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Ljusternik-Schnirelman theory with local Palais-Smale condition and singular dynamical systems

by

P. MAJER

Scuola Normale Superiore,
Piazza dei Cavalieri, 7, 56126 Pisa, Italy

ABSTRACT. — We find infintely many T-periodic solutions to a system $\ddot{u}+\nabla_x V(t, u)=0$ with a singular, T-periodic potential V, whose behaviour at infinity is subjected to rather weak assumptions. In order to do so, we adapt the Ljusternik-Schnirelman method to handle a functional possibly unbounded from below and which possibly does not satisfy the Palais-Smale condition at any level.

RÉSUMÉ. — Nous trouvons un nombre infini de solutions T-périodiques d'un système $\ddot{u}+\nabla_x V(t, u)=0$ pour un potentiel singulier, T-périodique V dont le comportement à l'infini est sujet à des hypothèses très faibles. Pour ce faire, nous adaptons la méthode de Ljusternik-Schnirelman pour traiter une fonctionnelle même non bornée inférieurement et ne satisfaisant pas la condition de Palais-Smale à tout niveau.

Mots clés : Ljusternik-Schnirelman theory, singular dynamical systems, periodic solution.

Classification A.M.S. : 58 F 05, 58 E 05, 34 C 25.

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0. INTRODUCTION

In this paper we seek T-periodic solutions of second order systems of the type

$$(0.1) \qquad \ddot{u}+au+W'(t, u)=0,$$

where W is singular at $x=0$,

$$W(t+T, x)=W(t, x), \qquad \text{and} \qquad W'(t, x)=:\nabla_x W(t, x).$$

Problem (0.1) has been studied in [1] under the assumptions:

(i) $a=0$;

(ii) $W(t, x)$, $W'(t, x)\to 0$ as $|x|\to\infty$ *uniformly in t*;

(iii) W satisfies a "Strong Force condition" $\left(\text{namely } W\simeq -\dfrac{1}{|x|^\alpha},\ \alpha\geqq 2, \text{ at } x=0\right)$.

(*See* also [2], [4], [5] for other results in this direction.)

The prupose of this work is to extend the results of [1], retaining condition (iii), but weakening (i) and (ii). More precisely we assume that:

(j) $a<\left(\dfrac{\pi}{T}\right)^2$;

(jj) there exist constants c, $\theta<2$, $r>0$ such that for $|x|\geqq r$ and for all $t\in\mathbf{R}$

$$W(t, x)\leqq c|x|^\theta, \qquad W'(t, x).x-2W(t, x)\leqq c|x|^\theta,$$

and we show that (0.1) has infinitely many T-periodic solutions u with $u(t)\neq 0\ \forall t$.

From the abstract point of view, the solutions of (0.1) are critical points of the action integral

$$(0.1) \qquad f(u)=\int_0^T\left\{\frac{1}{2}|\dot{u}|^2-\frac{a}{2}|u|^2-W(t, u)\right\}dt$$

on

$$\Lambda=\{u\in H^1(S^1_T, \mathbf{R}^N) : u(t)\neq 0,\ \forall t\in S^1_T\}.$$

Two difficulties arise in weakening the hypotheses (i), (ii). First, since we made rather weak assumptions on the derivatives of W at infinity, the Palais-Smale condition may possibly fail at any level (while it holds at any level but 0 under the hypotheses (i), (ii); *see* [1], Lemma 3.1). Second, if $a>0$ the functional f is no longer bounded from below.

In order to overcome these difficulties we prove in section 2 a Ljusternik-Schnirelman type theorem which establishes the existence of infinitely many critical points (Theorem 2.4). The main features of this theorem·

are:

(*a*) the Palais-Smale condition is not required on the whole domain of the functional;

(*b*) the functional need not be bounded from below;

(*c*) a certain control is required on the Ljusternik-Schnirelman category of the sublevel sets of the functional (conditions 2.4.iii and 2.4.iv).

