Ann. Inst. Henri Poincaré,

Vol. 9, n° 4, 1992, p. 433-464. *Analyse non linéaire*

The large deformation of nonlinearly elastic shells in axisymmetric flows

by

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ABSTRACT. — This paper treats the large deformation of closed nonlinearly elastic axisymmetric shells under an external pressure field generated by the steady, irrotational, axisymmetric flow of an incompressible, inviscid fluid. The flow is assumed to have a prescribed velocity U and pressure P at infinity. The deformation of the shells is described by a geometrically exact theory. The parameters U and P and the deformed shape of the shell uniquely determine the velocity field of the steady flow. The most difficult part of the analysis is to show that the velocity and pressure of the flow on the shell depend continuously and compactly on the function describing the shape. The pressure field on the shell is substituted into the equilibrium equations for the shell, yielding a system of ordinary functional-differential equations. These are converted into a fixed-point form, which is analyzed by a global implicit function theorem. The problem has technical difficulties that do not arise in problems with rigid obstacles.

Classification A.M.S. : 31 A 25, 24 B 15, 45 J 05, 47 H 17, 73 J 06, 73 L 99, 76 B 99.

Annales de l'Institut Henri Poincaré - Analyse non linéaire - 0294-1449
Vol. 9/92/04/433/32/$5.20/

Key words : Nonlinearly elastic shells, fluid-solid interactions, incompressible inviscid fluids, global implicit function theorem, compactness, Schauder estimates, potential theory.

RÉSUMÉ. — On s'intéresse aux grandes déformations d'une coque fermée axisymétrique soumise à l'action du champ de pression externe engendré par l'écoulement stationnaire et irrotationnel d'un fluide parfait incompressible autour de la coque. On utilise pour modéliser la coque une théorie géométriquement exacte avec une loi de comportement élastique non linéaire générale. La vitesse et la pression de l'écoulement à l'infini sont deux constantes données U et P. Le champ de vitesse du fluide est déterminé de façon unique par ces deux paramètres et par la forme de la coque dans sa configuration déformée. La partie la plus délicate de notre analyse consiste à montrer que les champs de vitesse et de pression sur la surface de la coque dépendent continûment et de façon compacte de la fonction qui décrit la forme de la coque. Substituant la pression dans les équations d'équilibre de la coque, on obtient un système d'équations fonctionnelles-différentielles ordinaires que l'on transforme ensuite en problème de point fixe. Ce problème est lui-même résolu à l'aide d'un théorème des fonctions implicites global. On rencontre ici les difficultés techniques qui ne se présentent pas dans les problèmes avec obstacle rigide.

1. INTRODUCTION

In this paper we study the large deformation of a closed nonlinearly elastic spherical shell produced by an external pressure field generated by the steady, irrotational, axisymmetric flow of an incompressible, inviscid fluid. (With very little additional work all our results can be extended to shells with any closed axisymmetric reference shape). The flow is assumed to have a prescribed velocity U and pressure P at infinity. We describe the deformation of these shells with a geometrically exact theory (*cf.* [19]) that accounts for flexure, compression, and shear. We allow the material properties of the shell to be described by a very general class of nonlinear constitutive relations.

We begin our analysis by observing that U, P, and the deformed shape of the shell uniquely determine the velocity field of the steady exterior flow. We show that the velocity of the flow on the shell depends continuously and compactly on the function describing the shape of the outer surface of the shell. We then use Bernoulli's theorem to express the pressure field on the shell in terms of U, P, and the velocity field on the shell. We substitute this pressure into the equilibrium equtions for the elastic shell. We transform these equations into a fixed-point equation for the shape, involving a family of compact operators and depending parametrically upon U and P. We apply a generalization of the Global Implicit Function Theorem of [3] to these equations to deduce the existence of connected families of solutions. In this program we encounter serious technical difficulties in showing that the pressure on the shell depends continuously and compactly on an appropriate function describing the shape and in constructing a suitable fixed point equation. To handle the first of these difficulties (which can be ignored in the study of flows past rigid bodies) we could develop and exploit refined results from potential theory. We are able to shortcut this lengthy process by using a variety of Schauder estimates (which, at bottom, rest on potential-theoretic arguments) together with a construction relying on a conformal mapping. In Section 7, we sketch the steps needed for a direct proof of compactness by using potential theory.

Thus we replace the coupled problem for the deformation of the shell and the external flow of the fluid with a single problem for the deformation of the shell in which the pressure field on it depends nonlocally on its shape. One of the goals of this paper is to develop effective methods for treating well-set nonlinear problems from mechanics with such nonlocal terms. (The corresponding problem for the two-dimensional flow past a ring was solved in [13] by using conformal mapping theory. Its mathematical treatment differs considerably from that used here.)

Notation

Vectors in Euclidean 3-space and n-tuples of real numbers are each denoted by bold-face lower-case Roman letters. Partial derivatives are denoted by subscripts and ordinary derivatives by primes. If f and g are functions of u and v, then $\dfrac{\partial(f, g)}{\partial(u, v)}$ denotes the *matrix* of partial derivatives of f and g with respect to u and v. We denote the closure of a set $\mathscr{E}$ by $\mathrm{cl}\,\mathscr{E}$, the boundary of $\mathscr{E}$ by $\partial\mathscr{E}$, and the set of elements belonging to set $\mathscr{A}$ and not belonging to set $\mathscr{B}$ by $\mathscr{A}\setminus\mathscr{B}$.

We denote the norm on a Banach space $\mathscr{X}$ by $\|\cdot, \mathscr{X}\|$. Let Ω be a bounded open connected subset of $\mathbb{R}^n$ with a boundary of class C^1. The space of m-times continuously differentiable functions on $\mathrm{cl}\,\Omega$ with its usual norm is denoted $C^m(\mathrm{cl}\,\Omega)$. The subspace of $C^m(\mathrm{cl}\,\Omega)$ whose functions have m-th derivatives that are Hölder continuous with exponent α are denoted $C^{m,\alpha}(\mathrm{cl}\,\Omega)$; if u is in $C^{0,\alpha}(\mathrm{cl}\,\Omega)$, then its α-Hölder quotient is

$|u|_\alpha \equiv \sup\left\{\frac{|u(x)-u(y)|}{|x-y|^\alpha}, x, y \in \mathrm{cl}\,\Omega, x \neq y\right\}$. The space $C^{m,\alpha}(\mathrm{cl}\,\Omega)$ is equipped with its usual norm:

$$\|u, C^{m,\alpha}\| \equiv \|u, C^m\| + \sum_{|\beta|=m} |D^\beta u|_\alpha$$

where $\beta=(\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$ and $|\beta|=\beta_1+\ldots+\beta_n$. $W^{m,p}(\Omega)$ denotes the Sobolev space of all functions in $L^p(\Omega)$ all of whose distributional derivatives up to order m are in $L^p(\Omega)$. If $\partial\Omega$ is of class C^m, then the Sobolev space $W^{s,p}(\partial\Omega)$ can be defined for each $s \in [0, m]$. (Consult [1], [7], [16] for details about these spaces.) The domain (Ω or $\mathrm{cl}\,\Omega$ or $\partial\Omega$) of the functions under consideration will not be indicated when it is evident from the context.

If $\mathscr{X}$ is a space of real-valued functions, then we simply write $\mathbf{u} \in \mathscr{X}$ if each component of the vector-valued function $\mathbf{u}$ is in $\mathscr{X}$. We typically use brackets to denote the value of a mapping f defined on a function space. Thus the value of f at a function u in its domain of definition is denoted $f[u]$. If $f[u]$ is a function on an interval, then its value at a point s on this interval is of course $f[u](s)$.

2. EQUILIBRIUM EQUATIONS FOR THE AXISYMMETRIC DEFORMATION OF NONLINEARLY ELASTIC SHELLS

Let $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ be a fixed right-handed orthonormal basis for Euclidean 3-space. For each real number φ we set

(2.1) $\mathbf{e}_1(\varphi)=\cos\varphi\,\mathbf{i}+\sin\varphi\,\mathbf{j}, \qquad \mathbf{e}_2(\varphi)=-\sin\varphi\,\mathbf{i}+\cos\varphi\,\mathbf{j}, \qquad e_3=\mathbf{k}.$

Geometry of deformation

To each $(s, \varphi) \in [0, \pi] \times [0, 2\pi]$ corresponds exactly one point on the sphere of radius 1 centered at the origin with position vector

(2.2) $$\mathbf{r}_*(s, \varphi)=\sin s\,\mathbf{e}_1-\cos s\,\mathbf{e}_3.$$

Note that s measures the arc length of $\mathbf{r}_*(\cdot, \varphi)$ from the *south* pole of the sphere to $\mathbf{r}_*(s, \varphi)$. We interpret the sphere (2.2) as the natural reference state of the outer surface of a thin three-dimensional shell. The coordinates (s, φ) identify material points on this surface.

The *axisymmetric configuration of a shell* that can suffer flexure, in-surface extension, and shear is determined by a pair of vector-valued functions $\tilde{\mathbf{r}}$ and $\mathbf{b}$ of s and φ of the form

$$\begin{cases} \tilde{\mathbf{r}}(s, \varphi) = r(s)\,\mathbf{e}_1(\varphi) + z(s)\,\mathbf{k}, \\ \mathbf{b}(s, \varphi) = -\sin\theta(s)\,\mathbf{e}_1(\varphi) + \cos\theta(s)\,\mathbf{k}. \end{cases} \tag{2.3}$$

The reference configuration of the shell is given by $\tilde{\mathbf{r}} = \mathbf{r}_*$, $\mathbf{b} = -\mathbf{r}_*$. The vector $\tilde{\mathbf{r}}(s, \varphi)$ is interpreted as the deformed position of the material point $\mathbf{r}_*(s, \varphi)$. The vector $\mathbf{b}(s, \varphi)$ is interpreted as characterizing the deformed configuration of the material fiber whose reference configuration is on the normal to the mid-surface through $\mathbf{r}_*(s, \varphi)$. We define

$$\mathbf{a}(s, \varphi) = \mathbf{e}_2(\varphi) \times \mathbf{b}(s, \varphi) = \cos\theta(s)\,\mathbf{e}_1(\varphi) + \sin\theta(s)\,\mathbf{k}. \tag{2.4}$$

We define the set of strain variables

$$\mathbf{q} \equiv (\tau, \nu, \eta, \sigma, \mu) \tag{2.5}$$

by

$$\begin{cases} \tilde{\mathbf{r}}_s(s, \varphi) \equiv \nu(s)\,\mathbf{a}(s, \varphi) + \eta(s)\,\mathbf{b}(s, \varphi), \qquad \tau(s) \equiv \dfrac{r(s)}{\sin s}, \\ \sigma(s) \equiv \dfrac{\sin\theta}{\sin s} - 1, \qquad \mu \equiv \theta' - 1. \end{cases} \tag{2.6}$$

The arc length from $\tilde{\mathbf{r}}(0, \varphi)$ to $\tilde{\mathbf{r}}(s, \varphi)$ along a deformed circle of longitude is

$$\lambda[\tilde{\mathbf{r}}](s) \equiv \int_0^s |\tilde{\mathbf{r}}_s(\xi, \varphi)|\, d\xi. \tag{2.7}$$

Let the reference configuration of a three-dimensional spherical shell occupy the region

$$\mathscr{B} \equiv \{(1-\xi)\,\mathbf{r}_*(s, \varphi) : s \in [0, \pi], \varphi \in [0, 2\pi], \xi \in [0, h]\} \tag{2.8}$$

where $h \in (0, 1)$ is the given constant thickness. We can interpret the vectors $\tilde{\mathbf{r}}$ and $\mathbf{b}$ as corresponding to the deformation that takes $(1-\xi)\,\mathbf{r}_*(s, \varphi)$ to $\tilde{\mathbf{r}}(s, \varphi) + \xi\,\mathbf{b}(s, \varphi)$. In this case, we find that the Jacobian of this transformation and its restriction to the $(\mathbf{e}_1(\varphi), \mathbf{k})$-plane for each φ are positive if and only if

$$\nu(s) > \max\{0, [\mu(s)+1]\,h\}, \qquad \tau(s) > \max\{0, [\sigma(s)+1]\,h\}. \tag{2.9}$$

To be specific, we adopt these requirements as characterizing deformations that preserve orientation. It is easy to handle far more general requirements stemming from a more general interpretation of $\tilde{\mathbf{r}}$ and $\mathbf{b}$.

Equilibrium equations

Let $\hat{\mathrm{N}}(s)\,\mathbf{a}(s, \varphi) + \hat{\mathrm{H}}(s)\,\mathbf{b}(s, \varphi)$ and $-\hat{\mathrm{M}}(s)\,\mathbf{e}_2(\varphi)$ denote the resultant contact force and contact couple per unit reference length of the circle

$\varphi \mapsto \mathbf{r}_*(s, \varphi)$ of radius $r_*(s)$ that are exerted across the deformed image of this material section at the material point $\mathbf{r}_*(s, \varphi)$. Let $\mathrm{T}(s)\mathbf{e}_2(\varphi)$ and $\hat{\Sigma}(s)\mathbf{a}(s, \varphi)$ denote the resultant contact force and contact couple per unit reference length of the curve $s \mapsto \mathbf{r}_*(s, \varphi)$ that are exerted across the deformed image of this material section at $\mathbf{r}_*(s, \varphi)$. These forms of the resultants reflect our assumption of axisymmetry. If the only external force applied to the shell is a hydrodynamical pressure of intensity $p(s)$ per unit deformed area at $\tilde{\mathbf{r}}(s, \varphi)$, then the classical form of the equilibrium equations have the form

$$(2.10)\quad \frac{d}{ds}[\sin s\, \hat{\mathrm{N}}(s)] - \hat{\mathrm{T}}(s)\cos\theta(s) - \sin s\, \hat{\mathrm{H}}(s)\,\theta'(s) - p(s)\,r(s)\,\eta(s) = 0,$$

$$(2.11)\quad \frac{d}{ds}[\sin s\, \hat{\mathrm{H}}(s)] + \hat{\mathrm{T}}(s)\sin\theta(s) + \sin s\, \hat{\mathrm{N}}(s)\,\theta'(s) + p(s)\,r(s)\,\nu(s) = 0,$$

$$(2.12)\quad \frac{d}{ds}[\sin s\, \hat{\mathrm{M}}(s)] - \hat{\Sigma}(s)\cos\theta(s) + \sin s\,[\nu(s)\,\hat{\mathrm{H}}(s) - \eta(s)\,\hat{\mathrm{N}}(s)] = 0.$$

Constitutive equations

Let

$$(2.13)\quad \mathcal{Q} \equiv \{\mathbf{q} \in \mathbb{R}^5 : \nu > \max\{0, h[\mu+1]\}, \tau > \max\{0, h[\sigma+1]\}\}$$

[*cf.* (2.9)]. The material of the shell is elastic (and homogeneous) if there are functions T, N, H, Σ, M : $\mathcal{Q} \to \mathbb{R}$ such that

$$(2.14)\quad \hat{\mathrm{T}}(s) = \mathrm{T}(\mathbf{q}(s)), \quad \text{etc.}$$

We assume that these functions are thrice continuously differentiable. We require that these functions satisfy the *monotonicity condition*: The *matrix*

$$(2.15)\quad \mathbf{D} \equiv \frac{\partial(\mathrm{N}, \mathrm{H}, \mathrm{M})}{\partial(\nu, \eta, \mu)} \text{ is positive-definite.}$$

This condition, which is a shell-theoretic analog of the strong ellipticity condition of three-dimensional elasticity, ensures that an increase in the bending strain μ is accompanied by an increase in the bending couple M, etc. We also require that extreme strains be enforced by corresponding extreme values of the stress resultants. Specific realizations of such growth conditions are given by [19]. We complement (2.15) with the requirement that

$$(2.16)\quad (h, \infty) \ni k \mapsto \mathrm{N}(k, k, 0, 0, 0) \quad \text{strictly increases from } -\infty \text{ to } \infty,$$

We require that the material meet the following minimal restrictions on its symmetry:

$$(2.17)\quad \mathrm{T}, \mathrm{N}, \Sigma, \mathrm{M} \text{ are even in } \eta, \ \mathrm{H} \text{ is odd in } \eta.$$

We further require that the shell meet the restricted *isotropy conditions*

$$(2.18)\qquad \begin{cases} N(\tau, \nu, 0, \sigma, \mu) = T(\nu, \tau, 0, \mu, \sigma), \\ M(\tau, \nu, 0, \sigma, \mu) = \Sigma(\nu, \tau, 0, \mu, \sigma). \end{cases}$$

To ensure that the reference configuration is stress-free we assume that

$$(2.19)\qquad T = N = H = \Sigma = M = 0 \quad \text{when } \mathbf{q} = (1, 1, 0, 0, 0).$$

We impose the boundary conditions

$$(2.20a)\qquad r(0) = 0 = r(\pi), \qquad z'(0) = 0 = z'(\pi),$$
$$(2.20b)\qquad \theta(0) = 0, \qquad \theta(\pi) = \pi,$$

which require that the deformation be regular at the poles, and the integral condition

$$(2.21)\qquad \int_0^\pi z(s)\, ds = 0,$$

which fixes translation in the **k**-direction.

