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Classical solvability in dimension two of the second boundary-value problem associated with the Monge-Ampère operator

by

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ABSTRACT. — Given two bounded strictly convex domains of $\mathbb{R}^n$ and a positive function on their product, all data being smooth, find a smooth strictly convex function whose gradient maps one domain onto the other with Jacobian determinant proportional to the given function. We solve this problem under the (technical) condition $n=2$.

Key words : Strictly convex functions, prescribed gradient image, Monge-Ampère operator, continuity method, *a priori* estimates.

RÉSUMÉ. — Soit deux domaines bornés strictement convexes de $\mathbb{R}^n$ et une fonction positive définie sur leur produit, ces données étant lisses, trouver une fonction lisse strictement convexe dont le gradient applique un domaine sur l'autre avec déterminant Jacobien proportionnel à la fonction donnée. Nous résolvons ce problème sous la condition (technique) $n=2$.

Classification A.M.S. : 35 J 65, 35 B 45, 53 C 45.

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I. INTRODUCTION

Let D and D* be bounded C^∞ strictly convex domains of $\mathbb{R}^n$. We denote by $S(D, D^*)$ the subset of $C^\infty(\bar{D})$ consisting of strictly convex real functions [1] whose gradient maps D onto D*. Given any $u \in C^\infty(\bar{D})$, we denote by $A(u)$ the Jacobian determinant of the gradient mapping $x \to du(x)$. The nonlinear second order differential operator A is called the *Monge-Ampère* operator on D. Basic features of A restricted to $S(D, D^*)$ are listed in the preliminary

PROPOSITION 1. — A *sends* $S(D, D^*)$ *into*

$$\Sigma := \{f \in C^\infty(\bar{D}), f > 0, \langle f \rangle = |D^*|/|D|\}$$

($\langle f \rangle$ denotes the average of f over D and $|D|$, the Lebesgue measure of D). *On* $S(D, D^*)$, A *is elliptic and its derivative is divergence-like. Given any defining function* h^* *of* D*, *the boundary operator* $u \to B(u) := h^*(du)|_{\partial D}$ *is co-normal with respect to* A *on* $S(D, D^*)$. *Furthermore, given any* $u \in S(D, D^*)$ *and any* $x \in \partial D$, *the co-normal direction at* x *with respect to the derivative of* A *at* u *is nothing but the normal direction of* ∂D^* *at* $du(x)$.

We postpone the proof of proposition 1 till the end of this section. The *second boundary-value problem* consists in showing that $A : S(D, D^*) \to \Sigma$ is *onto*. More generally, we wish to solve in $S(D, D^*)$ two kinds of equations namely

$$\operatorname{Log} A(u) = f(x, du) + \langle u \rangle \tag{1}$$

$$\operatorname{Log} A(u) = F(x, du, u) \tag{2}$$

where $f \in C^\infty(\bar{D} \times \bar{D}^*)$ and $F \in C^\infty(\bar{D} \times \bar{D}^* \times \mathbb{R})$, the latter being uniformly increasing in u. We aim at the following

THEOREM. — *Equations* (1) *and* (2) *are uniquely solvable in* $S(D, D^*)$ *provided* $n = 2$.

The second boundary-value problem was first posed and solved (with $n = 2$ but the methods, geometric in nature, extend to any dimension) in a *generalized sense* in [18] chapter V section 3 (*see* also [3] theorem 2, where the whole plane is taken in place of D). The elliptic Monge-Ampère operator with a quasilinear Neumann boundary condition is treated in [16], in any dimension, and it is further treated with a quasilinear *oblique* boundary condition in [21] provided $n = 2$. A general study of nonlinear oblique boundary-value problems for nonlinear second order *uniformly*

[1] Here the meaning of "strictly convex" is restricted to having a positive-definite hessian matrix, which rules out e.g. the strictly convex function $u(x) = |x - y|^4$ near $y \in D$, as pointed out to us by Martin Zerner.

elliptic equations is performed in [15]. Quite recently, the following problem was solved [5]: existence and regularity on a given bounded domain D of $\mathbb{R}^n$ (no convexity assumption, no restriction on n) of a *diffeomorphism* from $\bar{D}$ to itself, reducing to the identity on ∂D, with prescribed positive Jacobian determinant (of average 1 on D).

Remarks. — 1. The restriction $n=2$ is unsatisfactory but we could not draw second order boundary estimates without it. In May 1988, in Granada (Spain), Neil Trudinger informed us that Kai-Sing Tso had treated the problem in *any* dimension; however, from that time on, Tso's preprint has not been available due to a serious gap in his proof, as he himself wrote us [20]. In June 1989, John Urbas visited us in Antibes and he kindly advised us to submit our own 2-dimensional result; it is a pleasure to thank him for his thorough reading of the present paper. This may be the right place to thank also the Referee for pointing out a mistake at the end of the original proof of proposition 2 below, and a few inaccuracies (particularly one in remark 6).

2. We do *not* assume the non-emptiness of $S(D, D^*)$ to prove the theorem; we thus *obtain* it (when $n=2$) as a by-product of our proof. In fact, we found no straightforward way of exhibiting any member of $S(D, D^*)$ – except, of course, if $D=D^*$ –, although we can write down explicitely a $C^\infty(\bar{D})$ convex (but not *strictly* convex) function with gradient image D^*, constructed from any suitable *support function* for D^*. Provided non-emptiness, it is possible to prove that $S(D, D^*)$ is a locally closed Fréchet submanifold of the open subset of strictly convex functions in $C^\infty(\bar{D})$, as the fiber of a submersion.

