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Homogenization of almost periodic monotone operators

by

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ABSTRACT. — We determine some sufficient conditions for the G-convergence of sequences of quasi-linear monotone operators, together with an asymptotic formula for the G-limit. We then prove a homogenization theorem for quasiperiodic monotone operators and, eventually, extend this result to general almost periodic monotone operators using an approximation result and a closure lemma.

Key words : Homogenization, G-convergence, almost periodic functions, monotone operators, quasi-linear equations.

RÉSUMÉ. — Nous déterminons quelques conditions suffisantes pour la G-convergence de suites d'opérateurs monotones quasi linéaires, avec une

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formule asymptotique pour la G-limite. Ensuite nous démontrons un théorème d'homogénéisation pour les opérateurs monotones quasi-périodiques et, enfin, nous extendons ce résultat aux opérateurs monotones presque périodiques en utilisant un résultat d'approximation.

INTRODUCTION

In this paper we consider a class of quasi-linear operators $\mathcal{A}: \mathrm{H}_0^{1,p}(\Omega) \to \mathrm{H}^{-1,q}(\Omega)$ of the form

$$\mathcal{A} u = -\operatorname{div}(a(x, \mathrm{D}u)),$$

where Ω is a bounded open subset of $\mathbf{R}^n$, $1<p<+\infty$, $1/p+1/q=1$, and the function $a: \Omega \times \mathbf{R}^n \to \mathbf{R}^n$ satisfies suitable measurability, continuity, and monotonicity assumptions. By M_Ω we denote the set of such functions a. In order to study the behaviour of boundary value problems of the type

$$\begin{cases} -\operatorname{div}(a(x, \mathrm{D}u)) = f & \text{on } \Omega, \\ u \in \mathrm{H}_0^{1,p}(\Omega) \end{cases}$$

under perturbations of the function $a \in \mathrm{M}_\Omega$, a notion of G-convergence has been introduced in [16]. Its definition and main properties are recalled in Section 1.

The main purpose of Section 2 is to determine some conditions on a sequence of functions (a_h) in M_Ω which imply G-convergence. It turns out that one of them is simply the strong convergence in $(\mathrm{L}^1(\Omega))^n$ of the sequence $(a_h(\,.\,, \xi))$, for every $\xi \in \mathbf{R}^n$ (*see* Theorem 2.1). A necessary and sufficient condition involves the limit behaviour, as h tends to $+\infty$, of the integrals

$$\int_{\mathrm{A}} a_h(x, \mathrm{D}v_h + \xi)\, dx$$

for every $\xi \in \mathbf{R}^n$ and for every A in a suitable family of open sets, where the functions v_h are the solutions to the Dirichlet boundary value problems

$$\begin{cases} -\operatorname{div}(a_h(x, \mathrm{D}v_h + \xi)) = 0 & \text{on A}, \\ v_h \in \mathrm{H}_0^{1,p}(\mathrm{A}) \end{cases}$$

(*see* Theorem 2.3). The proof of this result relies on a representation formula for functions $a \in \mathrm{M}_\Omega$ given in Theorem 2.2.

Section 3 is concerned with the homogenization of operators defined by functions of the class M_Ω; *i.e.*, the G-convergence of sequences of functions (a_h) of the form

$$a_h(x, \xi) = a\left(\frac{x}{\varepsilon_h}, \xi\right),$$

where (ε_h) is a sequence of positive real numbers converging to 0. We suppose that the function $a: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ satisfies the usual measurability, continuity, and monotonicity assumptions, and a condition of quasiperiodicity with respect to x (*see* Definition 3.1). By using the results obtained in Section 2, we prove that there exists a monotone operator $b: \mathbf{R}^n \to \mathbf{R}^n$ such that, for every $f \in H^{-1,q}(\Omega)$, the solutions u_h and the momenta $a\left(\frac{x}{\varepsilon_h}, D u_h\right)$ of the Dirichlet boundary value problems

$$\begin{cases} -\operatorname{div}\left(a\left(\dfrac{x}{\varepsilon_h}, Du_h\right)\right) = f \quad \text{on } \Omega, \\ u_h \in H_0^{1,p}(\Omega) \end{cases}$$

converge, as (ε_h) tends to 0, to the solution u and the momentum $b(Du)$ of the homogenized problem

$$\begin{cases} -\operatorname{div}(b(Du)) = f \quad \text{on } \Omega, \\ u \in H_0^{1,p}(\Omega). \end{cases}$$

Theorem 3.4 gives also an asymptotic formula for the function b; its proof generalizes a construction used in [36].

In Section 4 we extend the homogenization result of Section 3 to almost periodic operators (in the sense of Besicovitch; *see* Definition 4.1) defined by functions of the class $M_{\mathbf{R}^n}$ using an approximation result (Lemma 4.4) and a closure lemma (Lemma 4.3).

The notion of G-convergence for second order linear elliptic operators was studied by E. De Giorgi and S. Spagnolo in the symmetric case (*see* [40], [41], [42], [21]), and then extended to the non-symmetric case by F. Murat and L. Tartar under the name of H-convergence (*see* [43], [44], and [34]). A further extension to higher order linear elliptic operators can be found in [47] together with an extensive bibliography on this subject. Results for the quasi-linear case are given, among others, in [46], [37], [23], [22] and [16].

For the related problems in homogenization theory under periodicity hypotheses on $a(., \xi)$, we refer to the books [2], [39], and [1], which contain a wide bibliography on this topic. Homogenization results for quasi-linear operators are obtained in [4], [5], [6], [25], [26], [17], while the almost periodic case for linear equations is studied in [28] and [36]. A corrector result for quasi-linear periodic equations has been obtained in

[18], whereas an analogous theorem for the almost periodic case will appear in [10].

From another point of view, the homogenization of a class of variational integrals of the form

$$F_h(u)=\int_\Omega f\left(\frac{x}{\varepsilon_h}, Du\right)dx,$$

which is related to the homogenization of the operators $-\operatorname{div}\left(\partial_\xi f\left(\frac{x}{\varepsilon_h},\xi\right)\right)$, has been studied in [32], [15], and [7], using the techniques of Γ-convergence introduced by E. De Giorgi. Homogenization results for variational integrals under almost periodicity assumptions have been proven in [8], [9], [11].

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1. NOTATIONS AND PRELIMINARY RESULTS

Let p be a real constant, $1<p<+\infty$, and let q be its dual exponent, $1/p+1/q=1$. The Euclidean norm and the scalar product in $\mathbf{R}^n$ are denoted by $|.|$ and $(.,.)$, respectively.

Definition 1.1. – Given four constants α, β, c_1, and c_2, such that $c_1>0$, $c_2>0$, $0\leqq\alpha\leqq 1\wedge(p-1)$, $p\vee 2\leqq\beta<+\infty$, we denote by $M(\alpha,\beta,c_1,c_2)$ the class of all functions $a:\mathbf{R}^n\to\mathbf{R}^n$ which fulfill the following conditions:

(i) $|a(0)|\leqq c_1$;

(ii) a satisfies the following inequalities of equicontinuity and strict monotonicity:

$$|a(\xi_1)-a(\xi_2)|\leqq c_1(1+|\xi_1|+|\xi_2|)^{p-1-\alpha}|\xi_1-\xi_2|^\alpha \tag{1.1}$$
$$(a(\xi_1)-a(\xi_2),\xi_1-\xi_2)\geqq c_2(1+|\xi_1|+|\xi_2|)^{p-\beta}|\xi_1-\xi_2|^\beta \tag{1.2}$$

for every ξ_1, $\xi_2\in\mathbf{R}^n$.

For every open subset $\mathcal{O}$ of $\mathbf{R}^n$, by $M_{\mathcal{O}}(\alpha,\beta,c_1,c_2)$ we denote the class of all functions $a:\mathcal{O}\times\mathbf{R}^n\to\mathbf{R}^n$ which satisfy the following conditions:

(iii) for a.e. $x\in\mathcal{O}$, $|a(x,.)|\in M(\alpha,\beta,c_1,c_2)$;

(iv) for every $\xi\in\mathbf{R}^n$, $a(.,\xi)$ is Lebesgue measurable.

It is easy to see that (iii) implies that there exist constants $c_3>0$, $c_4>0$, such that

$$|a(x,\xi)|\leqq c_3(1+|\xi|)^{p-1}, \tag{1.3}$$
$$|\xi|^p\leqq c_4(1+(a(x,\xi),\xi)) \tag{1.4}$$

for a.e. $x\in\mathcal{O}$, for every $\xi\in\mathbf{R}^n$. The proof of (1.3) is trivial. As for the proof of (1.4), we just observe that by Young's inequality we have

$$|\xi|^p\leqq\frac{p}{\beta}(1+|\xi|)^{p-\beta}|\xi|^\beta+\frac{\beta-p}{\beta}(1+|\xi|)^p.$$

In the case $\mathcal{O}=\mathbf{R}^n$, we simply use the notation $M_{\mathbf{R}^n}$ for $M_{\mathbf{R}^n}(\alpha,\beta,c_1,c_2)$.

Let us fix from now on a bounded open subset Ω of $\mathbf{R}^n$. Given $a\in M_\Omega(\alpha,\beta,c_1,c_2)$, it can be proved that for every $f\in H^{-1,q}(\Omega)$ there exists a unique solution $u\in H_0^{1,p}(\Omega)$ to the following Dirichlet boundary value problem

$$\left.\begin{aligned}-\operatorname{div}(a(x,Du))=f \quad \text{on } \Omega,\\ u\in H_0^{1,p}(\Omega).\end{aligned}\right\} \tag{1.5}$$

For a proof we refer, for instance, to [27], Chapter III, Corollary 1.8, or to [31], Chapter 2, Theorem 2.1. The solution to (1.5) satisfies a Meyers' regularity estimate (*see* [33]) that will be needed in the sequel in the particular case where Ω is a cube, as stated in the following theorem.

THEOREM 1.2. – *Let* Q *be a cube in* $\mathbf{R}^n$ *and let* $w\in H_0^{1,p}(Q)$ *be the weak solution to the equation*

$$\left.\begin{aligned}-\operatorname{div}(a(x,Dw))=0 \quad \textit{on } Q,\\ w\in H_0^{1,p}(Q).\end{aligned}\right\}$$

Then there exists $\eta>0$ *such that* $w\in H^{1,p+\eta}(Q)$ *and*

$$\|Dw\|_{(L^{p+\eta}(Q))^n}\leqq C(Q)\|Dw\|_{(L^p(Q))^n}. \tag{1.6}$$

The constant η *depends only on* c_1, c_2, n, p, *while* C(Q) *depends in addition on* Q. *Moreover, a simple rescaling argument shows that we can take*

$$C(tQ)=t^{-n/r}C(Q)$$

for all $t>0$, *where* $\frac{1}{r}+\frac{1}{q}+\frac{1}{p+\eta}=1.$

In order to study the behaviour of problem (1.5) under perturbations of the function a we make use of the following notion of G-convergence.

DEFINITION 1.3. – We say that a sequence (a_h) in $M_\Omega(\alpha,\beta,c_1,c_2)$ *G-convergences* to $a\in M_\Omega(\alpha,\beta,c_1,c_2)$ if, for every $f\in H^{-1,q}(\Omega)$ and for every sequence (f_h) converging to f strongly in $H^{-1,q}(\Omega)$, the solutions u_h to the equations

$$\left.\begin{aligned} -\operatorname{div}(a_h(x, \mathrm{D}\,u_h)) = f_h \quad \text{on } \Omega, \\ u_h \in \mathrm{H}_0^{1,p}(\Omega), \end{aligned}\right\} \tag{1.7}$$

satisfy the following conditions:

$$u_h \to u \quad \text{weakly in } \mathrm{H}_0^{1,p}(\Omega), \tag{1.8}$$

$$a_h(x, \mathrm{D}\,u_h) \to a(x, \mathrm{D}u) \quad \text{weakly in } (\mathrm{L}^q(\Omega))^n, \tag{1.9}$$

where u is the solution to the equation

$$\left.\begin{aligned} -\operatorname{div}(a(x, \mathrm{D}u)) = f \quad \text{on } \Omega, \\ u \in \mathrm{H}_0^{1,p}(\Omega). \end{aligned}\right\} \tag{1.10}$$

Remark 1.4. – It can be proved that this definition of G-convergence is independent of the boundary condition. More precisely, if $\varphi \in \mathrm{H}^{1,p}(\Omega)$, if the sequence $(a_h) \in \mathrm{M}_\Omega(\alpha, \beta, c_1, c_2)$ G-converges to $a \in \mathrm{M}_\Omega(\alpha, \beta, c_1, c_2)$, (f_h) converges to f strongly in $\mathrm{H}^{-1,q}(\Omega)$, and u_h are the solutions in $\mathrm{H}^{1,p}(\Omega)$ to the equations

$$\left.\begin{aligned} -\operatorname{div}(a_h(x, \mathrm{D}\,u_h)) = f_h \quad \text{on } \Omega, \\ u_h - \varphi \in \mathrm{H}_0^{1,p}(\Omega), \end{aligned}\right\}$$

then

$$u_h \to u \quad \text{weakly in } \mathrm{H}^{1,p}(\Omega),$$

$$a_h(x, \mathrm{D}\,u_h) \to a(x, \mathrm{D}u) \quad \text{weakly in } (\mathrm{L}^q(\Omega))^n,$$

where u is the solution to the equation

$$\left.\begin{aligned} -\operatorname{div}(a(x, \mathrm{D}u)) = f \quad \text{on } \Omega, \\ u - \varphi \in \mathrm{H}_0^{1,p}(\Omega). \end{aligned}\right\}$$

A proof of this fact can be found, for instance, in [16], Theorem 3.8.

