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Heteroclinic orbits for spatially periodic Hamiltonian systems

by

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ABSTRACT. — We study the existence of heteroclinic orbits for a Hamiltonian system

$$\dot{p} = -H_q(p, q)$$
$$\dot{q} = H_p(p, q)$$

where the Hamiltonian is periodic in the space variable q and superlinear in p. We use the Saddle Point Theorem to obtain existence of solutions for a finite time interval, and then we obtain heteroclinic orbits as limit of them. Our hypothesis on H are motivated by the second order Lagrangean systems on the torus.

Key words : Critical Point Theory, Hamiltonian System, Heteroclinic Orbits.

RÉSUMÉ. — On étudie l'existence des orbites hétérocliniques pour le système hamiltonien

$$\dot{p} = -H_q(p, q)$$
$$\dot{q} = H_p(p, q)$$

quand l'Hamiltonien est périodique par rapport à la variable q et superlinéaire en p. On utilise le théorème de Point Selle pour obtenir des solutions dans un intervalle de temps fini et on obtient donc les orbites hétérocliniques comme limites. Les hypothèses qu'on utilise sur H sont motivées par les systèmes Lagrangiens sur le torus.

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0. INTRODUCTION

In this paper we study the existence of heteroclinic orbits of some autonomous Hamiltonian systems. We generalize results obtained by Rabinowitz [8] for the equation

$$\ddot{q}+V'(q)=0 \tag{0.1}$$

with periodic potential. Rabinowitz studied the problem using a variational approach through a minimization procedure. In this work we consider a general Hamiltonian system

$$\left.\begin{aligned}\dot{p}&=-H_q(p, q)\\ \dot{q}&=H_p(p, q)\end{aligned}\right\} \quad \text{(HS)}$$

with Hamiltonian H periodic in the q variables. In the case of equation (HS) a minimization argument can not be applied.

Our method consists in studying approximate problems by letting the time interval being finite. This idea has been used by Tanaka [10] and Rabinowitz [9] in the study of homoclinic orbits for some second order Hamiltonian systems with singular potential. In order to study the approximate problem we use a version of the Saddle Point Theorem of Rabinowitz. We obtain estimates for the critical values, independent of the length of the time interval. We use then the estimates in passing to the limit. We note that the problem of heteroclinic orbits, due to the infinite time interval, lacks of the compactness one usually needs to use critical point theory.

At this point we mention the work of Coti Zelati and Ekeland [5] where the study of homoclinic orbits for Hamiltonian systems is undertaken. Their approach is based on convexity assumptions on the Hamiltonian, that allows to use a dual formulation, and the concentrated compactness of P. L. Lions. Hofer and Wysocki [6] generalized the results of [5], dropping the convexity assumption; they study the problem considering certain first order elliptic systems.

We describe our results now. We consider a Hamiltonian $H : \mathbb{R}^{2n} \to \mathbb{R}$, and we denote $H(0, q)=V(q)$. We note that in the case of system (0.1) the Hamiltonian is given by $H(p, q)=\frac{1}{2}|p|^2+V(q)$, so that $H(0, q)$ corresponds exactly to the potential.

We make the following hypotheses on the Hamiltonian

(H0) H is of class C^1.

(H1) $H(p, q)=H(p, q+2\pi k)$, $\forall p\in\mathbb{R}^n$, $\forall q\in\mathbb{R}^n$, $\forall k\in\mathbb{Z}^n$.

(H2) $\max_{q\in\mathbb{R}^n} V(q)=\bar{V}$ and the set $M=\{q\in\mathbb{R}^n/V(q)=\bar{V}\}$ is discrete.

(H3) There is a constant $\mu>1$ such that

$$0<\mu(H(p, q)-V(q))\leqq H_p(p, q).p, \qquad \forall (p, q)\in\mathbb{R}^{2n}, \quad p\neq 0.$$

(H4) There are constants $\varepsilon_0>0$ and $a_1>0$ such that every $\bar{q}\in M$

$$H(p, q)\geqq a_1|p|^{\mu}+V(q), \qquad \forall (p, q)\in \mathbb{R}^{2n}, \quad |p|\leqq \varepsilon_0, \quad |q-\bar{q}|\leqq \varepsilon_0.$$

(H5) For constants $s\leqq\mu$, $a_2>0$, $a_3>0$

$$|H_q(p, q)|\leqq a_2|p|^s+a_3, \qquad \forall (p, q)\in \mathbb{R}^{2n}.$$

Here and in the future we denote by . the usual inner product in $\mathbb{R}^n$ and by $|\,.\,|$ its norm. We normalize H so to have that $\bar{V}=0$ and that $V(0)=0$. We will prove the following theorem.

THEOREM 0.1. – *If* (H0)-(H5) *hold then* (*HS*) *possesses at least* 2 *heteroclinic orbits, one emanating from* 0 *and one terminating at* 0.

Remark 0.1. – In Theorem 0.1 we can replace 0 by any other point in M.

The method used to prove Theorem 0.1 can also be applied to a problem in which H is not periodic in q. Consider

(H1n) The function $H_p(p, q)$ is bounded on sets of the form $B_\delta\times\mathbb{R}^n$, where $B_\delta=\{p\in\mathbb{R}^n/|p|\leqq\delta\}$.

(H2n)

$$\limsup_{|q|\to\infty} V(q)<\bar{V}$$

and the set M as defined in (H2) is discrete and it contains at least two points.

Then we have

THEOREM 0.2. – *If* (H0), (H1n), (H2n), (H3)-(H5) *hold then* (*HS*) *possesses at least* 2 *heteroclinic orbits, one emanating from* 0 *and one terminating at* 0.

Coming back to a periodic Hamiltonian, let us consider

(H2′) $M=\{\bar{q}+2\pi k/k\in\mathbb{Z}^n\}$

then we have

THEOREM 0.3. – *If* H *satisfies* (H0), (H1), (H2′), (H3)-(H5) *then the* (HS) *possesses at least* $2n$ *heteroclinic orbits,* n *emanating from* 0 *and* n *terminating at* 0. *If we further assume*

(H6) $H(p, q)=H(-p, q)$, $\forall (p, q)\in\mathbb{R}2^n$, *then there are* $2n$ *additional heteroclinic orbits,* n *emanating from* 0 *and* n *terminating at* 0.

Remark 0.2. – If in Theorem 0.2 we assume (H6) then (HS) possesses at least 4 heteroclinic orbits.

As we mention above our results generalize the case

$$H(p, q)=\frac{1}{2}|p|^2+V(q) \tag{0.2}$$

corresponding to equation (0.1). Other case of interest, not covered in [8], is the Lagrangean system with Lagrangean

$$L(p, q) = Q(q)p.p - V(q). \tag{0.3}$$

The Hamiltonian

$$H(p, q) = Q(q)^{-1}p.p + V(q) \tag{0.4}$$

corresponds to (0.3) and it satisfies our hypothesis if Q and V are periodic in q, and Q is positive definite. The Lagrangean of the n-pendulum has the form (0.3), and then the Hamiltonian is given by (0.4) that also satisfies (H2′) and (H6). Thus Theorem 0.3 guarantees the existence of at least $4n$ heteroclinic orbits for the n-pendulum. This result complements recent works on the forced n-pendulum, see Fournier and Willem [4], Chang, Long and Zehnder [1] and Felmer [2].