We require that:

$$(2.22a)\qquad [0, \pi] \ni s \mapsto r(s)\, \mathbf{e}_1(\varphi) + z(s)\, \mathbf{k} \quad \text{is injective},$$
$$(2.22b)\qquad r(s) > 0 \quad \text{for} \quad s \in (0, \pi),$$
$$(2.22c)\qquad z(\pi) > z(0).$$

These conditions ensure that the outer surface of the shell is simple and has $\tilde{\mathbf{r}}(0, \varphi)$ as the south pole.

3. THE BOUNDARY VALUE PROBLEM FOR THE SHELL

In this section we formulate the boundary value problem for the shell when the pressure field has the form delivered by the analysis of the flow problem, which is carried out in Sections 6 and 7. We then convert the boundary value problem to a fixed-point problem, to which we apply a global implicit function theorem.

Since the dependence of $\tilde{\mathbf{r}}$, **a**, **b** on φ is determined by the form of these functions when $\varphi = 0$ [*cf.* (2.3)], we define

$$(3.1)\qquad \mathbf{r}(s) \equiv (r(s), z(s))$$

and henceforth use only these new functions.

To give a precise statement of our boundary value problem we must introduce certain sets of functions. For $\mathbf{r} \in C^1[0, \pi]$ we define

$$(3.2a)\qquad l_1[\mathbf{r}] \equiv \inf\left\{ \left| \frac{\mathbf{r}(s) - \mathbf{r}(t)}{s - t} \right| : s, t \in [0, \pi], s \neq t \right\},$$

$$(3.2b)\qquad l_2[\mathbf{r}]\equiv\inf\left\{\frac{r(s)}{\sin s}: s\in(0,\pi)\right\}.$$

For $\delta>0$, we set

$$(3.3a)\qquad \mathscr{R}(\delta)\equiv\{\mathbf{r}\in C^1:(2.20a),(2.21),(2.22c)\text{ hold}, l_1[\mathbf{r}]>\delta, l_2[\mathbf{r}]>\delta\},$$

$$(3.3b)\qquad \mathscr{R}\equiv\bigcup_{\delta>0}\mathscr{R}(\delta).$$

(3.4) LEMMA. – *$\mathscr{R}$ is the set of those functions $\mathbf{r}$ in the linear space*

$$(3.5)\qquad \mathscr{M}\equiv\{\mathbf{r}\in C^1:(2.20a),(2.21)\text{ hold}\}$$

that satisfy (2.22) *and have stretches* $|\mathbf{r}'|$ *that are everywhere positive. Moreover,* $\mathscr{R}(\delta)$ *(and consequently* $\mathscr{R}$*) are open in* $\mathscr{M}$.

We omit the proof of this lemma, because it is a straightforward variant of those of Lemmas 4.11 and 4.13 of [13].

We introduce the space

$$(3.6)\qquad \mathscr{X}\equiv\{f\in C^1[0,\pi]\cap C^2(0,\pi): f''(.)\sin^{1/2}(.)\in C^0[0,\pi]\}$$

equipped with the norm

$$(3.7)\qquad \|f,\mathscr{X}\|\equiv\|f,C^1\|+\max_{[0,\pi]}|f''(s)\sin^{1/2}s|.$$

(3.8) LEMMA. – *$\mathscr{X}$ is a Banach space. It is continuously embedded in* $C^{1,1/2}[0,\pi]$.

Proof. – It is easy to verify that $\|.,\mathscr{X}\|$ is a norm. We now prove that $\mathscr{X}$ is complete. Let $\{f_n\}$ be a Cauchy sequence in $\mathscr{X}$. Since it is a Cauchy sequence in C^1, there exists an $f\in C^1$ such that $\{f_n\}$ converges to f in C^1. Since (3.7) implies that $\{f_n''(.)\sin^{1/2}(.)\}$ is a Cauchy sequence in C^0, there exists a $g\in C^0$ to which it converges uniformly. Thus f' is continuously differentiable on $(0,\pi)$, $f''(.)\sin^{1/2}(.)\in C^0$, and $\{f_n\}$ converges to f in $\mathscr{X}$.

The statement about embedding is an immediate consequence of the following inequality (for $t<s$):

$$(3.8)\qquad \|f,C^{1,1/2}\|\equiv\|f,C^1\|+\sup_{s\neq t}\left\{\frac{|f'(s)-f'(t)|}{|s-t|^{1/2}}\right\}$$
$$\leqq\|f,C^1\|+\sup_{s\neq t}\left\{\frac{1}{|s-t|^{1/2}}\int_t^s|\sin^{-1/2}\xi f''(\xi)\sin^{1/2}\xi|\,d\xi\right\}$$
$$\leqq\|f,C^1\|+\|f''(.)\sin^{1/2}(.)\|_0\sup_{s\neq t}\left\{\frac{1}{|s-t|^{1/2}}\int_t^s\sin^{-1/2}\xi\,d\xi\right\}$$
$$\leqq\text{Const}\,\|f,\mathscr{X}\|.\quad\square$$

We assume that the shell is subjected to a pressure field generated by the axisymmetric, irrotational flow of an inviscid, incompressible fluid. The fluid velocity at the point $\mathbf{r}(s)$ on the shell, denoted $\mathbf{u}[\mathbf{r},U](s)$, will be shown to depend on the shape $\mathbf{r}$ of the shell and the constant velocity U (in the $\mathbf{k}$-direction) at infinity and to be independent of the pressure P at infinity. Let the fluid have constant unit density. Bernoulli's theorem implies that the pressure on the shell at $\mathbf{r}(s)$ is

$$p[\mathbf{r}, \mathrm{U}, \mathrm{P}](s) \equiv \mathrm{P} + \frac{\mathrm{U}^2}{2}\{1 - |\mathrm{U}^{-1}\mathbf{u}[\mathbf{r}, \mathrm{U}](s)|^2\}. \tag{3.9}$$

Our basic theorem about the flow, proved in Sec. 6, is

(3.10) THEOREM. – *The scaled speed* $|\mathrm{U}^{-1}\mathbf{u}[\mathbf{r}, \mathrm{U}]|$ *is independent of* U (*and is thus determined solely by* $\mathbf{r}$.) *The operator*

$$\mathscr{X} \cap \mathscr{R} \ni \mathbf{r} \mapsto |\mathrm{U}^{-1}\mathbf{u}[\mathbf{r}, \mathrm{U}]| \in \mathrm{C}^0$$

is continuous and its restriction to $\mathscr{X} \cap \mathscr{R}(\delta)$ *is compact for every* $\delta > 0$. *The operator* $p[., ., .]: \mathscr{X} \cap \mathscr{R} \times \mathbb{R}^2 \to \mathrm{C}^0$ *is continuous and its restriction to* $\mathscr{X} \cap \mathscr{R}(\delta) \times \mathbb{R}^2$ *is compact for every* $\delta > 0$.

Our boundary value problem BVP is to find $\mathbf{r} \in \mathscr{X}$ and $\theta \in \mathscr{X}$ satisfying (2.1), (2.3), (2.4), (2.6) (with $\varphi = 0$), (2.9)-(2.12), (2.14), (2.20)-(2.22). We now follow [19] in transforming BVP to a fixed-point form involving compact operators to which we can apply a global implicit function theorem.

Let L denote the Legendre differential operator defined by

$$(\mathrm{L}u)(s) \equiv \frac{d}{ds}[u'(s)\sin s] - \frac{u(s)}{\sin s}, \tag{3.11}$$

which is associated with the linearization of the governing equations about a spherical state and which captures the behavior of the polar singularities. We introduce new variables $\mathbf{v} \equiv (v_1, v_2, v_3)$ by

$$v_1(s)\sin^{1/2} s = (\mathrm{L}r)(s), \tag{3.12 a}$$

$$v_2(s)\sin^{1/2} s = \frac{d}{ds}[z'(s)\sin s], \tag{3.12 b}$$

$$v_3(s)\sin^{1/2} s = (\mathrm{L}\psi)(s), \qquad \psi(s) \equiv \theta(s) - s. \tag{3.12 c}$$

The following two results are obtained by a straightforward (but lengthy) computation.

(3.13) PROPOSITION. – *The linear operator* Y *defined by*

$$\left\{\begin{aligned} &(\mathrm{Y}u)(s) \equiv \int_0^\pi \mathrm{K}(s,t)\sin^{1/2} t\, u(t)\,dt, \\ &\mathrm{K}(s,t) \equiv \begin{cases} -\dfrac{1}{2}\dfrac{1+\cos s}{1+\cos t}\dfrac{\sin t}{\sin s} & \textit{if} \quad t \leqq s, \\ -\dfrac{1}{2}\dfrac{1+\cos t}{1+\cos s}\dfrac{\sin s}{\sin t} & \textit{if} \quad s \leqq t \end{cases}\end{aligned}\right. \tag{3.14}$$

is continuous from $C^0[0,\pi]$ *to* $\mathscr{X}$. *Moreover,*

(3.15) $$(\mathrm{LY}\,u)(s)=u(s)\sin^{1/2}s,\qquad (\mathrm{Y}\,u)(0)=0=(\mathrm{Y}\,u)(\pi).$$

(3.16) PROPOSITION. – *The linear operator* Z *defined by*

(3.17) $$\left\{\begin{aligned}&(\mathrm{Z}\,u)(s)\equiv\int_0^\pi\frac{\pi^{-1}y-1+\chi(y,s)}{\sin y}\int_0^y u(t)\sin^{1/2}t\,dt\,dy,\\ &\chi(y,s)\equiv\begin{cases}1 & \text{if } y\in[0,s],\\ 0 & \text{if } y\in(s,\pi]\end{cases}\end{aligned}\right.$$

is continuous from $C^0[0,\pi]\cap\left\{u:\int_0^\pi u(t)\sin^{1/2}t\,dt=0\right\}$ *to* $\mathscr{X}$. *Moreover,*

(3.18) $$\left\{\begin{aligned}&\frac{d}{ds}[(\mathrm{Z}\,u)'(s)\sin s]=u(s)\sin^{1/2}s,\\ &(\mathrm{Z}\,u)'(0)=(\mathrm{Z}\,u)'(\pi)=0\\ &\forall\,u\in C^0[0,\pi]\cap\left\{u:\int_0^\pi u(t)\sin^{1/2}t\,dt=0\right\}.\end{aligned}\right.$$

Let

(3.19) $$\mathscr{V}\equiv C^0\times\left[C^0\cap\left\{u:\int_0^\pi u(t)\sin^{1/2}t\,dt=0\right\}\right]\times C^0.$$

$\mathscr{V}$ equipped with the norm $\|.,C^0\|$ is clearly a Banach space.

We can use (3.12)-(3.18) to represent our geometric variables in terms of $\mathbf{v}$ by means of the following functions:

(3.20) $$\left\{\begin{aligned}&\hat{\mathbf{r}}[\mathbf{v}](s)\equiv(\mathrm{Y}\,v_1)(s)\,\mathbf{i}+(\mathrm{Z}\,v_2)(s)\,\mathbf{k},\\ &\hat{\theta}[\mathbf{v}](s)\equiv s+(\mathrm{Y}\,v_3)(s),\\ &\hat{\tau}[\mathbf{v}](s)\equiv(\mathrm{Y}\,v_1)(s)/\sin s,\\ &\hat{\nu}[\mathbf{v}](s)\equiv(\mathrm{Y}\,v_1)'(s)\cos(s+(\mathrm{Y}\,v_3)(s))\\ &\qquad\qquad+(\mathrm{Z}\,v_2)'(s)\sin(s+(\mathrm{Y}\,v_3)(s)),\\ &\hat{\eta}[\mathbf{v}](s)\equiv-(\mathrm{Y}\,v_1)'(s)\sin(s+(\mathrm{Y}\,v_3)(s))\\ &\qquad\qquad+(\mathrm{Z}\,v_2)'(s)\cos(s+(\mathrm{Y}\,v_3)(s)),\\ &\hat{\sigma}[\mathbf{v}](s)\equiv[\sin(s+(\mathrm{Y}\,v_3)(s))/\sin s]-1,\\ &\hat{\mu}[\mathbf{v}](s)\equiv(\mathrm{Y}\,v_3)'(s),\\ &\hat{\mathbf{q}}[\mathbf{v}](s)\equiv(\hat{\tau}[\mathbf{v}](s),\hat{\nu}[\mathbf{v}](s),\hat{\eta}[\mathbf{v}](s),\hat{\sigma}[\mathbf{v}](s),\hat{\mu}[\mathbf{v}](s)).\end{aligned}\right.$$

We now turn to the equilibrium equations. We substitute (2.14) into (2.10)-(2.12), carry out the differentiations, and use Cramer's rule [valid by virtue of (2.15)] to obtain