Then in section 3 we show (Theorem 3.5) that if (j), (jj), and (iii) hold, f satisfies the hypotheses of Theorem 2.4. So, whereas checking the Palais-Smale condition (2.4.v) becomes much simpler, more care is needed in verifying conditions 2.4.iii and 2.4.iv. Roughly, the idea is to show that if $f(u) \leqq \lambda$, then $\|\dot{u}\|_2 / \inf |u(t)| \leqq k(\lambda)$. This allows us to deformate the sublevel sets in compact sets (hence with finite category) via a convolution operator.

Theorem 3.5 is completed by two examples. In the former we show a case in which $a=0$, $W(x) \to 0$ as $x \to \infty$ *and* f does not satisfy the usual Palais-Smale condition at any positive level.

In the latter we show that if $a > \left(\frac{\pi}{T}\right)^2$, the category of every sublevel set $\{f \leqq \lambda\}$ can actually be infinite, so that Theorem 2.4 cannot be applied.

1. NOTATIONS

If f is a real-valued function on some set Λ and $\lambda \in \mathbf{R}$, $\{f \leqq \lambda\}$ denotes the set $\{u \in \Lambda : f(u) \leqq \lambda\}$; similar meaning has $\{f \geqq \lambda\}$ and so on. If X is a metric space with metric d, and if $x \in X$ and $\rho \in \mathbf{R}$, $B(x, \rho)$ is the ball $\{y \in X : d(x, y) < \rho\}$. If $x, y \in \mathbf{R}^N$, $|x$ and $x.y$ are respectively the euclidean norm of x and the scalar product of x, y. S^1_T denotes $\mathbf{R}/T\mathbf{Z}$. Finally, $\|u\|_2 = \left(\int_0^T |u(t)|^2 \, dt\right)^{1/2}$ and $\|u\|_{1,2} = (\|u\|_2^2 + \|\dot{u}\|_2^2)^{1/2}$ denote respectively the L^2-norm and the H^1-norm of $u \in L^2([0, T], \mathbf{R}^N)$, respectively $u \in H^1([0, T], \mathbf{R}^N)$.

Hereafter SF, LS and PS means respectively Strong Force, Ljusternik-Schnirelman, Palais-Smale.

2. A THEOREM OF LJUSTERNIK-SCHNIRELMAN TYPE

We first recall some definitions and basic results on Critical Point Theory. Let Λ be a topological space, and let $\mathscr{K}(\Lambda)$ be the family of the closed subsets of Λ which are contractible in Λ; if $A \subset \Lambda$, the LS category of A relatively to Λ is the number (possibly $+\infty$)

$$\mathrm{Cat}_\Lambda(\mathrm{A}) = \inf\left\{k \in \mathbf{N} : \mathrm{A} \subset \bigcup_{i=1}^{k} \mathrm{X}_i \in \mathscr{K}(\Lambda)\right\}.$$

In the following proposition we list some properties of the category.

2.1. Proposition. – *Let* Λ *be a topological space and* A, B $\subset \Lambda$. *Then*

$$\mathrm{Cat}_\Lambda(\mathrm{A} \cup \mathrm{B}) \leqq \mathrm{Cat}_\Lambda(\mathrm{A}) + \mathrm{Cat}_\Lambda(\mathrm{B}). \tag{2.1}$$

If A *is closed and there exists a deformation of* A *in* B, i.e., *a continuous map* $h : [0, 1] \times \mathrm{A} \to \Lambda$ *such that* $h(0, .) = 1_\mathrm{A}$ *and* $h(1, \mathrm{A}) \subset \mathrm{B}$ (*in particular if* A $\subset$ B), *then*

$$\mathrm{Cat}_\Lambda(\mathrm{A}) \leqq \mathrm{Cat}_\Lambda(\mathrm{B}). \tag{2.2}$$

If Λ *is regular and locally contractible every compact subset of* Λ *has finite category.*

If Λ *is arcwise connected,* $\{\mathrm{A}_i\}_{i \in \mathrm{I}}$ *is a locally finite family of pairwise disjoint closed subsets of* Λ *and* $\mathrm{A} = \bigcup_{i \in \mathrm{I}} \mathrm{A}_i$, *then*

$$\mathrm{Cat}_\Lambda(\mathrm{A}) = \sup_{i \in \mathrm{I}} \mathrm{Cat}_\Lambda(\mathrm{A}_i). \tag{2.3}$$

Proof. – See [7] for the first three properties. Since we have no references for the last, we report here a proof.

