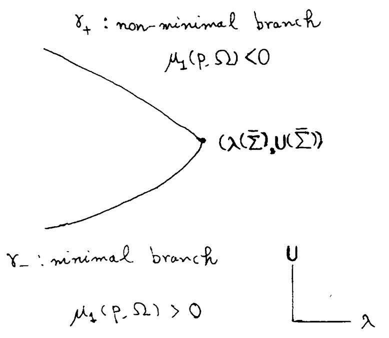

FIG. 3

$(\Sigma, h)=\left(0, \begin{pmatrix}0\\0\end{pmatrix}\right)$ to approach the hyperplane $\Sigma=8\pi$. On that hyperplane $\Sigma=8\pi$ lies the singlar limit $\left(8\pi, \begin{pmatrix}8\pi G(., \kappa)\\0\end{pmatrix}\right)$.

Therefore, in the case that Σ tends to 8π from below, the branch C connects the trivial solution $\left(0, \begin{pmatrix}0\\0\end{pmatrix}\right)$ and the singular limit $\left(8\pi, \begin{pmatrix}8\pi G(., \kappa)\\0\end{pmatrix}\right)$. More precisely, $C=\{\Sigma, h(\Sigma))\,|\,0<\Sigma<8\pi\}$ with $\lim_{\Sigma\downarrow 0} h(\Sigma)=\begin{pmatrix}0\\0\end{pmatrix}$ and $\lim_{\Sigma\uparrow\pi} h(\Sigma)=\begin{pmatrix}8\pi G(., \kappa)\\0\end{pmatrix}$ (*Fig*. 2).

Now the problem arises to represent $\left\{h(\Sigma)=\begin{pmatrix}u(\Sigma)\\\lambda(\Sigma)\end{pmatrix}\,\middle|\,0<\Sigma<8\pi\right\}$ in $\lambda-u$ plane. Lamma 3 and the implicit function theorem imply that u is locally parametrized by λ unless $\mu_1(\bar{p}, \Omega)=0$, where $\bar{p}=\lambda(\bar{\Sigma})\,e^{u(\bar{\Sigma})}$ with some $\bar{\Sigma}\in(0, 8\pi)$. However, at the degenerating point $\mu_1(\bar{p}, \Omega)=0$ works well the theory of Crandall-Rabinowitz [5]. That is, on account of the convexity of $f(u)=e^u$ and the positivity of the first eigenfunction of the linearized operator $-\Delta_D(\Omega)-\bar{p}$, the family $\{(\lambda(\Sigma), u(\Sigma))\}$ forms a bending branch around $\Sigma=\bar{\Sigma}$ which changes the solutions from the minimal to the nonminimal (*Fig*. 3).

Regarding the uniqueness of minimal solutions, we see that only one possibility of such a degeneracy $\mu_1(\bar{p}, \Omega)$ is permitted, and the proof of Theorem 2 will have been completed.

We prove those lemmas in later sections.

3. *A PRIORI* ESTIMATES FOR SOLUTIONS AND EIGENVALUES

The following proposition implies Lemma 1:

PROPOSITION 1. — *If* $h=\begin{pmatrix} u \\ \lambda \end{pmatrix}$ *solves* (P) *with* $\Sigma=\int_\Omega \lambda e^u dx<8\pi$, *then the inequality*

$$|u|_{L^\infty(\Omega)}\leqq -2\log\left(1-\frac{\Sigma}{8\pi}\right) \tag{13}$$

follows.

In fact we have the elliptic estimate for u and the existence of an upper bound for λ described as Fact 1 in Section 1. Therefore, the *a priori* estimate (13) for $|u|_{L^\infty}(\Omega)$ implies the compactness of the solution set $\{h\,|\,\Psi(h,\Sigma)=0 \textit{ for some } \Sigma\in[0,8\pi-\delta]\}$ in $C_0^{2+\alpha}(\bar\Omega)\times\mathbf{R}$ throught the bootstrap argument.

On the other hand Lemma 3 is proven by the following proposition:

PROPOSITION 2. — *If the positive function* $p\in C^2(\Omega)\cap C^0(\bar\Omega)$ *satisfies*

$$-\Delta\log p\leqq p \qquad (\textit{in } \Omega\subset\mathbf{R}^2) \tag{14}$$

and $\Sigma\equiv\int_\Omega p\,dx<4\pi$, *then*

$$\nu_1(p,\Omega)\equiv\inf\left\{\int_\Omega|\nabla v|^2dx\,\Big|\,v\in H_0^1(\Omega),\ \int_\Omega pv^2dx=1\right\}>1. \tag{15}$$

We note that if $h=\begin{pmatrix} u \\ \lambda \end{pmatrix}$ solves (P), then $p=\lambda e^u$ satisfies (14). Then the following corollary to Proposition 2 implies Lemma 3.

COROLLARY. — *If* p *satisfies* (14) *with* $\Sigma=\int_\Omega p\,dx<8\pi$, *then* $\mu_2(p,\Omega)>0$ *follows.*

Proof of Corollary. — First, we note that $\nu_1(p,\Omega)>1$ is equivalent to $\mu_1(p,\Omega)>0$ because of the Dirichlet principle for $\mu_1(p,\Omega)$.

By the argument of Å. Pleijel [17], the second eigenfunction ψ_2 of $-\Delta_D(\Omega)-p$ has two nodal domains $\Omega_\pm=\{\pm\psi_2>0\}$. That is, both Ω_+ and Ω_- are open connected sets. Their boundaries consist of a number of piecewise C^2 Jordan curves by a theorem due to Cheng [5]. In fact, extending ψ_2 outside Ω through a suitable reflection, we can regard $\partial\Omega$ as a portion of its nodal lines $\{\psi_2=0\}$.

By means of Jacobi's argument we can show that

$$\mu_1(p, \Omega_\pm) \equiv \inf\left\{ \int_{\Omega_\pm} |\nabla v|^2\, dx \,\Big|\, v \in H_0^1(\Omega_\pm),\ \int_{\Omega_\pm} pv^2\, dx = 1 \right\} = \mu_2(p, \Omega).$$

In fact, for $\mu = \mu_2(p, \Omega)$ the function

$$\varphi_\pm = \pm \psi_2 |_{\Omega_\pm} \in C^2(\Omega_\pm) \cap C^0(\bar{\Omega}_\pm) \cap H_0^1(\Omega_\pm)$$

satisfies

$$-\Delta\varphi_\pm = \mu p \varphi_\pm,\ \varphi_\pm > 0 \quad (\mathit{in}\ \Omega_\pm) \qquad \mathit{and} \qquad \varphi_\pm = 0 \quad (\mathit{on}\ \partial\Omega_\pm).$$

Hence

$$\int_{\Omega_\pm} |\nabla \varphi_\pm|^2\, dx = \mu \int_{\Omega_\pm} p\, \varphi_\pm^2\, dx.$$

On the other hand for any $v_\pm \in C_0^\infty(\Omega_\pm)$, the function $\eta_\pm = v_\pm / \varphi_\pm \in C_0^2(\Omega_\pm)$ is well-defined so that

$$\begin{aligned}
\int_{\Omega_\pm} |\nabla v_\pm|^2\, dx
&= \int_{\Omega_\pm} \{ \varphi_\pm^2 |\nabla \eta_\pm|^2 + 2\varphi_\pm \eta_\pm \nabla \varphi_\pm . \nabla \eta_\pm + \eta_\pm^2 |\nabla \varphi_\pm|^2 \}\, dx \\
&= \int_{\Omega_\pm} \{ \varphi_\pm^2 |\nabla \eta_\pm|^2 + \varphi_\pm \nabla \eta_\pm^2 . \nabla \varphi_\pm + \eta_\pm^2 |\nabla \varphi_\pm|^2 \}\, dx \\
&= \int_{\Omega_\pm} \{ \varphi_\pm^2 |\nabla \eta_\pm|^2 - \eta_\pm^2 \nabla . (\varphi_\pm \nabla \varphi_\pm) + \eta_\pm^2 |\nabla \varphi_\pm|^2 \}\, dx \\
&= \int_{\Omega_\pm} \{ \varphi_\pm^2 |\nabla \eta_\pm|^2 - \eta_\pm^2 \varphi_\pm \Delta \varphi_\pm \}\, dx \\
&= \int_{\Omega_\pm} \varphi_\pm^2 \{ |\nabla \eta_\pm|^2 + \mu p \eta_\pm^2 \}\, dx \geqq \mu \int_{\Omega_\pm} pv_\pm^2\, dx.
\end{aligned}$$

Therefore, we have

$$\int_{\Omega_\pm} |\nabla v_\pm|^2\, dx \geqq \mu \int_{\Omega_\pm} pv_\pm^2\, dx \qquad [v_\pm \in H_0^1(\Omega_\pm)].$$

This means that

$$\mu \equiv \mu_2(p, \Omega) = \mu_1(p, \Omega_\pm).$$

On the other hand we have $\Sigma=\int_{\Omega_+} pdx+\int_{\Omega_-} pdx<8\pi$. Therefore, either $\int_{\Omega_+} pdx<4\pi$ or $\int_{\Omega_-} pdx<4\pi$ holds so that $\mu_2(p, \Omega)>0$ by Proposition 2.