(3.21 *a*) $$\frac{d}{ds}(\nu\sin s)=(\mathbf{D}^{-1}\mathbf{f})_1+\nu\cos s,$$

(3.21 *b*) $$\frac{d}{ds}(\eta\sin s)=(\mathbf{D}^{-1}\mathbf{f})_2+\eta\cos s,$$

$$(3.21\,c)\qquad \frac{d}{ds}(\mu\sin s)=(\mathbf{D}^{-1}\,\mathbf{f})_3+\mu\cos s,$$

where $\mathbf{D}$ is defined in (2.15), where

$$(3.21\,d)\ \mathbf{f}\equiv\begin{pmatrix}-\mathrm{N}_\tau(\mathbf{q})\,\tau'\sin s-\mathrm{N}_\sigma(\mathbf{q})\,\sigma'\sin s-\mathrm{N}(\mathbf{q})\cos s\\ \qquad+\mathrm{T}(\mathbf{q})\cos\theta+\sin s\,\mathrm{H}(\mathbf{q})\,\theta'+p\,[\mathbf{r},\mathrm{U},\mathrm{P}]\,r\,\eta\\ -\mathrm{H}_\tau(\mathbf{q})\,\tau'\sin s-\mathrm{H}_\sigma(\mathbf{q})\,\sigma'\sin s-\mathrm{H}(\mathbf{q})\cos s\\ \qquad-\mathrm{T}(\mathbf{q})\sin\theta-\sin s\,\mathrm{N}(\mathbf{q})\,\theta'-p\,[\mathbf{r},\mathrm{U},\mathrm{P}]\,r\,\nu\\ -\mathrm{M}_\tau(\mathbf{q})\,\tau'\sin s-\mathrm{M}_\sigma(\mathbf{q})\,\sigma'\sin s-\mathrm{M}(\mathbf{q})\cos s\\ \qquad+\Sigma(\mathbf{q})\cos\theta-\sin s\,[\nu\,\mathrm{H}(\mathbf{q})-\eta\,\mathrm{N}(\mathbf{q})]\end{pmatrix}.$$

and where $(\mathbf{D}^{-1}\,\mathbf{f})_k$ is the k-th component of $(\mathbf{D}^{-1}\,\mathbf{f})$. Now (2.6) and (3.11) imply that

$$(3.22\,a)\quad \mathrm{L}\,r=\cos\theta\frac{d}{ds}(\nu\sin s)+(\nu\sin s)\frac{d}{ds}\cos\theta$$
$$-\sin\theta\frac{d}{ds}(\eta\sin s)-(\eta\sin s)\frac{d}{ds}\sin\theta-\tau,$$

$$(3.22\,b)\quad \frac{d}{ds}(z'\sin s)=\sin\theta\frac{d}{ds}(\nu\sin s)+\nu\sin s)\frac{d}{ds}\sin\theta$$
$$+\cos\theta\frac{d}{ds}(\eta\sin s)+(\eta\sin s)\frac{d}{ds}\cos\theta,$$

$$(3.22\,c)\qquad \mathrm{L}(\theta-s)=\frac{d}{ds}(\mu\sin s)-\frac{\theta-s}{\sin s}.$$

We now replace the left-hand sides of (3.22) with the left-hand sides of (3.12), we substitute (3.21) into the right-hand sides of (3.22), we substitute (3.20) into the resulting form of the right-hand sides of (3.22), and we divide the resulting equations by $\sin^{1/2}s$ to obtain the operator equation

$$(3.23)\qquad \mathbf{v}=\mathbf{g}_1[\mathbf{v}]+p\,[(\mathrm{Y}\,v_1)\,\mathbf{i}+(\mathrm{Z}\,v_2)\,\mathbf{k},\mathrm{U},\mathrm{P}]\,\mathbf{g}_2[\mathbf{v}]\equiv\mathbf{g}[\mathbf{v},\mathrm{U},\mathrm{P}].$$

Our boundary value problem is equivalent to finding $\mathbf{v}\in\mathscr{V}$ such that $\hat{\mathbf{r}}[\mathbf{v}]\in\mathscr{R}$, $\mathbf{q}[\mathbf{v}](s)\in\mathscr{Q}$ for all $s\in[0,\pi]$, and (3.23) is satisfied.

Unfortunately the operator $\mathbf{g}$ does not map $\mathscr{V}\times\mathbb{R}^2$ into $\mathscr{V}$, because of the side condition in (3.19), and is consequently not in the fixed-point form we require. We accordingly replace (3.23) with a modified problem having the same solutions.

We introduce a projection $\mathbf{J}$ of C^0 onto $\mathscr{V}$ by

$$(3.24)\qquad \mathbf{Ju}\equiv\left(u_1,u_2-\frac{\displaystyle\int_0^\pi u_2(s)\sin^{1/2}s\,ds}{\displaystyle\int_0^\pi\sin^{1/2}s\,ds},u_3\right).$$

We define the operator $\mathbf{k}: C^0 \times \mathbb{R}^2 \to C^0$ by

(3.25) $$\mathbf{k}[\mathbf{v}, U, P] \equiv \mathbf{g}[\mathbf{Jv}, U, P].$$

We seek solutions $\mathbf{v} \in C^0$ of

(3.26) $$\mathbf{v} = \mathbf{k}[\mathbf{v}, U, P].$$

(3.27) PROPOSITION. — *For given* U *and* P *let* $\mathbf{v} \in C^0$ *and satisfy* (3.26), *let* $\hat{\mathbf{q}}[\mathbf{Jv}](s) \in \mathcal{Q}$ *for all* s, *and let* $\hat{\mathbf{r}}[\mathbf{Jv}] \in \mathcal{R}$. *Then* $\int_0^\pi v_2(s) \sin^{1/2} s\, ds = 0$, *so that* $\mathbf{v} \in \mathcal{V}$, $\mathbf{Jv} = \mathbf{v}$, *and* $\mathbf{v}$ *satisfies* (3.23).

To effect the proof, we observe that if $\mathbf{v} \in C^0$, then v_2 has the form $x_2 + y_2$ where

(3.28) $$x_2 = \frac{\int_0^\pi v_2(s) \sin^{1/2} s\, ds}{\int_0^\pi \sin^{1/2} s\, ds}, \qquad \int_0^\pi y_2(s) \sin^{1/2} s\, ds = 0.$$

We need only show that $x_2 = 0$. For this purpose, we merely have to trace through the steps leading to (3.26). We omit the details, which are straightforward.

Let us now study the problem in which $U = 0$, $P \geqq 0$. We seek *trivial* solutions in which the shell remains spherical, unsheared, and uniformly compressed, so that $\tau = \nu = k$ (Const.), $\eta = 0$, $\theta(s) = s$, $r(s) = k \sin s$, $z(s) = -k \cos s$. Under these conditions, the constitutive assumptions (2.17) and (2.18) reduce (2.10)–(2.12) to

(3.29) $$N(k, k, 0, 0, 0) \equiv T(k, k, 0, 0, 0) = -Pk^2/2.$$

Condition (2.16) ensures that for each $P \geqq 0$, equation (3.29) has a unique solution for k, denoted $k(P)$, with $k(.) : [0, \infty) \to (h, 1]$ twice continuously differentiable and strictly decreasing. (*Cf.* [19].)

Systems (3.23) and (3.26) admit the corresponding solution

(3.30) $$\bar{\mathbf{v}}[P](s) = (2k(P) \sin^{3/2} s, 2k(P) \sin^{1/2} s \cos s, 0)$$

with $\bar{\mathbf{v}} \in \mathcal{V}$.

4. EXISTENCE THEORY FOR THE SHELL

We analyze (3.26) with

(4.1) GLOBAL IMPLICIT FUNCTION THEOREM. — Let $\mathcal{Y}$ be a Banach space and let $\{\mathcal{O}(\varepsilon), 0 < \varepsilon < E\}$ *be a family of open sets (not necessarily bounded) in* $\mathcal{Y} \times \mathbb{R}^m$ *for which*

(4.2) $$(0, 0) \in \mathcal{O}(\varepsilon), \qquad 0 < \varepsilon < E,$$

$$\text{(4.3)} \qquad \mathrm{cl}\,\mathcal{O}(\varepsilon_2) \subset \mathcal{O}(\varepsilon_1) \quad \textit{for} \quad 0<\varepsilon_1<\varepsilon_2<\mathrm{E},$$

Let

$$\text{(4.4)} \qquad \mathcal{O} \equiv \bigcup_{0<\varepsilon<\mathrm{E}} \mathcal{O}(\varepsilon).$$

Let $\mathrm{F}:\mathcal{O}\to\mathcal{Y}$ *be continuous, let* $\mathrm{F}(0,0)=0$, *and let* $\mathrm{F}:\mathcal{O}(\varepsilon)\to\mathcal{Y}$ *be compact for* $0<\varepsilon<\mathrm{E}$, *where* E *is a given positive number. Let* I *denote the identity operator on* $\mathcal{Y}$. *Let the Fréchet derivative* $\mathrm{I}-\mathrm{F}_x(0,0):\mathcal{Y}\to\mathcal{Y}$ *of* $x\mapsto x-\mathrm{F}(x,\lambda)$ *at* $(0,0)$ *exist and be invertible. Let*

$$\mathcal{S} \equiv \{(x,\lambda)\in\mathcal{O} : x=\mathrm{F}(x,\lambda)\}$$

and let $\mathcal{S}_0$ *be the connected component of* $\mathcal{S}$ *containing* $(0,0)$. *Then one of the following statements is true:*

(i) $\mathcal{S}_0$ *is bounded and there is an* $\varepsilon^*\in(0,\mathrm{E})$ *such that* $\mathcal{S}_0\subset\mathcal{O}(\varepsilon^*)$. *There is an essential map* (i.e., *a continuous map not homotopic to a constant*) σ *from* $\mathcal{S}_0$ *onto the m-dimensional sphere* $\mathbb{S}^m$ *whose restriction to* $\mathcal{S}_0\setminus\{(0,0)\}$ *is inessential. Moreover,* $\mathcal{S}_0$ *contains a connected subset* $\mathcal{S}_{00}$ *that contains* $(0,0)$, *that has the same properties as* $\mathcal{S}_0$ *with respect to* σ, *and that has the property that each point of it has Lebesgue dimension at least m.*

(ii) $\mathcal{S}_0\setminus\mathcal{O}(\varepsilon)\neq\varnothing$ $\forall\varepsilon\in(0,\mathrm{E})$ *or* $\mathcal{S}_0$ *is unbounded. For each* $\varepsilon\in(0,\mathrm{E})$ *there is a modified equation* $x=\varphi(x,\lambda,\varepsilon)\,\mathrm{F}(x,\lambda)$ [*cf.* (7.4)] *defined on all of* $\mathcal{Y}\times\mathbb{R}^m$ *that agrees with* $x=\mathrm{F}(x,\lambda)$ *on* $\mathcal{O}(\varepsilon)$. *The one-point compactification* $\mathcal{S}_0^+(\varepsilon)$ *of the connected component* $\mathcal{S}_0(\varepsilon)$ *containing* $(0,0)$ *of the set of solution pairs of the modified equation may be unbounded, but otherwise has the same properties as* $\mathcal{S}_0$ *in statement* (i).

This theorem, a variant of that of [3], is proved in [13]. The statement about topological dimension follows from the treatment of [2].

Let

$$\text{(4.5)} \qquad \hat{\varepsilon}[\mathbf{v}] \equiv \min\begin{cases} l_1[\hat{\mathbf{r}}[\mathbf{Jv}]], \\ l_2[\hat{\mathbf{r}}[\mathbf{Jv}]], \\ \min_s\{\hat{\nu}[\mathbf{Jv}](s)-\max\{0,[\hat{\mu}[\mathbf{Jv}](s)+1]\,h\}\}, \\ \min_s\{\hat{\tau}[\mathbf{Jv}](s)-\max\{0,[\hat{\sigma}[\mathbf{Jv}](s)+1]\,h\}\}, \end{cases}$$

$$\text{(4.6)} \qquad \mathrm{E}\equiv\hat{\varepsilon}[\bar{\mathbf{v}}[0]]=\hat{\varepsilon}[(2\sin^{3/2}(.),2\sin^{1/2}(.)\cos(.),0)].$$

For each $\varepsilon\in[0,\mathrm{E})$ we define

$$\text{(4.7)} \qquad \begin{cases} \mathcal{O}(\varepsilon)\equiv\{(\mathbf{v},\mathrm{U},\mathrm{P})\in\mathrm{C}^0\times\mathbb{R}^2:\hat{\varepsilon}[\mathbf{v}]>\varepsilon,\ \hat{z}[\mathbf{Jv}](\pi)>\hat{z}[\mathbf{Jv}](0)\}, \\ \mathcal{O}(0)=\bigcup_{0<\varepsilon<\mathrm{E}}\mathcal{O}(\varepsilon). \end{cases}$$

It can easily be verified that $\mathcal{O}(\varepsilon)$ satisfies (4.2)−(4.4) provided we identify the point $(\bar{\mathbf{v}}[\mathrm{P}],(0,\mathrm{P}))$ associated with (4.7) with the point $(0,0)$ arising in Theorem 4.1.

(4.8) THEOREM. – *The operator* $\mathbf{k}$, *defined in* (3.25), *is continuous on* $\mathcal{O}(0)$ *and is continuously differentiable with respect to* $(\mathbf{v},\mathrm{P})$ *when* $\mathrm{U}=0$.

The restriction of $\mathbf{k}$ *to* $\mathcal{O}(\varepsilon)$ *is compact for every* $\varepsilon \in (0, \mathrm{E})$. *The Fréchet derivative* $\mathbf{I}-(\partial \mathbf{k}/\partial \mathbf{v})\,[\bar{\mathbf{v}}[\mathrm{P}], 0, \mathrm{P}]$ *of* $\mathbf{I}-\mathbf{k}$ *with respect to* $\mathbf{v}$ *at* $(\bar{\mathbf{v}}[\mathrm{P}], 0, \mathrm{P})$ *is an invertible linear mapping of* C^0 *onto itself for all* $\mathrm{P} \geqq 0$ *except for the set* $\mathscr{E}$ *of eigenvalues of the linearization of* (3.26) *about* (3.30). *Moreover,* (3.26) *is equivalent to BVP. (Specifically, if* $\mathbf{v} \in \mathrm{C}^0$ *and satisfies* (3.26), *if* $\hat{\mathbf{q}}[\mathbf{J}\mathbf{v}](s) \in \mathscr{Q}$ *for all s, and if* $\hat{\mathbf{r}}[\mathbf{J}\mathbf{v}] \in \mathscr{R}$, *then* $\mathbf{J}\mathbf{v}=\mathbf{v}$ *and* $(\hat{\mathbf{r}}[\mathbf{v}], \hat{\theta}[\mathbf{v}])$ *satisfies BVP, and conversely.)*

Proof. — Under a slightly specialized version of our constitutive hypotheses Shih and Antman [19] proved that the operators $\mathbf{g}_1$ and $\mathbf{g}_2$ of (3.23) are continuously differentiable on $\{\mathbf{v}:(\mathbf{v}, \mathrm{U}, \mathrm{P}) \in \mathcal{O}(0)\}$ and compact on $\{\mathbf{v}:(\mathbf{v}, \mathrm{U}, \mathrm{P}) \in \mathcal{O}(\varepsilon)\}$ for $\varepsilon>0$, that the Fréchet derivative $\mathbf{I}-(\partial \mathbf{g}/\partial \mathbf{v})\,[\bar{\mathbf{v}}[\mathrm{P}], 0, \mathrm{P}]$ of $\mathbf{I}-\mathbf{g}$ with respect to $\mathbf{v}$ at $(\bar{\mathbf{v}}[\mathrm{P}], 0, \mathrm{P})$ has a trivial null space in $\mathscr{V}$ for all $\mathrm{P} \geqq 0$ except for values in $\mathscr{E}$, and that (3.26) is equivalent to BVP. The proof of [19] carries over to our more general case. Since the linear operators Y, Z, and $\mathbf{J}$ are continuous, the differentiability and compactness of $\mathbf{k}$ follows from that of $\mathbf{g}$. Thus we need only prove the statement about $\mathbf{I}-(\partial \mathbf{k}/\partial \mathbf{v})\,[\bar{\mathbf{v}}[\mathrm{P}], 0, \mathrm{P}]$. The corresponding statement about $\mathbf{g}$, proved in [19], is equivalent to

$$\left\{\begin{array}{lll}\mathbf{v}=\mathbf{0} & \text{if} & \mathbf{I}-(\partial \mathbf{g}/\partial \mathbf{v})\,[\bar{\mathbf{v}}[\mathrm{P}], 0, \mathrm{P}]\,\mathbf{v}=\mathbf{0}, \\ & \mathbf{v} \in \mathscr{V}, & \mathrm{P} \in [0, \infty) \setminus \mathscr{E}.\end{array}\right. \tag{4.9}$$