3. From the proof below, it appears that, given any $\alpha \in (0, 1)$ $C^{2,\alpha}(\bar{D})$ solutions may be derived (by approximation) from the above theorem under the sole regularity assumptions: D and D^* are $C^{2,1}$, f and F are $C^{1,1}$. We did not study further 2-dimensional global regularity refinements as done in [19], [14] for the Dirichlet problem.

4. The uniqueness for (1) shows that, in general, the equation $\operatorname{Log} A(u) = f(x, du)$ is *not* well-posed on $S(D, D^*)$. The idea of introducing in (1) the average term goes back to [6] and it proved to be useful in various contexts ([2], [8], [9], [10]). If $u \in S(D, D^*)$ solves (1), then $v = u + \text{Const.}$ solves in $S(D, D^*)$ the equation $\operatorname{Log} A(v) = f(x, dv) + \langle u \rangle$, while the Legendre transform v^* of v solves in $S(D^*, D)$ the "dual" equation $\operatorname{Log} A(v^*) = -f(dv^*, x) - \langle u \rangle$. In case $f(x, x^*) = f_1(x) - f_2(x^*)$, the value of $\langle u \rangle$ is *a priori* fixed by the constraint (due to the "Jacobian" structure of A)

$$\int_{D^*} e^{f_2(x^*)}\, dx^* = e^{\langle u \rangle} \int_D e^{f_1(x)}\, dx.$$

The prescribed Gauss-curvature equation is an example of this type.

Proof of proposition 1. — By its very definition, as the *Jacobian* of the gradient mapping, A readily sends $S(D, D^*)$ into the submanifold Σ.

Let $u \in S(D, D^*)$. In euclidean co-ordinates $(x^1, \ldots, x^n)$, $A(u)$ reads

$$A(u) = \det(u_{ij})$$

and the derivative of A at u reads

$$\delta u \in C^\infty(\bar{D}) \to dA(u)(\delta u) = A^{ij}(\delta u)_{ij}$$

where

$$A^{ij} = A(u)\, u^{ij}$$

(indices denote partial derivatives, Einstein's convention holds, (u^{ij}) is the matrix inverse of (u_{ij}) and (A^{ij}), its co-matrix). Since u is strictly convex, A is indeed *elliptic* at u. Furthermore, one easily verifies the following identity: for any $\delta u \in C^\infty(\bar{D})$,

$$A^{ij}(\delta u)_{ij} \equiv [A^{ij}(\delta u)_i]_j.$$

So, as asserted, $dA(u)$ is *divergence-like*. The co-normal boundary operator associated with A at u is

$$\delta u \in C^\infty(\bar{D}) \to \beta(\delta u) = A^{ij} N^i (\delta u)_j \in C^\infty(\partial D),$$

N standing for the outward unit normal on ∂D. Fix a defining function h^* for D^* (*i.e.* $h^* \in C^\infty(\bar{D}^*)$ is strictly convex and vanishes on ∂D^*). Since $u \in S(D, D^*)$, the function $H := h^*(du) \in C^\infty(\bar{D})$ is negative inside D and vanishes on ∂D. Moreover, a straightforward computation yields in D:

$$u^{ij} H_{ij} - u^{ij} [\operatorname{Log} A(u)]_i H_j = u_{ij} h^*_{ij} > 0.$$

Hopf's lemma [12] implies that $H_N > 0$ on ∂D. Since

$$H_i = u_{ij} h^*_j$$

the boundary operators satisfy

$$A(u)\, dB(u) = H_N \beta.$$

So B is indeed *co-normal* with respect to A at u.

Last, the geometric interpretation of the co-normal direction β given at the end of proposition 1, simply follows from the fact that $dB(u)(x)$ equals the derivative in the direction of $dh^*[du(x)]$ which is precisely (outward) *normal* to ∂D^* at $du(x)$. □

II. THE CONTINUITY METHOD

Fix $(x_0, x_0^*) \in D \times D^*$ and $\lambda \in (0, 1]$ such that the gradient of

$$v_0 = \frac{\lambda}{2} |x - x_0|^2 + x_0^* . x$$

maps $\bar{D}$ *into* D^* ($|.|$ stands for the standard euclidean norm, . for the euclidean scalar product). Set $u_0 := v_0 - \langle v_0 \rangle$, $D_0 := du_0(D)$. A routine verification shows that D_0 is C^∞ *strictly convex*. Let $t \in [0, 1] \to D_t$ be a smooth path of bounded C^∞ strictly convex domains connecting D_0 to $D_1 = D^*$, with $D_t \subset D_{t'}$ for $t < t'$; fix $t \to h_t$ a smooth path of corresponding defining functions. For each $t \in [0, 1]$, consider in $S(D, D_t)$ the two following equations:

$$\operatorname{Log} A(u) = tf(x, du) + (1-t) n \operatorname{Log} \lambda + \langle u \rangle \tag{1.t}$$

$$\operatorname{Log} A(u) = t F(x, du, u) + (1-t)(u - u_0 + n \operatorname{Log} \lambda). \tag{2.t}$$

By construction u_0 solves both equations for $t=0$, so (for $i=1, 2$) the sets $T_i := \{ t \in [0, 1], (i.t) \text{ admits a solution in } S(D, D_t) \}$ are non-empty. Hereafter, we show that they are both relatively open and closed in [0, 1]: if so, by connectedness, they coincide with all of [0, 1]. The solutions for $t=1$ are those announced in the theorem; their uniqueness is established at the end of this section.