Propositions 1 and 2 are known when p is real analytic. For instance, see [18], Proposition 2 and [3], p. 108, respectively. That is enough for showing Lemmas 1 and 3. However, we shall perform the proof here, because it is necessary for us to describe that of Lemma 2.

4. ALEXANDROV-BOL'S INEQUALITY AND ITS CONSEQUENCES

It is well-known that the relation (14), *i.e.*,

$$-\Delta \log p \leqq p \qquad (in\, \Omega \subset \mathbf{R}^2) \tag{16}$$

implies Alexandrov-Bol's inequality

$$l(\partial\Omega)^2 \geqq \frac{1}{2}(8\pi-m(\Omega))\,m(\Omega), \tag{17}$$

where ds denotes the line element,

$$l(\partial\Omega)=\int_{\partial\Omega} p^{1/2}\,ds \qquad and \qquad m(\Omega)=\int_{\Omega} pdx. \tag{18}$$

An analytic proof is given in Bandle [2] when p is real analytic. In the present section, we just refine her argiment and show (17) even for non-real analytic p, to prove Proposition 1 in more general situations.

PROPOSITION 3. — *If a positive function* $p \in C^2(\Omega) \cap C^0(\bar{\Omega})$ *satisfies* (16), *then the inequality* (17) *holds.*

Proof of Proposition 3. — Let h be the harmonic lifting of $\log p$, that is,

$$-\Delta h=0 \qquad (in\, \Omega) \tag{19}$$

and

$$h=\log p \qquad (on\ \partial\Omega). \tag{20}$$

For each subdomain $\omega \subset \Omega$ with sufficiently smooth boundary, the inequality

$$\left\{\int_{\partial\omega} e^{h/2}\,ds\right\}^2 \geqq 4\pi \int_{\omega} e^h\,dx \tag{21}$$

holds true. This is essentially due to Z. Nehari [16].

In fact, there exists an analytic function $g=g(z)$ in Ω such that $|g'|^2=e^h$, and hence

$$\int_{\partial\omega} e^{h/2}\,ds=\int_{\partial\omega}|g'|\,ds=\textit{the length of } g(\partial\omega) \textit{ as an immersion}$$

and

$$\int_{\omega} e^{h}\,dx=\int_{\omega}|g'|^2\,dx=\textit{the area of } g(\partial\omega) \textit{ as an immersion.}$$

Therefore, (21) is nothing but an isoperimetric inequality for the flat Riemannian surface $g(\omega)$.

Now, we introduce the function $q=pe^{-h}$, which solves

$$-\Delta \log q \leqq qe^h \qquad (in\ \Omega) \tag{22}$$

and

$$q=1 \qquad (on\ \partial\Omega). \tag{23}$$

We shall derive a differential inequality satisfied by the right continuous and strictly decreasing functions

$$\mathrm{K}(t)=\int_{\{q>t\}} qe^h\,dx \tag{24}$$

and

$$\mu(t)=\int_{\{q>t\}} e^h\,dx. \tag{25}$$

In fact, co-area formula implies

$$-\mathrm{K}'(t)=\int_{\{q=t\}}\frac{qe^h}{|\nabla q|}ds=t\int_{\{q=t\}}\frac{e^h}{|\nabla q|}ds=-t\mu'(t) \qquad (a.e.\ t). \tag{26}$$

On the other hand, Green's formula gives

$$\int_{\{q>t\}}(-\Delta\log q)\,dx=\int_{\{q=t\}}\frac{|\nabla q|}{q}ds=\frac{1}{t}\int_{\{q=t\}}|\nabla q|\,ds \qquad (a.e.\ t>1),$$

because of Sard's lemma and the fact that $\partial\{q>t\}=\{q=t\}$ for $t>1$.

Hence we have from (22) that

$$\frac{1}{t}\int_{\{q=t\}}|\nabla q|\,ds\leqq\int_{\{q>t\}} qe^h\,dx=\mathrm{K}(t) \qquad (a.e.\ t>1), \tag{27}$$

Therefore, we get from Schwarz's inequality and (21) that

$$
\begin{aligned}
-\mathrm{K}(t)\,\mathrm{K}'(t) &\geqq \frac{1}{t}\int_{\{q=t\}} |\nabla q|\,ds\,.\,t\int_{\{q=t\}} \frac{e^h}{|\nabla q|}\,ds \\
&\geqq \left\{\int_{\{q=t\}} e^{h/2}\,ds\right\}^2 \geqq 4\pi\int_{\{q>t\}} e^h\,dx \\
&= 4\pi\mu(t) \qquad (a.e.\ t>1). \qquad (28)
\end{aligned}
$$

In particular,

$$
\begin{aligned}
\frac{d}{dt}&\left\{\mu(t)\,t-\mathrm{K}(t)+\frac{1}{8\pi}\mathrm{K}(t)^2\right\} \\
&= \mu(t)+\frac{1}{4\pi}\mathrm{K}(t)\,\mathrm{K}'(t)\leqq 0 \qquad (a.e.\ t>1). \qquad (29)
\end{aligned}
$$

Here, we note that

$$\mathrm{K}(t+0)=\mathrm{K}(t) \quad \text{and} \quad \mathrm{K}(t-0)\geqq \mathrm{K}(t).$$

On the other hand, the function

$$j(t)\equiv \mathrm{K}(t)-\mu(t)\,t=\int_{\{q>t\}}(q-t)\,e^h\,dx$$

is continuous as $j(t+0)=j(t)=j(t-0)$. In fact, $j(t+0)=j(t)$ is obvious and $j(t)-j(t-0)=\int_{\{q=t\}}(q-t)\,e^h\,dx=0$.

Therefore, (29) implies that

$$\left[\mu(t)\,t-\mathrm{K}(t)+\frac{1}{8\pi}\mathrm{K}(t)^2\right]_{t=1}^{t=\infty}=-\left\{\mu(1)-\mathrm{K}(1)+\frac{1}{8\pi}\mathrm{K}(1)^2\right\}\leqq 0. \qquad (30)$$

However, we have

$$\mathrm{K}(1)-\mu(1)=j(1)=\int_{\{q>1\}}(q-1)\,e^h\,dx\geqq\int_{\Omega}(q-1)\,e^h\,dx=m(\Omega)-\int_{\Omega}e^h\,dx$$

as well as

$$\mathrm{K}(1)^2\leqq m(\Omega)^2,$$

so that

$$m(\Omega)-\frac{1}{8\pi}m(\Omega)^2\leqq\int_{\Omega}e^h\,dx. \qquad (31)$$

Combining the inequalities (31) and (21) with $\omega=\Omega$, we see that

$$m(\Omega)-\frac{1}{8\pi}m(\Omega)^2\leqq\frac{1}{4\pi}\left\{\int_{\partial\Omega}p^{1/2}\,ds\right\}^2,$$

because $h=\log p$ on $\partial\Omega$. This is nothing but (17).

The following theorem implies Proposition 1 of the previous section. We have only to take $p=\lambda e^u$ for the solution $h=\begin{pmatrix} u \\ \lambda \end{pmatrix}$ of (P).

PROPOSITION 4. — *If a positive function* $p\in C^2(\Omega)\cap C^0(\bar{\Omega})$ *satisfies* (16) *and* $\Sigma\equiv\int_\Omega p\,dx<8\pi$, *then the inequality*

$$\max_{\bar{\Omega}} p \leqq \left(1-\frac{\Sigma}{8\pi}\right)^{-2} \max_{\partial\Omega} p \tag{32}$$

holds.