We must show that

$$\left\{\begin{array}{lll}\mathbf{v}=\mathbf{0} & \text{if} & \mathbf{I}-(\partial \mathbf{k}/\partial \mathbf{v})\,[\bar{\mathbf{v}}[\mathrm{P}], 0, \mathrm{P}]\,\mathbf{v}=0, \\ & \mathbf{v} \in \mathrm{C}^0, & \mathrm{P} \in [0, \infty) \setminus \mathscr{E}.\end{array}\right. \tag{4.10}$$

We reduce (4.10) to (4.9) by the simple device of showing that the hypotheses of (4.10) imply that $\mathbf{v} \in \mathscr{V}$, *i.e.*, that $\int_0^\pi v_2(s) \sin^{1/2} s\, ds=0$. We omit the details, which form a straightforward analog of those of the proof of Proposition 3.27. □

We define

$$\mathscr{P} \equiv \{\mathrm{P} \in \mathbb{R}: (3.25) \text{ has a solution } \mathbf{v}^*[\mathrm{P}] \text{ for } \mathrm{U}=0, \ (\mathbf{v}^*[\mathrm{P}], 0, \mathrm{P}) \in \mathcal{O}(0), \mathbf{I}-(\partial \mathbf{k}/\partial \mathbf{v})\,[\bar{\mathbf{v}}[\mathrm{P}], 0, \mathrm{P}] \text{ is invertible}\}. \tag{4.11}$$

Now under very mild additional constitutive restrictions it can be shown that $0 \notin \mathscr{E}$. A specific set of conditions are given by [19], Eq. (4.21). More general conditions can be constructed from the development of [5], Sec. VIII.5. Theorem 4.8 would then ensure that $\mathscr{P}$ contains a nonempty interval containing 0. We assume that this is the case. (Actually, all we require is that $\mathscr{P}$ not be empty.) Let

$$\mathscr{S} \equiv \{(\mathbf{v}, \mathrm{U}, \mathrm{P}) \in \mathcal{O}(0): (3.26) \text{ holds}\}. \tag{4.12}$$

Let $P_0 \in \mathscr{P}$ and let $\mathbf{v}_0 = \mathbf{v}^*[P_0]$. Let $\mathscr{S}_0$ to be the connected component of $\mathscr{S}$ containing $(\mathbf{v}_0, 0, P_0)$. Theorems 3.10, 4.1, and 4.8 together with the fact that $\mathbf{v} \mapsto \hat{\mathbf{r}}[\mathbf{J}\mathbf{v}]$ is continuous from C^0 to $\mathscr{X}$ and maps bounded sequences into bounded sequences immediately imply

(4.13) THEOREM. – *At least one of the following statements holds*:

$$(4.14a) \qquad \mathscr{S}_0 \cap [\mathscr{O} \setminus \mathscr{O}(\varepsilon)] \neq \varnothing \qquad \forall \varepsilon \in (0, E),$$

$$(4.14b) \qquad \exists \varepsilon_1 \in (0, E) \text{ such that } \mathscr{S}_0 \subset \mathscr{O}(\varepsilon_1),\ \mathscr{S}_0 \text{ is bounded,}$$

$$(4.14c) \qquad \mathscr{S}_0 \text{ is unbounded.}$$

If (4.14*b*) *holds, then there is an essential mapping* σ *from* $\mathscr{S}_0$ *to* $\mathbb{S}^2$ *whose restriction to* $\mathscr{S}_0 \setminus \{(\mathbf{v}_0, 0, P_0)\}$ *is inessential. Moreover,* $\mathscr{S}_0$ *contains a subset* $\mathscr{S}_{00}$ *each point of which has topological dimension at least* 2. *The restriction of* σ *to* $\mathscr{S}_{00}$ *is essential.*

As in [13] we obtain

4.15. COROLLARY. – *Let* $P_0 \in \mathscr{P}$. *Then there is a number* $U_1 > 0$, *depending on* P_0, *such that the set*

$$(4.16) \qquad \mathscr{S}_*(P_0) \equiv \bigcup_{|U| \leqq U_1} \{(\mathbf{v}, U, P_0) \in \mathscr{S}\}$$

contains a connected subset joining the planes $U = \pm U_1$. (*Thus there is a solution for each* U *with* $|U| \leqq U_1$.)

It is important to note that Proposition 3.27 implies that $\mathscr{S} \subset \mathscr{V}$ and that Propositions 3.13, 3.16, and the Open Mapping Theorem imply that the change of variables $(r, z, \psi) \mapsto \mathbf{v}$ [*cf.* (3.12)] is a linear homeomorphism of the Banach space $\{(r, z, \psi) \in \mathscr{X} : \psi(0) = 0 = \psi(\pi),\ (2.20a).\ (2.21) \text{ hold}\}$ onto $\mathscr{V}$ with inverse $\mathbf{v} \mapsto (Y v_1, Z v_2, Y v_3)$. Hence Theorem 4.13 and Corollary 4.15 can be stated for the boundary value problem in terms of the original variables.

For P held fixed at a positive $P_0 \in \mathscr{P}$, the continuity of $p[.,.,.]$, defined in (3.9), ensures that the pressure on the shell is everywhere positive if U is small enough. If the pressure becomes negative on part of the shell, then cavitation occurs. (Our analysis does not account for cavitation. *Cf.* [12].)

5. THE EXTERIOR FLOW PROBLEM

We study the steady, irrotational, axisymmetric flow of an incompressible, inviscid fluid of constant density ρ in the simply-connected domain $\mathscr{F}[\mathbf{r}]$ exterior to the shell. We denote a typical point in $\mathrm{cl}\,\mathscr{F}[\mathbf{r}]$ by $\mathbf{x}$, which we identify with the triple (x_1, x_2, x_3) of its coordinates with respect to the basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. In consonance with Section 3 we assume that $\partial\mathscr{F}$ is defined by

$$(5.1) \qquad \mathbf{r} \in \mathscr{R} \cap C^{1,\lambda}$$

where $\lambda \in (0, 1/2]$. Then $\partial \mathscr{F}$ is of class $C^{1,\lambda}$. The fluid is required to have prescribed pressure P and velocity $U\mathbf{k}$ at infinity. It is well known (*cf.* [18]) that the governing equations are equivalent to the following boundary value problem for the modified velocity potential Φ:

$$(5.2a) \qquad \Phi \in C^2(\mathscr{F}) \cap C^1(\mathrm{cl}\,\mathscr{F}),$$
$$(5.2b) \qquad \Delta\Phi(\mathbf{x}) = 0 \quad \text{for} \quad \mathbf{x} \in \mathscr{F},$$
$$(5.2c) \qquad \nabla\Phi(\mathbf{x}) . \boldsymbol{\nu}(\mathbf{x}) = -U\mathbf{k} . \boldsymbol{\nu} \quad \text{for} \quad \mathbf{x} \in \partial\mathscr{F},$$
$$(5.2d) \qquad \lim_{\mathbf{x} \to \infty} \Phi(\mathbf{x}) = 0$$

where $\boldsymbol{\nu}(\mathbf{x})$ is the inner unit normal to $\partial\mathscr{F}$ at $\mathbf{x}$. Problem (5.2) has a unique solution (*cf.* [20], Lectures 16, 19), which we denote by $\Phi[\mathbf{r}, U]$. The velocity $\mathbf{u}(\mathbf{x})$ and the pressure $p(\mathbf{x})$ of the fluid at $\mathbf{x} \in \mathrm{cl}\,\mathscr{F}$ are given by

$$(5.3a, b) \qquad \mathbf{u}(\mathbf{x}) = U\mathbf{k} + \nabla\Phi(\mathbf{x}), \qquad p(\mathbf{x}) = P + \frac{1}{2}U^2\{1 - U^{-2}|\mathbf{u}(\mathbf{x})|^2\}.$$

[Eq. (5.3 *b*) is Bernoulli's formula.]

The uniqueness of $\Phi[\mathbf{r}, U]$ yields

(5.4) LEMMA. – *Let* (5.1) *hold. Then the solution* $\Phi[\mathbf{r}, U]$ *of* (5.2) *is linear in* U *and is axisymmetric; in particular,*

$$(5.4) \qquad \Phi[\mathbf{r}, U] = U\,\Phi[\mathbf{r}, 1],$$
$$(5.5) \qquad \Phi[\mathbf{r}, U](x_1, x_2, x_3) = \Phi[\mathbf{r}, U](\sqrt{x_1^2 + x_2^2}, 0, x_3).$$

We define

$$(5.6) \qquad \mathscr{E}[\mathbf{r}] = \{(x_1, x_3) : (x_1, 0, x_3) \in \mathscr{F}[\mathbf{r}]\}.$$

6. COMPACTNESS

In this section we prove Theorem 3.10 by using Schauder estimates and conformal mappings. We first extend $\mathbf{r}$ of (3.1) to the interval $[-\pi, \pi]$ by setting $\mathbf{r}(-s) = (-r(s), z(s))$ for $s \in [0, \pi]$. We use the same notation for this extension. The extension operator is continuous from $\mathscr{X}$, defined in (3.6), to $W^{2,p}(-\pi, \pi)$ for $p \in (1, 2)$. We define $l[\mathbf{r}]$ by (3.2*a*) with $|s-t|$ replaced by $\sigma(s, t)$ where

$$(6.1) \qquad \sigma(s_1, s_2) \equiv \min\{|t_1 - t_2| : e^{it_\alpha} = e^{is_\alpha}, \alpha = 1, 2\}$$

and with $[0, \pi]$ replaced by $[-\pi, \pi]$. In place of $\mathscr{R}(\delta)$ and $\mathscr{R}$ we define for these extensions the sets

$$(6.2)\quad \mathscr{X}(\delta) \equiv \{\mathbf{r} \in C^1[-\pi, \pi] : (r(-s), z(-s)) = (-r(s), z(s)), \\ r(\pi) = 0, z'(\pi) = 0, r'(0) > 0, z(\pi) > z(0), l[\mathbf{r}] > \delta\},$$

$$(6.3)\quad \mathscr{X} \equiv \bigcup_{\delta > 0} \mathscr{X}(\delta).$$

A slight modification of Lemma 4.11 of [13] shows that $\mathscr{X}$ is precisely the set of all simple, planar, closed, positively oriented, continuously differentiable $\mathbf{r}$'s symmetric about the $\mathbf{i}$-axis with $|\mathbf{r}'| > 0$ everywhere and with $z(\pi) > z(0)$. (*Cf.* Lemma 3.4. Hence $\mathscr{X}$ is the set of positively oriented, symmetric extensions of the elements of $\mathscr{R}$.) Note that

$$(6.4)\quad l[\mathbf{r}] \leqq \min_s |\mathbf{r}'(s)|.$$

For $0 < h_1 < h_2$ we define

$$(6.5)\quad \mathscr{X}(h_1, h_2, \delta) \equiv \{\mathbf{r} \in \mathscr{X}(\delta) : h_1 < |\mathbf{r}| < h_2\}.$$

For $0 \leqq R_1 < R_2 \leqq \infty$ we define the open annular regions

$$(6.6a)\quad \mathscr{A}_n(R_1, R_2) \equiv \{\xi \in \mathbb{R}^n : R_1 < |\xi| < R_2\}.$$

For $0 < R \leqq \infty$ and $\mathbf{a} \in R^n$, we define the open ball

$$(6.6b)\quad \mathscr{D}_n(\mathbf{a}, R) \equiv \{\xi \in \mathbb{R}^n : |\xi - \mathbf{a}| < R\}.$$

We now regard $\mathbf{r}$ as a curve and $\mathscr{E}[\mathbf{r}]$ as a set in the complex (x_1, x_3)-plane. ($\mathscr{E}$ is defined in (5.6).) By the Riemann Mapping Theorem, for each such curve $\mathbf{r}$ there exists a unique biholomorphic mapping

$$(6.7)\quad \mathscr{A}_2(1, \infty) \ni (\xi_1, \xi_3) \mapsto \mathbf{f}[\mathbf{r}](\xi_1, \xi_3) \equiv (\rho[\mathbf{r}](\xi_1, \xi_3), \zeta[\mathbf{r}](\xi_1, \xi_3)) \in \mathscr{E}[\mathbf{r}]$$

such that

$$(6.8)\quad \mathbf{f}[\mathbf{r}](\infty) = \infty,$$

$$(6.9)\quad \begin{cases} \mathbf{f}[\mathbf{r}]'(\infty) \equiv \lim\limits_{(\xi_1, \xi_3) \to \infty} \mathbf{f}[\mathbf{r}]'(\xi_1, \xi_3) \\ \text{exists and is a positive real number.} \end{cases}$$

A theorem of Carathéodory (*cf.* [17], Thm. 9.10) states that $\mathbf{f}[\mathbf{r}]$ can be extended to $\mathrm{cl}\,\mathscr{A}_2(1, \infty)$ and that the extension is a homeomorphism onto $\mathrm{cl}\,\mathscr{E}[\mathbf{r}]$. A theorem of Warschawski (*cf.* [17], Thm. 10.2) states that $\mathbf{f}[\mathbf{r}]'$ can also be extended to $\mathrm{cl}\,\mathscr{A}_2(1, \infty)$ and that the extension vanishes nowhere. The uniqueness of $\mathbf{f}[\mathbf{r}]$, ensured by the Riemann Mapping Theorem, and the symmetry of $\partial\mathscr{E}[\mathbf{r}]$ about the imaginary x_3-axis imply that

$$(6.10)\quad (\rho(\xi_1, \xi_3), \zeta(\xi_1, \xi_3)) = (-\rho(-\xi_1, \xi_3), \zeta(-\xi_1, \xi_3)).$$

Since $\mathbf{f}[\mathbf{r}]$ preserves orientation and since (6.9) holds, we observe that

$$(6.11)\quad \rho(\xi_1,\xi_3)>0 \quad \text{if} \quad \xi_1>0,\quad \mathbf{r}\in\mathscr{Z}(h_1,h_2),\quad (\xi_1,\xi_3)\in\mathscr{A}_2(1,\infty).$$

We can now describe our basic strategy for proving Theorem 3.10. We use the mapping $\mathbf{f}[\mathbf{r}]$ to convert our original flow problem on a region depending on the unknown $\mathbf{r}$ to a new problem on a fixed domain, the exterior of a ball, with coefficients depending on $\mathbf{r}$. As our first step, we use the chain rule to prove

(6.12) LEMMA. – *Let* $\alpha\in(0,1]$ *and* $0<h_1<h_2<\infty$. *Let*

$$\mathbf{r}\in\mathscr{Z}(h_1,h_2)\cap C^{1,\alpha} \quad \textit{and} \quad U\in\mathbb{R}.$$