Let us show that T_1 is *relatively open* in [0, 1]; similar, more standard (due to the monotonicity assumption of F), reasonings hold for T_2. Fix $\alpha \in (0, 1)$ and denote by $U^{2,\alpha}$ the open subset of $C^{2,\alpha}(\bar{D})$ consisting of strictly convex functions. On $[0, 1] \times U^{2,\alpha}$, consider the smooth map (M, B) defined by

$$M(t, u) := \operatorname{Log} A(u) - tf(x, du) - (1-t) n \operatorname{Log} \lambda - \langle u \rangle,$$
$$B(t, u) := h_t(du)|_{\partial D},$$

and ranging in $C^{0,\alpha}(\bar{D}) \times C^{1,\alpha}(\partial D)$. Let $t_0 \in T$; there thus exists u_0 in $U^{2,\alpha}$ such that $(M, B)(t_0, u_0) = (0, 0)$. The proof is based on the Banach implicit function theorem applied to (M, B) at (t_0, u_0). We want to show that the map

$$(m, b) := [M_u(t_0, u_0), B_u(t_0, u_0)] : C^{2,\alpha}(\bar{D}) \to C^{0,\alpha}(\bar{D}) \times C^{1,\alpha}(\partial D)$$

is an *isomorphism*. Record the following expression of (m, b) in euclidean co-ordinates:

$$m(\delta u) = u_0^{ij} (\delta u)_{ij} - t_0 f_{u_i}(x, du_0)(\delta u)_i - \langle \delta u \rangle,$$
$$b(\delta u) = (h_t)_i (du_0)(\delta u)_i.$$

From proposition 1, we know that b is *oblique*; so Hopf's maximum principle [11] combined with Hopf's lemma [12] imply that any $\delta u \in \operatorname{Ker}(m, b)$ is *constant*, hence actually $\langle \delta u \rangle = 0$ and $\delta u \equiv 0$. Therefore (m, b) is *one-to-one*.

Now we fix $(\delta M_0, \delta B_0) \in C^{0,\alpha}(\bar{D}) \times C^{1,\alpha}(\partial D)$ and we look for δu in $C^{2,\alpha}(\bar{D})$ solving: $(m, b)(\delta u_0) = (\delta M_0, \delta B_0)$. Consider the auxiliary map

$$(m', b') := \{ A(u_0)(m + \langle . \rangle), [A(u_0)/H_N] b \},$$

where $H = h_t(du_0)$. It follows from proposition 1 that, given any $(\delta M', \delta B') \in C^{0,\alpha}(\bar{D}) \times C^{1,\alpha}(\partial D)$, the function $\delta u' \in C^{2,\alpha}(\bar{D})$ solves:

$$(m', b')(\delta u') = (\delta M', \delta B'), \tag{3}$$

if and only if, for every $\delta v' \in W^{1,2}(D)$,

$$L(\delta u', \delta v') = \int_{\partial D} \delta B' \, \delta v \, da - \int_D \delta M' \, \delta v \, dx$$

(da is the measure induced on ∂D by dx), where L is the continuous bilinear form on $W^{1,2}(D)$ given by

$$L(\delta u', \delta v') := \int_D A(u_0) [u_0^{ij} (\delta u')_i (\delta v')_j + t_0 f_{u_i}(x, du_0) (\delta u')_i \, \delta v'] \, dx.$$

Let us argue on (m', b') as in [6]. Combining the ellipticity of m' and the obliqueness of b' (asserted by proposition 1), with Hopf's maximum principle, Schauder's estimates and Fredholm's theory of *compact* operators, we know that the kernel of the adjoint of (m', b') (formally obtained by varying the first argument of L instead of the second, and by integrating by parts) is *one-dimensional*, let $\delta w \in C^{2,\alpha}(\bar{D})$ span it, and that (3) is solvable up to an additive constant if and only if

$$\int_{\partial D} \delta B' \, \delta w \, da - \int_D \delta M' \, \delta w \, dx = 0. \tag{4}$$

Observe that

$$\int_D A(u_0) \, \delta w \, dx \neq 0$$

since, otherwise, one could solve (3) with $(\delta M', \delta B') = [A(u_0), 0]$ contradicting the maximum principle. We may thus normalize δw by

$$\int_D A(u_0) \, \delta w \, dx = 1.$$

Then we can solve (3) with right-hand side equals:

$$\left\{ A(u_0) \left[\delta M_0 + \int_{\partial D} [A(u_0)/H_N] \, \delta B_0 \, \delta w \, da - \int_D A(u_0) \, \delta M_0 \, \delta w \, dx \right], [A(u_0)/H_N] \, \delta B_0 \right\}$$

since the latter satisfies (4). If $\delta u'_0$ is a solution, then

$$\delta u_0 = \delta u'_0 - \langle \delta u'_0 \rangle + \int_{\partial D} [A(u_0)/H_N] \, \delta B_0 \, \delta w \, da - \int_D A(u_0) \, \delta M_0 \, \delta w \, dx$$

solves the original equation

$$(m, b)(\delta u_0) = (\delta M_0, \delta B_0).$$

So (m, b) is also *onto*. The implicit function theorem thus implies the existence of a real $\delta>0$ and of a smooth map

$$t\in(t_0-\delta, t_0+\delta)\cap[0, 1]\rightarrow u_t\in U^{2,\alpha}$$

such that $(M, B)(t, u)=(0, 0)$. By proposition 1 and standard elliptic regularity [1], $u_t\in S(D, D_t)$, hence T_1 is relatively open. □