This paper is divided in four sections. In Section 1 we present a version of the Saddle Point Theorem that we use to study an approximate problem, for finite time intervals. In section 2 we consider the approximate problems and prove existence of solutions for every time interval. In Section 3 we obtain estimates on the critical values of the solutions to the approximate problems, that are independent of the length of the time interval. In Section 4 we let the length of the time interval to go to infinity and we prove Theorems 0.1, 0.2 and 0.3.

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1. SADDLE POINT THEOREM

The Saddle Point Theorem of Rabinowitz [7] provides a tool for finding critical points of functionals. Here we give a variation of that result that we will use in our application.

We consider a Hilbert space E with inner product $\langle .,. \rangle$ and norm $\| . \|$. We assume that E has a splitting $E = X \oplus Y$, where the subspaces X and Y are not necessarily orthogonal and both of them can be infinite dimensional.

Let $I : E \to \mathbb{R}$ be a functional having the following structure

$$I(z) = \langle Lz, z \rangle + b(z) \tag{1.1}$$

where

(I1) $L : E \to E$ is selfadjoint,

(I2) b' is compact.

Let us consider a family of bounded, linear operators $B(s) : E \to E$ where $s \in (0, 1]$. Assume that for some $s_0 \in (0, 1]$, $B(s_0) = \mathrm{id}_E$, and that B depends continuously on s. We will assume

(I3) If $s \in (0, 1]$ and $\nu \in \mathbb{R}_0^+$ then the linear operator

$$\tilde{B} = P_X \, exp(\nu L) B(s) : X \to X$$

is invertible, and its inverse depends continuously on s and ν.

Here P_X denotes the projection of E onto X induced by the splitting $E = X \oplus Y$. Let $R > 0$ and define $Q = \{x \in X / \|x\| \leqq R\}$ and $\partial Q = \{x \in X / \|x\| = R\}$. We define the class of functions

$$\Gamma = \{h \in C(E \times [0, 1], E)/h \text{ satisfies } \Gamma 1, \Gamma 2 \text{ and } \Gamma 3\}$$

where

(Γ1) h is given by

$$h(z, t) = \exp(\nu(z, t) L) B(s(z, t)) z + K(z, t)$$

where $s : E \times [0, 1] \to (0, 1]$ and $\nu : E \times [0,1] \to \mathbb{R}_0^+$ are continuous, ν transforms bounded sets into bounded sets, $s(E \times [0, 1])$ stays away from 0, $K : E \to E$ is compact and $K(u, 0) = 0$.

(Γ2) $h(x, t) = x$, $\forall x \in \partial Q$.

(Γ3) $h(x, 0) = x$, $\forall x \in Q$.

Taking $\nu \equiv 0$, $s \equiv s_0$ and $K \equiv 0$ we obtain that $h = \mathrm{id}_E$ belongs to Γ so it is not empty. It can also be proved that the class satisfies the following composition property: if $\eta \in \Gamma$ has the form

$$\eta(z, t) = \exp(\theta(z, t) L) z + K_\eta(z, t) \tag{1.2}$$

then $\eta(h(x, t), t) \in \Gamma$, $\forall h \in \Gamma$. Now we state the theorem

THEOREM 1.1 (Saddle Point Theorem). – *Let* $I : E \to \mathbb{R}$ *of class* C^1 *satisfying the Palais-Smale condition and* (I1), (I2) *and* (I3), *further assume*

(I4) *There are constants* $\alpha > w$ *such that*

(i) $I(y) \geqq \alpha$, $\forall y \in Y$.

(ii) $I(x) \leqq w$, $\forall x \in \partial Q$.

Then I *possesses at least one critical point with critical value* $c \geqq \alpha$, *characterized by*

$$c = \inf_{h \in \Gamma} \sup_{z \in Q} I(h(z, 1)).$$

Remark 1.1. – The only difference with Rabinowitz Saddle Point Theorem is that here the class Γ is bigger. Even though this version does not produce in general a smaller critical value, it makes the estimates in Section 3 somewhat easier.

Proof. – The proof goes in the standard way so that here we only mention the differences. Since Q is bounded we find that $c < \infty$. Now we show that $c \geqq \alpha$. By (I4) (i) we only need to show that

$$h(Q, 1) \cap Y \neq \varnothing, \qquad \forall h \in \Gamma. \tag{1.3}$$

Displaying the form of h and using (I2), (1.3) is equivalent to find $z\in Q$ such that

$$x+\{P_X \exp(\nu(x,1)L)B(s(x,1))\}^{-1}P_X K(x,1)=0. \qquad (1.4)$$

We define $\psi(x)$ as the left hand side of (1.4). Let

$$H(x,t)=x+\{P_X \exp(\nu(x,t)L)B(s(x,t))\}^{-1}P_X K(x,t)$$

for $x\in X$ and $t\in[0,1]$. Then we have that H is an admissible deformation and $H(x,0)=x$, $\forall x\in\partial Q$. Then by the homotopy invariance of the Leray-Schauder degree we have

$$1=\deg(H(x,0),0,Q)=\deg(H(x,1),0,Q)=\deg(\psi(x),0,Q).$$

Thus, (1.4) has at least one solution. This proves (1.3). To show that c is a critical value we proceed in the standard way, *see* [7]. We only note that the deformation provided by the Deformation Lemma has the form (1.2) so that $\eta\circ h\in\Gamma$ for every $h\in\Gamma$. □

2. THE APPROXIMATE PROBLEM

Given $k\in Z^n$, and $T>0$ we consider the Hamiltonian system

$$\left.\begin{aligned}&\dot z=JH_z(z)\\ &q(0)=0,\qquad q(T)=2\pi k.\end{aligned}\right\}\ (HS)_T$$

Let $\bar e=(2\pi k)/T$, $e(t)=\bar e t$ and $\xi(t)=(0,e(t))\in\mathbb{R}^{2n}$. If we define the constant forcing term $f=(\bar e,0)$, we can easily prove that if $z(t)$ satisfies

$$\left.\begin{aligned}&\dot z=J(H_z(z+\xi(t))+f)\\ &q(0)=0=q(T)\end{aligned}\right\}\ (KS)_T$$

then $\tilde z(t)=z(t)+\xi(t)$ satisfies $(HS)_T$. We devote this section to find solutions to $(KS)_T$. We consider the space E of functions $z:[0,T]\to\mathbb{R}^{2n}$, $z=(p,q)$ with Fourier series

$$p(t)=\sum_{l=0}^{\infty} b_l\cos\left(\frac{\pi}{T}lt\right),\qquad q(t)=\sum_{l=1}^{\infty} a_l\sin\left(\frac{\pi}{T}lt\right),\qquad (2.1)$$

where a_l, $b_l\in\mathbb{R}^n$ and

$$\sum_{l=1}^{\infty}(|a_l|^2+|b_l|^2)l+|b_0|^2<\infty. \qquad (2.2)$$

If $\zeta=(\varphi,\psi)\in E$ with Fourier series

$$\varphi(t)=\sum_{l=0}^{\infty}\beta_l\cos\left(\frac{\pi}{T}lt\right),\qquad \psi(t)=\sum_{l=1}^{\infty}\alpha_l\sin\left(\frac{\pi}{T}lt\right),\qquad (2.3)$$

then we define the inner product

$$\langle z, \xi \rangle = \sum_{l=1}^{\infty} (b_l . \beta_l + a_l . \alpha_l) l + b_0 . \beta_0, \tag{2.4}$$

that induces the norm

$$\| z \|_{\mathrm{E}}^2 = \sum_{l=1}^{\infty} (|b_l|^2 + |a_l|^2) l + |b_0|^2. \tag{2.5}$$