The next two theorems concern a localization property and a compactness result for G-convergence. Their proofs can be deduced from Theorem 6.1 and Theorem 4.1 in [16] respectively, by using Theorem 7.9 and Corollaries 7.10-7.12 therein.

Let Ω' be an open subset of Ω. For $a \in \mathrm{M}_\Omega(\alpha, \beta, c_1, c_2)$ we denote by a' the function of $\mathrm{M}_{\Omega'}(\alpha, \beta, c_1, c_2)$ defined by $a' = a_{|\Omega' \times \mathbf{R}^n}$. Then the following localization property holds.

THEOREM 1.5. – *Let (a_h) be a sequence in $\mathrm{M}_\Omega(\alpha, \beta, c_1, c_2)$ which G-converges to a in $\mathrm{M}_\Omega(\alpha, \beta, c_1, c_2)$. Then (a'_h) G-converges to a' in $\mathrm{M}_{\Omega'}(\alpha, \beta, c_1, c_2)$.*

THEOREM 1.6. – *Let (a_h) be a sequence in $\mathrm{M}_\Omega(\alpha, \beta, c_1, c_2)$. Then there exist suitable positive constants c'_1, c'_2 and a subsequence $(a_{\sigma(h)})$ of (a_h) which G-converges to a function a of the class $\mathrm{M}_\Omega\left(\dfrac{\alpha}{p-\alpha}, \beta, c'_1, c'_2\right)$.*

In order to simplify the notation, the classes $M_\Omega(\alpha, \beta, c_1, c_2)$ and $M_\Omega\left(\frac{\alpha}{p-\alpha}, \beta, c_1', c_2'\right)$ given by Theorem 1.6 will be denoted, from now on, by M_Ω and M'_Ω respectively.

Finally, we recall a lemma of compensated compactness type (*see* [35], [45]) which will be used in the sequel. For its proof *see*, for example, Lemma 3.4 in [16].

LEMMA 1.7. – *Let* (u_h) *be a sequence converging to* u *weakly in* $H^{1,p}(\Omega)$. *Let* (g_h) *be a sequence converging to* g *weakly in* $(L^q(\Omega))^n$ *with* $(\operatorname{div} g_h)$ *converging to* $\operatorname{div} g$ *strongly in* $H^{-1,q}(\Omega)$. *Then*

$$\int_\Omega (g_h, D u_h)\, \varphi\, dx \to \int_\Omega (g, Du)\, \varphi\, dx$$

for every $\varphi \in \mathscr{C}_0^\infty(\Omega)$.

2. SUFFICIENT CONDITIONS FOR THE G-CONVERGENCE AND REPRESENTATION FORMULA FOR MONOTONE OPERATORS

In this section we investigate some conditions on a sequence of functions (a_h) in M_Ω which imply G-convergence. Furthermore, we give a representation formula for functions $a \in M_\Omega$ showing that $a(x, \xi)$ can be determined by the knowledge of the solutions v to the Dirichlet boundary value problems

$$\left.\begin{array}{r} -\operatorname{div}(a(x, Dv + \xi)) = 0 \quad \text{on } \Omega', \\ v \in H_0^{1,p}(\Omega'), \end{array}\right\}$$

where $\xi \in \mathbf{R}^n$ and Ω' is an open subset of Ω. More precisely, we prove that $a(x, \xi)$ can be calculated by a differentiation process of the set function

$$\Omega' \to \int_{\Omega'} a(x, Dv(x) + \xi)\, dx$$

along a family of open subsets of Ω. Similar results for minima of variational functionals were proved in [21] and [24] for the quadratic case, and in [19] for the general case.

The following theorem shows that the strong convergence $(L^1(\Omega))^n$ of a sequence (a_h) in M_Ω implies the G-convergence.

THEOREM 2.1. – *Let* a_h *and* $a \in M_\Omega$. *Assume that* $(a_h(., \xi))$ *converges to* $a(., \xi)$ *strongly in* $(L^1(\Omega))^n$ *for every* $\xi \in \mathbf{R}^n$. *Then, the sequence* (a_h) *G-converges to* a.

Proof. — Let (f_h) be a sequence in $H^{-1,q}(\Omega)$ converging to f strongly in $H^{-1,q}(\Omega)$. Let u_h be the solution to the equation

$$\left.\begin{aligned} -\operatorname{div}(a_h(x, Du_h)) = f_h \quad \text{on } \Omega, \\ u_h \in H_0^{1,p}(\Omega). \end{aligned}\right\}$$

By the definition of G-convergence we have to prove that (1.8), (1.9) and (1.10) are satisfied. Since (f_h) is bounded in $H^{-1,q}(\Omega)$, condition (1.4) implies that (u_h) is bounded in $H_0^{1,p}(\Omega)$, hence $(a_h(.,Du_h(.)))$ is bounded in $(L^q(\Omega))^n$ by (1.3). Therefore, up to a subsequence,

$$\begin{gathered} u_h \to u \quad \text{weakly in } H_0^{1,p}(\Omega), \\ a_h(x, Du_h) \to g \quad \text{weakly in } (L^q(\Omega))^n, \end{gathered}$$

with $-\operatorname{div} g = f$. We shall show that $g(x) = a(x, Du(x))$ for a.e. $x \in \Omega$, hence u is the unique solution to (1.10). Therefore, the whole sequences (u_h) and $(a_h(.,Du_h(.)))$ converge, and the proof of our assertion is complete. By the strong convergence in $(L^1(\Omega))^n$ of the sequence $(a_h(.,\xi))$ to $a(.,\xi)$ and by the equicontinuity [*see* Definition 1.1 (iii)], there exists a subsequence, still denoted by (a_h), such that $(a_h(x,\xi))$ converges to $a(x,\xi)$ for a.e. $x \in \Omega$, for every $\xi \in \mathbf{R}^n$. Furthermore, by taking the equiboundedness condition [Definition 1.1 (iii)] for a_h into account, the dominated convergence theorem implies that

$$a_h(.,\xi) \to a(.,\xi) \quad \text{strongly in } (L^q(\Omega))^n, \qquad \text{for every } \xi \in \mathbf{R}^n.$$

Since $a_h(x, .)$ is monotone for a.e. $x \in \Omega$, we have

$$\int_\Omega (a_h(x, Du_h) - a_h(x,\xi), Du_h(x) - \xi)\, \varphi(x)\, dx \geqq 0$$

for every $\varphi \in \mathscr{C}_0^\infty(\Omega)$, $\varphi \geqq 0$. Passing to the limit as h tends to $+\infty$ we obtain by means of Lemma 1.7 that

$$\int_\Omega (g(x) - a(x,\xi), Du(x) - \xi)\, \varphi(x)\, dx \geqq 0 \tag{2.1}$$

holds for every $\varphi \in \mathscr{C}_0^\infty(\Omega)$, $\varphi \geqq 0$. By a standard density argument, (2.1) implies that

$$(g(x) - a(x,\xi), Du(x) - \xi) \geqq 0$$

for a.e. $x \in \Omega$, for every $\xi \in \mathbf{R}^n$ [remind that $a(x,\xi)$ is continuous with respect to ξ by (iii) in Definition 1.1]. By Minty's lemma (*see*, for example, [27], Chapter III, Lemma 1.5) it follows that

$$(g(x) - a(x, Du(x)), Du(x) - \xi) \geqq 0$$

for a.e. $x \in \Omega$, for every $\xi \in \mathbf{R}^n$. Hence, $g(x) = a(x, Du(x))$ for a.e. $x \in \Omega$, which completes the proof. □

In order to state the representation formula for functions in the class M_Ω, given $a \in M_\Omega$, let us define the function

$$\Psi(\xi, \Omega') = \int_{\Omega'} a(x, Dv(x) + \xi)\, dx \tag{2.2}$$

for every $\xi \in \mathbf{R}^n$, for every open subset Ω' of Ω, where the function v, depending on ξ and Ω', is the unique solution to

$$\left.\begin{array}{c} -\operatorname{div}(a(x, Dv + \xi)) = 0 \quad \text{on } \Omega', \\ v \in H_0^{1,p}(\Omega'). \end{array}\right\} \tag{2.3}$$

THEOREM 2.2. – *Let $a \in M_\Omega$. Then there exists a measurable subset* N *of* Ω *with* $|N| = 0$ *such that*

$$a(x_0, \xi) = \lim_{\rho \to 0+} \frac{\Psi(\xi, A_\rho(x_0))}{|A_\rho(x_0)|} \tag{2.4}$$

for every $\xi \in \mathbf{R}^n$, $x_0 \in \Omega \setminus N$, *where* $|.|$ *denotes the Lebesgue measure,* $A_\rho(x_0) = x_0 + \rho A$, *and* A *is any bounded open subset of* $\mathbf{R}^n$.

Proof. – By a standard density argument and by Lebesgue's differentiation theorem (*see*, for instance, [38], Theorem 8.8) there exists a measurable subset N of Ω with $|N| = 0$ such that

$$\lim_{\rho \to 0+} \frac{1}{|A_\rho(x_0)|} \int_{A_\rho(x_0)} |a(x, \xi) - a(x_0, \xi)|\, dx = 0 \tag{2.5}$$

for every $x_0 \in \Omega \setminus N$, for every $\xi \in \mathbf{R}^n$, where $A_\rho(x_0) = x_0 + \rho A$, and A is any bounded open subset of $\mathbf{R}^n$. Given $x_0 \in \Omega \setminus N$, $\rho > 0$, $\xi \in \mathbf{R}^n$ we denote by v the function depending on ξ and $A_\rho(x_0)$ which is the unique solution to the Dirichlet boundary value problem

$$\left.\begin{array}{c} -\operatorname{div}(a(x, Dv + \xi)) = 0 \quad \text{on } A_\rho(x_0), \\ v \in H_0^{1,p}(A_\rho(x_0)). \end{array}\right\} \tag{2.6}$$

By performing the change of variables $y = \dfrac{(x - x_0)}{\rho}$, problem (2.6) becomes

$$\left.\begin{array}{c} -\operatorname{div}_y(a(x_0 + \rho y, D_y u(y) + \xi)) = 0 \quad \text{on A}, \\ u(y) = \dfrac{1}{\rho} v(x_0 + \rho y) \in H_0^{1,p}(A). \end{array}\right\} \tag{2.7}$$

We may suppose that ρ runs through a sequence (ρ_h) which tends to 0^+ as h tends to $+\infty$. Let us set $a_h(y, \xi) = a(x_0 + \rho_h y, \xi)$ for every $y \in A$, $\xi \in \mathbf{R}^n$. By (2.5) we have

$$a_h(., \xi) \to a(x_0, \xi) \quad \text{strongly in } (L^1(A))^n,$$

which guarantees by Theorem 2.1 that

$$(a_h) \text{ G-converges to } a \text{ on A.} \tag{2.8}$$

Since $w=0$ is the unique solution to the Dirichlet problem

$$\left.\begin{aligned} -\operatorname{div}_y(a(x_0, D_y w(y)+\xi))=0 \quad \text{on A}, \\ w\in H_0^{1,p}(A), \end{aligned}\right\}$$

if u_h denotes the solution to (2.7) corresponding to $\rho=\rho_h$, the G-convergence condition (2.8) and Remark 1.4 imply that (u_h) converges to 0 weakly in $H_0^{1,p}(A)$ and

$$a(x_0+\rho_h y, D_y u_h(y)+\xi)\to a(x_0,\xi) \quad \text{weakly in } (L^q(A))^n. \tag{2.9}$$

By (2.9) we have then

$$a(x_0,\xi)=\lim_{h\to\infty}\frac{1}{|A|}\int_A a(x_0+\rho_h y, D_y u_h(y)+\xi)\,dy,$$

which by a change of variables proves (2.4). □

The aim of the next theorem is to obtain a necessary and sufficient condition for the G-convergence of a sequence $a_h\in M_\Omega$ by means of the convergence of the momenta related to the Dirichlet problems

$$\left.\begin{aligned} -\operatorname{div}(a_h(x, Dv+\xi))=0 \quad \text{on } A_\rho(x_0), \\ v\in H_0^{1,p}(A_\rho(x_0)), \end{aligned}\right\}$$

where $\xi\in\mathbf{R}^n$.