Assuming $n=2$, we shall carry out a $C^{2,\alpha}(\bar{D})$ *a priori* bound, independent of $t\in[0, 1]$, on the solutions in $S(D, D_t)$ of equations $(1.t)$ and $(2.t)$. Provided such a bound exists, the closedness of $T_i (i=1, 2)$ follows in a standard way from Ascoli's theorem combined with proposition 1 and elliptic regularity [1].

Last, let us prove that (1) admits *at most one* solution in $S(D, D^*)$; a similar argument holds for (2). By contradiction, let u_0 and u_1 be two distinct solutions of (1) in $S(D, D^*)$. Then, for $t\in[0, 1]$, $u_t:=tu_1+(1-t)u_0\in S(D, D^*)$ and $u:=u_1-u_0$ solves the linear boundary-value problem:

$$\left(\int_0^1 u_t^{ij}\,dt\right)u_{ij}-\left[\int_0^1 f_{u_i}(x, du_t)\,dt\right]u_i-\langle u\rangle=0 \quad \text{in } D,$$

$$\left[\int_0^1 (h_1)_i(du_t)\,dt\right]u_i=0 \quad \text{on } \partial D$$

which is elliptic inside D and oblique on ∂D by proposition 1. The maximum principle implies $u\equiv 0$, contradicting the assumption. □

III. PRELIMINARY *A PRIORI* ESTIMATES

In this section, we do *not* need yet the condition $n=2$. For any $v\in S(D, D_t)$, $dv\in D^*$, hence $|dv|$ is bounded above by $\rho(D^*):=\max_{x^*\in D^*}|x^*|$. Set $|f|_0=\max_{\bar{D}\times\bar{D}^*}|f(x, x^*)|$, and let $u\in S(D, D_t)$ solve $(1.t)$, then

$$e^{-|f|_0}A(u)\leqq e^{\langle u\rangle}\leqq e^{|f|_0+n|\operatorname{Log}\lambda|}A(u).$$

Integrating this over D yields for $\langle u\rangle$ the pinching:

$$\operatorname{Log}|D_0|-|f|_0\leqq\langle u\rangle\leqq\operatorname{Log}|D^*|+|f|_0+n|\operatorname{Log}\lambda|.$$

Since $|du|\leqq\rho(D^*)$, u is *a priori* bounded in $C^1(\bar{D})$ in terms of $|D^*|$, $\rho(D^*)$, $|f|_0$, $|D_0|$, λ and n.

By assumption, there exists $\mu\in(0, 1]$ such that $F_u\geqq\mu$ on $\bar{D}\times\bar{D}^*\times\mathbb{R}$. The right-hand side of equation $(2.t)$, let us denote it by

$$f(t, x, du, u),$$

thus satisfies $f_u \geqq \mu$ as well, on $[0, 1] \times \bar{D} \times \bar{D}^* \times \mathbb{R}$. Let $u \in S(D, D_t)$ solve $(2.t)$. Set

$$M := \max_{\bar{D}} (u), \qquad m := \min_{\bar{D}} (u)$$

$$M_0 := \max_{[0,1] \times \bar{D} \times \bar{D}^*} [f(t, x, x^*, 0)], \; m_0 := \min_{[0,1] \times \bar{D} \times \bar{D}^*} [f(t, x, x^*, 0)].$$

From the mean value theorem, we know that

$$M - m \leqq \rho(D^*) \delta(D),$$

$\delta(D)$ standing for the diameter of D. If $M \geqq 0$ and $m \leqq 0$, it implies $|u| \leqq \rho(D^*) \delta(D)$ and we are done. If not, say for instance $M < 0$, then $A(u) = \exp[f(t, x, du, u)] \leqq \exp[M_0 + \mu M]$. Integrating this over D yields: $\mu M \geqq [\mathrm{Log}(|D_0|/|D|) - M_0]$, hence under our assumption $[\mathrm{Log}(|D_0|/|D|) - M_0] < 0$ and

$$-m = \max_{\bar{D}} |u| \leqq \rho(D^*) \delta(D) + [M_0 - \mathrm{Log}(|D_0|/|D|)]/\mu.$$

Similarly, $m > 0$ yields $[\mathrm{Log}(|D^*|/|D|) - m_0] > 0$ and

$$M = \max_{\bar{D}} |u| \leqq \rho(D^*) \delta(D) + [\mathrm{Log}(|D^*|/|D|) - m_0]/\mu.$$

In any case, we obtain a $C^1(\bar{D})$ *a priori* bound on u in terms of $|D^*|$, $\rho(D^*)$, $|D|$, $\delta(D)$, $|D_0|$, M_0, m_0 and μ.