The space E with the inner product $\langle .,. \rangle$ is a Hilbert space, that can be isometrically embedded into $\mathrm{W}^{1/2,2}(\mathrm{S}^1, \mathbb{R}^{2n})$. This fact allows to prove that for every $1 \leqq s < \infty$ there is a constant α_s such that

$$\| z \|_s \leqq \alpha_s \mathrm{T}^{1/s} \| z \|_{\mathrm{E}}, \tag{2.6}$$

here, and in the future $\| . \|_s$ denotes the usual norm in $\mathrm{L}^s(0, \mathrm{T}; \mathbb{R}^{2n})$. We define the following subspaces of E

$$\mathrm{E}_q = \{ (0, q) \in \mathrm{E} \},$$
$$\mathrm{E}_p^0 = \mathrm{span}\{ e_1, \ldots, e_n \},$$

and

$$\mathrm{E}^- = \mathrm{span}\left\{ \cos\left(\frac{\pi}{\mathrm{T}} lt\right) e_j - \sin\left(\frac{\pi}{\mathrm{T}} lt\right) e_{j+n},\ 1 \leqq j \leqq n, j \geqq 1 \right\},$$

here $\{ e_1, \ldots, e_{2n} \}$ denotes the usual basis of $\mathbb{R}^{2n}$. It is easy to see that $\mathrm{E} = \mathrm{X} \oplus \mathrm{Y}$ where $\mathrm{X} = \mathrm{E}_p^0 \oplus \mathrm{E}^-$ and $\mathrm{Y} = \mathrm{E}_q$.

Let us define now some operators we need later. Let $z = (p, q)$ and $\zeta = (\varphi, \psi)$ be smooth functions in E, then we define

$$\mathscr{B}(z, \zeta) = \int_0^{\mathrm{T}} p . \dot{\psi} + \varphi . \dot{q}\, dt \tag{2.7}$$

and

$$\mathscr{A}(z) = \frac{1}{2} \mathscr{B}(z, z) = \int_0^{\mathrm{T}} p . \dot{q}\, dt. \tag{2.8}$$

The symmetric form $\mathscr{B}$ can be continuously extended to $\mathrm{E} \times \mathrm{E}$, and it induces a linear, bounded, selfadjoint operator $\mathrm{L} : \mathrm{E} \to \mathrm{E}$ defined by

$$\langle \mathrm{L} z, \zeta \rangle = \mathscr{B}(z, \zeta), \tag{2.9}$$

thus we have

$$\mathscr{A}(z) = \frac{1}{2} \langle \mathrm{L} z, z \rangle. \tag{2.10}$$

Remark 2.1. — We observe that $\mathscr{A}$ is negative on E^-, and it vanishes on E_p^0 and E_q.

We define now the operator B appearing in the Saddle Point Theorem proved in Section 1. Given $s\in(0,1]$ we define $B(s):E\to E$ by

$$B(s)z=B(s)(p,q)=\left(\frac{1}{s}p,\ sq\right). \tag{2.11}$$

Certainly $B(s)$ is a bounded, linear and it depends continuously on s. We note that

$$\mathscr{A}(B(s)z)=\mathscr{A}(z),\qquad \forall z\in E. \tag{2.12}$$

LEMMA 2.1. — *For every* $\nu\in\mathbb{R}_0^+$ *and* $s\in(0,1]$ *the linear operator*

$$\tilde{B}=P_X\exp(\nu L)B(s):X\to X$$

is invertible and its inverse depends continuously on s *and* ν.

Proof. — By calculations using Fourier series we obtain explicit formula for $\tilde{B}$. If $z=z^-+z^0$, $z^-\in E^-$, $z^0\in E^0$ then

$$P_X\exp(\nu L)B(s)z=m(s,\nu)z^-+\frac{1}{s}z^0, \tag{2.13}$$

where

$$m(s,\nu)=\frac{1}{s}\cosh\left(\frac{\nu\pi}{2}\right)-s\sinh\left(\frac{\nu\pi}{2}\right).$$

The number $m(s,\nu)$ is certainly positive $\forall\nu\geqq 0$, $s\in(0,1]$ then $\tilde{B}$ is invertible, and its inverse is given by

$$\tilde{B}^{-1}z=\frac{1}{m(s,\nu)}z^-+sz^0$$

and it depends continuously on s and ν. □

After this preliminaries we consider the variational formulation of $(KS)_T$. Let us assume for the moment that the Hamiltonian H satisfies the following growth condition (G). There are constants $a>0$, $b>0$ and $s>1$ so that

$$|H_p(p,q)|\leqq a|p|^s+b,\qquad \forall(p,q)\in\mathbb{R}^{2n}.$$

Then we can define the functional

$$I_T(z)=\mathscr{A}(z)-\int_0^T H(z+\xi(t))\,dt-\int_0^T \bar{e}.p\,dt$$

on E. This functional is well defined and it has the form (1.1). If we define

$$\mathscr{H}(z)=\int_0^T H(z+\xi(t))+\bar{e}.p\,dt$$

then $\mathscr{H}(z)$ is of class C^1 and its derivative is compact. *See* [7]. Then by Lemma 2.1 we have that I_T satisfies (I1)-(I3). The following proposition relates the critical points of I_T with the solutions of $(KS)_T$.

PROPOSITION 2.1. – *If $z \in E$ is a critical point of I_T then $z(t)$ is of class C^1 and it satisfies* $(KS)_T$.

Proof. – See [7].

Now we study the existence of solutions of $(KS)_T$. Since the Hamiltonian H does not satisfy the growth condition (G) necessarily, the functional $\mathscr{H}$ may be not well defined in E. We will make a modification following a trick used by Rabinowitz.

Let $\kappa > \varepsilon_0$, where ε_0 is defined in (H4) and $\chi \in C^\infty(\mathbb{R}^+, \mathbb{R}^+)$ such that $\chi(y) = 1$ if $0 \leqq y \leqq \kappa$, $\chi(y) = 0$ if $y \geqq \kappa + 1$ and $\chi'(y) < 0$ for $\kappa < y < \kappa + 1$. Let M be a constant so that

$$M \geqq \max \left\{ \frac{H(p, q) - V(q)}{|p|^\mu} \Big/ \kappa \leqq |p| \leqq \kappa + 1,\ q \in \mathbb{R}^n \right\}. \tag{2.14}$$

We define

$$H_\kappa(p, q) = \chi(|p|) H(p, q) + (1 - \chi(|p|))(M|p|^\mu + V(q)). \tag{2.15}$$

The following lemma can be easily proved.