THEOREM 2.3. – *Let (a_h) be a sequence in M_Ω. Let Ψ_h be the function associated to a_h by* (2.2). *For every $\rho>0$ and $x_0\in\mathbf{R}^n$, let $A_\rho(x_0)=x_0+\rho A$, where A is any bounded open subset of $\mathbf{R}^n$. Let N be a measurable subset of Ω with $|N|=0$. Then, the following conditions are equivalent*:

(*a*) *the limit*

$$\lim_{h\to\infty}\Psi_h(\xi, A_\rho(x_0))$$

exists for every $\xi\in\mathbf{R}^n$ and for every $x_0\in\Omega\setminus N$;

(*b*) *there exists a function $a\in M'_\Omega$ such that (a_h) G-converges to a.*

Moreover, if the previous conditions are satisfied, then

$$a(x,\xi)=\lim_{\rho\to 0+}\lim_{h\to\infty}\frac{\Psi_h(\xi, A_\rho(x))}{|A_\rho(x)|} \tag{2.10}$$

for a. e. $x\in\Omega$, for every $\xi\in\mathbf{R}^n$.

Proof. – Assume (*a*). Let $(a_{\tau(h)})$ be a subsequence of (a_h). By the compactness theorem 1.6 there exist a further subsequence $(a_{\tau(\sigma(h))})$ and $a\in M'_\Omega$ such that

$$(a_{\tau(\sigma(h))}) \text{ G-converges to } a. \tag{2.11}$$

If we show that

$$a(x_0,\xi)=\lim_{\rho\to 0+}\lim_{h\to\infty}\frac{\Psi_{\tau(\sigma(h))}(\xi, A_\rho(x_0))}{|A_\rho(x_0)|} \tag{2.12}$$

for a. e. $x_0\in\Omega$, for every $\xi\in\mathbf{R}^n$, by the existence of the limit in (*a*) and the Urysohn property of the G-convergence (*see* [16], Remark 3.7) we may conclude that (*b*) and (2.10) hold.

Given $\xi\in\mathbf{R}^n$, $x_0\in\Omega\setminus N$, and $\rho>0$, we denote by v_h the solution to the Dirichlet problem

$$\left.\begin{aligned}-\operatorname{div}(a_{\tau(\sigma(h))}(x, Dv_h+\xi))=0 \quad \text{on } A_\rho(x_0),\\ v_h\in H_0^{1,p}(A_\rho(x_0)).\end{aligned}\right\}$$

Since v is the unique solution to the Dirichlet problem

$$\left.\begin{aligned}-\operatorname{div}(a(x, Dv+\xi))=0 \quad \text{on } A_\rho(x_0),\\ v\in H_0^{1,p}(A_\rho(x_0)),\end{aligned}\right\} \tag{2.13}$$

the G-convergence condition (2.11), Theorem 1.5 and Remark 1.4 imply

$$v_h\to v \quad \text{weakly in } H_0^{1,p}(A_\rho(x_0))$$
$$a_{\tau(\sigma(h))}(x, Dv_h+\xi)\to a(x, Dv+\xi) \quad \text{weakly in } (L^q(A_\rho(x_0)))^n.$$

Therefore

$$\Psi_{\tau(\sigma(h))}(\xi, A_\rho(x_0))\to\int_{A_\rho(x_0)} a(x, Dv+\xi)\,dx=\Psi(\xi, A_\rho(x_0)).$$

Now, by the representation Theorem 2.2 we conclude that

$$a(x_0,\xi)=\lim_{\rho\to 0+}\lim_{h\to\infty}\frac{\Psi_{\tau(\sigma(h))}(\xi, A_\rho(x_0))}{|A_\rho(x_0)|}$$

for a. e. $x_0\in\Omega$, for every $\xi\in\mathbf{R}^n$, proving (2.12).

Assume (*b*). Then, Theorem 1.5 and Remark 1.4 guarantee that the solutions v_h to

$$\left.\begin{aligned}-\operatorname{div}(a_h(x, Dv_h+\xi))=0 \quad \text{on } A_\rho(x_0),\\ v_h\in H_0^{1,p}(A_\rho(x_0))\end{aligned}\right\}$$

satisfy

$$v_h\to v \quad \text{weakly in } H_0^{1,p}(A_\rho(x_0))$$
$$a_h(x, Dv_h+\xi)\to a(x, Dv+\xi) \quad \text{weakly in } (L^q(A_\rho(x_0)))^n,$$

where v is the unique solution to (2.13). Hence, condition (*a*) follows immediately. □

Remark 2.4. – Theorem 2.3 provides a simple characterization of G-convergence in the special case of functions a_h in M_Ω satisfying

$$-\operatorname{div}_x(a_h(x,\xi))=0 \quad \text{on } \Omega$$

for every $h \in \mathbf{N}$ and $\xi \in \mathbf{R}^n$. In this case (a_h) G-converges to a function $a \in M_\Omega$ if and only if $(a_h(.,\xi))$ tends to $a(.,\xi)$ weakly in $(L^1(\Omega))^n$, for every $\xi \in \mathbf{R}^n$. According to Theorem 2.3 this condition is clearly sufficient, since in this case

$$\Psi_h(\xi, A_\rho(x_0)) = \int_{A_\rho(x_0)} a_h(x,\xi)\,dx,$$

while its necessity follows easily from the local character of the G-convergence (*see* Theorem 1.5 and Remark 1.4). In the linear case, the previous characterization was proved in [13]. *See* also [12] for a similar result in the case $a_h(x,\xi) = \partial_\xi f_h(x,\xi)$ with f_h convex in ξ.

3. HOMOGENIZATION OF QUASIPERIODIC OPERATORS

In this section we give a characterization of the G-limit of a sequence of functions a_h of the form

$$a_h(x,\xi) = a\left(\frac{x}{\varepsilon_h}, \xi\right),$$

with $a \in M_{\mathbf{R}^n}$ verifying suitable hypotheses of quasiperiodicity in the first variable (*see* Definition 3.1). This result will be used in Section 4 to derive the homogenization theorem for general almost periodic operators.

DEFINITION 3.1 (*see* [30] 3.3). – A continuous function $f: \mathbf{R}^n \to \mathbf{R}$ is *quasiperiodic* if there exist $m_1, \ldots, m_n \in \mathbf{N}$ and a continuous function $F: \mathbf{R}^N \to \mathbf{R}$, where $N = m_1 + \ldots + m_n$, $N \geq n$, such that

$$f(x_1, \ldots, x_n) = F(\underbrace{x_1, \ldots, x_1}_{m_1}, \underbrace{x_2, \ldots, x_2}_{m_2}, \ldots, \underbrace{x_n, \ldots, x_n}_{m_n}),$$

and F periodic of periods $\dfrac{2\pi}{\lambda_1^1}, \dfrac{2\pi}{\lambda_1^2}, \ldots, \dfrac{2\pi}{\lambda_1^{m_1}}, \dfrac{2\pi}{\lambda_2^1}, \ldots, \dfrac{2\pi}{\lambda_2^{m_2}}, \ldots, \dfrac{2\pi}{\lambda_n^{m_n}}$, with $\lambda_r^l \in]0, +\infty[$. It is not restrictive to assume that the frequencies $\lambda_r^1, \ldots, \lambda_r^{m_r}$ are linearly independent on $\mathbf{Z}$ for every $r = 1, \ldots, n$. This will be done constantly in the sequel. Under this assumption, Kronecker's lemma (*see* Appendix, Section A) guarantees that F is uniquely determined by f.

Given $m_1, \ldots, m_n$ as above and given $\lambda \in \mathbf{R}^{m_1} \times \ldots \times \mathbf{R}^{m_n}$, $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_r = (\lambda_r^1, \ldots, \lambda_r^{m_r})$ for $r = 1, \ldots, n$, we denote by $QP(\lambda)$ the set of all quasiperiodic functions $f: \mathbf{R}^n \to \mathbf{R}$ with frequencies λ. Furthermore, by $\mathrm{Trig}(\lambda)$ we indicate the set of all trigonometric polynomials with frequencies λ; *i.e.*, finite sums of terms of the form

$$\mathrm{P}(x)=\mathrm{Re}\,(c\exp(i\sum_{l,r}k_r^l\lambda_r^l x_r)), \tag{3.1}$$

where $k\in\mathbf{Z}^N$, $c\in\mathbf{C}$.

Remark 3.2. — Trigonometric polynomials are obviously quasiperiodic functions. Moreover, it can be proved that every function f of $\mathrm{QP}(\lambda)$ is the uniform limit of a sequence of trigonometric polynomials belonging to $\mathrm{Trig}(\lambda)$ (*see* [30] 3.3).

For every $s>0$, for every $z\in\mathbf{R}^n$, let $\mathrm{Q}_s(z)$ be the cube of side length s and center z. For every $f\in\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^n)$ we define

$$⨍ f(x)\,dx=\limsup_{s\to\infty}\frac{1}{|\mathrm{Q}_s(0)|}\int_{\mathrm{Q}_s(0)}f(x)\,dx.$$

It can be seen easily that for every function f in $\mathrm{QP}(\lambda)$ we have

$$⨍ f(x)\,dx=\lim_{s\to\infty}\frac{1}{|\mathrm{Q}_s(z)|}\int_{\mathrm{Q}_s(z)}f(x)\,dx$$

uniformly with respect to z. This limit is called the *mean value* of f (on $\mathbf{R}^n$) (*see* [30], 2.3).

In the sequel it will be useful to consider the function $j:\mathbf{R}^n\to\mathbf{R}^N$ defined by

$$j(x)=(\underbrace{x_1,\ldots,x_1}_{m_1},\underbrace{x_2,\ldots,x_2}_{m_2},\ldots,\underbrace{x_n,\ldots,x_n}_{m_n}).$$

Then, introducing the variables

$$(y_1,\ldots,y_N)=(y_1^1,\ldots,y_1^{m_1},\ldots,y_n^1,\ldots,y_n^{m_n})$$

on $\mathbf{R}^N$, for every $v\in\mathscr{C}^1(\mathbf{R}^N)$ and $u(x)=v(j(x))$ we get

$$\mathrm{D}_r u(x)=\sum_{l=1}^{m_r}\frac{\partial v}{\partial y_r^l}(j(x))\quad\text{for every } r=1,\ldots,n,$$

or briefly, $\mathrm{D}u(x)=(\partial v)\circ j(x)$, where $\partial=(\partial_1,\ldots,\partial_n)$ with $\partial_r=\sum_{l=1}^{m_r}\frac{\partial}{\partial y_r^l}$.

Let us denote by $\mathrm{Trig}_0(\lambda)$ the set of all trigonometric polynomials in $\mathrm{Trig}(\lambda)$ with mean value 0. By Birkhoff's theorem (*see* Appendix, Theorem A) it is easy to prove that

$$\|u\|_{1,p}=\left(\sum_{r=1}^{n}⨍|\mathrm{D}_r u|^p\,dx\right)^{1/p}$$

is a norm on $\mathrm{Trig}_0(\lambda)$. Since $(\mathrm{Trig}_0(\lambda), \|\cdot\|_{1,p})$ is not complete, we study its completion. To this aim let us introduce

$$T = \prod_{l, r} \left]0, \frac{2\pi}{\lambda_t^l}\right[\subset \mathbf{R}^N, \tag{3.2}$$

and let us denote by Trig(T) the set of all trigonometric polynomials in $\mathbf{R}^N$ with period T. Let us also define the set $\mathrm{Trig}_0(T)$ of all functions v in Trig(T) with mean value zero. On $\mathrm{Trig}_0(T)$ we consider the norm

$$\|v\|_{\mathscr{W}} = \|\partial v\|_{(L^p(T))^n} = \left(\sum_{r=1}^{n} \frac{1}{|T|} \int_T |\partial_r v|^p \, dy\right)^{1/p};$$

in fact, if $\partial v = 0$, then $u = v \circ j$ satisfies $Du = 0$. Hence, u is constant. Since v depends uniquely on u (*see* Definition 3.1), v is constant too; hence $v = 0$. By Birkhoff's theorem (*see* Appendix, Theorem A) the linear map

$$\mathrm{Re}\,(c \exp(i \sum_{l, r} k_r^l \lambda_r^l x_r)) \mapsto \mathrm{Re}\,(c \exp(i \sum_{l, r} k_r^l \lambda_r^l y_r^l))$$

is a bijective isometry between $(\mathrm{Trig}_0(\lambda), \|\cdot\|_{1,p})$ and $(\mathrm{Trig}_0(T), \|\cdot\|_{\mathscr{W}})$, and will be denoted by J.