For simplicity, let us give a unified treatment of higher order *a priori* estimates for equations $(1.t)$ and $(2.t)$ by rewriting these equations into a single general form

$$\mathrm{Log}\, A(u) = \Gamma(t, x, du, u, \langle u \rangle). \qquad (*)$$

Let $u \in S(D, D_t)$ solve $(*)$. In this section, a constant will be said *under control* provided it depends only on the following quantities: $|u|_1$, *i.e.* the $C^1(\bar{D})$-norm of u, on the C^2-norm of Γ on

$$K := [0, 1] \times \bar{D} \times \bar{D}^* \times I \times I,$$

where $I = [-|u|_1, |u|_1]$, on the $C^0([0, 1], C^2)$-norm of $t \to h_t$ (the fixed path of defining functions, *cf. supra*), and on the *positive* real

$$\sigma := \min_{t \in [0, 1]} \sigma(t)$$

where $\sigma(t)$ is the smallest eigenvalue of $[(h_t)_{ij}]$ over $\bar{D}_t$.

Since u is convex, a $C^2(\bar{D})$ bound on u follows from a bound on

$$M_2 := \max_{(x, \theta) \in \bar{D} \times S} [u_{\theta\theta}(x)]$$

S standing for the unit sphere of R^n. Set $H := h_t(du)$ and consider

$$Q : (c, \theta, x) \in (0, \infty) \times S \times \bar{D} \to Q(c, \theta, x) = \mathrm{Log}[u_{\theta\theta}(x)] + c\, H(x).$$

PROPOSITION 2. — *There exists* $C \in (0, \infty)$ *under control such that, if* $\max_{(\theta, x) \in S \times \bar{D}} [Q(C, \theta, x)]$ *occurs at* $(z, x_0) \in S \times D$ *with* x_0 interior *to* D, *then* M_2 *is under control.*

This proposition does not refer to any boundary condition and constitutes by no means an interior estimate (it is rather the type of argument suited on a compact manifold). A similar proposition (with Δu and $|du|^2$, respectively in place of $u_{\theta\theta}$ and H) is lemma 2 of [13], later (and independently) reproved in [7] (p. 694); a similar argument is used in [4] (p. 398). Here proposition 2 may serve for the higher dimensional theorem, due to the special form of Q; so for completeness, we provide a detailed proof of it.

Proof. — Fix $(c, \theta) \in (0, \infty) \times S$ and consider Q as a function of x only. Let us record some auxiliary formulae: differentiating twice equation (∗) in the θ-direction yields,

$$u^{ij} u_{\theta ij} = (\Gamma)_\theta \equiv \Gamma_\theta + \Gamma_u u_\theta + \Gamma_{u_i} u_{\theta i} \tag{5}$$

$$u^{ij} u_{\theta\theta ij} = (\Gamma)_{\theta\theta} + u^{ik} u^{jm} u_{\theta ij} u_{\theta km} \tag{6}$$

with

$$(\Gamma)_{\theta\theta} \equiv \Gamma_{u_i} u_{\theta\theta i} + \Gamma_{u_i u_j} u_{\theta i} u_{\theta j} + 2(\Gamma_{\theta u_i} + u_\theta \Gamma_{u u_i}) u_{\theta i} + \Gamma_u u_{\theta\theta} + [\Gamma_{\theta\theta} + 2 u_\theta \Gamma_{\theta u} + \Gamma_{uu} (u_\theta)^2].$$

Differentiating twice H yields (with the subscript t, of h, dropped),

$$H_i = h_k u_{ik} \tag{7}$$

$$H_{ij} = h_k u_{ijk} + h_{km} u_{ik} u_{jm} \tag{8}$$

and similarly for Q,

$$Q_i = (u_{\theta\theta i}/u_{\theta\theta}) + c H_i$$

$$Q_{ij} = (u_{\theta\theta ij}/u_{\theta\theta}) - [u_{\theta\theta i} u_{\theta\theta j}/(u_{\theta\theta})^2] + c H_{ij}.$$

Combining (8) with (5) and (7), we get

$$u^{ij} H_{ij} = h_i (\Gamma_i + \Gamma_u u_i) + \Gamma_{u_i} H_i + h_{ij} u_{ij} \tag{9}$$

while from (6) we get,

$$u^{ij} Q_{ij} = [(\Gamma)_{\theta\theta}/u_{\theta\theta}] + (1/u_{\theta\theta}) [u^{ik} u^{jm} u_{\theta ij} u_{\theta km} - (1/u_{\theta\theta}) u^{ij} u_{\theta\theta i} u_{\theta\theta j}] + c u^{ij} H_{ij}.$$

Expanding the square

$$(u_{\theta\theta} u_{\theta ij} - u_{\theta i} u_{\theta\theta j})(u_{\theta\theta} u_{\theta km} - u_{\theta k} u_{\theta\theta m}) u^{ik} u^{jm}$$

one immediately verifies the identity:

$$u^{ik} u^{jm} u_{\theta ij} u_{\theta km} \geqq (1/u_{\theta\theta}) u^{ij} u_{\theta\theta i} u_{\theta\theta j}.$$

So,

$$u^{ij} Q_{ij} \geqq [(\Gamma)_{\theta\theta}/u_{\theta\theta}] + c u^{ij} H_{ij}.$$