Let $\Phi[\mathbf{r},U]\in C^2(\mathscr{F}[\mathbf{r}])\cap C^1(\operatorname{cl}\mathscr{F}[\mathbf{r}])$ *satisfy* (5.5). *Then* $\Phi[\mathbf{r},U]$ *is a solution of boundary value problem* (5.2) *if and only if the function* $\omega[\mathbf{r},U]\in C^2(\mathscr{A}_3(1,\infty))\cap C^1(\operatorname{cl}\mathscr{A}_3(1,\infty))$ *defined by*

$$(6.13)\quad \omega[\mathbf{r},U](\xi_1,\xi_2,\xi_3) \equiv \Phi[\mathbf{r},U]\left(\rho[\mathbf{r}]\left(\sqrt{\xi_1^2+\xi_2^2},\xi_3\right),0,\zeta[\mathbf{r}]\left(\sqrt{\xi_1^2+\xi_2^2},\xi_3\right)\right)$$

satisfies the following boundary value problem:

$$(6.14a)\quad \Delta\omega[\mathbf{r},U]+\sum_{j=1}^{3}a_j[\mathbf{r}]\frac{\partial}{\partial\xi_j}\omega[\mathbf{r},U]=0 \quad \textit{for} \quad \xi\in\mathscr{A}_3(1,\infty),$$

$$(6.14b)\quad \xi\cdot\nabla\omega[\mathbf{r},U](\xi)=b[\mathbf{r},U](\xi) \quad \textit{for} \quad |\xi|=1,$$

$$(6.14c)\quad \omega[\mathbf{r},U](\xi)\to 0 \quad \textit{as} \quad |\xi|\to\infty$$

where

$$(6.15)\quad \begin{cases} a_\beta(\xi)\equiv \dfrac{\xi_\beta}{\sqrt{\xi_1^2+\xi_2^2}}\left[\dfrac{\partial\rho\left(\sqrt{\xi_1^2+\xi_2^2},\xi_3\right)/\partial\xi_1}{\rho\left(\sqrt{\xi_1^2+\xi_2^2},\xi_3\right)}-\dfrac{1}{\sqrt{\xi_1^2+\xi_2^2}}\right], \\ \qquad \beta=1,2, \\ a_3(\xi)\equiv\dfrac{1}{\rho\left(\sqrt{\xi_1^2+\xi_2^2},\xi_3\right)}\dfrac{\partial\rho}{\partial\xi_3}\left(\sqrt{\xi_1^2+\xi_2^2},\xi_3\right), \\ b(\xi)\equiv -U\left|\mathbf{f}'\left(\sqrt{\xi_1^2+\xi_2^2},\xi_3\right)\right| \\ \qquad \times\nu_3\left(\rho\left(\sqrt{\xi_1^2+\xi_2^2},\xi_3\right),0,\zeta\left(\sqrt{\xi_1^2+\xi_2^2},\xi_3\right)\right)\end{cases}$$

and where ν_3 *is the component of the unit normal* $\boldsymbol{\nu}$ *defined in Section* 5. [*The arguments* $\mathbf{r}$ *and* U *have been omitted in* (6.15).]

Our main effort in this section is to use standard interior and boundary estimates to prove

(6.16) LEMMA. – *Let* $3<q<2p<4$, $R>1$, $0<h_1<h_2$, $\delta>0$. *Then the operator* $(\mathbf{r},U)\mapsto\omega[\mathbf{r},U]$ *is continuous from*

$$[W^{2,p}(-\pi,\pi)\cap\mathscr{Z}(h_1,h_2,\delta)]\times\mathbb{R} \quad \textit{to} \quad W^{2,q}(\mathscr{A}_3(1,R))$$

and maps bounded sequences into bounded sequences.

We exploit (5.3 *a*) and (6.13) to obtain

(6.17) LEMMA. – *Let* $3<2p<4, \mathrm{R}>1, 0<h_1<h_2, \delta>0$. *Then the operator* $(\mathbf{r}, \mathrm{U}) \mapsto |\mathbf{u}[\mathbf{r}, \mathrm{U}]|$ *is continuous and compact from*

$$[\mathrm{W}^{2,p}(-\pi,\pi) \cap \mathscr{X}(h_1,h_2,\delta)] \times \mathbb{R} \quad \text{to} \quad \mathrm{C}^0[0,\pi].$$

Theorem 3.10 follows easily from this last result and Bernoulli's Theorem.

We begin this program by obtaining a technical result on conformal mappings.

For $\mathbf{r} \in \mathscr{X}$ let

$$\lambda[\mathbf{r}](s) \equiv \int_{-\pi}^{s} |\mathbf{r}'(\xi)| \, d\xi, \qquad s \in [-\pi,\pi] \tag{6.18}$$

be the arc-length parametrization of the curve $\mathbf{r}$. For $\mathbf{r} \in \mathscr{X}(h_1,h_2)$ let

$$t^*[\mathbf{r}](t) \equiv \int_{\pi/2}^{t} |\mathbf{f}[\mathbf{r}](e^{i\eta})| \, d\eta, \qquad t \in [\pi/2, 5\pi/2] \tag{6.19}$$

be the arc-length parametrization of the curve $t \mapsto \mathbf{f}[\mathbf{r}](e^{it})$. Since $\mathbf{f}$ preserves orientation and satisfies (6.9) and (6.10), it must satisfy

$$\mathbf{f}[\mathbf{r}](e^{-i\pi/2}) = \mathbf{r}(0) \qquad \text{for} \quad \mathbf{r} \in \mathscr{X}(h_1,h_2). \tag{6.20}$$

Our definitions imply that

$$\mathbf{r}(\lambda[\mathbf{r}]^{-1}(t^*[\mathbf{r}](t))) = \mathbf{f}[\mathbf{r}](e^{it}), \qquad t \in [\pi/2, 5\pi/2] \tag{6.21}$$

for $\mathbf{r} \in \mathscr{X}(h_1,h_2)$.

(6.22) PROPOSITION. – *Let* $3<2p<4$, $0<\beta<\alpha<1$, $0<h_1<h_2$, $1 \leqq \mathrm{R}_1 < \mathrm{R}_2$, $\delta>0$. *The operator* $\mathbf{r} \mapsto \mathbf{f}[\mathbf{r}]$ *is continuous from* $\mathrm{W}^{2,p}(-\pi,\pi) \cap \mathscr{X}(h_1,h_2,\delta)$ *to* $\mathrm{W}^{2,2p}(\mathscr{A}_2(\mathrm{R}_1,\mathrm{R}_2))$ *and maps bounded sequences into bounded sequences.*

Let $\mathscr{G}$ *be a bounded open subset of* $\mathscr{A}_2(1,\infty)$. *Then* $\mathbf{r} \mapsto \mathbf{f}[\mathbf{r}]$ *is continuous from* $\mathrm{C}^{1,\alpha}[-\pi,\pi] \cap \mathscr{X}(h_1,h_2,\delta)$ *to* $\mathrm{C}^{1,\beta}(\mathrm{cl}\,\mathscr{G})$ *and maps bounded sequences into bounded sequences.*

If $\mathrm{cl}\,\mathscr{G} \subset \mathscr{A}_2(1,\infty)$, *then* $\mathbf{r} \mapsto \mathbf{f}[\mathbf{r}]$ *is continuous from*

$$\mathrm{C}^{1,\alpha}[-\pi,\pi] \cap \mathscr{X}(h_1,h_2,\delta) \quad \text{to} \quad \mathrm{C}^k(\mathrm{cl}\,\mathscr{G})$$

and maps bounded sequences into bounded sequences for all $k \in \mathbb{N}$ *and* $\delta>0$.

If $\{\mathbf{r}_n\}$ *is a bounded sequence in* $\mathrm{C}^{1,\alpha}[-\pi,\pi] \cap \mathscr{X}(h_1,h_2,\delta)$, *then there exists a constant* $c>0$ *such that*

$$\frac{1}{c} < |\mathbf{f}[\mathbf{r}_n]'(\xi_1,\xi_3)| < c, \qquad (\xi_1,\xi_3) \in \mathscr{A}_2(1,\infty). \tag{6.23}$$

The first statement of this proposition is proved by [11]. The remaining statements are derived from the work of [22] by [13].

It is convenient to define an operator S that converts functions of the cylindrical coordinates $\sqrt{\xi_1^2+\xi_2^2}, \xi_3$ to axisymmetric functions of Cartesian coordinates ξ_1, ξ_2, ξ_3 by

$$S[w](\xi_1,\xi_2,\xi_3)=w(\sqrt{\xi_1^2+\xi_2^2},\xi_3). \tag{6.24}$$

A straightforward computation yields

(6.25) LEMMA. — *Let $3<q$ and let $1\leqq R_1<R_2$. For each $k\in\mathbb{N}$, let $u_n\to u$ and let $\{v_n\}$ be bounded in $\{w\in C^k(\mathrm{cl}\,\mathscr{A}_2(R_1,R_2)) : w(-\xi,\xi_3)=w(\xi,\xi_3)\}$. Then $S[u_n]\to S[u]$ and $\{S[v_n]\}$ is bounded in $C^k(\mathrm{cl}\,\mathscr{A}_3(R_1,R_2))$ for each k. S is linear and continuous from from*

$$\{w\in W^{1,q}(D_2(0,1)) : w(-\xi,\xi_3)=w(\xi,\xi_3)\}$$

to $W^{1,q}(D_3(0,1))$ and from

$$\{w\in W^{1-1/q,q}(\partial D_2(0,1)) : w(-\xi,\xi_3)=w(\xi,\xi_3)\} \text{ to } W^{1-1/q,q}(\partial D_3(0,1)).$$

Our next lemma gives important technical properties of the coefficients and data of the modified problem (6.14).

(6.26) LEMMA. — *Let $3<q<2p<4$, $0<\alpha<1$, $\delta>0$, $0<h_1<h_2$, $1<R_1<R_2$, $k\in\mathbb{N}$. Then*

(i) *The mappings $\mathbf{r}\mapsto a_i[\mathbf{r}]$, defined in (6.15), are continuous from $W^{2,p}(-\pi,\pi)\cap\mathscr{Z}(h_1,h_2,\delta)$ and from $C^{1,\alpha}[-\pi,\pi]\cap\mathscr{Z}(h_1,h_2,\delta)$ to $C^k(\mathscr{A}_3(R_1,R_2))$ and map bounded sequences into bounded sequences.*

(ii) *The mappings $\mathbf{r}\mapsto a_i[\mathbf{r}]$ are continuous from*

$$W^{2,p}(-\pi,\pi)\cap\mathscr{Z}(h_1,h_2,\delta) \quad to \quad L^q(\mathscr{A}_3(1,R_1))$$

and map bounded sequences into bounded sequences

(iii) *Let $\mathscr{G}$ be a bounded open subset of $\mathscr{A}_3(1,\infty)$ with $(0,0,\pm 1)\notin\mathrm{cl}\,\mathscr{G}$. Then*

$$\left\{\begin{array}{l} \qquad\qquad a_i[\mathbf{r}]\in C^{0,\gamma}(\mathrm{cl}\,\mathscr{G}) \\ \textit{for} \\ \quad \gamma\in\left\{\begin{array}{ll}\left(0,\dfrac{p-1}{p}\right) & \textit{if}\ \ \mathbf{r}\in W^{2,p}(-\pi,\pi)\cap\mathscr{Z}(h_1,h_2), \\ (0,\alpha] & \textit{if}\ \ \mathbf{r}\in C^{1,\alpha}[-\pi,\pi]\cap\mathscr{Z}(h_1,h_2).\end{array}\right.\end{array}\right. \tag{6.27}$$

(iv) *the mapping $b[.,.]$, defined in (6.15), is continuous from $[W^{2,p}(-\pi,\pi)\cap\mathscr{Z}(h_1,h_2,\delta)]\times\mathbb{R}$ to $W^{1-1/2p,2p}(\partial\mathscr{A}_3(1,\infty))$ and maps bounded sequences into bounded sequences.*

We define a^* and a_3^* by

$$\left\{\begin{array}{ll} a_\beta[\mathbf{r}](\xi)\equiv\xi_\beta a^*[\mathbf{r}](\sqrt{\xi_1^2+\xi_2^2},\xi_3), & \beta=1,2, \\ a_3[\mathbf{r}](\xi)\equiv a_3^*[\mathbf{r}](\sqrt{\xi_1^2+\xi_2^2},\xi_3). & \end{array}\right. \tag{6.28}$$

Proposition 6.22, Lemma 6.25, and the fact that $\rho(\xi,\xi_3)=0$ only if $\xi=0$ imply that the proof of statements (i) and (ii) follows from

(6.29) LEMMA. — *Let the hypotheses of Lemma* 6.26 *hold. Let* $\mathcal{N}(\varepsilon)\equiv\{(\xi,\xi_3)\in\mathbb{R}^2: 1+\varepsilon\leqq|\xi_3|\leqq 1+\frac{1}{\varepsilon}, |\xi|\leqq\varepsilon\}$ *for* $\varepsilon\in(0,1)$. *Then there exists an* $\varepsilon\in(0,1)$ *such that* $\mathbf{r}\mapsto a^*[\mathbf{r}]$ *and* $\mathbf{r}\mapsto a_3^*[\mathbf{r}]$ *are continuous from* $C^{1,\alpha}[-\pi,\pi]\cap\mathscr{Z}(h_1,h_2,\delta)$ *and from* $W^{2,p}(-\pi,\pi)\cap\mathscr{Z}(h_1,h_2,\delta)$ *to* $C^k(\mathcal{N}(\varepsilon))$ *and map bounded sequences into bounded sequences for each* k.

(6.30) LEMMA. — *Let the hypotheses of Lemma* 6.26 *hold. Then there exists an* $\varepsilon\in(0,1)$ *such that the mappings taking* $\mathbf{r}$ *to the functions with values* $\sqrt{\xi_1^2+\xi_2^2}\,a^*[\mathbf{r}](\sqrt{\xi_1^2+\xi_2^2},\xi_3)$ *and* $a_3^*[\mathbf{r}](\sqrt{\xi_1^2+\xi_2^2},\xi_3)$ *are continuous from*

$$W^{2,p}(-\pi,\pi)\cap\mathscr{Z}(h_1,h_2,\delta)$$

to

$$L^q([\mathscr{D}_3((0,0,1),\varepsilon)\cup\mathscr{D}_3((0,0,-1),\varepsilon)]\setminus \mathrm{cl}\,\mathscr{D}_3(\mathbf{0},1))$$

and map bounded sequences into bounded sequences.