Let us denote by $\mathscr{W}$ the completion of $\mathrm{Trig}_0(T)$ with respect to the norm $\|\cdot\|_{\mathscr{W}}$, which we can identify with the completion of $\mathrm{Trig}_0(\lambda)$ with respect to the norm $\|\cdot\|_{1,p}$. Finally, let us remark that the isometry

$$\begin{aligned} \partial: \quad & \mathrm{Trig}_0(T) \to (L^p(T))^n \\ & u \mapsto (\partial_1 u, \partial_2 u, \ldots, \partial_n u) \end{aligned}$$

can be extended in a unique way to an isometry between $\mathscr{W}$ and a closed subspace of $(L^p(T))^n$, which makes $\mathscr{W}$ a reflexive Banach space.

Now, let us fix the frequencies λ, with $\lambda_r^l, \ldots, \lambda_r^{m_r}$ linearly independent on $\mathbf{Z}$ for every $r = 1, \ldots, n$, and let us fix $a \in M_{\mathbf{R}^n}$ such that

$$a(., \xi) \in \mathrm{QP}(\lambda) \quad \text{for every } \xi \in \mathbf{R}^n. \tag{3.3}$$

By definition it follows that there exists a unique function $\tilde{a}(., \xi): \mathbf{R}^N \to \mathbf{R}^n$ such that $a(x, \xi) = \tilde{a}(j(x), \xi)$ for every $x \in \mathbf{R}^n$ and for every $\xi \in \mathbf{R}^n$, $\tilde{a}(., \xi)$ is T-periodic and continuous for every $\xi \in \mathbf{R}^n$. It follows that $\tilde{a}(y, .)$ satisfies (1.1) and (1.2) for every $y \in \mathbf{R}^N$. This is obvious for $y = j(x)$, $x \in \mathbf{R}^n$, whereas the conclusion for a general $y \in \mathbf{R}^N$ comes from Kronecker's lemma (*see* Appendix A) and the continuity of $\tilde{a}(., \xi)$.

The next proposition is an extension to the case $p \neq 2$ of Lemma 1 in [36].

PROPOSITION 3.3. — *Let $\xi \in \mathbf{R}^n$ be fixed, let $a \in M_{\mathbf{R}^n}$ satisfying* (3.3) *and let $\tilde{a}$ be the corresponding* T-*periodic function. Then, there exists a unique* *solution* $w^\xi \in \mathscr{W}$ *to the problem*

$$\left.\begin{array}{l}\dfrac{1}{|\mathrm{T}|}\displaystyle\int_{\mathrm{T}} (\tilde{a}(y, \partial w^\xi + \xi), \partial h)\, dy = 0 \quad \text{for every } h \in \mathscr{W}, \\ w^\xi \in \mathscr{W}.\end{array}\right\} \tag{3.4}$$

Furthermore, for any $\delta > 0$ *there exist* $u_\delta^\xi \in \mathrm{Trig}_0(\lambda)$ *and a vector function* $g_\delta^\xi \in (\mathrm{QP}(\lambda))^n$ *such that*

$$-\operatorname{div}(a(x, \mathrm{D}u_\delta^\xi + \xi)) = -\operatorname{div} g_\delta^\xi \quad in\ \mathscr{D}'(\mathbf{R}^n), \tag{3.5}$$

$$\lim_{\delta \to 0+} \| \mathrm{J}(u_\delta^\xi) - w^\xi \|_{\mathscr{W}} = 0, \tag{3.6}$$

$$\lim_{\delta \to 0+} ⨍ | g_\delta^\xi |^q\, dx = 0 \tag{3.7}$$

hold.

Proof. — Given $\xi \in \mathbf{R}^n$, let $\tilde{\mathscr{A}}^\xi : \mathscr{W} \to \mathscr{W}'$ be the operator defined by

$$\langle \tilde{\mathscr{A}}^\xi w, h \rangle = \frac{1}{|\mathrm{T}|} \int_{\mathrm{T}} (\tilde{a}(y, \partial w + \xi), \partial h)\, dy$$

for every w, $h \in \mathscr{W}$, where $\langle .,. \rangle$ denotes here the duality pairing between $\mathscr{W}$ and its dual space $\mathscr{W}'$. By the coerciveness and continuity properties of $\tilde{\mathscr{A}}^\xi$ on $\mathscr{W}$, the theory of monotone operators on reflexive Banach spaces (*see*, for example [27]) implies immediately that there exists a unique $w^\xi \in \mathscr{W}$ satisfying $\tilde{\mathscr{A}}^\xi w^\xi = 0$, which implies (3.4). By the density of $\mathrm{J}(\mathrm{Trig}_0(\lambda))$ in $\mathscr{W}$ there exists $u_\delta^\xi \in \mathrm{Trig}_0(\lambda)$ such that (3.6) is satisfied. By using (3.4) and the equicontinuity assumption on $\tilde{a}$ we get

$$\begin{aligned}\| \tilde{\mathscr{A}}^\xi \mathrm{J}(u_\delta^\xi) \|_{\mathscr{W}'} &= \| \tilde{\mathscr{A}}^\xi \mathrm{J}(u_\delta^\xi) - \tilde{\mathscr{A}}^\xi w^\xi \|_{\mathscr{W}'} \\ &\leqq c(1 + |\xi| + \| \mathrm{J}(u_\delta^\xi) \|_{\mathscr{W}} + \| w^\xi \|_{\mathscr{W}})^{p-1-\alpha} \| \mathrm{J}(u_\delta^\xi) - w^\xi \|_{\mathscr{W}}^\alpha.\end{aligned}$$

By (3.6) it follows that

$$\lim_{\delta \to 0+} \| \tilde{\mathscr{A}}^\xi \mathrm{J}(u_\delta^\xi) \|_{\mathscr{W}'} = 0. \tag{3.8}$$

Since $x \mapsto a(x, \mathrm{D}u_\delta^\xi + \xi)$ belongs to $(\mathrm{QP}(\lambda))^n$, by Remark 3.2 there exist a function $f_\delta^\xi \in (\mathrm{Trig}(\lambda))^n$ and a quasiperiodic function $h_\delta^\xi \in (\mathrm{QP}(\lambda))^n$ such that

$$a(x, \mathrm{D}u_\delta^\xi + \xi) = f_\delta^\xi + h_\delta^\xi, \tag{3.9}$$

and

$$\lim_{\delta \to 0+} ⨍ | h_\delta^\xi |^q\, dx = 0. \tag{3.10}$$

Since $a(x, \mathrm{D}u_\delta^\xi + \xi) = \tilde{a}(j(x), \partial \mathrm{J}(u_\delta^\xi)(j(x)) + \xi)$, by (3.8) and (3.10) we get

$$\lim_{\delta \to 0+} \left\| \sum_{r=1}^{n} \partial_r (\mathrm{F}_\delta^\xi)_r \right\|_{\mathscr{W}'} = 0, \tag{3.11}$$

where $(F^\xi_\delta)_r$ is $J((f^\xi_\delta)_r)$, and

$$\left\| \sum_{r=1}^{n} \partial_r (F^\xi_\delta)_r \right\|_{\mathcal{W}'} = \sup_{\substack{v \in \mathcal{W} \\ \|v\|_{\mathcal{W}} \leq 1}} \frac{1}{|T|} \sum_{r=1}^{n} \int_T (F^\xi_\delta)_r \, \partial_r v \, dy.$$

Being $\operatorname{div} f^\xi_\delta \in \operatorname{Trig}_0(\lambda)$, we can write

$$\operatorname{div} f^\xi_\delta(x) = \operatorname{Re}\left(\sum_k c_k \exp\left(i \sum_{l,r} k^l_r \lambda^l_r x_r\right)\right),$$

where the sum runs over a finite set of non-zero vectors $k \in \mathbf{Z}^N$. It turns out that the function

$$w^\xi_\delta(x) = \operatorname{Re}\left(\sum_k b_k \exp\left(i \sum_{l,r} k^l_r \lambda^l_r x_r\right)\right),$$

with $b_k = -c_k / \left(\sum_r \left(\sum_l k^l_r \lambda^l_r\right)^2\right)$, is the unique solution in $\operatorname{Trig}_0(\lambda)$ to

$$\Delta w^\xi_\delta = \operatorname{div} f^\xi_\delta \quad \text{on } \mathbf{R}^n.$$

If $W^\xi_\delta = J(w^\xi_\delta) \in \operatorname{Trig}(T)$, then

$$\sum_{r=1}^{n} \partial^2_r W^\xi_\delta = \sum_{r=1}^{n} \partial_r (F^\xi_\delta)_r \quad \text{in } \mathbf{R}^N$$

holds. By elliptic regularity (*see* Appendix, Section B)

$$\left(\frac{1}{|T|} \sum_{r=1}^{n} \int_T |\partial_r W^\xi_\delta|^q \, dx \right)^{1/q} \leq c \left\| \sum_{r=1}^{n} \partial_r (F^\xi_\delta)_r \right\|_{\mathcal{W}'},$$

with $c > 0$ independent of ξ and δ. We have then by Birkhoff's theorem

$$\left(\fint |D w^\xi_\delta|^q \, dx \right)^{1/q} \leq c \left\| \sum_{r=1}^{n} \partial_r (F^\xi_\delta)_r \right\|_{\mathcal{W}'}. \tag{3.12}$$

Hence, by setting

$$g^\xi_\delta = D w^\xi_\delta + h^\xi_\delta$$

we have

$$\operatorname{div}(a(x, D u^\xi_\delta + \xi)) = \operatorname{div} g^\xi_\delta$$

in the sense of distributions on $\mathbf{R}^n$, proving (3.5). Furthermore, by (3.10)-(3.12) we obtain (3.7). □

Let us consider the following Dirichlet boundary value problem

$$\left. \begin{array}{c} -\operatorname{div}\left(a\left(\dfrac{x}{\varepsilon_h}, D u_h\right)\right) = f \quad \text{on } \Omega, \\ u_h \in H^{1,p}_0(\Omega), \end{array} \right\} \tag{3.13}$$

where $f \in H^{-1,q}(\Omega)$ and (ε_h) is a sequence of positive real numbers converging to 0.

In this section we prove the convergence, as (ε_h) tends to 0^+, of the solutions u_h of (3.13) to the solution u of the homogeneized problem

$$\left.\begin{aligned} -\operatorname{div}(b(\mathrm{D}u))=f \quad \text{on } \Omega,\\ u\in \mathrm{H}_0^{1,p}(\Omega). \end{aligned}\right\} \tag{3.14}$$

Furthermore, we give an asymptotic formula for the homogenized function b in terms of the solutions v_s^ξ to the following Dirichlet boundary value problems

$$\left.\begin{aligned} -\operatorname{div}(a(x,\mathrm{D}v_s^\xi+\xi))=0 \quad \text{on } \mathrm{Q}_s(z),\\ v_s^\xi\in \mathrm{H}_0^{1,p}(\mathrm{Q}_s(z)). \end{aligned}\right\} \tag{3.15}$$

The convergence result above mentioned will follow from the next two theorems.