Combining the expression of $(\Gamma)_{\theta\theta}$ with that of Q_i and (9) yields,

$$u^{ij} Q_{ij} - \Gamma_{u_i} Q_i \geqq c h_{ij} u_{ij} + (1/u_{\theta\theta}) \Gamma_{u_i u_j} u_{\theta i} u_{\theta j} + (2/u_{\theta\theta})(\Gamma_{\theta u_i} + u_\theta \Gamma_{u u_i}) u_{\theta i} + \Gamma_u + c h_i (\Gamma_i + \Gamma_u u_i) + (1/u_{\theta\theta})[\Gamma_{\theta\theta} + 2 u_\theta \Gamma_{\theta u} + \Gamma_{uu} (u_\theta)^2]. \quad (10)$$

Introducing the constant σ (defined above) we get

$$(1/u_{\theta\theta}) \Gamma_{u_i u_j} u_{\theta i} u_{\theta j} + \frac{1}{3} c h_{ij} u_{ij} \geqq (1/u_{\theta\theta}) u_{ik} u_{jm} \left(\theta^k \theta^m \Gamma_{u_i u_j} + \frac{1}{3} c \sigma u_{\theta\theta} \delta_{ij} u^{km} \right) \geqq (1/u_{\theta\theta}) u_{\theta i} u_{\theta j} \left(\Gamma_{u_i u_j} + \frac{1}{3} c \sigma \delta_{ij} \right)$$

this last inequality being obtained by noting that, identically for u strictly convex, $u_{\theta\theta} u^{km} \geqq \theta^k \theta^m$. Hence our first requirement on c is:

$$\left(\Gamma_{u_i u_j} + \frac{1}{3} c \sigma \delta_{ij} \right) \geqq 0$$

in the sense of symmetric matrices, over K. To express our second requirement on c, we first note that the inequality between the arithmetic and the geometric means of n positive numbers applied to the eigenvalues of (u_{ij}) and combined with (*), yields on D:

$$\Delta u \geqq n \exp\left(\frac{1}{n} \min_{K} \Gamma \right) = : \gamma.$$

Then we take c such that

$$2 \min [\Gamma_{y u_y}(r) + u_y(x) \Gamma_{u u_y}(r)] + \frac{1}{3} c \sigma \gamma \geqq 0$$

the minimum being taken on $(r, x, y) \in K \times \bar{D} \times S$. From now on, c has a fixed value under control, C, meeting both requirements and we take $(\theta, x) = (z, x_0)$ as defined in proposition 2. In particular, $u_{zz}(x_0)$ is now the *maximum* eigenvalue of $[u_{ij}(x_0)]$; diagonalizing the latter and using the *second* requirement on C, we obtain at x_0:

$$\frac{1}{3} C h_{ij} u_{ij} + (2/u_{zz})(\Gamma_{z u_i} + u_z \Gamma_{u u_i}) u_{zi} \geqq \frac{1}{3} C \sigma \Delta u + 2 (\Gamma_{z u_z} + u_z \Gamma_{u u_z}) \geqq 0.$$

Now (10) yields for $x \to Q = Q(C, z, x)$ at x_0,

$$u^{ij} Q_{ij} - \Gamma_{u_i} Q_i \geqq C \left(\frac{1}{3} \sigma u_{zz} - C' \right) - C'' (1/u_{zz}), \quad (11)$$

for some positive constants under control C′, C″. Since Q(C, z, .) assumes its *maximum* at $x_0 \in D$, (11) implies a controlled bound from above on

$u_{zz}(x_0)$, hence also on $(\theta, x) \to Q(C, \theta, x)$ and on $(\theta, x) \to u_{\theta\theta}(x)$. Therefore M_2 is under control. □

According to proposition 2, we may assume, without loss of generality, that the point x_0 above lies on ∂D, hence a $C^2(\bar{D})$ *a priori* bound on u follows from an *a priori* bound on $u_{zz}(x_0)$ which, in turn, coincides with $\max_{(\theta,\, x) \in S \times \partial D} [u_{\theta\theta}(x)]$.

IV. *A PRIORI* ESTIMATES OF SECOND DERIVATIVES ON THE BOUNDARY ($n=2$)

In this section we fix a defining function of D, denoted by k, and we include in the definition of constants *under control* the possible dependence on $|k|_3$, on $\tau := \min_{\partial D} k_N > 0$ and on the minimum over $\bar{D}$ of the smallest eigenvalue of (k_{ij}), denoted by $s > 0$.

We still let $u \in S(D, D_t)$ solve equation $(*)$. According to proposition 1 $H = h_t(du)$ which vanishes on ∂D, satisfies there $H_N > 0$; moreover, (7) implies on ∂D (dropping the subscript t of h):

$$h_i[du(x)] = H_N u^{ij} N^j(x). \tag{12}$$

In particular, the function on ∂D

$$\varphi(x) := N^i(x) h_i[du(x)]$$

is *positive*. Fix an arbitrary point $x_0 \in \partial D$ and a direct system of euclidean co-ordinates (O, x^1, x^2) satisfying $N(x_0) = \partial/\partial x^2$. Then (12) reads at x_0,

$$\left.\begin{aligned} u_{11}(x_0) &= (e^\Gamma/H_N)\,\varphi(x_0) \\ u_{12}(x_0) &= -(e^\Gamma/H_N)(x_0)\, h_1[du(x_0)] \end{aligned}\right\} \tag{13}$$

while equation $(*)$ itself provides for $u_{22}(x_0) = u_{NN}(x_0)$,

$$\varphi\, u_{22}(x_0) = H_N(x) + (e^\Gamma/H_N)(x_0)\{h_1[du(x_0)]\}^2. \tag{14}$$

We thus need positive lower bounds under control on $H_N(x_0)$ and $\varphi(x_0)$, as well as a controlled upper bound on $H_N(x_0)$.