Proof of Lemma 6.29. — The symmetry condition (6.10) and the inequality (6.23) imply that

$$\xi\mapsto\rho[\mathbf{r}](\xi,\xi_3)\quad\text{is odd }\ \forall\,\xi_3\in\mathbb{R},\tag{6.31 a}$$

$$\begin{cases}\dfrac{\partial}{\partial\xi}\rho[\mathbf{r}](0,\xi_3)\neq 0 & \text{for }\ |\xi_3|\geqq 1,\\[2mm] \dfrac{\partial^2}{\partial\xi^2}\rho[\mathbf{r}](0,\xi_3)=0 & \text{for }\ |\xi_3|>1,\end{cases}\tag{6.31 b}$$

$$\varepsilon<\left|\frac{\partial}{\partial\xi}\rho[\mathbf{r}](\xi,\xi_3)\right|<\frac{1}{\varepsilon}\quad\text{for }\ |\xi_3|<1+\varepsilon,\ \ |\xi|\geqq 1,\ \ |\xi|\leqq\varepsilon\tag{6.31 c}$$

for ε sufficiently small. The continuity of $\frac{\partial}{\partial\xi}\rho[\mathbf{r}]$ and condition (6.31 *b*) enable us to get from the Mean Value Theorem representations like

$$\rho[\mathbf{r}](\xi,\xi_3)=\xi\int_0^1\frac{\partial}{\partial\xi}\rho[\mathbf{r}](t\xi,\xi_3)\,dt\tag{6.32}$$

for $(\xi,\xi_3)\in\mathcal{N}(\varepsilon)$. We substitute this representation and similar representations for $\rho[\mathbf{r}](\xi,\xi_3)-\xi\frac{\partial}{\partial\xi}\rho[\mathbf{r}](\xi,\xi_3)$ and $\xi^{-1}\frac{\partial}{\partial\xi_3}\rho[\mathbf{r}](\xi,\xi_3)$ into the expressions for a^* and a_3^* obtained from (6.28) and (6.15). We invoke Proposition 6.22 to complete the proof. □

Proof of Lemma 6.30. — We first observe that for $|\xi_3|<1$ we cannot use the Mean Value Theorem as in the proof of Lemma 6.29 because $\mathrm{cl}\,\mathscr{A}_2(1,\infty)$ is not convex. To circumvent this difficulty, we modify a standard extension theorem (*cf.* [21], Sec. 1.2, e.g.) to show that there exists a symmetrized extension $(\tilde{\rho},\tilde{\zeta})$ of any pair (ρ,ζ) satisfying (6.10) to the entire complex plane with

$$(6.33)\quad \{(\rho,\zeta)\in C^{1,\beta}(\mathrm{cl}\,\mathscr{A}_2(1,R_2)):(6.10)\text{ holds}\}\ni(\rho,\zeta) \mapsto(\tilde{\rho},\tilde{\zeta})\in\{(\rho,\zeta)\in C^{1,\beta}(\mathrm{cl}\,\mathscr{D}_2(0,R_2)):(6.10)\text{ holds}\}$$

continuous for each $\beta\in(0,1)$ and each $R_2\in(1,\infty)$. We denote the resulting symmetrized extension of $\mathbf{f}[\mathbf{r}]\equiv(\rho[\mathbf{r}],\zeta[\mathbf{r}])$ by $\tilde{\mathbf{f}}[\mathbf{r}]\equiv(\tilde{\rho}[\mathbf{r}],\tilde{\zeta}[\mathbf{r}])$. (Note that in general $\tilde{\mathbf{f}}[\mathbf{r}]$ is not holomorphic on $\mathscr{D}_2(\mathbf{0},1)$.)

Let $\{\mathbf{r}_j\}$ be a bounded sequence in $W^{2,p}(-\pi,\pi)\cap\mathscr{X}(h_1,h_2,\delta)$. Proposition 6.22, the compactness of the embedding of $W^{2,p}(-\pi,\pi)$ into $C^{1,\alpha}(-\pi,\pi)$ for $0<\alpha<1-\dfrac{1}{p}$, and the boundedness of (6.33) imply that the family $\left\{\dfrac{\partial}{\partial\xi}\tilde{\rho}[\mathbf{r}_j],\dfrac{\partial}{\partial\xi_3}\tilde{\rho}[\mathbf{r}_j]\right\}$ is equicontinuous in $C^0(\mathrm{cl}\,\mathscr{D}_2(0,R_2))$ for $R_2\in(1,\infty)$. Hence condition (6.10) and inequality (6.23) imply that there exists an $R_0\in(0,1)$ and an $\varepsilon>0$ independent of j such that

$$(6.34)\quad \varepsilon<\left|\frac{\partial}{\partial\xi}\tilde{\rho}[\mathbf{r}_j](\xi,\xi_3)\right|<\frac{1}{\varepsilon}\qquad\text{for}\quad R_0\leqq|\xi|\leqq R_2,\quad |\xi|\leqq\varepsilon.$$

By virtue of the extension we had effected, we can use the Mean Value Theorem to write

$$(6.35)\quad \xi a^*[\mathbf{r}_j](\xi,\xi_3)=-\frac{\displaystyle\int_0^1\left\{\frac{\partial}{\partial\xi}\tilde{\rho}[\mathbf{r}_j](t\xi,\xi_3)-\frac{\partial}{\partial\xi}\tilde{\rho}[\mathbf{r}_j](\xi,\xi_3)\right\}dt}{\displaystyle\xi\int_0^1\frac{\partial}{\partial\xi}\tilde{\rho}[\mathbf{r}_j](t\xi,\xi_3)\,dt}$$
$$\text{for}\quad 1\leqq|\xi|\leqq R_2,\quad |\xi|\leqq\varepsilon.$$

Thus

$$(6.36)\quad |\xi a^*[\mathbf{r}_j](\xi,\xi_3)| \leqq\frac{\xi^{\beta-1}}{\varepsilon}\left\{\int_0^1(1-t)\,dt\right\}\left\|\frac{\partial}{\partial\xi}\tilde{\rho}[\mathbf{r}_j],C^{0,\beta}(\mathrm{cl}\,\mathscr{D}(\mathbf{0},R_2))\right\|$$
$$\text{for}\quad \beta\in\left(0,\frac{p-1}{p}\right),\quad 1\leqq|\xi|\leqq R_2,\quad |\xi|\leqq\varepsilon.$$

Since $(\xi_1^2+\xi_2^2)^{(-1+\beta)/2}\in L^q(\mathscr{D}_3(\mathbf{0},1+\varepsilon))$ if $q<\dfrac{2}{1-\beta}$, we conclude that there is a positive constant c such that

$$(6.37)\quad \left\|\sqrt{\xi_1^2+\xi_2^2}\,a^*[\mathbf{r}_j]\left(\sqrt{\xi_1^2+\xi_2^2},\xi_3\right), L^q([\mathscr{D}_3((0,0,1),\varepsilon)\cup\mathscr{D}_3((0,0,-1),\varepsilon)]\setminus\mathrm{cl}\,\mathscr{D}_3(\mathbf{0},1))\right\|$$
$$\leqq c\left\|\frac{\partial}{\partial\xi}\tilde{\rho}[\mathbf{r}_j],C^{0,\beta}(\mathrm{cl}\,\mathscr{D}(\mathbf{0},R_2))\right\|\qquad\text{for}\quad q<\frac{2}{1-\beta},\quad \beta\in\left(0,\frac{p-1}{p}\right).$$

Proposition 6.22 implies that the mapping that takes $\mathbf{r}$ to the function with values $\sqrt{\xi_1^2+\xi_2^2}\, a^*[\mathbf{r}](\sqrt{\xi_1^2+\xi_2^2}, \xi_3)$ maps bounded sequences into bounded sequences as in Lemma 6.30.

Now let Let $\{\mathbf{r}_j\}$ converge to $\mathbf{r}$ in $W^{2,p}(-\pi,\pi)\cap \mathscr{X}(h_1,h_2,\delta)$. Lemma 6.26(i) implies that

$$\lim_{j\to\infty} a^*[\mathbf{r}_j](\sqrt{\xi_1^2+\xi_2^2}, \xi_3) = a^*[\mathbf{r}](\sqrt{\xi_1^2+\xi_2^2}, \xi_3) \tag{6.38}$$

pointwise on $[\mathscr{D}_3((0,0,1),\varepsilon)\cup \mathscr{D}_3((0,0,-1)\,\varepsilon)]\setminus \mathrm{cl}\,\mathscr{D}_3(\mathbf{0},1)$. Inequality (6.36) and Proposition 6.22 enable us to use the Lebesgue Dominated Convergence Theorem to prove that $a^*[\mathbf{r}_j]$ converges to $a^*[\mathbf{r}]$ as in Lemma 6.30. The proof of the analogous results for a_3^* is identical. □

Statement (iii) of Lemma 6.26 follows directly from (6.15), statement (i), and Proposition 6.22.

Proof of Lemma 6.26 (iv). – By computing ν_3 of (6.15) we obtain from (6.15) that

$$b(\xi_1,0,\xi_3) = -U\left\{\frac{\partial\rho}{\partial\xi_1}\xi_3 + \frac{\partial\zeta}{\partial\xi_1}\xi_1\right\} \quad \text{for} \quad \xi_1^2+\xi_3^2=1. \tag{6.39}$$

(Here we suppress the argument $\mathbf{r}$.) Proposition 6.22, Lemma 6.25, and the fact that $W^{1-(1/q),q}(\partial\mathscr{D}_3(\mathbf{0},1))$ is a Banach algebra if $q>3$ implies the conclusion. □

Thus the proof of Lemma 6.26 is complete. Having determined crucial properties of the coefficients in (6.14) we are now ready to obtain *apriori* estimates on its solution. Our next result gives a maximum principle and an associated uniqueness theorem.

(6.40) LEMMA. – *Let*

$3<2p$, $0<h_1<h_2$, $U\in\mathbb{R}$, *and* $\mathbf{r}\in W^{2,p}(-\pi,\pi)\cap\mathscr{X}(h_1,h_2)$.

If $\omega\in C^2(\mathscr{A}_3(1,\infty))\cap C^1(\mathrm{cl}\,\mathscr{A}_3(1,\infty))$ *satisfies boundary value problem* (6.14) *and is axisymmetric, then*

$$\sup\{|\omega(\xi)| : |\xi|\geqq 1\} \leqq \max\{|\omega(\xi)| : |\xi|=1\}. \tag{6.41}$$

Problem (6.14) *has at most one such solution.*

Proof. – Since the coefficients $a_i[\mathbf{r}]$ are continuous on $\mathscr{A}_3(1,\infty)$ and since (6.14*c*) must hold, a corollary of the Maximum Principle yields (6.41). The uniqueness then follows in the standard way. □

(6.42) LEMMA. – *Let* $3<q<2p<4, 0<h_1<h_2$, $1<R_1$, $c_1>0$, *and* $\mathbf{r}\in W^{2,p}(-\pi,\pi)\cap\mathscr{X}(h_1,h_2)$ *with* $c_1\geqq \| a_i[\mathbf{r}], L^q(\mathscr{A}_3(1,R_1))\|$. *Then there exists a constant* $c>0$ *depending only on* q, c_1, R_1 *such that*

$$\begin{aligned}\|u, W^{2,q}(\mathscr{A}_3(1,R_1))\| &\leqq c(\|\xi.\nabla u, W^{1-(1/q),q}(\partial\mathscr{D}_3(\mathbf{0},1))\| \\ &+ \|u, W^{2-(1/q),q}(\partial\mathscr{D}_3(\mathbf{0},R_1))\| + \|u, L^q(\mathscr{A}_3(1,R_1))\|)\end{aligned} \tag{6.43}$$

for every solution $u \in W^{2,q}(\mathscr{A}_3(1, R_1))$ *of* (6.14 *a*) *on* $\mathscr{A}_3(1, R_1)$.

Proof. — The one difficulty with this proof is that the coefficients of the lower-order terms in (6.14 *a*) need not be continuous. We use standard methods. By combining Theorem 8.2 of [14] with Theorem 3.28 of [21] we obtain the standard *apriori* estimate for the Laplace operator that there is a constant $c>0$ such that

$$(6.44)\quad \| u, W^{2,q}(\mathscr{A}_3(1, R_1)) \| \leq c (\| \xi . \nabla u, W^{1-(1/q), q}(\partial \mathscr{D}_3(\mathbf{0}, 1)) \| + \| u, W^{2-(1/q), q}(\partial \mathscr{D}_3(\mathbf{0}, R_1)) \| + \| \Delta u, L^q(\mathscr{A}_3(1, R_1)) \|)$$

for every $u \in W^{2,q}(\mathscr{A}_3(1, R_1))$. Since $q>3$, it follows that when $\lambda \in (0, 1-(3/q))$ the space $W^{2,q}(\mathscr{A}_3(1, R_1))$ can be continuously embedded into $C^{1,\lambda}(\mathrm{cl}\,\mathscr{A}_3(1, R_1))$, which in turn is compactly embedded into $C^1(\mathrm{cl}\,\mathscr{A}_3(1, R_1))$. By imitating Troianello's proof of Lemma 1.37 on page 61, we can thus derive the following interpolation inequality: For every $\varepsilon>0$ there exists a positive constant $c(\varepsilon)$ such that

$$(6.45)\quad \sum_{i=1}^{3} \left\| \frac{\partial u}{\partial \xi_i}, L^\infty(\mathscr{A}_3(1, R_1)) \right\| \leq \varepsilon \sum_{i,j=1}^{3} \left\| \frac{\partial^2 u}{\partial \xi_i \partial \xi_j}, L^q(\mathscr{A}_3(1, R_1)) \right\| + c(\varepsilon) \| u, L^q(\mathscr{A}_3(1, R_1)) \| \quad \forall u \in W^{2,q}(\mathscr{A}_3(1, R_1)).$$

We apply (6.44) to the equation $\Delta u = -\sum_{j=1}^{3} a_j[\mathbf{r}] \frac{\partial u}{\partial \xi_j}$ in $\mathscr{A}_3(1, R_1)$ to show that there is a constant $c>0$, depending only on q and R_1, such that

$$(6.46)\quad \| u, W^{2,q}(\mathscr{A}_3(1, R_1)) \| \leq c \left(\sum_{i=1}^{3} \| a_i[\mathbf{r}], L^q(\mathscr{A}_3(1, R_1)) \| . \left\| \frac{\partial u}{\partial \xi_i}, L^\infty(\mathscr{A}_3(1, R_1)) \right\| + \| \xi . \nabla u, W^{1-(1/q), q}(\partial \mathscr{D}_3(\mathbf{0}, 1)) \| + \| u, W^{2-(1/q), q}(\partial \mathscr{D}_3(\mathbf{0}, R_1)) \| \right).$$

We now set $\varepsilon = 1/2cc_1$ and use (6.45) to complete the proof. □

(6.47) LEMMA. — *Let* $3<2p<4$, $0<h_1<h_2$, $\delta>0$. *Let* $\{(\mathbf{r}_j, U_j)\}$ *be a bounded sequence in* $[W^{2,p}(-\pi, \pi) \cap \mathscr{Z}(h_1, h_2, \delta)] \times \mathbb{R}$. *Then the sequence* $\{\max\{|\omega[\mathbf{r}_j, U_j](\xi)| : |\xi| = 1\}\}$ *is bounded.*

Proof. — We set

$$(6.48)\quad \mathscr{A}^k \equiv \mathscr{A}_3\left(1 + \frac{1}{k+1}, 2+k\right).$$

Lemma 6.26 implies that there exists a sequence of real numbers $\{c_k\}$ and a number $\beta \in (0, 1]$ such that

$$(6.49\,a)\qquad \| a_i[\mathbf{r}_j], L^q(\mathscr{A}_3(1, R_1)) \| < c_0,$$
$$(6.49\,b)\qquad \| b[\mathbf{r}_j, U_j], W^{1-(1/q), q}(\partial\mathscr{D}_3(\mathbf{0}, 1)) \| < c_0,$$
$$(6.49\,c)\qquad \| a_i[\mathbf{r}_j], C^{0, \beta}(\operatorname{cl}\mathscr{A}^k) \| < c_k$$

for $3 < q < 2p$, and for all j and k.