LEMMA 2.2. – *For every $\kappa > \varepsilon_0$, the Hamiltonian H_κ satisfies the analogues to* (H0), (H1), (H3)-(H5), *with exactly the same constants.*

The following inequalities follow from (H3) and the fact that $\kappa > \varepsilon_0$. There are constants a_3 and a_4 independent of κ so that

$$H_\kappa(p, q) \geqq a_3 |p|^\mu - a_4, \qquad \forall (p, q) \in \mathbb{R}^{2n} \tag{2.16}$$

and

$$H_\kappa(p, q) \geqq a_3 |p|^\mu + V(q), \qquad \forall |p| \geqq \varepsilon_0, \quad \forall q \in \mathbb{R}^n. \tag{2.17}$$

On the other hand, the definition of $H_\kappa(p, q)$ implies that there are constants a_5 and a_6 so that

$$|H_{\kappa z}(p, q)| \leqq a_5 |p|^{\mu - 1} + a_6 \tag{2.18}$$

here a_5 and a_6 may depend on κ. From (2.18) we can define the functional

$$I_T^\kappa(z) = \mathscr{A}(z) - \int_0^T H_\kappa(z + \xi(t)) + \bar{e}.p\, dt. \tag{2.19}$$

We note that if $z = (p, q)$ with $|p(t)| \leqq \kappa$, $\forall t \in [0, T]$ then

$$I_T^\kappa(z) = \mathscr{A}(z) - \int_0^T H(z + \xi(t)) + \bar{e}.p\, dt.$$

LEMMA 2.3. – *The functional I_T^κ possesses at least one critical point z_T^κ.*

Proof. — By (2.18) we can use the same argument given in Lemma 3.2 in [3] to show that $I_T^\varkappa$ satisfies the Palais-Smale condition. By the structure of $I_T^\varkappa$ discussed above, and Lemma 2.2 we only need to prove (I4). For $z \in E_q \equiv Y$, $z=(0, q)$ we have

$$I_T^\varkappa(z) = \mathscr{A}(z) - \int_0^T H_\varkappa(z+\xi(t)) + \bar{e}.p\,dt$$
$$= -\int_0^T V(q+e(t))\,dt \geqq 0.$$

Now, if $z = z^- + z^0 = (p, q) \in E^- + E_p^0 = X$ we have

$$I_T^\varkappa(z^- + z^0) = \mathscr{A}(z^-) - \int_0^T H_\varkappa(z^- + z^0 + \xi(t)) + \bar{e}.p\,dt. \qquad (2.20)$$

From (2.16), the definition of $\|\,.\,\|_E$ and the Schwarz inequality we have

$$I_T^\varkappa(z^- + z^0) \leqq -\pi \| z^- \|_E^2 - a_3 \| p \|_\mu^\mu + T\,a_4 + |\bar{e}|\,\| p \|_1$$

Using (2.6) and the projection from $E^- \oplus E_p^0$ onto E_p^0 we find

$$I_T^\varkappa(z^- + z^0) \leqq -\pi \| z^- \|_E^2 + a_7 \| z^- \|_E - a_8 | z^0 |^\mu + a_7 | z^0 | + T\,a_4$$

Then, for $R \geqq R_0$, with R_0 big enough

$$I_T^\varkappa(z^- + z^0) \leqq -1, \qquad \forall \| z^- + z^0 \|_E = R. \qquad (2.21)$$

From (2.20) and (2.21) we obtain (I4), then we apply the Saddle Point Theorem to obtain the result. □

Remark 2.2. — Any $R \geqq R_0$ will make the hypothesis (I4) of the Saddle Point Theorem to be satisfied. In the next Lemma we will precise how to choose R. This lemma will be used in the limit process and we postpone its proof to the next section.

LEMMA 2.4. — *For every* T *there is* R *depending only on* T *so that for a constant c independent of* T *and* κ

$$I_T^\varkappa(z_T^\varkappa) \leqq c.$$

We use Lemma 2.4 to prove the following proposition

PROPOSITION 2.2. — *For every* $T>0$ *there is a solution* z_T *of the system* $(KS)_T$ *and*

$$0 \leqq I_T(z_T) = \mathscr{A}(z_T) - \int_0^T H(z_T + \xi(t)) + \bar{e}.p_T\,dt \leqq c. \qquad (2.22)$$

Proof. — The proof consists in showing that given R defined in Lemma 2.4 there is κ large enough so that for $z_T^\varkappa = (p_T^\varkappa, q_T^\varkappa)$ we have $|p_T^\varkappa(t)| \leqq \kappa$, $\forall t \in [0, T]$. Thus, by the definition of $H_\varkappa$ we see that

$H_\varkappa(z_T^\varkappa+\xi(t))=H(z_T^\varkappa+\xi(t))$ and then

$$0\leqq I_T(z_T)=I_T^\varkappa(z_T^\varkappa)\leqq c,$$

and $z_T^\varkappa$ is a solution of $(KS)_T$.

Now we choose κ. Since $z_T^\varkappa\equiv z=(p, q)$ is critical point of $I_T^\varkappa$, and using (H 3) for $H_\varkappa$ and (2.16) we have

$$\begin{aligned} c &\geqq I_T^\varkappa(z)-I_T^{\varkappa'}(z).p \\ &=\int_0^T -H_\varkappa(z+\xi(t))+H_{\varkappa p}(z+\xi(t)).p\,dt \\ &\geqq \int_0^T (\mu-1)H_\varkappa(z+\xi(t))-\mu V(q+e(t))\,dt \\ &\geqq (\mu-1)a_3\|p\|_\mu^\mu-a_9 T. \end{aligned}$$

But this implies that there is a constant a_{10} independent of κ such that

$$\|p\|_\mu^\mu\leqq a_{10}.$$

Following from here a standard argument, and noting that $H_\varkappa$ satisfies (H 5) with constant independent of κ we obtain that $|p(t)|$ is bounded independent of κ. See for example [3]. □

Remark 2.3. — The constant κ may depend on T, however it is independent of R. Thus we are free to choose $R\geqq R_0$ without changing our conclusions.

3. ESTIMATES ON THE CRITICAL VALUE $c_T=I_T^\varkappa(z_T^\varkappa)$

In this section we provide a proof of Lemma 2.4. Since our estimates only depend on (H 0)-(H 5) and their consequences (2.16) and (2.17) that are independent of the value of κ, we drop it from the indices. For every T we find $R\geqq R_0$ and we construct $h\in\Gamma$ such that

$$\sup_{z\in Q} I_T(h_T(z, 1))\leqq c$$

for a constant c independent of κ and T. Let us consider a C^∞ function $\tilde{e}:[0, T]\to\mathbb{R}^n$ such that

$$\tilde{e}(t)=e(t)=\frac{2\pi}{T}kt \qquad \text{for} \quad t\in[0, T-1], \quad \tilde{e}(T)=0$$

and

$$|\dot{\tilde{e}}(t)|\leqq 4\pi|k|\frac{T-1}{T} \qquad \text{for} \quad t\in[T-1, T]$$

and $\tilde{\xi}(t)=(0, \tilde{e}(t))\in \mathrm{E}$. Let us also consider γ, $s:\mathbb{R}^+ \to \mathbb{R}^+$, C^∞ functions, such that

$$s(\tau)=\begin{cases} 1 & \mathrm{R}-1\leqq\tau\leqq\mathrm{R} \\ \varepsilon & \tau\leqq\mathrm{R}-2, \end{cases}$$

$$\gamma(\tau)=\begin{cases} 0 & \mathrm{R}-1\leqq\tau\leqq\mathrm{R} \\ 1 & \tau\leqq\mathrm{R}-2 \end{cases}$$

with $\varepsilon\leqq s(\tau)\leqq 1$ and $0\leqq\gamma(\tau)\leqq 1$. The constants $\mathrm{R}>0$ and $\varepsilon>0$ will be determined later. We define

$$h(z, t)=\mathrm{B}(1-t+ts(\|z\|_{\mathrm{E}}))z-t\gamma(\|z\|_{\mathrm{E}})\tilde{\xi}.$$

Clearly the function h belongs to Γ.