THEOREM 3.4. — *Let $a\in \mathrm{M}_{\mathbf{R}^n}$ satisfying* (3.3) *and let $\tilde{a}$ be the corresponding* T*-periodic function. Let $b:\mathbf{R}^n\to\mathbf{R}^n$ be the function defined by*

$$b(\xi)=\frac{1}{|\mathrm{T}|}\int_{\mathrm{T}}\tilde{a}(y,\partial w^\xi+\xi)\,dy,$$

where w^ξ is the solution to (3.4). *Then, for any family $(z_s)_{s>0}$ in $\mathbf{R}^n$, we have*

$$b(\xi)=\lim_{s\to\infty}\frac{1}{s^n}\int_{\mathrm{Q}_s(z_s)}a(x,\mathrm{D}v_s^\xi+\xi)\,dx,$$

where v_s^ξ is the unique solution to (3.15) *with $z=z_s$.*

THEOREM 3.5. — *Let $a\in \mathrm{M}_{\mathbf{R}^n}$ satisfying* (3.3). *Let (ε_h) be a sequence of positive real numbers converging to* 0 *and let $a_h(x,\xi)=a\left(\frac{x}{\varepsilon_h},\xi\right)$ for every $x\in\Omega$ and for every $\xi\in\mathbf{R}^n$. Then (a_h) G-converges to b, where b is the function defined in Theorem* 3.4.

The proofs of these theorems are quite technical and are therefore given at the end of the section as a consequence of the next proposition and some remarks stated in the sequel.

For every $s>0$, and for every family (z_s) in $\mathbf{R}^n$, let us define

$$g_s(\xi)=\frac{1}{s^n}\int_{\mathrm{Q}_s(z_s)}a(x,\mathrm{D}v_s^\xi+\xi)\,dx, \tag{3.16}$$

where v_s^ξ is the solution to (3.15) on $\mathrm{Q}_s(z_s)$. By (iii) and (iv) in Definition 1.1, using Young's and Hölder's inequalities, it turns out that the function v_s^ξ satisfies the following estimates

$$\left.\begin{aligned} \|\mathrm{D}v_s^\xi+\xi\|_{(\mathrm{L}^p(\mathrm{Q}_s(z_s)))^n}^p\leq cs^n(1+|\xi|^p),\\ \|a(x,\mathrm{D}v_s^\xi+\xi)\|_{(\mathrm{L}^q(\mathrm{Q}_s(z_s)))^n}^q\leq cs^n(1+|\xi|^p) \end{aligned}\right\} \tag{3.17}$$

with c independent of z_s. Here and henceforth, we will denote by c any constant depending at most on c_1, c_2, α, β, n, p, that can change from line to line. By (3.17), we have

$$|g_s(\xi)| \leqq c(1+|\xi|^{p/q}).$$

PROPOSITION 3.6. – *Let $\xi \in \mathbf{R}^n$ be fixed. Let u_δ^ξ be as in Proposition 3.3 and let $v_s^\xi \in H_0^{1,p}(Q_s(z_s))$ be the unique solution to (3.15) with z replaced by z_s. Then*

$$\lim_{\delta \to 0+} \limsup_{s \to \infty} \frac{1}{s^n} \int_{Q_s(z_s)} |Dv_s^\xi - Du_\delta^\xi|^p \, dx = 0 \tag{3.18}$$

for every family $(z_s)_{s>0}$ in $\mathbf{R}^n$.

Proof. – Given a family $(z_s)_{s>0}$ we consider

$$\begin{aligned} I_{s,\delta} \equiv \frac{1}{s^n} \int_{Q_s(z_s)} & (a(x, Dv_s^\xi + \xi) - a(x, Du_\delta^\xi + \xi), Dv_s^\xi - Du_\delta^\xi) \, dx \\ &= \frac{1}{s^n} \int_{Q_s(z_s)} (a(x, Dv_s^\xi + \xi), Dv_s^\xi) \, dx \\ &+ \frac{1}{s^n} \int_{Q_s(z_s)} (a(x, Du_\delta^\xi + \xi), Du_\delta^\xi) \, dx \\ &- \frac{1}{s^n} \int_{Q_s(z_s)} (a(x, Dv_s^\xi + \xi), Du_\delta^\xi) \, dx \\ &- \frac{1}{s^n} \int_{Q_s(z_s)} (a(x, Du_\delta^\xi + \xi), Dv_s^\xi) \, dx \\ &\equiv I_s^1 + I_{s,\delta}^2 - I_{s,\delta}^3 - I_{s,\delta}^4. \end{aligned}$$

By (3.15), $I_s^1 = 0$. On the other hand,

$$\lim_{s \to \infty} I_{s,\delta}^2 = ⨍ (a(x, Du_\delta^\xi + \xi), Du_\delta^\xi) \, dx.$$

By Birkhoff's theorem we get

$$\lim_{s \to \infty} I_{s,\delta}^2 = \frac{1}{|T|} \int_T (\tilde{a}(y, \partial J(u_\delta^\xi) + \xi), \partial J(u_\delta^\xi)) \, dy = \langle \tilde{\mathscr{A}}^\xi J(u_\delta^\xi), J(u_\delta^\xi) \rangle,$$

where $\tilde{\mathscr{A}}^\xi$ is the operator defined in the proof of Proposition 3.3. Since by (3.6) we have

$$J(u_\delta^\xi) \to w^\xi \quad \text{strongly in } \mathscr{W},$$

the continuity of $\tilde{\mathscr{A}}^\xi$ and the equality $\tilde{\mathscr{A}}^\xi w^\xi = 0$ implies that

$$\lim_{\delta \to 0+} \lim_{s \to \infty} I_{s,\delta}^2 = 0.$$

Since

$$\frac{1}{|Q_s(z_s)|}\int_{Q_s(z_s)} |Du_\delta^\xi(x)|^p\,dx \to ⨍ |Du_\delta^\xi(x)|^p\,dx,$$

and $w_s(x)=\frac{1}{s}u_\delta^\xi(sx+z_s)$ converges to 0, as $s\to+\infty$, we obtain that the sequence (w_s) converges weakly to 0 in $H^{1,p}(Q_1(0))$. Applying Lemma 1.7, we obtain that the sequence of functions

$$\zeta_s(x)=(a(sx+z_s, Dv_s^\xi(sx+z_s)+\xi), Du_\delta^\xi(sx+z_s))$$

converges to 0 in $\mathscr{D}'(Q_1(0))$ as $s\to+\infty$. On the other hand, by (1.6) and (3.17) we have

$$\| Dv_s^\xi(sx+z_s)+\xi\|^{p+\eta}_{(L^{p+\eta}(Q_1(0)))^n}\leqq c(1+|\xi|^p)^{(p+\eta)/p};$$

by (1.3) it follows that the sequence of functions

$$x\mapsto a(sx+z_s, Dv_s^\xi(sx+z_s)+\xi)$$

is uniformly bounded in some $(L^\tau(Q_1(0)))^n$ for some $\tau>q$. This implies that there exists $\sigma>1$ such that

$$\|\zeta_s\|_{L^\sigma(Q_1(0))}\leqq c,$$

where c is independent of s. Since (ζ_s) converges to 0 in $\mathscr{D}'(Q_1(0))$, the above inequality implies that (ζ_s) converges to 0 weakly in $L^\sigma(Q_1(0))$, as $s\to+\infty$. This fact gives then

$$\lim_{s\to\infty} I^3_{s,\delta}=0.$$

Let us show that

$$\lim_{\delta\to 0+}\ \limsup_{s\to\infty} I^4_{s,\delta}=0.$$

Let (g_δ^ξ) be as in Proposition 3.3. By applying Hölder's inequality and (3.17) we obtain

$$|I^4_{s,\delta}|\leqq c\left(\frac{1}{s^n}\int_{Q_s(z_s)}|g_\delta^\xi|^q\,dx\right)^{1/q}\left(\frac{1}{s^n}\int_{Q_s(z_s)}|Dv_s^\xi|^p\,dx\right)^{1/p}$$
$$\leqq c\left(\frac{1}{s^n}\int_{Q_s(z_s)}|g_\delta^\xi|^q\,dx\right)^{1/q}(1+|\xi|).$$

Now, by taking the limit first as s tends to $+\infty$ and then as δ tends to 0^+, (3.7) implies that

$$\lim_{\delta\to 0}\ \limsup_{s\to\infty}|I^4_{s,\delta}|=0.$$

Since the equicoerciveness assumption in Definition 1.1 (iii) guarantees

$$|I_{s,\delta}| \geqq c_2 \frac{1}{s^n} \int_{Q_s(z_s)} (1+|Dv_s^\xi+\xi|+|Du_\delta^\xi+\xi|)^{p-\beta} |Dv_s^\xi - Du_\delta^\xi|^\beta \, dx,$$

we obtain immediately

$$c\,|I_{s,\delta}|^{p/\beta} \left(\frac{1}{s^n} \int_{Q_s(z_s)} (1+|Dv_s^\xi+\xi|+|Du_\delta^\xi+\xi|)^p \, dx \right)^{(\beta-p)/\beta} \geqq \left(\frac{1}{s^n} \int_{Q_s(z_s)} |Dv_s^\xi - Du_\delta^\xi|^p \, dx \right).$$

Hence, by passing to the limit first as s tends to $+\infty$, and then as $\delta \to 0^+$, we get (3.18) and the proof of Proposition 3.6 is accomplished. □

Proof of Theorem 3.5. – Let $x_0 \in \Omega$, and let $\xi \in \mathbf{R}^n$. Consider

$$\Psi_h(\xi, Q_\rho(x_0)) = \int_{Q_\rho(x_0)} a\left(\frac{x}{\varepsilon_h}, Dz_{h,\rho}^\xi(x) + \xi \right) dx,$$

where $z_{h,\rho}^\xi$ is the unique solution to

$$\left.\begin{aligned} -\operatorname{div}\left(a\left(\frac{x}{\varepsilon_h}, Dz_{h,\rho}^\xi + \xi \right) \right) = 0 \quad \text{on } Q_\rho(x_0), \\ z_{h,\rho}^\xi \in H_0^{1,p}(Q_\rho(x_0)). \end{aligned}\right\}$$

It follows immediately that

$$\Psi_h(\xi, Q_\rho(x_0)) = (\varepsilon_h)^n \int_{Q_{\rho/\varepsilon_h}(x_0/\varepsilon_h)} a(x, Dw_{h,\rho}^\xi + \xi) \, dx,$$

where $w_{h,\rho}^\xi$ is the unique solution to

$$\left.\begin{aligned} -\operatorname{div}(a(x, Dw_{h,\rho}^\xi(x) + \xi)) = 0 \quad \text{on } Q_{\rho/\varepsilon_h}\left(\frac{x_0}{\varepsilon_h} \right), \\ v_{h,\rho}^\xi \in H_0^{1,p}\left(Q_{\rho/\varepsilon_h}\left(\frac{x_0}{\varepsilon_h} \right) \right). \end{aligned}\right\}$$

By Theorem 3.4 we conclude that

$$\lim_{h \to \infty} \Psi_h(\xi, Q_\rho(x_0)) = \rho^n b(\xi)$$

for every $\rho > 0$. Hence, the limit

$$\lim_{h \to \infty} \frac{\Psi_h(\xi, Q_\rho(x_0))}{\rho^n} = b(\xi)$$

is independent on ρ. This implies by Theorem 2.3 that (a_h) G-converges to b and concludes the proof. □

4. HOMOGENIZATION OF ALMOST PERIODIC OPERATORS

In this section we prove the homogenization theorem for general almost periodic monotone operators defined by functions of the class $M_{\mathbf{R}^n}$.

DEFINITION 4.1. – A function $f \in L^1_{loc}(\mathbf{R}^n)$ is *almost periodic* (in the sense of Besicovitch [3]) if there exists a sequence of trigonometric polynomials $P_h : \mathbf{R}^n \to \mathbf{R}$ such that

$$\lim_{h \to \infty} ⨍ | P_h(x) - f(x) | \, dx = 0. \tag{4.1}$$

It is easy to see that for every almost periodic function f and for every $z \in \mathbf{R}^n$ we have

$$⨍ f(x) \, dx = \lim_{s \to \infty} \frac{1}{|Q_s(z)|} \int_{Q_s(z)} f(x) \, dx.$$

The limit is called the *mean value* of f (on $\mathbf{R}^n$) and is, in general, not uniform with respect to z.

THEOREM 4.2. – *Let $a \in M_{\mathbf{R}^n}$ such that $a(.,\xi)$ is almost periodic for all $\xi \in \mathbf{R}^n$ and let (ε_h) be a sequence of positive real numbers converging to* 0. *Let us define $a_h(x, \xi) = a\left(\frac{x}{\varepsilon_h}, \xi\right)$ for every $x \in \Omega$ and $\xi \in \mathbf{R}^n$. Denote by v^ξ_s the solution to*

$$\left.\begin{array}{r} -\operatorname{div}(a(x, Dv^\xi_s + \xi)) = 0 \quad on \ Q_s(0), \\ v^\xi_s \in H^{1,p}_0(Q_s(0)). \end{array}\right\} \tag{4.2}$$

Then, for every $\xi \in \mathbf{R}^n$ there exists the limit

$$b(\xi) = \lim_{s \to \infty} \frac{1}{s^n} \int_{Q_s(0)} a(x, Dv^\xi_s + \xi) \, dx. \tag{4.3}$$

Moreover, the map b belongs to $M(\alpha, \beta, c_1, c_2)$ *and* (a_h) G-*converges to* b.