Let us assume for contradiction that there is a subsequence of $\{(\mathbf{r}_j, U_j)\}$, denoted the same way, such that $M_j \equiv \max\{|\omega[\mathbf{r}_j, U_j](\xi)| : |\xi| \geqq 1\} \geqq j$. Then $\chi_j \equiv \omega[\mathbf{r}_j, U_j]/M_j$ satisfies

$$(6.50\,a)\qquad \Delta\chi_j + \sum_{i=1}^{3} a_i[\mathbf{r}_j]\frac{\partial\chi_j}{\partial\xi_i} = 0 \qquad \text{for} \quad \xi \in \mathscr{A}_3(1, \infty),$$
$$(6.50\,b)\qquad \xi . \nabla\chi_j(\xi) = b[\mathbf{r}_j, U_j](\xi)/M_j \qquad \text{for} \quad |\xi| = 1,$$
$$(6.50\,c)\qquad \chi_j(\xi) \to 0 \quad \text{as } |\xi| \to \infty,$$
$$(6.50\,d)\qquad \chi_j(\xi) = \chi_j(\sqrt{\xi_1^2 + \xi_2^2}, 0, \xi_3),$$
$$(6.50\,e)\qquad |\chi_j(\xi)| \leqq 1 \qquad \forall j, \quad \xi \in \operatorname{cl}\mathscr{A}_3(1, \infty).$$

Furthermore, Lemma 6.40 implies that there exists a $\xi_j \in \partial\mathscr{D}_3(\mathbf{0}, 1)$ such that

$$(6.51)\qquad |\chi_j(\xi_j)| = 1.$$

We now use a diagonalization process to construct a convergent subsequence of $\{\chi_j\}$. Inequalities (6.49 *c*) and (6.50 *e*) and standard Schauder interior estimates (*cf.* [7], Thm. 6.2) imply that

$$(6.52)\qquad \sup_j \| \chi_j, C^{2, \beta}(\operatorname{cl}\mathscr{A}^k) \| < \infty$$

for each $k \in \mathbb{N}$. Thus there is a subsequence $\{\chi_j^1\}$ of $\{\chi_j\}$ and a χ^1 in $C^2(\operatorname{cl}\mathscr{A}^1)$ such that $\chi_j^1 \to \chi^1$ in $C^2(\operatorname{cl}\mathscr{A}^1)$. By induction, there is a subsequence $\{\chi_j^k\}$ of $\{\chi_j^{k-1}\}$ and an element χ^k such that $\chi_j^k \to \chi^k$ in $C^2(\operatorname{cl}\mathscr{A}^k)$. It is easy to see that there exists a $\chi \in C^2(\mathscr{A}_3(1, \infty))$ such that

$$(6.53\,a)\qquad \chi(\xi) = \chi^k(\xi) \qquad \text{for} \quad \xi \in \operatorname{cl}\mathscr{A}^k, \quad k \geqq 1,$$
$$(6.53\,b)\qquad \chi_j^j \to \chi \quad \text{in } C^2(\operatorname{cl}\mathscr{A}_3(R_1, R_2))$$

for every R_1 and R_2 with $1 < R_1 < R_2 < \infty$. Inequality (6.52) and the continuity of the embedding of $C^{2, \beta}(\operatorname{cl}\mathscr{A}(R, R+1))$ into $W^{2, p}(\mathscr{A}(R, R+1))$ imply that

$$(6.54)\qquad \sup_j \| \chi_j, W^{2-(1/q), q}(\partial\mathscr{D}_3(\mathbf{0}, R)) \| < \infty$$

for each $R > 1$. Hence Lemma 6.42 and inequalities (6.49 *b*), (6.50 *e*), and (6.54) imply that

$$(6.55)\qquad \sup_j \| \chi_j^j : W^{2, q}(\mathscr{A}_3(1, R)) \| < \infty.$$

Using the compactness of the embedding of $W^{2,q}(\mathscr{A}_3(1,R))$ into $C^{1,\beta}(\text{cl}\,\mathscr{A}_3(1,R))$ we deduce that for every $R>1$ and for every $\beta\in\left(0,1-\frac{3}{q}\right)$ there exists a $\chi^*\in C^{1,\beta}(\text{cl}\,\mathscr{A}_3(1,R))$ such that

$$\chi_j^j\to\chi^* \quad \text{in } C^{1,\beta}(\text{cl}\,\mathscr{A}_3(1,R)). \tag{6.56}$$

Clearly $\chi=\chi^*$ on $\mathscr{A}_3(1,R)$. Thus $\chi\in C^2(\mathscr{A}_3(1,\infty))\cap C^1(\text{cl}\,\mathscr{A}_3(1,\infty))$.

Let $\{(\mathbf{r}_j^j,U_j^j)\}$ be the subsequence of $\{(\mathbf{r}_j,U_j)\}$ corresponding to χ_j^j. By the compactness of the embedding of $W^{2,p}(-\pi,\pi)$ into $C^{1,\lambda}[-\pi,\pi]$ for $\lambda\in\left(0,\frac{p-1}{p}\right)$ we may assume that $(\mathbf{r}_j^j,U_j^j)$ converges to $(\mathbf{r},U)$ in $C^{1,\lambda}[-\pi,\pi]\times\mathbb{R}$ for such λ's. Hence Lemma 6.26 implies that $a_i[\mathbf{r}_j^j]\to a_i[\mathbf{r}]$ uniformly on compact subsets of $\mathscr{A}_3(1,\infty)$. The boundedness of the embedding of $W^{1-(1/q),q}(\partial\mathscr{D}_3(\mathbf{0},1))$ into $C^0(\partial\mathscr{D}_3(\mathbf{0},1))$ for $q>3$ and inequality (6.49 *b*) imply that $\frac{b[\mathbf{r}_j^j,U_j^j]}{M_j}\to 0$ pointwise on $\partial\mathscr{D}_3(0,1)$. Thus χ satisfies

$$\Delta\chi+\sum_{i=1}^{3}a_i[\mathbf{r}]\frac{\partial\chi}{\partial\xi_i}=0 \qquad \text{for} \quad \xi\in\mathscr{A}_3(1,\infty), \tag{6.57 a}$$

$$\xi\,.\,\nabla\chi(\xi)=0 \qquad \text{for} \quad |\xi|=1, \tag{6.57 b}$$

$$\chi(\xi)=\chi\left(\sqrt{\xi_1^2+\xi_2^2},0,\xi_3\right). \tag{6.57 c}$$

We now prove that

$$\chi(\xi)\to 0 \quad \text{as } \xi\to\infty. \tag{6.58}$$

For this purpose we introduce the mapping $\mathbf{g}$ defined by

$$\mathbf{g}[\mathbf{r}](\xi_1,\xi_2,\xi_3)\equiv\left(\frac{\xi_1\,\rho[\mathbf{r}]\left(\sqrt{\xi_1^2+\xi_2^2},\xi_3\right)}{\sqrt{\xi_1^2+\xi_2^2}},\ \frac{\xi_2\,\rho[\mathbf{r}]\left(\sqrt{\xi_1^2+\xi_2^2},\xi_3\right)}{\sqrt{\xi_1^2+\xi_2^2}},\ \zeta[\mathbf{r}]\left(\sqrt{\xi_1^2+\xi_2^2},\xi_3\right)\right), \tag{6.59}$$

which takes $\text{cl}\,\mathscr{A}_3(1,\infty)$ onto $\mathscr{F}[\mathbf{r}]$. We abbreviate $\mathbf{g}[\mathbf{r}]$ and $\mathbf{g}[\mathbf{r}_j]$ by $\mathbf{g}$ and $\mathbf{g}_j$ and use analogous notation for other functions. We make some preliminary observations.

We choose R so large that $\partial\mathscr{F}[\mathbf{r}_j]\subset\mathscr{D}_3(\mathbf{0},R-1)$. By applying a theorem of Carathéodory (*cf.* [15]) to the sequence $\left\{z\mapsto\frac{1}{f_j^{-1}(1/z)}\right\}$ we easily show that $\mathbf{f}_j^{-1}\to\mathbf{f}^{-1}$ uniformly on $\partial\mathscr{D}_2(\mathbf{0},R)$. It then easily follows that $\mathbf{g}[\mathbf{r}_j]^{-1}\to\mathbf{g}[\mathbf{r}]^{-1}$ uniformly on $\partial\mathscr{D}_3(\mathbf{0},R)$. This uniform convergence implies that there is no loss of generality in assuming that R_1 and R_2 with $1<R_1<R_2$ are chosen so that $\mathscr{A}_3(R_1,R_2)\supset\mathbf{g}_j^{-1}(\partial\mathscr{D}_3(\mathbf{0},R))$ and

implies that there is a $c>0$ depending only on $\mathscr{A}_3(R_1, R_2)$ such that

$$(6.60) \quad \sup\{|\chi_j(\mathbf{g}_j^{-1}(\mathbf{x}))-\chi(\mathbf{g}^{-1}(\mathbf{x}))| : \mathbf{x}\in\partial\mathscr{D}_3(\mathbf{0}, R)\} \leqq c\sup\{|\nabla\chi_j(\xi)| : \xi\in\mathscr{A}_3(R_1, R_2)\} \times\sup\{|\mathbf{g}_j^{-1}(\mathbf{x})-\mathbf{g}^{-1}(\mathbf{x})| : \mathbf{x}\in\partial\mathscr{D}_3(\mathbf{0}, R)\} + \sup\{|\chi_j(\xi)-\chi(\xi)| : \xi\in\mathscr{A}_3(R_1, R_2)\}$$

for j sufficiently large. In view of the convergence of χ_j and $\mathbf{g}_j^{-1}$ we deduce from (6.60) that

$$(6.61) \qquad \chi_j(\mathbf{g}_j^{-1})\rightarrow\chi(\mathbf{g}^{-1})$$

uniformly on $\partial\mathscr{D}_3(\mathbf{0}, R)$.

Now let $\varepsilon>0$. Condition (6.61) implies that there is a number K such that

$$(6.62) \qquad \sup_{\mathbf{x}\in\mathscr{D}_3(\mathbf{0}, R)}|\chi_j(\mathbf{g}_j^{-1}(\mathbf{x}))-\chi_l(\mathbf{g}_l^{-1}(\mathbf{x}))|<\frac{\varepsilon}{3} \qquad \forall j, l\geqq K.$$

Since

$$(6.63) \qquad \mathbf{g}[\mathbf{r}_j]^{-1}(\mathbf{x})\rightarrow\infty \quad \text{as } \mathbf{x}\rightarrow\infty,$$

since (6.50 c) holds, and since the $\chi_j(\mathbf{g}_j^{-1})$ are harmonic on $\mathscr{A}_3(R, \infty)$, we can use a corollary of the Maximum Principle to deduce that

$$(6.64) \qquad \sup_{|\mathbf{x}|\geqq R}|\chi_j(\mathbf{g}_j^{-1})-\chi_l(\mathbf{g}_l^{-1})|<\frac{\varepsilon}{3} \qquad \forall j, l\geqq K.$$

Let $l\geqq K$. By choosing R_1 so large that $\sup_{|\mathbf{x}|\geqq R_1}|\chi_l(\mathbf{g}_l^{-1})|<\frac{\varepsilon}{3}$ for $|\mathbf{x}|\geqq R_1$ [*cf.* (6.50 c)] and by using (6.61) and (6.64) we obtain that $|\chi(\mathbf{g}^{-1}(\mathbf{x}))|<\varepsilon$ for $|\mathbf{x}|\geqq R_1$. Thus (6.58) holds. From (6.57), (6.58), and Lemma 6.40 we deduce that $\chi=0$, in contradiction to (6.51) and (6.56). □

(6.65). LEMMA. — *Let* $3<q<2p<4$, $R>1$, $0<h_1<h_2$, $\delta>0$. *The mapping* $(\mathbf{r}, U)\mapsto\omega[\mathbf{r}, U]$ *is continuous from* $[W^{2,p}(-\pi, \pi)\cap\mathscr{Z}(h_1, h_2, \delta)]\times\mathbb{R}$ *to* $W^{2,q}(\mathscr{A}_3(1, R))$ *and maps bounded sequences into bounded sequences.*

Proof. — Since

$$\omega[\mathbf{r}, U]\in C^2(\mathscr{A}_3(1, R))\cap C^1(\text{cl}\,\mathscr{A}_3(1, R)),$$

$\nabla\omega[\mathbf{r}, U]\in W^{1-(1/q), q}(\partial\mathscr{D}_3(\mathbf{0}, 1))$, $\omega[\mathbf{r}, U]\in W^{2-(1/q), q}(\partial\mathscr{D}_3(\mathbf{0}, 1))$, and $a_i[\mathbf{r}]\in L^q(\mathscr{A}_3(1, R))$, we can use (6.14 a, b) and standard results specifying how the regularity of solutions of boundary value problems for Poisson's equation depend on the data to deduce that $\omega[\mathbf{r}, U]\in W^{2,q}(\mathscr{A}_3(1, R))$.

Let $\{(\mathbf{r}_j, U_j)\}$ be a bounded sequence in

$$[W^{2,p}(-\pi, \pi)\cap\mathscr{Z}(h_1, h_2, \delta)]\times\mathbb{R}.$$

Using Lemmas 6.26 and 6.47, and standard interior estimates (*cf.* [7], Thm. 6.2) we easily deduce that

$$(6.66)\qquad \sup_j \| \omega[\mathbf{r}_j, U_j], C^{2,\beta}(\mathrm{cl}\,\mathscr{A}_3(R, R+1)) \| < \infty$$

for some $\beta \in (0, 1)$, whence

$$(6.67)\qquad \sup_j \| \omega[\mathbf{r}_j, U_j], W^{2-(1/q), q}(\partial \mathscr{D}_3(\mathbf{0}, R)) \| < \infty.$$

Lemmas 6.26, 6.42, and 6.47 then imply that

$$\{ \| \omega[\mathbf{r}_j, U_j], W^{2,q}(\mathscr{A}_3(1, R)) \| \}$$

is bounded.

Now let $(\mathbf{r}_j, U_j) \to (\mathbf{r}, U)$. It suffices to prove that every subsequence of $\{\omega[\mathbf{r}_j, U_j]\}$ has a subsequence converging to $\omega[\mathbf{r}, U]$. For this purpose we follow the proof of Lemma 6.47 to show that there is a $\psi \in C^2(\mathscr{A}_3(1, \infty)) \cap C^1(\mathrm{cl}\,\mathscr{A}_3(1, \infty))$ such that $\omega[\mathbf{r}_j, U_j] \to \psi$ in $C^2(\mathrm{cl}\,\mathscr{A}^k)$ for all k and in $W^{2,q}(\mathscr{A}_3(1, R))$ for all $R > 1$. Then we show that ψ satisfies the same boundary value problem as $\omega[\mathbf{r}, U]$ and accordingly equals it. □

Lemma 6.41 of [13] is

(6.68) LEMMA. – *Let* $0 < \beta < \alpha \leq 1$ *and let m be a nonnegative integer. Let* $\{f_k\}$ *be a sequence of functions on* $[-L, L]$ *that is bounded in* $C^{m,\alpha}$. *If* f_k *converges to* $f \in C^0$ *either pointwise a.e. or in the sense of distributions on* $(-L, L)$, *then* $f \in C^{m,\alpha}$ *and* f_k *converges to f in the norm of* $C^{m,\beta}$.

(6.69) LEMMA. – *Let* $3 < 2p < 4$, $0 < h_1 < h_2$, $\delta > 0$. *The mapping* $(\mathbf{r}, U) \mapsto |\mathbf{u}[\mathbf{r}, U]| \equiv |U\mathbf{k} + \nabla\Phi[\mathbf{r}, U](r(.), 0, z(.))|$ *(delivering the speed) is continuous and compact from* $[W^{2,p}(-\pi, \pi) \cap \mathscr{Z}(h_1, h_2, \delta)] \times \mathbb{R}$ *to* $C^0[0, \pi]$.