Let us start with $H_N(x_0)$. Aside from (9), H also satisfies in D [still by combining (8), (5), (7)],

$$u^{ij} H_{ij} - u^{ij}(\Gamma)_i H_j = h_{ij} u_{ij}. \tag{15}$$

Set $T = u_{11} + u_{22}$, $T^* = u^{11} + u^{22}$, and note the identity: $T^* = A(u)T$. It implies the existence of positive constants under control, α, β, such that

$$\alpha T^* \leqq T \leqq \beta T^*, \tag{16}$$

which we simply denote by: $T \simeq T^*$. Consider the function

$$(c, x) \in (0, \infty) \times \bar{D} \to w(c, x) = H(x) - ck(x).$$

From (9) and $T \geqq \gamma$ (*cf. supra*), we infer

$$u^{ij}[w(c,.)]_{ij} \leqq -T\left[\frac{1}{2}c(s/\beta)-(u_{ij}/T)(h_{ij}+\Gamma_{u_i}h_j)\right] \\ -\left[\frac{1}{2}\gamma c(s/\beta)-h_i(\Gamma_i+\Gamma_u u_i)\right],$$

and there readily exists $c=C>1$, under control, such that the latter right-hand side is non-positive. Similarly (15) (16) yield:

$$u^{ij}[w(c,.)]_{ij}-u^{ij}(\Gamma)_i[w(c,.)]_j \geqq \frac{1}{2}\sigma T \\ -T\max\{0,(c/\alpha)u^{ij}/T^*)[k_{ij}-k_i(\Gamma_j+\Gamma_u u_j)]\}+\left(\frac{1}{2}\sigma\gamma+ck_i\Gamma_{u_i}\right),$$

(σ was defined at the beginning of section III) and there exists $c\in(0,1)$ under control such that the right-hand side is nonnegative. Since w identically vanishes on $(0,\infty)\times\partial D$, Hopf's maximum principle [11] implies the following pinching under control on ∂D:

$$c\tau \leqq ck_N \leqq H_N \leqq Ck_N \leqq C|k|_1. \tag{17}$$

Combined with (13), it implies a controlled upper bound on $|u_{11}(x_0)|+|u_{12}(x_0)|$. Furthermore, combined with (14), it implies also (the notation $\simeq$ is defined at (16))

$$u_{22}(x_0) \simeq 1/\varphi(x_0). \tag{18}$$

We now turn to a lower bound on $\varphi(x_0)$ and consider the function

$$(c,x)\in(0,\infty)\times\bar{D}\to P(c,x)=\psi-ck,$$

where

$$\psi(x):=k_i(x)h_i[du(x)].$$

A routine computation using (5) yields in D:

$$u^{ij}\psi_{ij}=k_i h_{ij}(\Gamma_j+\Gamma_u u_j+\Gamma_{u_m}u_{jm})+2k_{ij}h_{ij}+k_i h_{ijm}u_{jm}+u^{ij}k_{ijm}h_m.$$

It implies the existence of a constant c_1 under control such that, in D,

$$u^{ij}P_{ij} \leqq c_1(1+T)-c(s/\beta)T=-\left[\frac{1}{2}c\gamma(s/\beta)-c_1\right]-T\left[\frac{1}{2}c(s/\beta)-c_1\right];$$

let us choose $c=C_0:=2c_1\beta/s\min(1,\gamma)$, so that $u^{ij}[P(C_0,.)]_{ij}\leqq 0$ in D. By Hopf's maximum principle [11], $P(C_0,.)$ necessarily assumes its *minimum* over $\bar{D}$ at a boundary point y_0 where

$$\psi_N \leqq C_0 k_N. \tag{19}$$

Pick a euclidean system of co-ordinates (O,y^1,y^2) such that $N(y_0)=\partial/\partial y^2$. Then $dk(y_0)=k_N\,\partial/\partial y^2$ while, using (13) (17):

$$|u_{12}(y_0)| \leqq C_1:=e^{|\Gamma|_0}|h|_1/c\tau$$

is under control, and (19) reads:

$$u_{22}(y_0)\,k_N(y_0)\,h_{22}[du(y_0)] \leqq C_0\,k_N(y_0) - k_{2i}(y_0)\,h_i[du(y_0)] - k_N(y_0)\,u_{12}(y_0)\,h_{12}[du(y_0)].$$

It implies

$$\sigma\gamma\,u_{22}(y_0) \leqq C_0\,|k|_1 + |h|_1\,|k|_2 + C_1\,|k|_1\,|h|_2$$

i.e. a controlled bound from above on $u_{22}(y_0)$. Recalling (18), it means a controlled positive bound from below, λ, on $\varphi(y_0)$. Since on ∂D, $P(C_0, .) \equiv k_N\,\varphi$, and since $P(C_0, .)$ assumes its *minimum* at y_0, we infer on ∂D:

$$\varphi(x) \geqq \lambda\,k_N(y_0)/k_N(x) \geqq \lambda\tau/|k|_1.$$

Using (18) again, we obtain a controlled upper bound on $u_{22}(x_0)$. The second derivatives of u are thus *a priori* bounded on ∂D. □

Remarks. — 5. Proposition 1 and (12) show that the lower bound $\varphi \geqq \lambda$ ensures *a priori* the *uniform obliqueness* of the boundary operator at u. Geometrically, it implies another positive lower bound on the scalar product of the outward unit normals, to ∂D at x and to ∂D_t at $du(x)$.