Proof of Lemma 2.4. — In doing our estimates we consider two cases (i) $z\in \mathrm{Q}$ and $\|z\|_{\mathrm{E}}\leqq\mathrm{R}-2$ and (ii) $z\in\mathrm{Q}$ and $\mathrm{R}-2\leqq|z|\leqq\mathrm{R}$. We will use b_i to denote several constants independent of κ and T.

Case (i). Since $\|z\|_{\mathrm{E}}\leqq\mathrm{R}-2$ we have $h(z, 1)=\mathrm{B}(\varepsilon)z-\xi$, then

$$\mathrm{I}(h(z, 1))=\mathscr{A}(\mathrm{B}(\varepsilon)z-\tilde{\xi})-\int_0^{\mathrm{T}}\frac{p}{\varepsilon}.\bar{e}\,dt-\int_0^{\mathrm{T}}\mathrm{H}\left(\frac{p}{\varepsilon}, \varepsilon q-\tilde{e}+e\right)dt. \quad (3.1)$$

We analyse first the quadratic term. Since $\tilde{\xi}$ has p-component equal to 0 by (2.8), (2.12) and definition of $\tilde{\xi}$ we have

$$\mathscr{A}(\mathrm{B}(\varepsilon)z-\tilde{\xi})-\int_0^{\mathrm{T}}\frac{p}{\varepsilon}.\bar{e}\,dt\leqq-\int_0^{\mathrm{T}}\frac{p}{\varepsilon}.\dot{\tilde{e}}\,dt-\int_0^{\mathrm{T}}\frac{p}{\varepsilon}.\bar{e}\,dt$$
$$\leqq b_1\left|\frac{p_0}{\varepsilon}\right|+b_2\int_{\mathrm{T}-1}^{\mathrm{T}}\left|\frac{p}{\varepsilon}\right|dt. \quad (3.2)$$

We analyse now the last term in (3.1). By the definition of $\tilde{e}$

$$\int_0^{\mathrm{T}}\mathrm{H}\left(\frac{p}{\varepsilon}, \varepsilon q-\tilde{e}+e\right)dt$$
$$=\int_0^{\mathrm{T}-1}\mathrm{H}\left(\frac{p}{\varepsilon}, \varepsilon q\right)dt+\int_{\mathrm{T}-1}^{\mathrm{T}}\mathrm{H}\left(\frac{p}{\varepsilon}, \varepsilon q-\tilde{e}+e\right)dt. \quad (3.3)$$

By (2.16) we have

$$\int_{\mathrm{T}-1}^{\mathrm{T}}\mathrm{H}\left(\frac{p}{\varepsilon}, \varepsilon q-\tilde{e}+e\right)dt\geqq a_3\int_{\mathrm{T}-1}^{\mathrm{T}}\left|\frac{p}{\varepsilon}\right|^{\mu}dt-a_4 \quad (3.4)$$

we recall that a_3 and a_4 are independent of κ. Now we estimate the first integral in (3.3). By (2.6) we see that $\|q\|_1\leqq\alpha_1\mathrm{RT}$. Choose ε_1 so that $0<\varepsilon_1<\varepsilon_0$ and $|\mathrm{V}(q)|\leqq 1/(\mathrm{T}-1)$ if $|q|\leqq\varepsilon_1$. Given $\varepsilon>0$, we define

$$\mathrm{A}^+_{q,\varepsilon}=\{t\in[0, \mathrm{T}-1]/|\varepsilon q(t)|\geqq\varepsilon_1\}$$

and

$$A_{q,\varepsilon}^{-}=\{t\in[0,\ T-1]/|\varepsilon q(t)|<\varepsilon_1\}.$$

Since $\|q\|_1\leqq\alpha_1 TR$ we have

$$m(A_{q,\varepsilon}^{+})\leqq\frac{\alpha_1 TR}{\varepsilon_1}\varepsilon=b_3 TR\varepsilon. \tag{3.5}$$

here m represents the Lebesgue measure. By (2.17) and (H 4) if $t\in A_{q,\varepsilon}^{-}$ we have

$$H\left(\frac{p(t)}{\varepsilon},\ \varepsilon q(t)\right)\geqq b_4\left|\frac{p(t)}{\varepsilon}\right|^{\mu}+V(\varepsilon q(t)) \tag{3.6}$$

and by (2.16) if $t\in A_{q,\varepsilon}^{+}$ we have

$$H\left(\frac{p(t)}{\varepsilon},\varepsilon q(t)\right)\geqq a_3\left|\frac{p(t)}{\varepsilon}\right|^{\mu}-a_4. \tag{3.7}$$

Then, by (3.6), (3.7) and by the choice of ε_1 we obtain

$$\int_0^{T-1}H\left(\frac{p}{\varepsilon},\ \varepsilon q\right)\geqq b_5\int_0^{T-1}\left|\frac{p}{\varepsilon}\right|^{\mu}dt-b_6 TR\varepsilon-1. \tag{3.8}$$

By (3.3), (3.4) and (3.8) we have

$$I_T(h(z,\ 1))\leqq b_1\left|\frac{p_0}{\varepsilon}\right|+b_2\int_{T-1}^{T}\left|\frac{p}{\varepsilon}\right|dt-b_7\int_0^{T}\left|\frac{p}{\varepsilon}\right|^{\mu}dt+b_6 TR\varepsilon+b_8. \tag{3.9}$$

By Hölder inequality and for $T>1$ we obtain

$$b_1\left|\frac{p_0}{\varepsilon}\right|-\frac{b_7}{2}\int_0^{T}\left|\frac{p}{\varepsilon}\right|^{\mu}dt\leqq b_9\left\|\frac{p}{\varepsilon}\right\|_{\mu}-\frac{b_7}{2}\left\|\frac{p}{\varepsilon}\right\|_{\mu}^{\mu}. \tag{3.10}$$

By Hölder inequality again we obtain

$$b_2\int_{T-1}^{T}\left|\frac{p}{\varepsilon}\right|dt-\frac{b_7}{2}\int_0^{T}\left|\frac{p}{\varepsilon}\right|^{\mu}dt$$

$$\leqq b_2\left(\int_{T-1}^{T}\left|\frac{p}{\varepsilon}\right|^{\mu}dt\right)^{1/\mu}-\frac{b_7}{2}\int_{T-1}^{T}\left|\frac{p}{\varepsilon}\right|^{\mu}dt. \tag{3.11}$$

Thus, from (3.9), (3.10) and (3.11), for a constant b_{10} we have

$$I_T(h(z,\ 1))\leqq b_6 TR\varepsilon+b_{10}. \tag{3.12}$$

Case (ii). We consider now that $R\geqq\|z\|_E\geqq R-2$. As before we obtain

$$\mathscr{A}(B(s)z-\gamma\tilde{\xi})-\int_0^{T}\frac{p}{s}.\gamma\bar{e}\,dt$$

$$\leqq-\pi\|z^-\|_E^2+b_1\left|\frac{p_0}{s}\right|+b_2\int_{T-1}^{T}\left|\frac{p}{s}\right|dt, \tag{3.13}$$

we recall that $\gamma \leqq 1$. By (2.16) we obtain

$$\int_0^T H\left(\frac{p}{s}, sq+e-\gamma \tilde{e}\right) dt \geqq a_3 \left\| \frac{p}{s} \right\|^{\mu} - a_4 T. \tag{3.14}$$