Proof. – Let

$$(6.70\,a)\qquad \Omega[\mathbf{r}, U](\xi_1, \xi_3) \equiv \omega[\mathbf{r}, U](\xi_1, 0, \xi_3).$$

We define

$$(6.70\,b)\qquad \Lambda[\mathbf{r}, U](t) \equiv \nabla\Omega[\mathbf{r}, U](e^{it})\, \mathbf{J}[\mathbf{r}](t)$$

where $\mathbf{J}[\mathbf{r}](t)$ is the inverse of the matrix of partial derivatives of $\rho[\mathbf{r}]$ and $\zeta[\mathbf{r}]$ with respect to ξ_1 and ξ_3 evaluated at e^{it} [*cf.* (6.13)]. It is easy to see that Proposition 6.22 and the continuity of the embedding of $W^{2,p}(\pi/2, 5\pi/2)$ into $C^{1,\beta}[\pi/2, 5\pi/2]$ for $\beta \in (0, 1-1/p)$ implies that $\mathbf{r} \mapsto \mathbf{J}[\mathbf{r}]$ is continuous from $W^{2,p}(-\pi, \pi) \cap \mathscr{Z}(h_1, h_2, \delta)$ to $C^{0,\beta}[\pi/2, 5\pi/2]$ and maps bounded sequences into bounded sequences. Lemma 6.65, the continuity of the embedding of $W^{1,q}(\mathscr{A}_3(1, R))$ into $C^{0,\beta}(\mathrm{cl}\,\mathscr{A}_3(1, R))$ for $R > 1$ and $0 < \beta < 1 - 3/q < 1 - 1/p$, and the continuity of the trace operator from $C^{0,\beta}(\mathrm{cl}\,\mathscr{A}_3(1, R))$ to the Banach algebra $C^{0,\beta}(\{(\xi_1, \xi_3) : \xi_1^2 + \xi_3^2 = 1\})$ implies that $(\mathbf{r}, U) \mapsto \Lambda[\mathbf{r}, U]$ is continuous from $[W^{2,p}(-\pi,\pi)\cap \mathscr{Z}(h_1,h_2,\delta)]\times\mathbb{R}$ to $C^{0,\beta}[\pi/2,5\pi/2]$ for $0<\beta<1-3/q$ and maps bounded sequences into bounded sequences.

By imitating the proof of Lemma 5.44 of [12] we show that $W^{2,p}(-\pi,\pi)\cap \mathscr{Z}(h_1,h_2,\delta)\ni \mathbf{r}\mapsto t^*[\mathbf{r}]^{-1}(\lambda[\mathbf{r}](.))\equiv t^{**}[\mathbf{r}]\in C^{0,\alpha}[-\pi,\pi]$ is continuous for $\alpha\in(0,1)$ and maps bounded sequences into bounded sequences. By using Lemma 6.68 as in the proof of Lemma 5.38 of [12] we find that the composition operator

$$C^{0,\beta}[\pi/2,5\pi/2]\times\{f\in C^{0,\alpha}[-\pi,\pi]:f([-\pi,\pi])\subset[\pi/2,5\pi/2]\}\ni(u,v)\mapsto u(v(.))\in C^{0,\gamma}[\pi,\pi],$$

with $0<\gamma<\beta\alpha$, is continuous and maps bounded sequences into bounded sequences. Combining this result with the properties of Λ and t^{**} and using the compactness of the embedding of $C^{0,\gamma}[-\pi,\pi]$ into $C^0[-\pi,\pi]$ we complete the proof. □

Proof of Theorem 3.10. — Let $(\mathbf{r}_j,U_j)$ be a bounded sequence of $\mathscr{X}\cap\mathscr{R}(\delta)\times\mathbb{R}$. The invariance of (5.2) under translations in the $\mathbf{k}$-direction implies that $\mathbf{u}[\mathbf{r}+a\mathbf{k},U]=\mathbf{u}[\mathbf{r},U]$ for $a\in\mathbb{R}$. Since elements of $\mathscr{X}\cap\mathscr{R}$ intersect the $\mathbf{k}$-axis only at $s=0,\pi$, we may assume that

$$(6.71)\qquad \mathbf{0}\notin \operatorname{cl}\mathscr{F}[\mathbf{r}_j],\qquad z_j(0)+z_j(\pi)=0.$$

Simple arguments by contradiction shows that there exist h_1, h_2, and $\delta_1>0$ such that

$$(6.72\ a,b)\quad 0<h_1<|\mathbf{r}_j(s)|<h_2<\infty\quad\text{for } j\in\mathbb{N},\ s\in[0,\pi],\ l[\mathbf{r}_j]>\delta_1.$$

Since

(6.73) $\mathscr{X}$ is continuously embedded in $W^{2,p}(-\pi,\pi)$ for $p\in(1,2)$

(in the sense of the extension introduced at the beginning of this section), we can therefore assume that there are h_1 and h_2 with $0<h_1<h_2$ such that $\{(\mathbf{r}_j,U_j)\}$ is a bounded sequence in $\mathscr{Z}(h_1,h_2,\delta)\cap W^{2,p}(-\pi,\pi)$ for $p\in(1,2)$. The continuity and compactness properties of $|\mathbf{u}|$ and p follow from Lemma 6.69, from (6.73), and from Bernoulli's Theorem (5.3 b). The independence of $|\mathbf{u}/U|$ of U immediately follows from (5.4). □

7. COMPACTNESS VIA POTENTIAL THEORY

In this section we outline the steps necessary to prove Theorem 3.10 directly by the methods of potential theory. We consider the boundary value problem (5.2). If there is an $\alpha\in(0,1]$ such that $\mathbf{r}\in(C^{1,\alpha}[0,\pi])\cap\mathscr{R}$, then $\partial\mathscr{F}[\mathbf{r}]$ is smooth enough for us to apply the Fredholm method of integral equations to solve (5.2). It implies that this problem has a unique solution, which can be expressed as

$$\Phi[\mathbf{r}, \tilde{\sigma}[\mathbf{r}, \mathrm{U}]](.)$$

where

$$(7.1) \qquad \Phi[\mathbf{r}, \sigma](\mathbf{x}) = \int_{\partial\mathscr{F}[\mathbf{r}]} \frac{\sigma(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\, da(\mathbf{y}) \qquad \text{for} \quad \mathbf{x} \in \mathrm{cl}\,\mathscr{F}[\mathbf{r}].$$

Here $da(\mathbf{y})$ is the differential surface area of $\partial\mathscr{F}[\mathbf{r}]$ at $\mathbf{y}$ and $\tilde{\sigma}[\mathbf{r}, \mathrm{U}]$ is the unique Hölder continuous solution of the integral equation

$$(7.2) \quad -\mathrm{U}\mathbf{k}.\mathbf{v}(\mathbf{x}) = -2\pi\tilde{\sigma}(\mathbf{x}) + \int_{\partial\mathscr{F}[\mathbf{r}]} \nabla_{\mathbf{x}}\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right)\mathbf{v}(\mathbf{x})\,\tilde{\sigma}(\mathbf{y})\, da(\mathbf{y}) \qquad \text{for} \quad \mathbf{x} \in \partial\mathscr{F}[\mathbf{r}].$$

Note that uniqueness implies that $\tilde{\sigma}[\mathbf{r}, \mathrm{U}] = \mathrm{U}\,\tilde{\sigma}[\mathbf{r}, 1]$. We specialize (7.1) to the variables introduced in Section 2 to obtain

$$(7.3) \quad \Phi[\mathbf{r}, \tilde{\sigma}[\mathbf{r}, \mathrm{U}]](\mathbf{x}) = \int_0^{\pi}\int_0^{2\pi} \frac{\tilde{\sigma}[\mathbf{r}, \mathrm{U}](\mathbf{r}(s,\varphi))\, r(s)\,|\mathbf{r}'(s)|\, ds\, d\varphi}{\{[x_1 - r(s)\cos\varphi]^2 + [x_2 - r(s)\sin\varphi]^2 + [x_3 - z(s)]^2\}^{1/2}}.$$

Since it is readily shown that $\tilde{\sigma}[\mathbf{r}, \mathrm{U}](r(s)\mathbf{e}_1(\varphi) + z(s)\mathbf{k})$ is independent of φ, we define

$$(7.4) \qquad \sigma[\mathbf{r}, \mathrm{U}](s) \equiv \tilde{\sigma}[\mathbf{r}, \mathrm{U}](r(s)\mathbf{e}_1(\varphi) + z(s)\mathbf{k}).$$

Then we can write (7.2) as

$$(7.5a) \qquad \mathrm{U}\frac{r'(s)}{|\mathbf{r}'(s)|} = -2\pi\sigma(s) + \mathrm{A}[\mathbf{r}, \sigma](s)$$

where

$$(7.5b) \quad \mathrm{A}[\mathbf{r}, \sigma](s) \equiv \int_0^{\pi} dt \int_0^{2\pi} \times \frac{[z'(s)\mathbf{e}_1(0) - r'(s)\mathbf{k}].[\mathbf{r}(t,\varphi) - \mathbf{r}(s,0)]\,\sigma(t)\, r(t)\,|\mathbf{r}'(t)|\, d\varphi}{|\mathbf{r}'(s)|\,|\mathbf{r}(t,\varphi) - \mathbf{r}(s,0)|^3}.$$

Analogously we specialize (7.3) by

$$(7.6) \qquad \hat{\Phi}[\mathbf{r}, \sigma](s) \equiv \int_0^{\pi} dt \int_0^{2\pi} \frac{\sigma(t)\, r(t)\,|\mathbf{r}'(t)|\, d\varphi}{|\mathbf{r}(t,\varphi) - \mathbf{r}(s,0)|}.$$

Our aim is to show that $s \mapsto \hat{\mathbf{u}}[\mathbf{r}, \mathrm{U}](s) \equiv \mathrm{U}\mathbf{k} + \nabla\hat{\Phi}[\mathbf{r}, \tilde{\sigma}[\mathbf{r}, \mathrm{U}]]\,(\mathbf{r}(s,0))$ depends continuously and compactly on $\mathbf{r}$. [*Cf.* (5.3).]. We do this by carrying out the following steps.

We first use a modification of Theorem 4(2.X) of [9], p. 363, which gives criteria on kernels that ensure the compactness of integral operators from L^p to $\mathrm{C}^{0,\alpha}$, to prove

(7.7) LEMMA. — *Let* $0<\alpha<1/4$, $\mathbf{r}\in\mathcal{X}\cap\mathcal{R}$, *and* $\sigma\in C^0$. *Then* $A[\mathbf{r},\sigma]\in C^{0,1/4}$. *Let* $\{\mathbf{r}_k\}$ *be a bounded sequence in* $\mathcal{X}\cap\mathcal{R}$ *converging to* $\mathbf{r}^+\in(C^1[0,\pi])\cap\mathcal{R}$ *in the* C^1*-norm. Then*

$$\sup_{m,n\in\mathbb{N}}\|A[\mathbf{r}_m,\sigma_n]\|_{0,\alpha}\leqq\infty. \tag{7.8}$$

We combine Lemmas 7.7 and 6.68 to obtain

(7.9) LEMMA. — *Let* $0<\alpha_1<1/2$, *let* $\{\mathbf{r}_n\}$ *be a bounded sequence in* $\mathcal{X}\cap\mathcal{R}$ *converging to* $\mathbf{r}^+\in(C^{1,\alpha_1})\cap\mathcal{R}$ *in the* C^{1,α_1}*-norm, and let* σ_n *be a sequence in* C^0 *converging in this space to* σ^+. *Then* $A[\mathbf{r}^+,\sigma^+]\in C^{1,\alpha_2}$ *and* $\{A[\mathbf{r}_n,\sigma_n]\}$ *converges to* $A[\mathbf{r}^+,\sigma^+]$ *in the* C^{0,α_2}*-norm for all* $\alpha_2\in(0,1/4)$.

Our first basic result is

(7.10) THEOREM. — *The operator* A *is continuous from* $(\mathcal{X}\cap\mathcal{R})\times C^0$ *to* $C^{0,\alpha}$ *for* $0<\alpha<1/4$. *There exists a unique continuous operator* $\check{\sigma}[.,U]$ $\mathcal{X}\cap\mathcal{R}$ *to* $C^{0,\alpha}$ *such that* $(\mathbf{r},\check{\sigma}[\mathbf{r},U])$ *satisfies* (7.5 *a*). *Furthermore,* $\check{\sigma}[.,U]$ *is compact from* $\mathcal{X}\cap\mathcal{R}(\delta)$ *to* $C^{0,\beta}$ *for all* $\beta\in(0,\alpha)$ *and for all* $\delta>0$.

The well-known existence and uniqueness of $\check{\sigma}[.,U]$ can be proved by integral equation methods (*cf.* [8], e. g.). The continuity follows from the version of the classical Implicit Function Theorem given by [6].

We now prove our second basic result:

(7.11) THEOREM. — *Let* $0<\alpha_1<\alpha<1$. *Define*

$$V[\mathbf{r},\sigma](s)\equiv\frac{1}{|\mathbf{r}'(s)|}\frac{d}{ds}\int_0^\pi dt\int_0^{2\pi}\frac{\sigma(t)\,r(t)\,|\mathbf{r}'(t)|\,d\varphi}{|\mathbf{r}(t,\varphi)-\mathbf{t}(s,0)|}. \tag{7.12}$$

Then $V[.,.]$ *is continuous from* $(C^{1,\alpha}\cap\mathcal{R})\times C^{0,\alpha}$ *to* C^{0,α_1}.

Sketch of Proof. — It follows from (7.12) that if $(\mathbf{r}_n,\sigma_n)\to(\mathbf{r},\sigma)$ in the $C^{1,\alpha}\times C^{0,\alpha}$-norm, then $V[\mathbf{r}_n,\sigma_n]\to V[\mathbf{r},\sigma]$ in the sense of distributions on $(0,\pi)$. Since

$$V[\mathbf{r},\sigma](s)=\nabla\Phi[\mathbf{r},\sigma](\mathbf{r}(s)).[r'(s)\,\mathbf{e}_1(0)+z'(s)\,\mathbf{k}]\,|\mathbf{r}'(s)|^{-1}, \tag{7.13}$$

we can follow Günter's [8], App. 1, treatment of gradients of single-layer potentials to show that

$$\sup_n\|V[\mathbf{r}_n,\sigma_n]\|_{0,\alpha}<\infty \tag{7.14}$$

if $(\mathbf{r}_n,\sigma_n)\to(\mathbf{r},\sigma)$ in $(C^{1,\alpha}[0,\pi]\cap\mathcal{R})\times C^{0,\alpha}$. We complete the proof by using Lemma 6.68. □

We easily deduce Theorem 3.10 from the combination of (5.3) and (7.13) by invoking Theorems 7.10 and 7.11 and using the compactness of the embeddings of $C^{1,\alpha}$ into $C^{1,\beta}$ and $C^{0,\alpha}$ into $C^{0,\beta}$ for $0<\beta<\alpha$.

The details of this proof, which are lengthy, are given by [10], Chap. 11.

ACKNOWLEDGEMENTS

The work reported here, which was supported in part by NSF Grant DMS-9001777, AFOSR-URI Grant 90-0015, and by the CNR of Italy, represents a significant extension of part of the doctoral dissertation of Lanza de Cristoforis (1987) at the University of Maryland.

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(*Manuscript received October* 4;
revised December 20, 1990.)