6. Let (T, N) and (T*, N*) be direct orthonormal moving frames on ∂D and on ∂D_t respectively (N* stands for the outward unit normal on ∂D_t) and let z_0 be a critical point of: $x \in \partial D \to N(x) . N^*[du(x)]$. Denote by $J\,du$ the Jacobian (or differential) of the gradient mapping du. With the help of Frénet's formulae, one verifies that

$$|J\,du[T(z_0)]| = (R_0^*/R_0), \tag{20}$$

R_0 (resp. R_0^*) standing for the curvature radius of ∂D at z_0 [resp. of ∂D_t at $du(z_0)$]. Equation (∗) implies that the area of the parallelogram [$J\,du(T)$, $J\,du(N)$] equals $\exp(\Gamma)$, in particular, it is uniformly bounded *below* by a positive constant. What happens if we drop the *strict* convexity of ∂D at z_0, but keep that of ∂D_t at $du(z_0)$, *i.e.* if we let R_0 go to infinity and R_0^* remain bounded? From (20), $|J\,du[T(z_0)]|$ goes to zero hence $|J\,du[N(z_0)]|$ goes to infinity. In a direct system of euclidean co-ordinates $(0, x^1, x^2)$ such that $N(z_0) = \partial/\partial x^2$, it implies that $|u_{11}(z_0)| + |u_{12}(z_0)|$ goes to zero while $|u_{22}(z_0)|$ blows up like R_0 *i.e.* the control on $u_{NN}(z_0)$ is lost.

V. HIGHER ORDER *A PRIORI* ESTIMATES

Let $u \in S(D, D_t)$ solve equation (∗). Fix a generic point $x \in \bar{D}$ and choose a euclidean co-ordinates system which puts $[u_{ij}(x)]$ into a *diagonal* form.

Observe that for each $i \in \{1, \ldots, n\}$,

$$u_{ii}(x) = \mathrm{A}(u)/\prod_{j \neq i} u_{jj}(x) \geqq \gamma/(|u|_2)^{n-1}. \tag{21}$$

In case $n=2$, the $C^2(\bar{D})$ *a priori* estimate drawn on u in the two preceding sections thus implies the controlled *uniform* ellipticity of $d[\operatorname{Log} \mathrm{A}(u)]$ on $\bar{D}$. Given $\alpha \in (0, 1)$, a $C^{2,\alpha}(\bar{D})$ *a priori* bound on u now follows from the general theory of [15] (section 6); however, this bound is so straightforward for $n=2$ that we include it for completeness.

First of all, given any interior subdomain D' of D and any $z \in S$, the 2-dimensional regularity theory of [17] applied to u_z, which satisfies (5) in D', yields a $C^{1,\alpha}(\bar{D}')$ *a priori* bound under control on u_z, hence, since z is arbitrary, a controlled $C^{2,\alpha}(\bar{D}')$ *a priori* bound on u. The theory of [17] also applies to H which satisfies (9) in D and vanishes on ∂D: it yields a $C^{1,\alpha}(\bar{D})$ *a priori* bound under control on H. Solving for u_{11}, u_{12} and u_{22} the 3×3 system given by (7) and equation (*), we get (dropping the subscript t of h):

$$\left.\begin{aligned} u_{11} &= [(\mathrm{H}_1)^2 + (h_2)^2 e^{\Gamma}]/\Delta \\ u_{12} &= (\mathrm{H}_1 \mathrm{H}_2 - h_1 h_2 e^{\Gamma})/\Delta \\ u_{22} &= [(\mathrm{H}_2)^2 + (h_1)^2 e^{\Gamma}]/\Delta, \end{aligned}\right\} \tag{22}$$

where

$$\Delta(x) := \mathrm{H}_i(x)\, h_i[du(x)].$$

Note that (7) and (21) imply

$$\Delta(x) \geqq (\gamma/|u|_2)\, |dh[du(x)]|^2. \tag{23}$$

Given any small enough $\delta \in (0, 1)$, let

$$\mathrm{D}_\delta := \{x \in \mathrm{D}, \operatorname{dist}(x, \partial \mathrm{D}) < \delta\}.$$

From the $C^2(\bar{D})$ *a priori* estimate precedingly drawn on u, it follows that the gradient image $du(\mathrm{D}_\delta)$ is contained in $(\mathrm{D}_t)_{\mathrm{C}\delta}$ for some positive constant C under control. If $\tau^* := \min_{t \in [0,1]} (\min_{\partial_t \mathrm{D}} |dh_t|)$, then there readily exists $\delta_0 \in (0, 1)$ under control such that, for any $x \in \mathrm{D}_{\delta_0}$, $|dh_t[du(x)]| \geqq \tau^*/2$. Therefore (22) and (23) imply a $C^\alpha(\bar{D}_{\delta_0})$ *a priori* bound under control on the second derivatives of u. A $C^{2,\alpha}(\bar{D})$ *a priori* bound on u follows.

Actually, a straightforward "bootstrap" argument now provides $C^{k,\alpha}(\bar{D})$ *a priori* bounds on u for each integer $k > 2$.

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