Proof of Proposition 4. — As in (30), we can derive from (29) the estimate

$$j(t)\equiv K(t)-\mu(t)\,t \leqq \frac{1}{8\pi} K(t)^2 \qquad (t\geqq 1) \tag{33}$$

or equivalently,

$$\mu(t)\geqq\frac{K(t)^2}{t}\left\{\frac{1}{K(t)}-\frac{1}{8\pi}\right\} \qquad (t\geqq 1). \tag{34}$$

Setting

$$J(t)=\frac{\mu(t)}{K(t)}-\frac{\mu(t)}{8\pi}, \tag{35}$$

we have

$$J(t+0)=J(t). \tag{36}$$

Furthermore, we note that

$$K(t-0)-K(t)=\int_{\{q=t\}} q e^h\,dx = t\,(\mu(t-0)-\mu(t))\geqq 0$$

to deduce

$$J(t-0)-J(t)=(\mu(t)-\mu(t-0))\left\{\frac{j(t)}{K(t)\,K(t-0)}-\frac{1}{8\pi}\right\}\geqq 0 \tag{37}$$

because of (33) and $\mu(t)-\mu(t-0)=-\int_{\{q=t\}} e^h\,dx\leqq 0.$

On the other hand we have

$$J'(t)=\mu'(t)\left\{\frac{1}{K(t)}-\frac{1}{8\pi}\right\}-\mu(t)\frac{K'(t)}{K(t)^2}$$
$$\geqq \frac{t\,\mu(t)\,\mu'(t)}{K(t)^2}-\frac{\mu(t)\,K'(t)}{K(t)^2}=0 \qquad (a.e.\, t>1) \quad (38)$$

by (34) and (26). The relations (36)-(38) imply for $t_0=\max_{\bar\Omega} q$ that

$$\lim_{t\uparrow t_0} J(t)=\lim_{t\uparrow t_0}\frac{\mu'(t)}{K'(t)}=1/t_0\geqq J(t)=\mu(t)\left\{\frac{1}{K(t)}-\frac{1}{8\pi}\right\} \qquad (1\leqq t\leqq t_0). \quad (39)$$

However,

$$J(t)\geqq\frac{K(t)^2}{t}\left\{\frac{1}{K(t)}-\frac{1}{8\pi}\right\}^2$$

by (34), of which right hand side tends to $\left(1-\frac{K(1)}{8\pi}\right)^2\geqq\left(1-\frac{\Sigma}{8\pi}\right)^2$ as $t\downarrow 1$.

In this way, we obtain

$$t_0=\max_{\bar\Omega} pe^{-h}\leqq\left(1-\frac{\Sigma}{8\pi}\right)^{-2},$$

so that

$$\max_{\bar\Omega} p\leqq\left(1-\frac{\Sigma}{8\pi}\right)^{-2}\max_{\bar\Omega} e^h\leqq\left(1-\frac{\Sigma}{8\pi}\right)^{-2}\max_{\partial\Omega} e^h=\left(1-\frac{\Sigma}{8\pi}\right)^{-2}\max_{\partial\Omega} p,$$

where the maximal principle for the harmonic function h is utilized.

5. SPHERICALLY DECREASING REARRANGEMENT

In this section we shall give an outline of the proof of Proposition 2 described in paragraph 3. We follow the idea of C. Bandle, employing some new arguments.

Proof of Proposition 2:

STEP 1. – We introduce the cannonical surface $h^*=\begin{pmatrix}u^*\\ \lambda^*\end{pmatrix}$ for given $\Sigma\in(0,\,8\pi)$, as $u^*\in C^2(\Omega^*)\cap C^0(\bar\Omega^*)$ and $\lambda^*>0$ solve

$$-\Delta u^*=\lambda^* e^{u^*} \qquad (in\,\Omega^*\equiv\{\,|x|<1\,\}\subset R^2), \quad (40)$$
$$u^*=0 \qquad (on\ \partial\Omega) \quad (41)$$

and

$$\int_{\Omega^*} \lambda^* e^* \, dx = \Sigma. \tag{42}$$

Here, we note the following proposition.

PROPOSITION 5. — *The solution* $h^* = \begin{pmatrix} u^* \\ \lambda^* \end{pmatrix}$ *of* (40) *with* (41) *is parametrized by* $\Sigma = \int_{\Omega^*} \lambda^* e^{u^*} dx \in (0, 8\pi)$. *In other words for each* $\Sigma \in (0, 8\pi)$ *there exists a unique zero* $h^* = \begin{pmatrix} u^* \\ \lambda^* \end{pmatrix}$ *of*

$$\Psi^* = \Psi^*(., \Sigma) : \begin{matrix} C_0^{2+\alpha}(\bar{\Omega}^*) \\ \times \\ \mathbf{R} \end{matrix} \to \begin{matrix} C^\alpha(\bar{\Omega}^*) \\ \times \\ \mathbf{R} \end{matrix}$$

with

$$\Psi^*(h^*, \Sigma) = \begin{pmatrix} \Delta u^* + \lambda^* e^{u^*} \\ \int_{\Omega^*} \lambda^* e^{u^*} dx - \Sigma \end{pmatrix}.$$

This fact is well-known. We can give the solution $h^* = \begin{pmatrix} u^* \\ \lambda^* \end{pmatrix}$ explicitly. In fact,

$$p^*(x) \equiv \lambda^* e^{u^*}(x) = \frac{8\mu}{(|x|^2 + \mu)^2} \tag{43}$$

holds for some $\mu = \mu(\Sigma) \in (0, +\infty)$. We furthermore have

$$\lim_{\Sigma \downarrow 0} \mu(\Sigma) = +\infty, \quad \lim_{\Sigma \uparrow 8\pi} \mu(\Sigma) = 0 \tag{44}$$

and

$$-\Delta \log p^* = p^* \qquad (in\ \Omega^*). \tag{45}$$

See [18] for the proof, for instance.

By virtue of (45), we can show that the equality holds in Alexandrov-Bol's inequality (17), when $p = p^*$ and Ω is a concentric disc of Ω^*.

STEP 2. — We shall adopt the procedure of spherically decreasing rearrangement. Namely, given a domain $\Omega \subset \mathbf{R}^2$ and a positive function

$p \in C^2(\Omega)$ satisfying (16) and $m(\Omega)=\int_\Omega p\,dx<8\pi$, we prepare the cannonical surface $h^*=\begin{pmatrix} u^* \\ \lambda^* \end{pmatrix}$ such that $\Sigma=\int_{\Omega^*} p^*\,dx=m(\Omega)$.

For given non-negative function $v=v(x)$ in Ω and a positive constant $t>0$ we put $\Omega_t=\{v>t\}$. We can define an open concentric ball Ω_t^* of Ω^* through

$$\int_{\Omega_t^*} p^*\,dx=\int_{\Omega_t} p\,dx \equiv a(t) \in [0, 8\pi). \tag{46}$$

Then, Bandle's spherically decreasing rearrangement v^* of v is a non-negative function in Ω^* defined as

$$v^*(x)=\sup\{t \mid x \in \Omega_t^*\}. \tag{47}$$

It is a kind of equi-measurable rearrangement, and the relation

$$\int_\Omega pv^2\,dx=\int t^2\,d(-a(t))=\int_{\Omega^*} p^* v^{*2}\,dx \tag{48}$$

holds true.

The following proposition can be proven, which is referred to as the decrease of Dirichlet integral:

PROPOSITION 6. – *If $v=v(x)$ is Lipschitz continuous on $\bar\Omega$, $v \geqq 0$ in Ω, and $v=0$ on $\partial\Omega$, then the inequality*

$$\int_\Omega |\nabla v|^2\,dx \geqq \int_{\Omega^*} |\nabla v^*|^2\,dx \tag{49}$$

holds.

From this proposition, it follows that

$$\begin{aligned} \nu_1(p, \Omega) &= \inf\left\{\int_\Omega |\nabla v|^2\,dx \,\Big|\, v \in H_0^1(\Omega),\ \int_\Omega pv^2\,dx=1\right\} \\ &\geqq \nu_1(p^*, \Omega^*)=\inf\left\{\int_{\Omega^*} |\nabla v|^2\,dx \,\Big|\, v \in H_0^1(\Omega^*),\ \int_\Omega pv^{*2}\,dx=1\right\}, \end{aligned} \tag{50}$$

provided that $m(\Omega)\equiv\int_\Omega p\,dx=\Sigma=\int_{\Omega^*} p^* \in (0, 8\pi)$. Hence the proof of Proposition 2 will be reduced to the radial case.