As we did for (3.11) and (3.12) we obtain

$$b_1 \left| \frac{p_0}{s} \right| - \frac{a_3}{3} \int_0^T \left| \frac{p}{s} \right|^{\mu} dt \leqq b_{11} \tag{3.15}$$

and

$$b_2 \int_{T-1}^{T} \left| \frac{p}{s} \right| dt - \frac{a_3}{3} \int_0^T \left| \frac{p}{s} \right|^{\mu} dt \leqq b_{12}. \tag{3.16}$$

And since $s \leqq 1$

$$T \left| p_0 \right|^{\mu} \leqq \left\| \frac{p}{s} \right\|_{\mu}^{\mu}. \tag{3.17}$$

Then, from (3.13)-(3.17) we obtain

$$I_T(h(z, 1)) \leqq -b_{13}(\| z^- \|_E^2 + |p_0|^{\mu}) + b_{14} + a_4 T. \tag{3.18}$$

If R is large enough then for all $z \in Q$ with $\| z \|_E \geqq R-2$ we see from (3.18) that

$$I_T(h(z, 1)) \leqq 0. \tag{3.19}$$

Now that we have chosen R we choose ε in such a way that $b_3 TR \varepsilon \leqq 1$, then from (3.12) and (3.19) we obtain

$$I_T(h(z, 1)) \leqq c \tag{3.20}$$

with c independent of T. □

4. THE LIMIT PROCESS AND PROOF OF THE THEOREMS

In this section we study the sequence $\{z_T\}$ as T goes to infinity, and we give a proof to the theorems presented in the introduction.

By Propositions 2.1 and 2.2, for every $T>0$ there is a solution z_T of $(KS)_T$ and

$$0 \leqq I_T(z_T) \leqq c, \qquad \forall T>0. \tag{4.1}$$

If $z_T=(p_T, q_T)$ we define $\tilde{z}_T=(p_T, q_T+e)$ then $\tilde{z}_T$ is a solution of $(HS)_T$, *i.e.*

$$\dot{\tilde{z}}_T = JH_z(\tilde{z}_T) \tag{4.2}$$

with boundary condition

$$\tilde{q}_T(0)=0, \qquad \tilde{q}(T)=2\pi k, \tag{4.3}$$

and by (4.1)

$$0 \leqq \int_0^T \tilde{p}_T \cdot \dot{\tilde{q}}_T - H(\tilde{p}_T, \tilde{q}_T)\, dt \leqq c. \tag{4.4}$$

We assume from now on that $k \neq 0$. In the arguments that follow we will only use $\tilde{z}_T$, so that no confusion will rise by dropping the tilde.

Since the system $(HS)_T$ is autonomous, the energy is conserved along the solutions, then there is a constant E_T such that

$$H(p_T(t), q_T(t)) = E_T, \qquad \forall t \in [0, T]. \tag{4.5}$$

In what follows we will denote by c_i various constants that are independent of T.

LEMMA 4.1:

(i) $\lim_{T \to \infty} E_T = 0$

(ii) $\int_0^T -V(q_T(t))\, dt \leqq c_1$

(iii) $|p_T(t)| \leqq c_2$, $\forall t \in [0, T]$.

Proof. – Since z_T satisfies (4.2), by (4.4) and using (H 3) we obtain

$$\begin{aligned} c &\geqq I_T(z_T) - I'_T(z_T) p_T \\ &= \int_0^T -H(p_T, q_T) + H_p(p_T, q_T) \cdot p_T\, dt \\ &\geqq \int_0^T -\mu V(q_T)\, dt + \int_0^T (\mu - 1) H(p_T, q_T)\, dt. \end{aligned} \tag{4.6}$$

From (4.5) with $t = 0$, (4.3) and hypothesis (H 3) we find

$$H(p_T(0), q_T(0)) = H(p_T(0), 0) \geqq 0$$

then $E_T \geqq 0$. Recalling that $V(q) \leqq 0$ we obtain from (4.6) that

$$0 \leqq (\mu - 1) T E_T \leqq c$$

from where statement (i) follows. Also from (4.6) (ii) follows. To show (iii) we note that by (2.17) if $|p_T(t)| \geqq \varepsilon_0$ we have

$$H(p_T(t), q_T(t)) \geqq a_3 |p_T(t)|^\mu + V(q_T(t))$$

then

$$a_3 |p_T(t)|^\mu \leqq E_T - V(q_T(t)) \leqq c_3. \tag{4.7}$$

Then from (4.7)

$$|p_T(t)| \leqq \max \left\{ \frac{c_3{}^{1/\mu}}{a_3}, \varepsilon_0 \right\} = c_2, \qquad \forall t \in [0, T]. \quad \square$$

COROLLARY 4.1:

$$\lim_{T\to\infty} p_T(0) = \lim_{T\to\infty} p_T(T) = 0.$$

Proof. — Since $q_T(0)=0$ we have from (H 4) and (2.17)

$$\min\{a_3, a_1\}\,|p_T(0)|^{\mu} \leqq H(p_T(0), q_T(0)) = E_T$$

then since $E_T \to \infty$ the conclusion follows. For $p_T(T)$ we can give a similar argument. □

Let us define

$$d = \min\{\,|q_1 - q_2|/q_1, q_2 \in M, q_1 \neq q_2\}.$$

The following lemma will be used repeteadly later.

LEMMA 4.2. — *Let $0 < a < d/2$, then there exist constants $c(a) > 0$ and ε_1 so that if $\bar{q} \in M$ and $|\bar{q} - q_T(t_0)| = a$ then*

$$\int_{t_0-\varepsilon_1}^{t_0+\varepsilon_1} -V(q_T(t))\,dt \geqq c(a).$$

Proof. — Since z_T satisfies (4.2), from Lemma 4.1 and the periodicity of H in q, we have

$$|\dot{q}_T(t)| \leqq \left|\frac{\partial H}{\partial p}(p_T, q_T)\right| \leqq \sup_{|p| \leqq c_2,\, q \in \mathbb{R}^n} \left|\frac{\partial H}{\partial p}(p, q)\right| = c_4. \tag{4.8}$$

Then, from (4.8), we have

$$|q_T(t) - q_T(t_0)| = \left|\int_{t_0}^{t} \dot{q}_T\,dt\right| \leqq c_4\,|t - t_0|. \tag{4.9}$$

Choose ε_1 so that $c_4 \varepsilon_1 = a/2$, then

$$\frac{3a}{2} \geqq |q_T(t) - \bar{q}| \geqq \frac{a}{2}, \qquad \forall\, t \in [t_0 - \varepsilon_1, t_0 + \varepsilon_1].$$

Now choose $c(a)$ so that

$$0 < c(a) \leqq \frac{1}{\varepsilon_1} \min\left\{-V(q)/\frac{a}{2} \leqq |q - \bar{q}| \leqq \frac{3a}{2}\right\}$$

then

$$\int_{t_0-\varepsilon_1}^{t_0+\varepsilon_1} -V(q_T(t))\,dt \geqq 2\varepsilon_1 \min_{t \in [t_0-\varepsilon_1,\, t_0+\varepsilon_1]} \{-V(q_T(t))\} \geqq c(a).$$

We note that since V is periodic in q and the set M is discrete the constant $c(a)$ can be chosen independent of $\bar{q}$. □