We show that $\mathrm{Cat}_\Lambda(\mathrm{A}) \leqq \sup_{i \in \mathrm{I}} \mathrm{Cat}_\Lambda(\mathrm{A}_i)$, since the converse inequality follows immediately from (2.2). We can assume $\sup_{i \in \mathrm{I}} \mathrm{Cat}_\Lambda(\mathrm{A}_i) = m < \infty$, for otherwise there is nothing to prove. Thus $\forall\, i = \bigcup_{j=1}^{m} \mathrm{X}_{i,j}$, with $\mathrm{X}_{i,j} \in \mathscr{K}(\Lambda)$. Since Λ is arcwise connected, for every (i, j) there exists a deformation $h_{i,j}$ of $\mathrm{X}_{i,j}$ in a common base point $x_0 \in \Lambda$. For any $j \leqq m$ set $\mathrm{Y}_j = \bigcup_{i \in \mathrm{I}} \mathrm{X}_{i,j}$, and let $h_j : [0, 1] \times \mathrm{Y}_j \to \Lambda$ be the map defined by

$$h_{j \mid [0,1] \times \mathrm{X}_{i,j}} = h_{i,j}, \qquad \forall\, i \in \mathrm{I}:$$

the definition makes sense because the $\{\mathrm{X}_{i,j}\}_{i \in \mathrm{I}}$ are pairwise disjoint. Moreover, since $\{\mathrm{X}_{i,j}\}_{i \in \mathrm{I}}$ is a locally finite family of closed sets, one has that each Y_j is closed and h_j is continuous, whence $\mathrm{Y}_j \in \mathscr{K}(\Lambda)$. Therefore $\mathrm{Cat}_\Lambda(\mathrm{A}) \leqq m$.

Q.E.D.

Now let Λ be an open subset of some Banach space X. For $f \in \mathscr{C}^1(\Lambda)$ we set $\mathrm{Z}_f = \{u \in \Lambda : f'(u) = 0\}$ and $\tilde{\Lambda} = \Lambda \setminus \mathrm{Z}_f$. In the proof of the main theorem (2.4) we need some technical lemmas. First of all we recall the following proposition

2.2. PROPOSITION. – *Let* $f \in \mathscr{C}^1(\Lambda)$, *and* $\alpha \in]0, 1[$: *then there exists a locally Lipschitz continuous map* $V : \tilde{\Lambda} \to X$ *such that* $\forall u \in \tilde{\Lambda}$

$$(2.4)\qquad \begin{cases} \|V(u)\| \leq \frac{1}{\alpha}\|f'(u)\|, \\ \langle f'(u), V(u)\rangle \geq \|f'(u)\|^2. \end{cases}$$

Proof. – *See* [7] or [8] $\left(\text{there } \Lambda = X \text{ and } \alpha = \frac{1}{2}\right.$, but the same construction works without changes in the case of Λ open subset of X, $\left.\alpha \in]0, 1[.\right)$

Q.E.D.

Maps like V, the so-called Pseudogradient vector fields, are used to establish a Deformation Lemma (*see* [7] or [8]). Actually, for our specific purposes, a statement slightly different from the usual ones is needed.

2.3. LEMMA. – *Let* $\alpha \in]0,1[$ *and let* $f \in \mathscr{C}^1(\Lambda)$ *be such that*

$$(2.5)\qquad \forall u_n \to u \in \partial\Lambda, \qquad f(u_n) \to \infty,$$

and suppose there exists a locally lipschitz map $h : \Lambda \to \mathbf{R}$ *such that* $Z_f \subset \{f < h - 1\}$.