Originally, Proposition 6 was proven for real analytic functions by C. Bandle.

Proof of Proposition 6. — The function $a=a(t)$ in (46) is right-continuous and strictly decreasing in $t>0$. Since $v\geqq 0$ in Ω, co-area formula gives

$$-a'(t)=\int_{\{v=t\}}\frac{p}{|\nabla v|}\,ds \qquad (a.e.\ t>0) \tag{51}$$

and

$$-\frac{d}{dt}\int_{\Omega_t}|\nabla v|^2\,dx=\int_{\{v=t\}}|\nabla v|\,ds \qquad (a.e.\ t>0). \tag{52}$$

From Schwarz's and Alexandrov-Bol's inequalities we obtain

$$\begin{aligned}-\frac{d}{dt}\int_{\Omega_t}|\nabla v|^2\,dx &\geqq \left(\int_{\{v=t\}}p^{1/2}\,ds\right)^2\Big/\left(\int_{\{v=t\}}\frac{p}{|\nabla v|}\,ds\right)\\ &= l(\{v=t\})^2/-a'(t)\geqq(8\pi-a(t))\,a(t)/-a'(t) \qquad (a.e.\ t>0).\end{aligned} \tag{53}$$

Here, the function $j(t)=-\int_{\Omega_t}|\nabla v|^2\,dx$ is continuous and strictly decreasing in $t\in \mathrm{I}$. To see this, we have only to note that v is Lipschitz continuous so that

$$j(t)-j(t-0)=\int_{\{v=t\}}|\nabla v|^2\,dx=0.$$

Therefore, $j(t)$ is absolutely continuous and hence

$$\begin{aligned}\int_{\Omega}|\nabla v|^2\,dx&=\int_0^{\infty}dt\left(-\frac{d}{dt}\int_{\Omega_t}|\nabla v|^2\,dx\right)\\ &\geqq\int_0^{\infty}dt\,(8\pi-a(t))\,a(t)/-a'(t).\end{aligned} \tag{54}$$

On the other hand, $v^*=v^*(x)$ is a decreasing function of $r=|x|$. Therefore, equalities hold at each step in (53) for $\Omega=\Omega^*$ and $v=v^*$. Therefore, we have the equality in (54) for this case, in other words,

$$\begin{aligned}\int_{\Omega^*}|\nabla v^*|^2\,dx&=\int_0^{\infty}dt\left(-\frac{d}{dt}\int_{\Omega^*}|\nabla v^*|^2\,dx\right)\\ &=\int_0^{\infty}dt\,(8\pi-a(t))\,a(t)/-a'(t).\end{aligned} \tag{55}$$

This means (49).

STEP 3. — We finally show that $\nu_1(p^*, \Omega^*)>1$ in (50), or equivelently, $\mu_1(p^*, \Omega^*)\equiv$ *the first eigenvalue of* $-\Delta_D(\Omega^*)-p^*>0$, under the assumption of $\Sigma=\int_{\Omega^*}p^*\,dx<4\pi$.

This has been done by C. Bandle from the study of

$$-\Delta\varphi=\Lambda p^*(r)\varphi \qquad (in\ \Omega^*=\{|x|<1\subset \mathrm{R}^2) \tag{56}$$

and

$$\varphi=0 \qquad (on\ \partial\Omega^*) \tag{57}$$

for $p^*(r)=\dfrac{8\mu}{(r^2+\mu)^2}$ with $r=|x|$. By the separation $\varphi=\Phi(r)e^{im\theta}$ and the transformation $\xi=(\mu-r^2)/(\mu+r^2)$ of variables, the problem is reduced to the associated Legendre equation

$$\left.\begin{array}{c}[(1-\xi^2)\Phi_\xi]_\xi+[2\Lambda-m^2/(1-\xi^2)]\Phi=0\\ \left(\xi_\mu=\dfrac{\mu-1}{\mu+1}<\xi<1\right)\end{array}\right\} \tag{58}$$

with

$$\Phi(1):\quad bounded\ and\ \Phi(\xi_\mu)=0 \tag{59}$$

Let Φ solves (58) with $\Lambda=1$, $m=0$ and $\Phi(1)=1$. Then, the relation $\nu_1(p^*, \Omega^*)>1$ is equivalent to $\Phi(\xi)>0\,(\xi_\mu<\xi<1)$. Since such a Φ is given as $P_0(\xi)=\xi$, we see that $\nu_1(p^*, \Omega^*)>1$ is equivalent to $\xi_\mu>0$, or $\mu>1$. From (43) we see that $\mu>1$ if and only if $\Sigma=\int_{\Omega^*}p^*\,dx<4\pi$. Thus the proof has been completed.

This fact, however, can been proven more easily if we note that the bending of $\left\{h^*=\begin{pmatrix}u^*\\ \lambda^*\end{pmatrix}\right\}$ in $\lambda-u$ plane occurs at $\Sigma=4\pi$ in Proposition 3. In fact, then $\Sigma<4\pi$ indicates that μ is a minimal solution, from which follows $\mu_1(p^*, \Omega^*)>0$.

Concluding the present section we show that the inequality (50) can be regarded as an isoperimetric inequality for the Laplace-Beltrami operators. In fact, take a round two-dimensional sphere S of area 8π. Its cannonical metric and the volume element are denoted by $d\sigma$ and $d\mathrm{V}$, respectively. Let $\iota:\mathrm{S}\to\mathrm{R}^2\cup\{\infty\}$ be the stereographic projection from the north pole $n\in\mathrm{S}$ onto $\mathrm{R}^2\cup\{\infty\}$ tangent to the south pole $s\in\mathrm{S}$.

Let $\omega^*\subset\mathrm{S}$ be a ball (or boul, more precisely,) with the center $s\in\mathrm{S}$, and $-\Delta_\mathrm{S}(\omega^*)$ be the Laplace-Beltrami operator in ω^* under the Dirichlet boundary condition on $\partial\omega^*$. Then the injection $\iota:\omega^*\to\mathrm{R}^2$ transforms $-\Delta_\mathrm{S}(\omega^*)$ into $-\Delta/p^*$ in $\Omega^*=\iota(\omega^*)$ under the Durichlet boundary condition.

Since the Gaussian curvature of S is 1/2, the radial function $p^*=p^*(|x|)$ satisfies

$$-\Delta\log p^*=p^* \qquad (in\ \Omega^*), \tag{60}$$

and

$$\Sigma = \int_{\Omega^*} p^* \, dx = \int_{\omega^*} d\mathrm{V}. \tag{61}$$

If ι^* denotes the pull-back by ι, the relations

$$\int_{\omega^*} |d(\iota^* v)|^2 \, d\mathrm{V} = \int_{\Omega^*} |\nabla v|^2 \, dx \tag{62}$$

and

$$\int_{\omega^*} (\iota^* v)^2 \, d\mathrm{V} = \int_{\Omega^*} p^* v^{*2} \, dx \tag{63}$$

hold, and hence $\nu_1(p^*, \omega^*)$ is nothing but the first eigenvalue of $-\Delta_S(\omega^*)$ under the Dirichlet boundary condition on $\partial\omega^*$. This consideration reveals the reason why the assoicated Legendre equation has arisen in the study of (56).