The proof of this theorem follows from the homogenization Theorem 3.5 for quasiperiodic functions, by means of an approximation result and a closure lemma, which are stated below.

With a slight change of notation we will write

$$b(x, \xi) = G - \lim_{\varepsilon \to 0+} a\left(\frac{x}{\varepsilon}, \xi\right)$$

meaning that $b(x, \xi)$ is the G-limit of $a\left(\frac{x}{\varepsilon_h}, \xi\right)$ for every sequence (ε_h) which tends to 0^+.

The following lemma states that homogenization is preserved under passage to the limit in the mean value.

LEMMA 4.3. — *Let* (a_h) *be a sequence of functions in* $M_{\mathbf{R}^n}$, *such that for every* $h \in \mathbf{N}$ *the limit*

$$\mathrm{G}-\lim_{\varepsilon \to 0+} a_h\left(\frac{x}{\varepsilon}, \xi\right) = b_h(\xi) \tag{4.4}$$

exists and is independent of x, *and let* $a \in M_{\mathbf{R}^n}$ *such that for every* $R > 0$

$$\limsup_{h \to \infty} \int\!\!\!\!\!\!- \sup_{|\xi| \leq R} |a_h(x, \xi) - a(x, \xi)|\, dx = 0. \tag{4.5}$$

Then, the limit

$$b(\xi) = \lim_{h \to \infty} b_h(\xi)$$

exists and

$$\mathrm{G}-\lim_{\varepsilon \to 0+} a\left(\frac{x}{\varepsilon}, \xi\right) = b(\xi).$$

The proof of Theorem 4.2 will be completed by the following approximation result.

LEMMA 4.4. — *Let a be a function of the class* $M_{\mathbf{R}^n}$ *such that* $a(\cdot, \xi)$ *is almost periodic for every* $\xi \in \mathbf{R}^n$. *Then, there exists a sequence* (a_h) *in* $M_{\mathbf{R}^n}$ *of quasiperiodic functions satisfying* (3.3) (*with* λ *possibly depending on* h) *such that for every* $R \geq 0$ *we have*

$$\limsup_{h \to \infty} \int\!\!\!\!\!\!- \sup_{|\xi| \leq R} |a_h(x, \xi) - a(x, \xi)|\, dx = 0. \tag{4.6}$$

Throughout this section the letter c will denote a positive constant depending at most on p, n, c_1, c_2, α, β, and possibly on a fixed vector $\xi \in \mathbf{R}^n$. Its value can vary from line to line.

We begin by proving Lemma 4.4.

Proof of Lemma 4.4.

Step 1 (*discretization of the function a on bounded sets*). — Let (γ_h) be a sequence of natural numbers, and (μ_h) a sequence of positive real numbers. For every $h \in \mathbf{N}$, let us set

$$I_h = \{\mathbf{z} = (z_1, \ldots, z_n) \in \mathbf{Z}^n : -h\gamma_h \leq z_j < h\gamma_h \text{ for all } j = 1, \ldots, n\}.$$

For every $\mathbf{z} \in I_h$, let us define

$$\xi_{\mathbf{z}}^h = \frac{1}{\gamma_h}\mathbf{z}, \qquad Q_{\mathbf{z}}^h = \xi_{\mathbf{z}}^h + \left[0, \frac{1}{\gamma_h}\right[^n, \qquad a_{\mathbf{z}}^h(x) = a(x, \xi_{\mathbf{z}}^h),$$

and let us choose a trigonometric polynomial $P_{\mathbf{z}}^h : \mathbf{R}^n \to \mathbf{R}^n$ such that

$$-\!\!\!\!\!\!\int | P_{\mathbf{z}}^h(x) - a_{\mathbf{z}}^h(x) | \, dx \leqq \frac{1}{\mu_h}. \tag{4.7}$$

The choice of the sequences $(\gamma_h) \subset \mathbf{N}$ and $(\mu_h) \subset \mathbf{R}$ will be made in the sequel. Then, taking into account that

$$\bigcup_{\mathbf{z} \in I_h} Q_{\mathbf{z}}^h = [-h, h[^n,$$

we define the function $\alpha_h : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ in the following way:

$$\alpha_h(x, \xi) = \begin{cases} P_{\mathbf{z}}^h(x) & \text{if there exists } \mathbf{z} \in I_h \text{ such that } \xi \in Q_{\mathbf{z}}^h, \\ 0 & \text{if } \xi \notin [-h, h[^n. \end{cases}$$

Then, for every $\xi \in \mathbf{R}^n$, the function $\alpha_h(., \xi)$ is quasiperiodic.

Step 2 (*projection of the function* α_h *on* $M_{\mathbf{R}^n}$). — For every $h \in \mathbf{N}$, let us define the function $f_h : [0, +\infty[\to [0, +\infty[$ by

$$f_h(\rho) = \min(1, e^{h!\,(h-\rho)}).$$

We also define the Hilbert space L_h^2 of measurable functions $u : \mathbf{R}^n \to \mathbf{R}^n$, provided with the norm

$$\| u \|_h = \left(\int_{\mathbf{R}^n} | u(\xi) |^2 f_h^2(| \xi |) \, d\xi \right)^{1/2}.$$

The set $M(\alpha, \beta, c_1, c_2)$ is a closed convex subset of L_h^2. We can define the projection $\pi_h : L_h^2 \to M(\alpha, \beta, c_1, c_2)$, and for every $x \in \mathbf{R}^n$ the function

$$a_h(x, .) = \pi_h(\alpha_h(x, .)). \tag{4.8}$$

Step 3 (*quasiperiodicity of the function* a_h). — Let us consider for every $y \in \mathbf{R}^N$, $N = N(h) \in \mathbf{N}$, the function $\tilde{a}_h(y, .) : \mathbf{R}^n \to \mathbf{R}^n$ defined by

$$\tilde{a}_h(y, .) = \pi_h(\tilde{\alpha}_h(y, .))$$

where, for every $\xi \in \mathbf{R}^n$, $\tilde{\alpha}_h(., \xi)$ is the periodic function given by the quasiperiodicity of $\alpha_h(., \xi)$. It follows clearly that

$$a_h(x, \xi) = \tilde{a}_h(j(x), \xi),$$

where j is the function related to the quasiperiodicity of a_h as in Section 3. Since the function $\tilde{\alpha}_h(., \xi)$ is a periodic function, such is also $\tilde{a}_h(., \xi)$. Moreover, being $\tilde{a}_h(y, .)$ uniformly continuous with respect to ξ, it remains only to prove that $\tilde{a}_h(., \xi)$ is continuous for every $\xi \in \mathbf{R}^n$.

Let $\xi, \eta \in \mathbf{R}^n$; for all $u, v \in M(\alpha, \beta, c_1, c_2)$ we have

$$| u(\xi) - v(\xi) | \leqq 2 c_1 (1 + | \xi | + | \eta |)^{p-1-\alpha} | \xi - \eta |^\alpha + | u(\eta) - v(\eta) |.$$

Let $0<\delta\leqq 1$; then an integration over $B_\delta(\xi)$ gives

$$
\begin{aligned}
|u(\xi)-v(\xi)|\,|B_\delta(\xi)| &\leqq 2c_1(1+2|\xi|+\delta)^{p-1-\alpha}\delta^\alpha|B_\delta(\xi)| \\
&\quad+\int_{B_\delta(\xi)}|u(\eta)-v(\eta)|\,d\eta \\
&\leqq c(1+|\xi|)^{p-1-\alpha}\delta^\alpha|B_\delta(\xi)| \\
&\quad+\left(\int_{B_\delta(\xi)}|u(\eta)-v(\eta)|^2 f_h^2(|\eta|)\,d\eta\right)^{1/2}\left(\int_{B_\delta(\xi)}(f_h(|\eta|))^{-2}\,d\eta\right)^{1/2},
\end{aligned}
$$

so that we obtain

$$
|u(\xi)-v(\xi)|\leqq c\left[(1+|\xi|)^{p-1-\alpha}\delta^\alpha+\frac{\|u-v\|_h}{\delta^{n/2}f_h(|\xi|+1)}\right].
$$

In particular, since π_h is a Lipschitz function with constant 1 in L_h^2, for all $u, v\in L_h^2$ and for all $\xi\in\mathbf{R}^n$ we obtain

$$
|\pi_h u(\xi)-\pi_h v(\xi)|\leqq c\left[(1+|\xi|)^{p-1-\alpha}\delta^\alpha+\frac{\|u-v\|_h}{\delta^{n/2}f_h(|\xi|+1)}\right]. \tag{4.9}
$$

Let us remark that the function $\tilde{\alpha}_h=\tilde{\alpha}_h(y,\xi)$ is uniformly continuous in y, uniformly with respect to ξ. Fixed $\varepsilon>0$, let $\rho>0$ such that if $|\tau|<\rho$, then

$$
|\tilde{\alpha}_h(y+\tau,\xi)-\tilde{\alpha}_h(y,\xi)|<\varepsilon
$$

for all $y\in\mathbf{R}^N$ and $\xi\in\mathbf{R}^n$. Then, for such a τ we have

$$
\begin{aligned}
&\|\tilde{\alpha}_h(y+\tau,\,.\,)-\tilde{\alpha}_h(y,\,.\,)\|_h \\
&\qquad\leqq\left(\int_{]-h,\,h[^n}|\tilde{\alpha}_h(y+\tau,\eta)-\tilde{\alpha}_h(y,\eta)|^2\,d\eta\right)^{1/2}\leqq\varepsilon(2h)^{n/2},
\end{aligned}
$$

so that, by (4.9) [with $u(\xi)=\tilde{\alpha}_h(y+\tau,\xi)$ and $v(\xi)=\tilde{\alpha}_h(y,\xi)$]

$$
|\tilde{a}_h(y+\tau,\xi)-\tilde{a}_h(y,\xi)|\leqq c\left[(1+|\xi|)^{p-1-\alpha}\delta^\alpha+\frac{\varepsilon(2h)^{n/2}}{\delta^{n/2}f_h(|\xi|+1)}\right] \tag{4.10}
$$

for all $\xi\in\mathbf{R}^n$. Since ε and δ can be chosen independently arbitrarily small, (4.10) proves that $\tilde{a}_h(\,.\,,\xi)$ is uniformly continuous.

Step 4 [*proof of* (4.6)]. – Fixed $R\geqq 0$, let us consider for every $x\in\mathbf{R}^n$ and $h\in\mathbf{N}$ the function

$$
g_h(x,R)=\sup_{|\xi|\leqq R}|a_h(x,\xi)-a(x,\xi)|. \tag{4.11}
$$

Since we are interested in the limit as h tends to $+\infty$, we will suppose $h>R+1$, so that $f_h(|\xi|)=1$ for $|\xi|\leqq R$.