LEMMA 4.3. — *q_T is uniformly bounded independent of* T, i.e. *there is a constant c_5 so that*

$$|q_T(t)| \leqq c_5, \qquad \forall t \in [0, T], \quad \forall T > 0.$$

Proof. — The idea is that if q_T is not uniformly bounded then $q_T(t)$ spends too much time outside M contradicting Lemma 4.1 (ii). We formalize this. By Lemma 4.1 (iii), p_T is uniformly bounded, then since z_T satisfies (4.2) we have

$$|q_T(t) - q_T(s)| \leqq c_6 |t - s|. \tag{4.10}$$

Assume there is $t_0 \in [0, T]$ and $N \in \mathbb{N}$ so that $|q(t_0)| = 2Nd$ and $|q(t)| < 2Nd$, $\forall t < t_0$. Since $q_T(0) = 0$ there is $t_1 < t_0$ such that

$$2(N-1)d < |q_T(t)| < 2Nd, \qquad \forall t \in (t_1, t_0)$$

and $|q_T(t_1)| = 2(N-1)d$. By (4.10) $|t_1 - t_0| \geqq 2d/c_6$. By continuity of q_T and the definition of d there is $t_1^* \in (t_1, t_0)$ so that

$$\operatorname{dist}(q_T(t_1^*), M) \geqq d/3.$$

if $t_2 < t_1$ is such that $|q_T(t_2)| = 2(N-2)d$ and $|q_T(t)| < 2(N-2)d$, $\forall t \leqq t_2$ then, by Lemma 4.2 there is a constant c_7 and $\varepsilon_1 > 0$ such that $(t_1^* - \varepsilon_1, t_1^* + \varepsilon_1) \subset (t_2, t_0)$ and

$$\int_{t_1^* - \varepsilon_1}^{t_1^* + \varepsilon_1} - V(q_T(t))\, dt \geqq c_7.$$

We can repeat this argument N/4 times to obtain finally

$$\int_0^{t_0} - V(q_T(t))\, dt \geqq \frac{N c_7}{4}.$$

By Lemma 4.2 (ii) follows that N has to be bounded, completing the proof. □

We now begin the limit process. Let us consider a sequence $\{T_m\}_{m \in \mathbb{N}}$ such that $T_m \to \infty$. Let us denote $z_m = z_{T_m}$. Assume we have two sequences $\{t_m^-\}_{m \in \mathbb{N}}$ and $\{t_m^+\}_{m \in \mathbb{N}}$ such that

(s 1) $t_m^+, t_m^- \in [0, T_m]$

(s 2) $\lim_{m \to \infty} t_m^+ - t_m^- = \lim_{m \to \infty} T_m + t_m^+ = \infty$

(s 3) $|q_m(t_m^-)| < \varepsilon_0/2$, $|q_m(t_m^+)| = \varepsilon_0$ and $|q_m(t)| < \varepsilon_0$, $\forall t \in (t_m^-, t_m^+)$.

Let us now define a sequence of functions

$$\zeta_m(t) = \begin{cases} z_m(t + t_m^+), & t \in [-t_m^+, T_m - t_m^+] \\ (p_m(0), 0), & t \in (-\infty, -t_m^+) \\ (p_m(T_m), 2\pi k), & t \in (T_m - t_m^+, \infty). \end{cases}$$

Since system (4.2) is autonomous the function $\zeta_m(t)$ is a solution for (4.2) for $t \in [-t_m^+, T_m - t_m^+]$. Given $N \in \mathbb{N}$ we consider the sequence $\{\zeta_m\}$ restricted to the interval $[-N, N]$. By Lemmas 4.1 and 4.3, and the definition of ζ_m we see that ζ_m is uniformly bounded. By equation (4.2) we have that also $\dot{\zeta}_m$ is uniformly bounded. Since for m large enough $[-N, N] \subset [-t_m^+, T_m - t_m^+]$, by the Arzela-Ascoli theorem we find a subsequence $\{\zeta_{m_N}\}$ uniformly convergent to a function $\zeta_N : [-N, N] \to \mathbb{R}^{2n}$ and this function satisfies (4.2) in $[-N, N]$.

Proceeding by induction, for every $N \in \mathbb{N}$ we can do the anterior procedure in such a way that $\{\zeta_{m_{N+1}}\}$ is a subsequence of $\{\zeta_{m_N}\}$. Then by taking the "diagonal" subsequence we obtain a subsequence of $\{\zeta_m\}$ we call $\{z_m^1\}$ and a function $z^1 : \mathbb{R} \to \mathbb{R}^{2n}$ so that z_m^1 converges to z^1 locally uniformly, and z^1 is a solution of (HS). We note that by (s 3) $|q^1(0)| = \varepsilon_0$ so that z^1 is not trivial.

LEMMA 4.4:

(i) $\lim_{t \to -\infty} p^1(t) = 0 = \lim_{t \to -\infty} q^1(t)$

(ii) $\lim_{t \to \infty} p^1(t) = 0$ and $\lim_{t \to \infty} q^1(t) \in M$.

Proof. — Let $\zeta_m = (\varphi_m, \psi_m)$. For every $N \in \mathbb{N}$, by Lemma 4.1 (ii)

$$\int_{-N}^{N} -V(\psi_{m_N}(t))\, dt \leqq \int_0^{T_{m_N}} -V(q_{m_N}(t))\, dt \leqq c,$$

taking the limit when $m_N \to \infty$, and then taking limit when $N \to \infty$ we find

$$\int_{-\infty}^{\infty} -V(q^1(t))\, dt \leqq c, \tag{4.11}$$

where $z^1 = (p^1, q^1)$. By (s 3) and the definition of ζ_m we see that

$$|\psi_m(t)| \leqq \varepsilon_0, \qquad t \in (t_m^- - t_m^+, 0)$$

and since $\lim_{m \to \infty} t_m^+ - t_m^- = \infty$ *we obtain that* $|q^1(t)| \leqq \varepsilon_0$, $\forall\, t \in (-\infty, 0)$. Let us assume that $\lim_{t \to -\infty} q^1(t) \neq 0$, then there exist a sequence $t_n \to -\infty$ such that $|q^1(t_n)| \geqq a > 0$. Taking $a < d/2$, and $c(a)$, ε_1 as in Lemma 4.2, we obtain that

$$\int_{t_n - \varepsilon_1}^{t_n + \varepsilon_1} -V(q^1(t))\, dt \geqq c(a), \qquad \forall\, n \in \mathbb{N}. \tag{4.12}$$

Assuming without lost of generality that $|t_n - t_{n+1}| > 2\varepsilon_1$ we see that (4.12) contradicts (4.11). This proves the second part of (i). We note that Lemma 4.2 was proved only for q_T, but the same argument allows to prove it for q^1.

By conservation of energy and Lemma 4.1, after taking the limit we find

$$H(p^1(t), q^1(t)) = 0, \qquad \forall\, t \in \mathbb{R},$$

then since $\lim_{T \to -\infty} q^1(t)=0$, by (H 4) we conclude that for t large enough

$$0 \geqq \min\{a_1, a_3\} |p^1(t)|^\mu + V(q^1(t))$$

from where $\lim_{T \to -\infty} p^1(t)=0$ follows. By a similar argument we show (ii). □

PROPOSITION 4.1. – *Equation* (HS) *possesses a heteroclinic orbit starting at* 0 *and terminating in* $M \setminus \{0\}$.