In fact, let us take the variable $y = x/\mathrm{R}$, where R denotes the radius of: $\Omega^* = \{ |x| < \mathrm{R} \}$. Then the positive radial function $p_1^*(y) = \mathrm{R}^2 p^*(x)$ satisfies for $\Omega_1^* = \{ |x| < 1 \} \subset \mathrm{R}^2$ that

$$-\Delta \log p_1^* = p_1^* \qquad (in\ \Omega_1^*) \tag{64}$$

Furthermore, we have for $v \in \mathrm{H}_0^1(\Omega^*)$ that

$$\frac{\int_{\Omega^*} |\nabla v|^2 \, dx}{\int_{\Omega^*} p^* v^2 \, dx} = \frac{\int_{\Omega_1^*} |\nabla v_1|^2 \, dy}{\int_{\Omega_1} p_1^* v_1^2 \, dy}, \tag{65}$$

where $v_1(y) = v(x)$. Hence the eigenvalue problem for $-\Delta_S(\omega^*)$ is reduced to that of $-\Delta/p^*$ on the domain of unit ball:

$$-\Delta\varphi = \Lambda p_1^* \varphi \qquad (in\ \Omega_1^* = \{ |y| < 1 \} \subset \mathrm{R}^2) \tag{66}$$

and

$$\varphi = 0 \qquad (on\ \partial\Omega_1^*). \tag{67}$$

Hence we note that

$$\int_{\Omega^*} p_1^* \, dy = \int_{\Omega^*} p^* \, dx = \Sigma \in (0, 8\pi). \tag{68}$$

For $\lambda^* = p_1^* |_{\partial\Omega_1^*} \in \mathrm{R}_+$, the function p_1^* is realized as $p_1^* = \lambda^* e^{u^*}$, where $u^* = u^*(|y|)$ solves

$$-\Delta u^* = \lambda^* e^{u^*} \qquad (in\ \Omega_1^*) \tag{69}$$

by (61) and

$$u^*=0 \qquad (on\ \partial\Omega_1^*). \tag{70}$$

Therefore, it holds that

$$p_1^*(y)=\frac{8\mu}{(|y|^2+\mu)^2} \tag{71}$$

with a constant $\mu>0$, as we have seen in Proposition 5. The parameter μ is determined by (68).

In this way the eigenvalue problem for $-\Delta_S(\omega^*)$ is reduced to (66) with (67) through those transformations. Via the usual separation of variables for $-\Delta_S(\omega^*)$, or three-dimensional Laplacian

$$-\Delta=-\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right),$$

the associated Legendre equation arises, and so does in (66) if the inverse transformation, $\xi=(\mu-r^2)/(\mu+r^2)$, is applied.

From those considerations, we see that the spherically decreasing rearrangement described here is nothing but the Schwarz symmetrization on the round sphere. Namely, given a domain $\Omega\subset\mathbb{R}^2$ with sufficiently smooth boundary and given a positive function $p\in C^2(\Omega)\cap C^0(\bar{\Omega})$ satisfying

$$-\Delta\log p\leqq p \qquad (in\ \Omega) \tag{72}$$

and

$$\Sigma=\int_\Omega p\,dx<8\pi, \tag{73}$$

we take a ball $\omega^*\subset S$ so that

$$\int_{\omega^*} dV=\Sigma. \tag{74}$$

Then, for a non-negative function v in Ω we define a function $v^*:\omega^*\to(-\infty,+\infty]$ through the relation

$$v^*(x)=\sup\{t\,|\,x\in\omega_t\}, \tag{75}$$

where $\omega_t\subset S$ denotes the open concentric ball of ω^* such that

$$\int_{\omega_t} dV=\int_{\{v>t\}} p\,dx. \tag{76}$$

The following propositions are the consequences of the present section:

PROPOSITION 7. — *For each continuous function* $\psi:\mathbb{R}\to\mathbb{R}$ *we have*

$$\int_\Omega p\,\varphi\circ v\,dx=\int_{\omega^*}\psi\circ v^*\,dV \tag{77}$$

PROPOSITION 8. — *If v is Lepschitz continuous on $\bar{\Omega}$, $v \geqq 0$ in Ω and $v=0$ on $\partial\Omega$, then the relation*

$$\int_{\Omega} |\nabla v|^2 \, dx \geqq \int_{\omega^*} |dv^*|^2 \, dV \tag{78}$$

holds.

6. ANNULAR INCREASING REARRANGEMENT

Now we can perform the proof of Lemma 2.

First we note that the lemma is obvious when $\Sigma=0$. In fact, then $h=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ follows from $\Psi(h, 0)=0$ and hence

$$d_h \Psi(h, 0) = \begin{pmatrix} \Delta + \lambda e^u & e^u \\ \int_{\Omega} \lambda e^u . dx & \int_{\Omega} e^u \, dx \end{pmatrix} = \begin{pmatrix} \Delta & 1 \\ 0 & |\Omega| \end{pmatrix} : \begin{matrix} C_0^{2+\alpha}(\bar{\Omega}) \\ \times \\ R \end{matrix} \to \begin{matrix} C^{\alpha}(\bar{\Omega}) \\ \times \\ R \end{matrix} .$$

In the case of $\Sigma>0$ we have $\lambda>0$ from $\Psi(h, \Sigma)=0$ with $h=\begin{pmatrix} u \\ \lambda \end{pmatrix}$, so that $\Psi(h, \Sigma)=0$ is equivalent to $\Phi(h, \Sigma)=0$, where

$$\Phi = \Phi(., \Sigma) : \begin{matrix} C_0^{2+\alpha}(\bar{\Omega}) \\ \times \\ R_+ \end{matrix} \to \begin{matrix} C^{\alpha}(\bar{\Omega}) \\ \times \\ R \end{matrix}$$

with

$$\Phi(h, \Sigma) = \begin{pmatrix} \Delta u + \lambda e^u \\ \int_{\Omega} e^u \, dx - \dfrac{\Sigma}{\lambda} \end{pmatrix} \quad \textit{for} \quad h = \begin{pmatrix} u \\ \lambda \end{pmatrix}. \tag{79}$$

Hence the isomorphy of $d_h \Psi(h, \Sigma)$ is reduced to that of $d_h \Phi(h, \Sigma)$ when $\Sigma>0$.

The linearized operator

$$d_h \Phi(h, \Sigma) = \begin{pmatrix} \Delta + \lambda e^u & e^u \\ \int_{\Omega} e^u . dx & \dfrac{\Sigma}{\lambda^2} \end{pmatrix} : \begin{matrix} C_0^{2+\alpha}(\bar{\Omega}) \\ \times \\ R \end{matrix} \to \begin{matrix} C^{\alpha}(\bar{\Omega}) \\ \times \\ R \end{matrix} \tag{80}$$

has a natural self-adjoint extension in $\begin{matrix} L^2(\Omega) \\ \times \\ \mathbf{R} \end{matrix}$, which is denoted by $-T$ with the domain $D(-T) = \begin{matrix} H_0^1(\Omega) \cap H^2(\Omega) \\ \times \\ \mathbf{R} \end{matrix}$. By wirtue of the elliptic regularity the isomorphy of $d_h \Phi(h, \Sigma)$ is equivalent to that of T.

The operator T is associated with the bilinear form $A = A(.,.)$ on $\begin{matrix} H_0^1(\Omega) \\ \times \\ \mathbf{R} \end{matrix}$:

$$A(\xi, \eta) = \int_\Omega \nabla v . \nabla w dx - \int_\Omega \{ \lambda e^u v w + e^u k w + e^u v \rho \} dx - \Sigma \kappa \rho / \lambda^2, \qquad (81)$$

where $\xi = \begin{pmatrix} v \\ \kappa \end{pmatrix}$ and $\eta = \begin{pmatrix} w \\ \rho \end{pmatrix}$ are in $\begin{matrix} H_0^1(\Omega) \\ \times \\ \mathbf{R} \end{matrix}$. That is,

$$A(\xi, \eta) = \langle T \xi, \eta \rangle$$

holds for $\xi \in D(T)$ and $\eta \in \begin{matrix} H_0^1(\Omega) \\ \times \\ \mathbf{R} \end{matrix}$, where

$$\langle \xi, \eta \rangle = \int_\Omega v w dx + k \rho$$

for $\xi = \begin{pmatrix} v \\ \kappa \end{pmatrix}$ and $\eta = \begin{pmatrix} w \\ \rho \end{pmatrix}$. *See* Kato [7], for instance, as for the bilinear for associated with a self-adjoint operator.