Let us remark that $a(x,\,.\,)\in M(\alpha,\beta,c_1,c_2)$ for almost every $x\in\mathbf{R}^n$, since $a\in M_{\mathbf{R}^n}$, so that

$$
a(x,\xi)=\pi_h(a(x,\xi))
$$

for a. e. $x \in \mathbf{R}^n$, for every $\xi \in \mathbf{R}^n$, and for every $h \in \mathbf{N}$. By (4.9), we have for $|\xi| \leqq \mathrm{R}$ and $0 < \delta \leqq 1$

$$|a_h(x, \xi) - a(x, \xi)| = |\pi_h(\alpha_h(x, \xi)) - \pi_h(a(x, \xi))| \leqq c[(1+\mathrm{R})^{p-1-\alpha}\delta^\alpha + \delta^{-n/2}\|\alpha_h(x, .) - a(x, .)\|_h]. \quad (4.12)$$

Thus, we have to estimate

$$\|\alpha_h(x, .) - a(x, .)\|_h^2 \leqq \int_{[-h, h]^n} |\alpha_h(x, \eta) - a(x, \eta)|^2 \, d\eta + \int_{\{|\eta| > h\}} |\alpha_h(x, \eta) - a(x, \eta)|^2 f_h^2(|\eta|) \, d\eta \equiv \mathrm{J}_h^1 + \mathrm{J}_h^2. \quad (4.13)$$

By (1.3) we have

$$\mathrm{J}_h^2 \leqq c \int_{\{|\eta| > h\}} |\eta|^{2p-2} f_h^2(|\eta|) \, d\eta \leqq c \int_h^{+\infty} \rho^{2p+n-3} f_h^2(\rho) \, d\rho, \quad (4.14)$$

so that $\lim\limits_{h \to \infty} \mathrm{J}_h^2 = 0$. By the definition of α_h, (1.3), and the Hölder inequality we obtain

$$\begin{aligned} \mathrm{J}_h^1 &= \sum_{z \in \mathrm{I}_h} \int_{\mathrm{Q}_z^h} |\alpha_h(x, \eta) - a(x, \eta)|^2 \, d\eta = \sum_{z \in \mathrm{I}_h} \int_{\mathrm{Q}_z^h} |\mathrm{P}_z^h(x) - a(x, \eta)|^2 \, d\eta \\ &\leqq \sum_{z \in \mathrm{I}_h} \int_{\mathrm{Q}_z^h} 2(|\mathrm{P}_z^h(x) - a(x, \xi_z^h)|^2 + |a(x, \xi_z^h) - a(x, \eta)|^2) \, d\eta \\ &\leqq \sum_{z \in \mathrm{I}_h} \int_{\mathrm{Q}_z^h} 2(|\mathrm{P}_z^h(x) - a_z^h(x)|^2 + [c_1(1 + |\xi_z^h| + |\eta|)^{p-1-\alpha}|\xi_z^h - \eta|^\alpha]^2) \, d\eta \\ &\leqq \sum_{z \in \mathrm{I}_h} 2\left(\frac{1}{\gamma_h}\right)^n |\mathrm{P}_z^h(x) - a_z^h(x)|^2 + c h^{n+2p-2-2\alpha}\left(\frac{1}{\gamma_h}\right)^{2\alpha}. \end{aligned} \quad (4.15)$$

Taking (4.12)-(4.15) into account, we obtain for $|\xi| \leqq \mathrm{R}$ and $0 < \delta \leqq 1$

$$|a_h(x, \xi) - a(x, \xi)| \leqq c(1+\mathrm{R})^{p-1-\alpha}\delta^\alpha + c\,\delta^{-n/2}(h^{n/2+p-1-\alpha}(\gamma_h)^{-\alpha} + (\gamma_h)^{-n/2} \sum_{z \in \mathrm{I}_h} |\mathrm{P}_z^h(x) - a_z^h(x)| + (\mathrm{J}_h^2)^{1/2});$$

an integration gives

$$\begin{aligned} ⨍ g_h(x, \mathrm{R}) \, dx &\leqq c(1+\mathrm{R})^{p-1-\alpha}\delta^\alpha \\ &\quad + c\,\delta^{-n/2}\left(h^{n/2+p-1-\alpha}(\gamma_h)^{-\alpha} + (\gamma_h)^{-n/2} \sum_{z \in \mathrm{I}_h} ⨍ |\mathrm{P}_z^h - a_z^h| \, dx + (\mathrm{J}_h^2)^{1/2}\right) \\ &\leqq c(1+\mathrm{R})^{p-1-\alpha}\delta^\alpha \\ &\quad + c\,\delta^{-n/2}\left(h^{n/2+p-1-\alpha}(\gamma_h)^{-\alpha} + h^n \frac{(\gamma_h)^{n/2}}{\mu_h} + (\mathrm{J}_h^2)^{1/2}\right). \end{aligned} \quad (4.16)$$

Now, we can choose the sequences (γ_h) and (μ_h) in such a way that

$$\lim_{h \to \infty} h^{n/2+p-1-\alpha}(\gamma_h)^{-\alpha} = \lim_{h \to \infty} h^n \frac{(\gamma_h)^{n/2}}{\mu_h} = 0$$

(for example, $\gamma_h = h!$ and $\mu_h = h^{nh}$). Then, (4.16) gives

$$\limsup_{h \to \infty} \fint g_h(x, R)\, dx \leq c(1+R)^{p-1-\alpha} \delta^\alpha.$$

By the arbitrariness of δ, the proof is completed. □

Proof of Lemma 4.3. – By the representation Theorem 2.3 it is enough to prove that for every $\xi \in \mathbf{R}^n$ and for every cube Q in $\mathbf{R}^n$ the limit

$$\lim_{\varepsilon \to 0+} \frac{1}{|Q|} \int_Q a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right) dx \tag{4.17}$$

exists and is independent of Q, where v_ε is the solution to the following boundary value problem

$$\left.\begin{aligned} -\operatorname{div}\left(a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right)\right) = 0 \quad \text{on } Q, \\ v_\varepsilon \in H_0^{1,p}(Q). \end{aligned}\right\} \tag{4.18}$$

Given a cube Q in $\mathbf{R}^n$ and $\xi \in \mathbf{R}^n$ let us prove that

$$\lim_{h \to \infty} \limsup_{\varepsilon \to 0+} \int_Q \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right| dx = 0, \tag{4.19}$$

where v_ε^h is the solution to the following boundary value problem

$$\left.\begin{aligned} -\operatorname{div}\left(a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right)\right) = 0 \quad \text{on } Q, \\ v_\varepsilon^h \in H_0^{1,p}(Q). \end{aligned}\right\} \tag{4.20}$$

For the sake of simplicity we drop in the notation any explicit dependence on ξ of v_ε and v_ε^h throughout this section. We can write

$$\begin{aligned} &\int_Q \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right| dx \\ &\quad \leq \left| \int_Q \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right) - a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right| dx \right| \\ &\quad + \left| \int_Q \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right| dx \right| \equiv J_{\varepsilon,h}^1 + J_{\varepsilon,h}^2. \end{aligned}$$

The estimate of $J_{\varepsilon,h}^1$ and $J_{\varepsilon,h}^2$ will be carried out in the following four steps.

Step 1. – Let us fix $\xi \in \mathbf{R}^n$. As in (3.17) one gets

$$\| Dv_\varepsilon^h + \xi \|_{(L^p(Q))^n}^p \leq c\,|Q|\,(1 + |\xi|^p).$$

This estimate will be used frequently in the sequel. By the Meyers estimate (1.6) we have then

$$\| Dv_\varepsilon^h + \xi \|_{(L^{p+\eta}(Q))^n} \leqq c.$$

Let us fix $R \geqq 0$, $h \in N$ and $\varepsilon > 0$, and let us define the set

$$U_R = \{ x \in Q : | Dv_\varepsilon^h(x) + \xi | > R \}.$$

We have then

$$| U_R | R^p \leqq \int_{U_R} | Dv_\varepsilon^h + \xi |^p \, dx,$$

so that, using the Hölder inequality, we obtain the estimate

$$\int_{U_R} (1 + | Dv_\varepsilon^h + \xi |^p) \, dx \leqq | U_R | + | U_R |^{\eta/(p+\eta)} \| Dv_\varepsilon^h + \xi \|^p_{(L^{p+\eta}(Q))^n}$$
$$\leqq c (R^{-p} + R^{-p\eta/(p+\eta)}). \quad (4.21)$$

Step 2 (*estimate of* $J^2_{\varepsilon,h}$). — Given $R > 0$ by (1.3) and (4.12) we have

$$(J^2_{\varepsilon,h})^q \leqq c \int_Q \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right|^q dx$$
$$= c \int_{U_R} \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right|^q dx$$
$$+ c \int_{Q \setminus U_R} \left| a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) \right|^q dx$$
$$\leqq c \int_{U_R} (1 + | Dv_\varepsilon^h + \xi |^p) \, dx + c \int_{Q \setminus U_R} g_h\left(\frac{x}{\varepsilon}, R\right)^q dx$$
$$\leqq c \int_{U_R} (1 + | Dv_\varepsilon^h + \xi |^p) \, dx + c (1 + R) \int_Q g_h\left(\frac{x}{\varepsilon}, R\right) dx.$$

By Step 1 we conclude then

$$J^2_{\varepsilon,h} \leqq c \left((R^{-p} + R^{-p\eta/(p+\eta)}) + (1 + R) \int_Q g_h\left(\frac{x}{\varepsilon}, R\right) dx \right)^{1/q}. \quad (4.22)$$

Step 3 (*estimate of* $\| Dv_\varepsilon^h - Dv_\varepsilon \|_{(L^p(Q))^n}$). — By (4.18) and (4.20) we have

$$\int_Q \left(a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right) - a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right), Dv_\varepsilon - Dv_\varepsilon^h \right) dx = 0,$$

so that by (iii) and (iv) in Definition 1.1 and the Hölder inequality we have

$$0 = \int_Q \left(a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right) - a\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right), Dv_\varepsilon - Dv_\varepsilon^h \right) dx$$

$$
\begin{aligned}
&+\int_Q \left(a\left(\frac{x}{\varepsilon}, \mathrm{D}v_\varepsilon^h+\xi\right)-a_h\left(\frac{x}{\varepsilon}, \mathrm{D}v_\varepsilon^h+\xi\right), \mathrm{D}v_\varepsilon-\mathrm{D}v_\varepsilon^h\right) dx \\
&\geqq c_2 \int_Q (1+|\mathrm{D}v_\varepsilon+\xi|+|\mathrm{D}v_\varepsilon^h+\xi|)^{p-\beta}|\mathrm{D}v_\varepsilon-\mathrm{D}v_\varepsilon^h|^\beta dx \\
&-\int_Q \left|a\left(\frac{x}{\varepsilon}, \mathrm{D}v_\varepsilon^h+\xi\right)-a_h\left(\frac{x}{\varepsilon}, \mathrm{D}v_\varepsilon^h+\xi\right)\right| |\mathrm{D}v_\varepsilon-\mathrm{D}v_\varepsilon^h| dx \\
&\qquad \geqq c(|\mathrm{Q}|^{1/p}+\|\mathrm{D}v_\varepsilon+\xi\|_{(\mathrm{L}^p(\mathrm{Q}))^n} \\
&\qquad +\|\mathrm{D}v_\varepsilon^h+\xi\|_{(\mathrm{L}^p(\mathrm{Q}))^n})^{p-\beta}\|\mathrm{D}v_\varepsilon-\mathrm{D}v_\varepsilon^h\|^\beta_{(\mathrm{L}^p(\mathrm{Q}))^n} \\
&\qquad -\left(\int_Q \left|a\left(\frac{x}{\varepsilon}, \mathrm{D}v_\varepsilon^h+\xi\right)-a_h\left(\frac{x}{\varepsilon}, \mathrm{D}v_\varepsilon^h+\xi\right)\right|^q dx\right)^{1/q} \\
&\qquad \qquad \times\left(\int_Q |\mathrm{D}v_\varepsilon-\mathrm{D}v_\varepsilon^h|^p dx\right)^{1/p},
\end{aligned}
$$

so that by (4.22) we obtain

$$
\begin{aligned}
&\|\mathrm{D}v_\varepsilon^h-\mathrm{D}v_\varepsilon\|^{\beta-1}_{(\mathrm{L}^p(\mathrm{Q}))^n} \\
&\qquad \leqq c\left((\mathrm{R}^{-p}+\mathrm{R}^{-p\eta/(p+\eta)})+(1+\mathrm{R})\int_Q g_h\left(\frac{x}{\varepsilon}, \mathrm{R}\right) dx\right). \qquad (4.23)
\end{aligned}
$$

Step 4 (*estimate of* $\mathrm{J}^1_{\varepsilon, h}$). – By the equicontinuity condition of a and Hölder's inequality we have

$$
\begin{aligned}
\mathrm{J}^1_{\varepsilon, h}&=\left|\int_Q \left(a\left(\frac{x}{\varepsilon}, \mathrm{D}v_\varepsilon+\xi\right)-a\left(\frac{x}{\varepsilon}, \mathrm{D}v_\varepsilon^h+\xi\right)\right) dx\right| \\
&\leqq c_1 \int_Q (1+|\mathrm{D}v_\varepsilon+\xi|+|\mathrm{D}v_\varepsilon^h+\xi|)^{p-1-\alpha}|\mathrm{D}v_\varepsilon-\mathrm{D}v_\varepsilon^h|^\alpha dx \\
&\leqq c_1|\mathrm{Q}|^{1/p}\left(\int_Q (1+|\mathrm{D}v_\varepsilon+\xi|+|\mathrm{D}v_\varepsilon^h+\xi|)^p dx\right)^{(p-1-\alpha)/p} \\
&\qquad \times\left(\int_Q |\mathrm{D}v_\varepsilon-\mathrm{D}v_\varepsilon^h|^p dx\right)^{\alpha/p}.
\end{aligned}
$$