Proof. – Consider $T_m = m$, $m \in \mathbb{N}$, and define $t_m^+ \in [0, T_m]$ so that

$$|q_m(t_m^+)| = \varepsilon_0 \quad \text{and} \quad |q_m(t)| < \varepsilon_0, \quad \forall t < t_m^+$$

and define $t_m^- = 0$. We claim that the sequences $\{t_m^+\}$ and $\{t_m^-\}$ so defined satisfy (s 1), (s 2) and (s 3). We only need to prove (s 2): Consider the initial value problem

$$\dot{p} = -\frac{\partial H}{\partial q}(p, q)$$
$$\dot{q} = \frac{\partial H}{\partial p}(p, q)$$
$$p(0)=a, \qquad q(0)=0.$$

By continuous dependence on initial data, and noting that $p \equiv 0$, $q \equiv 0$ is the solution for $a=0$, for every K there is $\varepsilon > 0$ so that $|a| \leqq \varepsilon$ implies $q(t) \in B_{\varepsilon_0}(0)$, $\forall\, 0 \leqq t \leqq K$. By Corollary 4.1, for every $\varepsilon > 0$ there is T so that $|p_T(0)| \leqq \varepsilon$, from where we conclude the first part of (s 2). A similar argument can be given to prove the second part, again we use Corollary 4.1.

Using the limit procedure described before Lemma 4.4, and Lemma 4.4 we find a solution $z^1 = (p^1, q^1)$ of (HS) that is a heteroclinic orbit of (HS) if $\lim_{T \to \infty} q^1(t) \in M \setminus \{0\}$. If this is not the case we end with a homoclinic orbit. Let us assume we are in this adverse situation.

Let $t^1 \in \mathbb{R}$ such that $q^1(t) \in B_{\varepsilon_0/3}(0)$, $\forall t \geqq t^1$. We note that $t^1 > 0$ because $|q^1(0)| = \varepsilon_0$. Let us consider the sequence $z_m^1 = (q_m^1, q_m^1)$ defined in the limit procedure. Since $q_m^1(t)$ reaches $2\pi k$ eventually there numbers $\tau_m^- < \tau_m^+$ so that $0 < \tau_m^- < t^1 < \tau_m^+$,

$$|q_m^1(\tau_m^-)| < \frac{\varepsilon_0}{2}, \qquad |q_m^1(\tau_m^+)| = \varepsilon_0$$

and $|q_m^1(t)| < \varepsilon_0$, $\forall t \in (\tau_m^-, \tau_m^+)$. We define $t_m^{1-} = t_m^+ + \tau_m^-$ and $t_m^{1+} = t_m^+ + \tau_m^+$. We claim that the sequences $\{t_m^{1+}\}$ and $\{t_m^{1-}\}$ satisfy (s 1), (s 2) and (s 3). (s 1) and (s 3) are clearly true. Let us show that (s 2) is also satisfied. Taking the subsequence of $\{z_m\}$ corresponding to $\{z_m^1\}$, to show that $\lim_{m \to \infty} T_m - t_m^{1+} = \infty$ we proceed as we did before.

We only need to show that $\tau_m^+ - \tau_m^-$ goes to infinity, and since $0 < \tau_m^- < t^1$, it is enough to show that τ_m^+ goes to infinity. Suppose it is bounded, then τ_m^+ and τ_m^- converge through a subsequence to τ^- and τ^+, with $0 < \tau^- \leqq t^1 \leqq \tau^+$, but then $|q^1(\tau^+)| = \varepsilon_0$ that contradicts the definition of t^1.

Since $\{t_m^{1+}\}$ and $\{t_m^{1-}\}$ satisfy (s 1)-(s 3) we can repeat the procedure to obtain a solution $z^2 = (p^2, q^2)$ of (HS), that is a heteroclinic orbit $\lim_{T \to \infty} q^2(t) \in M \setminus \{0\}$.

On the contrary, if $\lim_{T \to \infty} q^2(t) = 0$ we obtain a second homoclinic orbit. Repeating this procedure and assuming in each case we find a homoclinic orbit, we obtain a sequence of homoclinic orbits. We claim that this is impossible.

In fact we will generate sequences $\{t_m^+\}, \{t_m^{1+}\}, \ldots, \{t_m^{i+}\}, \ldots$ where

$$\lim_{m \to \infty} t_m^{i+} - t_m^{i-1+} = \infty \qquad |q_m^i(t_m^{i+})| = \varepsilon_0, \qquad \forall i \in \mathbb{N}$$

and this fact together with Lemma 4.2 contradicts assertion (ii) of Lemma 4.1. □

Proof of Theorem 0.1. – By Proposition 4.1 there is at least one heteroclinic orbit emanating from 0 and terminating in $M \setminus \{0\}$.

If in problem $(HS)_T$ we change the boundary condition by

$$q(0) = 2\pi k, \qquad q(T) = 0$$

and we modify the arguments accordingly we obtain a heteroclinic orbit emanating in $M \setminus \{0\}$ and terminating in 0. □

Remark 4.1. – Theorem 0.2 can be proved in the same way Theorem 0.1 was, with minor modifications.

Proof of Theorem 0.3. – Here we base the argument in the idea used by Rabinowitz in proving Proposition 3.33 in [8]. This together with an analysis similar to the one given in Proposition 4.1 will build the proof. We will be scketchy.

We are assuming (H 2′), thus M consist of integer translations of a single point. We can assume that $M = \mathbb{Z}^n$. Let B denote the set of $q \in M \setminus \{0\}$ so that 0 and q are connected by a heteroclinic orbit. By Theorem 0.1 B is not empty.

Let Λ be the set of linear combinations of elements in B with coefficients in $\mathbb{Z}$. We claim that $\Lambda = M$. If this is not the case then $S = M \setminus \Lambda \neq \varnothing$. Take $2\pi k \in S$ and consider the problem $(HS)_T$ with the boundary condition

$$q(0) = 0, \qquad q(T) = 2\pi k.$$

As in Proposition 4.1 we find solutions of (HS) in $\mathbb{R}$, z^1, z^2, ... Let $z^i=(p^i, q^i)$ the first one so that $\bar{q}^i=\lim_{T\to\infty} q^i(t)\neq 0$. By definition of S $\bar{q}^i\in\Lambda$ and then $\bar{q}^i\neq 2\pi k$. Then by going a further translation as in Proposition 4.1 we find a solution of (HS) z^{i+1} so that $\lim_{T\to-\infty} q^{i+1}(t)=\bar{q}^i$. If $\lim_{T\to\infty} q^{i+1}(t)=\bar{q}^{i+1}\notin\Lambda$, then $\bar{q}^{i+1}-\bar{q}^i\notin\Lambda$, but this is impossible since $z^{i+1}(t)-(0, \bar{q}^i)$ is an orbit joining 0 with $\bar{q}^{i+1}-\bar{q}^i$. Consequently $\bar{q}^{i+1}\in\Lambda$.

Now we continue generating orbits of (HS) whose end points will always be in Λ by the argument given above. Since we can not continue this procedure indefinitely and $2\pi k\in S$, for some j $\bar{q}^j\notin\Lambda$ contradicting the hypothesis. Thus $\Lambda=M$, and then we can find at least n orbits emanating from 0 and terminating in $M\setminus\{0\}$. The n heteroclinic orbits terminating at 0 are obtained similarly. This proves the first assertion of Theorem 0.3.

For the second we assume also (H 6), and we note that if $z(t)=(p(t), q(t))$ is a heteroclinic orbit joining 0 to $\bar{q}$ then $\bar{z}(t)=(-p(-t), q(-t)-\bar{q})$ joins $-\bar{q}$ *to* O. □

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