Since $\Sigma = \int_\Omega \lambda e^u dx$, we have

$$A(\xi, \eta) = \int_\Omega \nabla v . \nabla w dx - \int_\Omega \lambda e^u \left(v + \frac{\kappa}{\lambda} \right) \left(w + \frac{\rho}{\lambda} \right) dx \qquad (82)$$

for $\xi = \begin{pmatrix} v \\ \kappa \end{pmatrix}$ and $\eta = \begin{pmatrix} w \\ \rho \end{pmatrix}$. We put

$$H_c^1(\Omega) = \{ v \in H^1(\Omega) \mid v = \text{Const. on } \partial\Omega \},$$

to see the isomorphy of the mapping

$$\xi = \begin{pmatrix} v \\ \kappa \end{pmatrix} \in \begin{matrix} H_0^1(\Omega) \\ \times \\ \mathbf{R} \end{matrix} \mapsto v + \frac{\kappa}{\lambda} \in H_c^1(\Omega) \qquad (83)$$

for each $\lambda>0$. Therefore, $T=-d_h\Phi(h, \Sigma)$ is an isomorphism if and only if $\hat{A}_p$ has no zero spectrum, where A_p denotes the self-adjoint operator in $L^2(\Omega)$ associated with the bilinear form $B|_{H_c^1(\Omega)\times H_c^1(\Omega)}$ with

$$B(v, w)=\int_\Omega \nabla v.\nabla w dx-\int_\Omega pvwdx \tag{84}$$

for $p=\lambda e^u$.

The spectrum of $\hat{A}_p$ is composed of eigenvalues: $\sigma(\hat{A}_p)=\{\hat{\mu}_j(p, \Omega)\}_{j=1}^\infty$ with $-\infty<\hat{\mu}_1(p, \Omega)\leqq\hat{\mu}_2(p, \Omega)\leqq\ldots$ We have

$$\hat{\mu}_1(p, \Omega)=\inf\{B(v, v)\,|\,v\in H_c^1(\Omega), |v|_{L^2}=1\}\leqq-\frac{1}{|\Omega|}\int_\Omega pdx<0, \tag{85}$$

because $\zeta=\frac{1}{|\Omega|^{1/2}}\in H_c^1(\Omega)$ and $|\zeta|_{L^2}=1$.

On the other and we can prove the following proposition.

PROPOSITION 9. — *If a positive function* $p\in C^2(\Omega)\cap C^0(\bar{\Omega})$ *satisfies*

$$-\Delta\log p\leqq p \qquad (in\ \Omega) \tag{86}$$

with

$$\Sigma\equiv\int_\Omega pdx<8\pi, \tag{87}$$

then we have

$$K\equiv\inf\left\{\int_\Omega|\nabla v|^2dx\,\Big|\,v\in H_c^1(\Omega), \int_\Omega pv^2dx=1, \int_\Omega pvdx=0\right\}>1. \tag{88}$$

By virtue of Courant's mini-max principle, (88) implies that

$$\hat{\mu}_2(p, \Omega)=\sup_{X_1\subset H_c^1(\Omega),\ \mathrm{codim}\,X_1=1} \times\inf\left\{\int_\Omega|\nabla v|^2dx-\int_\Omega pv^2dx\,\Big|\,v\in X_1, |v|_{L^2}=1\right\}>0.$$

Hence Lemma 2 follows.

We finally give the

Proof of Proposition 9. — First we note that K in (88) is nothing but the second eigenvalue for the following eigenvalue problem (E.P.): To find $\varphi\in H_c^1\setminus\{0\}$ and $K\in R$ such that

$$\int_\Omega\nabla\varphi.\nabla vdx=K\int_\Omega p\varphi vdx \quad for\ any\ v\in H_c^1(\Omega). \tag{89}$$

In fact, the first eigenvalue and eigenfunction of (E.P.) are 0 and $\zeta\equiv\text{Const.}\neq 0$, respectively. Hence the second eigenvalue of (E.P.) is given by the value K of (88).

In particular, the minimizer $\varphi\in H_c^1(\Omega)$ of K in (88) satisfies

$$-\Delta\varphi = K p \varphi \qquad (\textit{in } \Omega) \tag{90}$$

and

$$\varphi = \text{Const.} \quad (\textit{on } \partial\Omega), \qquad \int_{\partial\Omega} \frac{\partial\varphi}{\partial\nu}\, ds = 0, \tag{91}$$

where ν denotes the outer unit normal vector on $\partial\Omega$.

Let $\{\Omega_i\}_{i\in I}$ be the nodal domains of φ, that is, the set of connected components of $\{\varphi\neq 0\}$. Then, $\partial\Omega_i$ consists of a number of piecewise C^2 Jordan curves by Cheng's theorem [4]. We have

$$\int_{\partial\Omega_i} \frac{\partial\varphi}{\partial\nu}\varphi\, ds = 0 \quad \textit{for each } i\in I \tag{92}$$

from (91).

In fact, each $\partial\Omega_i$ is composed of some portions of nodal lines $\{\varphi=0\}$ and/or the boundary $\partial\Omega$. That is, $\partial\Omega_i=\gamma_0\cup\gamma_1$ with $\gamma_0\subset\partial\Omega$ and $\gamma_1\subset\{\varphi=0\}$. If $\bar\gamma_0\cap\bar\gamma_1\neq\varnothing$, then $\gamma_0\subset\{\varphi=0\}$ and hence $\varphi=0$ on $\partial\Omega_i$. Therefore, it holds that $\int_{\partial\Omega_i}\varphi\frac{\partial\varphi}{\partial\nu}\,ds=0$. Otherwise, $\gamma_0=\partial\Omega$ follows from $\gamma_0\neq\varnothing$ because $\Omega\subset R^2$ is simply connected and $\partial\Omega$ has only one component. Therefore, in this case we have also that

$$\int_{\partial\Omega_i}\varphi\frac{\partial\varphi}{\partial\nu}\,ds = \int_{\gamma_0}\varphi\frac{\partial\varphi}{\partial\nu}\,ds = \varphi\,|_{\partial\Omega}\int_{\partial\Omega}\frac{\partial\varphi}{\partial\nu}\,ds = 0.$$

By virtue of Pleijel's argument [17], this fact (92) implies $\sharp I=2$. In fact φ cannot be definite because of $\int_\Omega p\varphi\,dx=0$. Suppose that there exist three nodal domains Ω_1, Ω_2. and Ω_3 of φ. Each zero extension to Ω of $\varphi\,|_{\Omega_i}(i=1, 2, 3)$, which is denoted by φ_i, satisfies

$$-\Delta\varphi_i = K p \varphi_i \qquad (\textit{in } \Omega_i) \tag{93}$$

and

$$\int_{\partial\Omega_i}\frac{\partial\varphi_i}{\partial\nu}\varphi_i\,ds = 0. \tag{94}$$

There exist constants a_1 and a_2 such that

$$\int_\Omega F p dx = 0 \qquad \textit{and} \qquad F\neq 0, \tag{95}$$

where $F=a_1\varphi_1+a_2\varphi_2$. Obviously, $F\in H^1(\Omega)$ satisfies

$$F=\text{Const.}\qquad (on\ \partial\Omega). \tag{96}$$

Furthermore, (93) and (94) imply that

$$\int_\Omega |\nabla F|^2\,dx=a_1^2\int_{\Omega_1}|\nabla\varphi_1|^2\,dx+a_2^2\int_{\Omega_2}|\nabla\varphi_2|^2\,dx$$
$$=K\left\{a_1^2\int_{\Omega_1}p\varphi_1^2\,dx+a_2^2\int_{\Omega_2}p\varphi_2^2\,dx\right\}=K\int_\Omega pF^2\,dx. \tag{97}$$

These equalities (95)-(97) indicate that $F\in H_c^1(\Omega)$ is a second eigenfunction for the eigenvalue problem (89). Therefore, F is smooth and satisfies

$$-\Delta F=KpF\qquad (in\ \Omega). \tag{98}$$

However, $F|_{\Omega_3}=0$ implies $F\equiv 0$ (*in* Ω) by the unique continuation theorem, a contradiction.

Thus, each of $\Omega_\pm\equiv\{\pm\varphi>0\}$ becomes a nodal domain of φ, that is, open connected set of which boundary consists of a number of piecewise smooth Jordan curves.

Let $\kappa=\varphi|_{\partial\Omega}\in R$.

In the case of $\kappa=0$, $\varphi\in H_c^1(\Omega)$ satisfies

$$-\Delta\varphi=Kp\varphi,\ \pm\varphi>0\qquad (in\ \Omega_\pm) \tag{99}$$

and

$$\varphi=0\qquad (on\ \partial\Omega_\pm). \tag{100}$$

Now (87) implies either $\int_{\Omega_+}p\,dx<4\pi$ of $\int_{\Omega_-}p\,dx<4\pi$ so that $K>1$ follows from Proposition 2 of section 3.