Hence, by (4.23) we get

$$
\mathrm{J}^1_{\varepsilon, h} \leqq c\left((\mathrm{R}^{-p}+\mathrm{R}^{-p\eta/(p+\eta)})+(1+\mathrm{R})\int_Q g_h\left(\frac{x}{\varepsilon}, \mathrm{R}\right) dx\right)^{\alpha/(\beta-1)}. \qquad (4.24)
$$

Taking into account that there exists $s>0$ such that $\mathrm{Q} \subset \mathrm{Q}_s(0)$ and

$$
\begin{aligned}
&\int_Q g_h\left(\frac{x}{\varepsilon}, \mathrm{R}\right) dx \leqq \int_{\mathrm{Q}_s(0)} g_h\left(\frac{x}{\varepsilon}, \mathrm{R}\right) dx \\
&\qquad \leqq \varepsilon^n \int_{\mathrm{Q}_{s/\varepsilon}(0)} g_h(x, \mathrm{R})\, dx \leqq \frac{c}{|\mathrm{Q}_{s/\varepsilon}(0)|}\int_{\mathrm{Q}_{s/\varepsilon}(0)} g_h(x, \mathrm{R})\, dx,
\end{aligned}
$$

by passing in (4.22) and (4.24) first to the limit as ε tends to 0^+, then as h tends to $+\infty$, and eventually as R tends to $+\infty$ we obtain (4.19). By (4.4) and Theorem 2.3 this implies that

$$\lim_{\varepsilon \to 0+} \frac{1}{|Q|}\int_Q a\left(\frac{x}{\varepsilon}, Dv_\varepsilon + \xi\right) dx = \lim_{h\to\infty} \lim_{\varepsilon \to 0+} \frac{1}{|Q|}\int_Q a_h\left(\frac{x}{\varepsilon}, Dv_\varepsilon^h + \xi\right) dx = \lim_{h\to\infty} b_h(\xi) = b(\xi). \quad (4.25)$$

Since the first limit does not depend on Q, the proof of Lemma 4.3 can be concluded by applying again the representation Theorem 2.3. □

Proof of Theorem 4.2. – Let $a \in M_{\mathbf{R}^n}$ such that $a(.,\xi)$ is almost periodic for every $\xi \in \mathbf{R}^n$. By Lemma 4.4 there exists a sequence (a_h) in $M_{\mathbf{R}^n}$ satisfying (3.3) (with λ possibly depending on h) such that condition (4.5) is satisfied. By the homogenization Theorem 3.5 we obtain that for every $h \in \mathbf{N}$ the limit

$$G - \lim_{\varepsilon \to 0+} a_h\left(\frac{x}{\varepsilon}, \xi\right) = b_h(\xi)$$

exists and is independent of x. Hence, by Lemma 4.3 the limit

$$G - \lim_{\varepsilon \to 0+} a\left(\frac{x}{\varepsilon}, \xi\right) = b(\xi)$$

exists. Finally, the representation formula follows from (4.25). □

APPENDIX

A. Birkhoff's theorem for quasiperiodic functions

Before we give a direct proof of Birkhoff's Ergodic theorem in the quasiperiodic case, we state Kronecker's lemma (*see*, for instance, [30] 3.1).

KRONECKER'S LEMMA. – *The set of vectors which are equivalent modulo* T *to vectors of the form* $j(x)$, *with* $x \in \mathbf{R}^n$, *is dense in* $\mathbf{R}^N$.

THEOREM A. – *Let* $f: \mathbf{R}^n \to \mathbf{R}$ *be a quasiperiodic function in* QP(λ) (see *Section* 3). *Then*

$$⨍ f(x)\, dx = \frac{1}{|T|}\int_T F(y)\, dy,$$

where T *is defined by* (3.2).

Proof. — Let $X = j(\mathbf{R}^n)$ and $Z = X^{\perp}$. Let us identify $\mathbf{R}^N = X \oplus Z$. Moreover, let us introduce

$$P_s^N(0) = j(Q_s(0)) \times B_s^Z(0),$$

where $B_s^Z(0)$ is the ball in Z of radius s and center 0, and define the set S by $S = \{ z \in B_s^Z(0) \colon z \text{ is equivalent modulo T to vectors of the type } j(\tau) \text{ with } \tau \in \mathbf{R}^n \}$. By Kronecker's lemma the set S is dense in $B_s^Z(0)$. Then, given $z \in S$ there exists $\tau = \tau(z) \in \mathbf{R}^n$ such that

$$F(j(x) + z) = f(x + \tau)$$

for every $x \in \mathbf{R}^n$. Since the limit

$$\lim_{s \to +\infty} \frac{1}{s^n} \int_{Q_s(\tau)} f(x)\, dx = ⨍ f(x)\, dx$$

exists uniformly with respect to τ (*see* [30] 2.3), given $\varepsilon > 0$ we have

$$\left| \frac{1}{s^n} \int_{Q_s(0)} f(x + \tau)\, dx - ⨍ f(x)\, dx \right| < \varepsilon$$

for every $\tau \in \mathbf{R}^n$, for s sufficiently large, so that for $z \in S$

$$\left| \frac{1}{s^n} \int_{Q_s(0)} F(j(x) + z)\, dx - ⨍ f(x)\, dx \right| < \varepsilon \tag{A.1}$$

for s sufficiently large. By the uniform continuity of F and the density of S, the estimate (A.1) holds for every $z \in B_s^Z(0)$. By Fubini's theorem we have

$$\begin{aligned} \frac{1}{|P_s^N(0)|} \int_{P_s^N(0)} F(y)\, dy &= \frac{1}{s^n} \int_{Q_s(0)} \left(\frac{1}{|B_s^Z(0)|} \int_{B_s^Z(0)} F(j(x) + z)\, dz \right) dx \\ &= \frac{1}{|B_s^Z(0)|} \int_{B_s^Z(0)} \left(\frac{1}{s^n} \int_{Q_s(0)} F(j(x) + z)\, dx \right) dz, \end{aligned}$$

which by (A.1) yields that

$$\left| \frac{1}{|P_s^N(0)|} \int_{P_s^N(0)} F(y)\, dy - ⨍ f(x)\, dx \right| < \varepsilon.$$

This implies together with the periodicity of F that

$$\frac{1}{|T|} \int_T F(y)\, dy = \lim_{s \to +\infty} \frac{1}{|P_s^N(0)|} \int_{P_s^N(0)} F(y)\, dy = ⨍ f(x)\, dx$$

and concludes the proof. □

B. Regularity of the quasiperiodic solutions of $\Delta u = \operatorname{div} f$

Remark B.1. – Let $\mathscr{W}$ be the completion of $\mathrm{Trig}_0(\mathrm{T})$ with respect to the norm $\|.\|_{\mathscr{W}}$ introduced in Section 3. Let us denote by $\mathscr{W}'$ its dual. It turns out that for every $\mathrm{S} \in \mathscr{W}'$ there exists $\mathrm{G} \in (\mathrm{L}^q(\mathrm{T}))^n$ such that

$$\langle \mathrm{S}, v \rangle = \frac{1}{|\mathrm{T}|} \sum_{r=1}^{n} \int_{\mathrm{T}} \mathrm{G}_r \, \partial_r v \, dy \quad \text{for every } v \in \mathscr{W}. \tag{B.1}$$

In fact, since by the map ∂ the space $\mathscr{W}$ is isometric to a subspace of $(\mathrm{L}^p(\mathrm{T}))^n$, by the Hahn-Banach and Riesz theorems there exists a function $\mathrm{G} \in (\mathrm{L}^q(\mathrm{T}))^n$ such that (B.1) and

$$\|\mathrm{S}\|_{\mathscr{W}'} = \left(\frac{1}{|\mathrm{T}|} \sum_{r=1}^{n} \int_{\mathrm{T}} |\mathrm{G}_r|^q \, dy \right)^{1/q}$$

hold. We will write $\|\mathrm{S}\|_{\mathscr{W}'} = \| \sum_{r=1}^{n} \partial_r \mathrm{G}_r \|_{\mathscr{W}'}$.

THEOREM B. – *Let* $\mathrm{F} \in (\mathrm{Trig}(\mathrm{T}))^n$ *and let* $\mathrm{W} \in \mathrm{Trig}_0(\mathrm{T})$ *be the unique solution to*

$$\sum_{r=1}^{n} \partial_r^2 \mathrm{W} = \sum_{r=1}^{n} \partial_r \mathrm{F}_r \quad \textit{in } \mathbf{R}^{\mathrm{N}}. \tag{B.2}$$

Then,

$$\left(\frac{1}{|\mathrm{T}|} \sum_{r=1}^{n} \int_{\mathrm{T}} |\partial_r \mathrm{W}|^q \, dy \right)^{1/q} \leq c \left\| \sum_{r=1}^{n} \partial_r \mathrm{F}_r \right\|_{\mathscr{W}'}, \tag{B.3}$$

with $c = c(\mathrm{T}) > 0$ *and* $\partial_r = \sum_{l=1}^{m_r} \frac{\partial}{\partial y_r^l}$ *as defined in Section* 3.

Proof. – Let us define $\mathrm{S} \in \mathscr{W}'$ by

$$\langle \mathrm{S}, v \rangle = \frac{1}{|\mathrm{T}|} \sum_{r=1}^{n} \int_{\mathrm{T}} \mathrm{F}_r \, \partial_r v \, dy \quad \text{for every } v \in \mathscr{W}.$$

By Remark B.1 there exists $\mathrm{G} \in (\mathrm{L}^q(\mathrm{T}))^n$ such that

$$\langle \mathrm{S}, v \rangle = \frac{1}{|\mathrm{T}|} \sum_{r=1}^{n} \int_{\mathrm{T}} \mathrm{G}_r \, \partial_r v \, dy \quad \text{for every } v \in \mathscr{W}$$

and

$$\|\mathrm{G}\|_{(\mathrm{L}^q(\mathrm{T}))^n} = \|\mathrm{S}\|_{\mathscr{W}'}. \tag{B.4}$$

By extending G on $\mathbf{R}^N$ by periodicity we have that

$$\sum_{r=1}^{n} \partial_r^2 W = \sum_{r=1}^{n} \partial_r G_r \quad \text{in } \mathscr{D}'(\mathbf{R}^N). \tag{B.5}$$

Let us introduce on $\mathbf{R}^N$ a new orthogonal base $(E_l)_{1 \leqq l \leqq N}$ with

$$E_1 = (\underbrace{1, \ldots, 1}_{m_1}, 0, \ldots, 0)$$

$$\vdots$$

$$E_n = (0, \ldots, 0, \underbrace{1, \ldots, 1}_{m_n}).$$

We shall denote the new coordinates by

$$(x, z) = (x_1, \ldots, x_n, z_1, \ldots, z_{N-n}) \in \mathbf{R}^N.$$

It turns out that

$$D_{x_l} u(x, z) = \partial_l u(y) \quad \text{for every } u \in \mathscr{W}, \quad \text{for every } l = 1, \ldots, n,$$

where $y = (x, z)$. Hence by (B.5) the function $W \in \mathscr{C}^\infty(\mathbf{R}^n \times \mathbf{R}^{N-n})$ satisfies for almost every $z \in \mathbf{R}^{N-n}$

$$\Delta_x W(., z) = \operatorname{div}_x G(., z) \quad \text{in } \mathscr{D}'(\mathbf{R}^n).$$

By elliptic regularity (*see*, for instance [14]) for every $A' \subset\subset A \subset\subset \mathbf{R}^n$ there exists a constant $c = c(A', A)$, such that

$$\int_{A'} |D_x W(x, z)|^q \, dx \leqq c \int_A |G(x, z)|^q \, dx.$$

By Fubini's theorem we get

$$\int_B \int_{A'} |D_x W(x, z)|^q \, dx \, dz \leqq c \int_B \int_A |G(x, z)|^q \, dx \, dz$$

for every $B \subset\subset \mathbf{R}^{N-n}$. Choosing A′, A and B such that

$$T \subset B \times A' \subset B \times A \subset \prod_{l, r} \left] -\frac{2\pi}{\lambda_r^l}, \frac{4\pi}{\lambda_r^l} \right[,$$

and using finally the periodicity of G, we get

$$\int_T |D_x W(x, z)|^q \, dx \, dz \leqq c \int_T |G(x, z)|^q \, dx \, dz.$$

By (B.4) we obtain (B.3), and we can conclude the proof. □

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