In the case $\kappa\neq 0$, any nodal line of φ cannot touch $\partial\Omega$. Hence either Ω_+ or Ω_- is simply connected.

Without loss of generality we suppose that Ω_- is simply connected and $\partial\Omega_+\supset\partial\Omega$ (*Fig.* 4). We put $\Sigma_\pm=\int_{\Omega_\pm}p\,dx$. Since $\Sigma_++\Sigma_-=\Sigma<8\pi$, we have either

$$\Sigma_-<4\pi \tag{101}$$

or

$$\Sigma_+<4\pi\leqq\Sigma_-. \tag{102}$$

In the case of (101), $K>1$ follows again from Proposition 2 because $\varphi_-\equiv\varphi|_{\Omega_-}\in H_0^1(\Omega_-)$ holds by the topological assumption for $\Omega_\pm$.

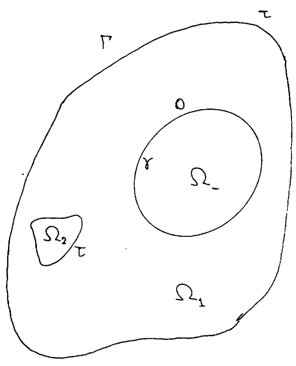

FIG. 4

Supposing the case (102), we put $\Gamma=\partial\Omega$ and $\gamma=\partial\Omega_-$. The function $\varphi_+=\varphi|_{\Omega_+}$ solves

$$-\Delta\varphi_+=\mathrm{K}p\,\varphi_+,\qquad \varphi_+>0\qquad (in\ \Omega_+), \tag{103}$$

$$\varphi_+=0\qquad (on\ \gamma), \tag{104}$$

and

$$\varphi_+=\text{Const.}\quad (on\ \Gamma),\qquad \int_\Gamma \frac{\partial\varphi_+}{\partial\nu}\,ds=0. \tag{105}$$

Therefore, it holds that

$$\mathrm{K}=\inf\left\{\int_{\Omega_+}|\nabla v|^2\,dx\ \middle|\ v\in\mathrm{H}^1(\Omega_+),\ v=0\ (on\ \gamma),\ v=\text{Const.}\ (on\ \Gamma),\ \int_{\Omega_+}pv^2\,dx=1\right\}. \tag{106}$$

The minimizer $\psi>0$ of K in (106) is defined in Ω_+ and is given by Const. $\times\,\varphi_+$.

Putting $\tau=\psi|_\Gamma$, we set $\Omega_1=\{\psi\leqq\tau\}$ and $\Omega_2=\{\psi>\tau\}$. The latter might be empty, but otherwise we take the spherically decreasing rearrangement ψ_2^* of $\psi_2=\psi|_{\Omega_2}$ described in section 5. Namely, let S be the round sphere with area 8π, and let $d\sigma^2$ and $d\mathrm{V}$ be its cannonical metric and volume element, respectively. We prepare a ball $\mathrm{B}\subset\mathrm{S}$ with

$$\int_\mathrm{B} d\mathrm{V}=\int_{\Omega_2} p\,dx. \tag{107}$$

Then the function ψ_2^* on B is defined through

$$\psi_2^*(x)=\sup\{t \mid x\in\omega_t\},$$

where ω_t denotes the open concentric ball of B satisfying

$$\int_{\omega_t} dV=\int_{\{\psi_2>t\}} p\,dx.$$

We have

$$\int_{\Omega_2} p\,\psi^2\,dx=\int_B (\psi_2^*)^2\,dV, \tag{108}$$

$$\int_{\Omega_2} |\nabla\psi_2|^2\,dx \geqq \int_B |d\psi_2^*|^2\,dV \tag{109}$$

and

$$\psi_2^*|_{\partial B}=\tau. \tag{110}$$

On the other hand we take the following procedure for $\psi_1=\psi|_{\Omega_1}$, which may be called the annular increasing rearrangement. Namely, we take open concentric balls $B_2\subset B_1\subset S$ so that

$$\int_{B_2} dV=\int_{\Omega_-} p\,dx\,(=\Sigma_-) \qquad \textit{and} \qquad \int_{B_1} dV=\int_{\Omega} p\,dx\ (=\Sigma). \tag{111}$$

The function ψ_{1*} on the annulus $A\equiv B_1\setminus\bar{B}_2$ is defined through

$$\psi_{1*}(x)=\inf\{t \mid x\in A_t\},$$

where A_t denotes the closed concentric annulus of A such that $A_t\cup\bar{B}_2\subset S$ is a closed ball and

$$\int_{A_t} dV=\int_{\{\psi_1\leqq t\}} p\,dx.$$

It is an equi-measurable rearrangement and the relation

$$\int_{\Omega_2} p\,\psi^2\,dx=\int_A (\psi_{1*})^2\,dV \tag{112}$$

follows. On the other hand the decrease of Dirichlet integral is derived from Alexandrov-Bol's inequality as in Proposition 8. Namely,

$$\int_{\Omega_2} |\nabla\psi|^2\,dx \geqq \int_A |d\psi_{1*}|^2\,dV, \tag{113}$$

because $0\leqq\psi\leqq\tau$ in Ω_2, $\psi=0$ on γ and $\psi=\tau$ on $\partial\Omega_2\setminus\gamma$. Finally, the relation

$$\psi_{1*}=0 \quad (\textit{on } \gamma^*) \qquad \textit{and} \qquad \psi_{1*}=\tau \quad (\textit{on } \Gamma^*) \tag{114}$$

is obvious, where $\gamma^* = \partial B_2$ and $\Gamma^* = \partial B_1$. We note that $\partial A = \gamma^* \cup \Gamma^*$. In this way we obtain

$$K \geqq K^* \equiv \inf \left\{ \int_{\omega+} |dv|^2 dV \,\middle|\, v \in H^1(\omega_+),\ v = 0 \ (on\ \gamma^*), \right.$$
$$\left. v|_{\Gamma^*} = v|_{\partial B} = \text{Const.} \int_{\omega+} v^2 dV = 1 \right\}, \quad (115)$$

where $\omega_+ = A \cup B \subset S$ is regarded as a disjoint sum. Therefore, the proposition has been reduced to showing $K^* > 1$.

Recalling the assumptions $\Sigma_+ < 4\pi \leqq \Sigma_-$ and $\Sigma_+ + \Sigma_- < 8\pi$ as well as the relation (107) and (111), we arrange the ball B and the annulus A so

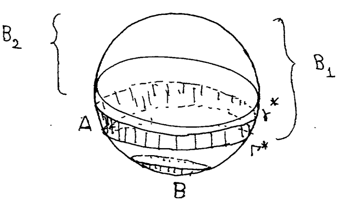

FIG. 5

that are concentric with the center $s \in S$ of the south pole (*Fig.* 5). Then, $\omega_+ = A \cup B$ is contained in a chemi-ball of S.

Though the stereographic projection $i: S \to R^2 \cup \{\infty\}$, the value K^* in (115) is realized as the first eigenvalue of the following problem. That is,

$$-\Delta\varphi = K p^* \varphi \quad (in\ \Omega) \quad (116)$$
$$\varphi = 0 \quad (on\ \Gamma_1) \quad (117)$$

and

$$\varphi|_{\Gamma_2} = \varphi|_{\Gamma_3} = \text{Const.}, \qquad \int_{\Gamma_2} \frac{\partial\varphi}{\partial\nu} + \int_{\Gamma_3} \frac{\partial\varphi}{\partial v}\, ds = 0. \quad (118)$$

Here, $\Omega^* = A^* \cup B^*$, $\Gamma_1 = i(\gamma^*)$, $\Gamma_2 = i(\Gamma^*)$, and $\Gamma_3 = i(\partial B)$ (*Fig.* 6). The function p^* comes from the transformation by the projection i of the Laplace-Beltrami operator $-\Delta_S$ on S. The annulus A^* and the ball B^* in the flat plane are concentric and disjoint.

Through the scalling transformation as we introduced at the end of the previous section, we may suppose that the outer radius of A^* is equal to one. Then, the function p^* in (116) is given as

$$p^*(x) = \frac{8\mu}{(|x|^2 + \mu)^2} \quad (119)$$

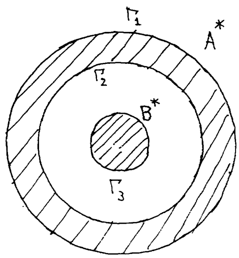

FIG. 6

with some constant $\mu>0$, which is determined by

$$\int_{\Omega^*} p^*\,dx=\int_{\omega_+} dV. \tag{120}$$

The set $\omega_+=A\cup B$ is contained in a chemi-ball of the round sphere S, and hence

$$\int_{\Omega^*} p^*\,dx<4\pi. \tag{121}$$

The first eigenfunction for (116)-(118) is radial and positive. Therefore, in terms of the variable $\xi=(\mu-r^2)/(\mu+r^2)$ it satisfies for some constants a and b in $a<b<1$ that

$$\left.\begin{array}{l}[(1-\xi^2)\Phi_\xi]_\xi+(2/K^*)\Phi=0\\ \left(\xi_\mu=\dfrac{\mu-1}{\mu+1}<\xi<a,\ b<\xi<1\right)\end{array}\right\} \tag{122}$$

with

$$\begin{array}{lll}\Phi(\xi)>0, & (\xi_\mu<\xi<a,\ b<\xi<1), & \\ \Phi(b)=\Phi(a), & \Phi'(b)=\Phi'(a), & \Phi(\xi_\mu)=0\end{array} \tag{123}$$

in the case of $\Omega_2=\varnothing$ and that

$$[(1-\xi^2)\Phi_\xi]_\xi+(2/K^*)\Phi=0 \qquad (\xi_\mu<\xi<a) \tag{124}$$

with

$$\Phi(\xi)>0\quad(\xi_\mu<\xi<a),\qquad \Phi'(a)=0,\qquad \Phi(\xi_\mu)=0 \tag{125}$$

in the case of $\Omega_2=\varnothing$, respectively.

Therefore, the desired inequality $K^*>1$ is proven if the relations

$$\Phi(\xi)>0\qquad(\xi_\mu<\xi<a,\ b<\xi<1) \tag{126}$$

and

$$\Phi(\xi)>0\qquad(\xi_\mu<\xi<a) \tag{127}$$

are obtained respectively, whenever Φ solves

$$[(1-\xi^2)\Phi_\xi]_\xi+2\Phi=0 \qquad (-1<\xi<a,\ b<\xi<1) \tag{128}$$

with

$$\Phi(1)=1, \qquad \Phi(b)=\Phi(a), \qquad \Phi'(b)=\Phi'(a) \tag{129}$$

and

$$[(1-\xi^2)\Phi_\varepsilon]_\xi+2\Phi=0 \qquad (-1<\xi<a) \tag{130}$$

with

$$\Phi(a)=1, \qquad \Phi'(a)=0. \tag{131}$$

The fundamental system of solutions for (128) or (130) is known. That is, $P_1(\xi)=\xi$ and $Q_1(\xi)=-1+\frac{\xi}{2}\log\frac{1+\xi}{1-\xi}$. Therefore, solutions for (127)-(128) and (129)-(130) are given as

$$\Phi(\xi)=\begin{cases} \xi & (b<\xi<1) \\ \left\{1+(b-a)\left(\dfrac{1-a^2}{2}\log\dfrac{1+a}{1-a}+a\right)\right\}\xi & \\ \quad -(1-a^2)(b-a)\left\{-1+\dfrac{\xi}{2}\log\dfrac{1+\xi}{1-\xi}\right\} & (-1<\xi<a) \end{cases} \tag{132}$$

and

$$\Phi(\xi)=(1-a^2)\left\{-\frac{\xi}{2}\log\frac{(1-a)(1+\xi)}{(1+a)(1-\xi)}+1\right\}+a\xi \qquad (-1<\xi<a), \tag{133}$$

respectively.

As we have seen in the previous section, the condition (121) implies that $\xi_\mu\equiv\dfrac{\mu-1}{\mu+1}>0$. Hence (126) and (127) follow from the elementary computations that

$$\Phi(\xi)\geqq\xi>0 \qquad (0<\xi<a,\ b<\xi<1) \tag{134}$$

and

$$\Phi(\xi)\geqq\Phi(0)=1-a^2>0 \qquad (0<\xi<a) \tag{135}$$

in (132) and (133), respectively. Thus the proof has been completed.

REFERENCES

[1] C. BANDLE, Existence Theorems, Qualitative Results and *a priori* Bounds for a Class of Nonlinear Dirichlet Problems, *Arch. Rat. Mech. Anal.*, Vol. **58**, 1975, pp. 219-238.

[2] C. BANDLE, On a Differential Inequality and its Application to Geometry, *Math. Z.*, Vol. **147**, 1976, pp. 253-261.

[3] C. BANDLE, *Isoperimetric Inequalities and Applications*, Pitman, Boston-London-Melbourne, 1980.

[4] S. Y. CHENG, Eigenfunctions and Nodal Sets, *Comment. Math. Helvetici*, Vol. **51**, 1976, pp. 43-55.

[5] M. G. CRANDALL et P. H. RABINOWITZ, Some Continuation and Variational Methods for Positive Solutions of Nonlinear Elliptic Eigenvalu Problems, *Arch. Rat. Mech. Anam.*, Vol. **58**, 1975, pp. 207-218.

[6] H. FUJITA, On the Nonlinear Equations $\Delta u + e^u = 0$ and $\partial v/\partial t = \Delta v + e^v$, *Bull. Am. Math. Soc.*, Vol. **75**, 1969, pp. 132-135.

[7] T. KATO, *Perturbation Theory for Linear Operators*, 2nd ed., Springer, Berlin-Heidelberg-New York, 1976.

[8] J. P. KEENER et H. B. KELLER, Positive Solutions of Convex Nonlinear Eigenvalue Problem, *J. Diff. Eq.*, Vol. **16**, 1974, pp. 103-125.

[9] H. B. KELLER et D. S. COHEN, Some Positive Problems Suggested by Nonlinear Heat Generation, *J. Math. Mech.*, **16**, 1967, pp. 1361-1376.

[10] T. LAETSCH, On the Number of Boundary Value Problems with Convex Nonlinearities, *J. Math. Anal. Appl.*, Vol. **35**, 1971, pp. 389-404.

[11] S. S. LIN, On Non-Radially Symmetric Bifurcations in the Annulus, *J. Diff. Eq.*, **80**, 1989, pp. 251-279.

[12] J. LIOUVILLE, Sur l'équation aux différences partielles $\partial^2 \operatorname{Log} \lambda/\partial u\, \partial v \pm \lambda/2\, a^2 = 0$, *J. Math.*, Vol. **18**, 1853, pp. 71-72.

[13] J. L. MOSELEY, A Two-Dimensional Dirichlet Problem with an Exponential Nonlinearily, *SIAM J. Math. Anal.*, Vol. **14**, 1983, pp. 934-946.

[14] K. NAGASAKI et T. SUZUKI, Radial and Nonradial Solutions for the Nonlinear Eigenvalue problem $\Delta u + \lambda e^u = 0$ on Annului in R^2, *J. Diff. Eq.*, Vol. **87**, 1990, pp. 144-168.

[15] K. NAGASAKI et T. SUZUKI, Asymptotic Analysis for Two-Dimensional Elliptic Eigenvalue Problems with Exponentially-Dominated Nonlinearities, *Asymptotic Analysis*, Vol. **3**, 1990, pp. 173-188.

[16] Z. NEHARI, On the Principal Frequency of a Membrane, *Pac. J. Math.*, Vol. **8**, 1958, pp. 285-293.

[17] Å. PLEIJEL, Remarks on Courant's Nodal Line Theorem, *Comm. Pure Appl. Math.*, Vol. **9**, 1956, pp. 543-550.

[18] T. SUZUKI et K. NAGASAKI, On the Nonlinear Eigenvalue Problem $\Delta u + \lambda e^u = 0$, *Trans. Am. Math. Soc.*, Vol. **309**, 1988, pp. 591-608.

[19] H. WENTE, Counterexample to a Conjecture of H. Hopf, *Pacific J. Math.*, Vol. **121**, 1986, pp. 193-243.

[20] V. H. WESTON, On the Asymptotic Solution of a Partial Differential Equation with an Exponential Nonlinearity, *SIAM J. Math. Anal.*, Vol. **9**, 1978, pp. 1030-1053.

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