Then there exists a continuous map $\eta : [0, \infty[\times \Lambda \to \Lambda$ *such that for any* $u \in \Lambda$ *one has*

(η i) $\eta(0, u) = u$;

(η ii) $\eta(., u)$ *is* $\mathscr{C}^1$ *with* $\|\dot{\eta}(t, u)\| \leq 1$;

(η iii) $f(\eta(., u))$ *is non-increasing;*

(η iv) *if* $f(\eta(t, u)) \geq h(\eta(t, u)$, *then*

$$(2.6)\qquad \frac{d}{dt}(f(\eta(t, u)) \leq -\alpha \|f'(\eta(t, u))\|.$$

Proof. – Let V be the pseudogradient for f constructed in Proposition 2.2 and let us define a map $F : \Lambda \to X$ by

$$(2.7)\qquad F(u) = \begin{cases} 0, & \text{if } f(u) \leq h(u) - 1; \\ \dfrac{V(u)}{\|V(u)\|}(f(u) - h(u) + 1), & \text{if } h(u) - 1 \leq f(u) \leq h(u); \\ \dfrac{V(u)}{\|V(u)\|} & \text{if } f(u) \geq h(u). \end{cases}$$

Consider the Cauchy problem

$$(2.8)\qquad \begin{cases} \dfrac{\partial \eta}{\partial t} = -F(\eta(t, u)) \\ \eta(0, u) = u, \qquad u \in \Lambda. \end{cases}$$

Since V is locally Lipschitz continuous in $\tilde{\Lambda}$ and F vanishes in a neighbourhood of Z_f, F is locally Lipshitz in Λ. In addition $\|F\| \leqq 1$ and, from (2.4), there results $\langle f'(u), F(u) \rangle \geqq 0$. Hence (2.8) has a unique solution $\eta(t, u)$ for any initial value $u \in \Lambda$; $\eta(., u)$ is of class $\mathscr{C}^1$ with $\|\dot{\eta}(t, u)\| \leqq 1$; $f(\eta(t, u))$ is not increasing in t, because

$$\frac{d}{dt} f(\eta(t, u)) = -\langle f'(\eta(t, u)), F(\eta(t, u)) \rangle \leqq 0.$$

Now with standard arguments of o.d.e. we have that $\eta = \eta(t, u)$ is defined and continuous on $[0, \infty[\times \Lambda$. Namely, if for some $u_0 \in \Lambda$ the maximal existence interval $I =]t_0, t_1[$ of $\eta(., u_0)$ is right-bounded, then there exists the limit u_1 of $\eta(t, u_0)$ as $t \nearrow t_1 . u_1$ belongs to Λ, otherwise from (2.5) $\lim_{t \nearrow t_1} f(\eta(t, u_0)) = \infty$, whereas $f(\eta(t, u_0))$ is not increasing. Then η can be continued for $t > t_1$ and I is not maximal, a contradiction. Thus η verifies (η i), (η ii) and (η iii). Finally suppose that $f(\eta(t, u)) \geqq h(\eta(t, u))$. Then from (2.7) one has

$$\frac{d}{dt} f(\eta(t, u)) = -\langle f'(\eta(t, u)), F(\eta(t, u) \rangle$$
$$= -\left\langle f'(\eta(t, u)), \frac{V(\eta(t, u)}{\|V(\eta(t, u)\|} \right\rangle.$$

Then (η iv) follows, since from (2.4)

$$-\left\langle f'(\eta(t, u)), \frac{V(\eta(t, u))}{\|V(\eta(t, u))\|} \right\rangle \leqq -\alpha \|f'(\eta(t, u))\|.$$

Q.E.D.

Lastly we recall the well known Palais-Smale condition. A sequence $\{u_n\} \subset \Lambda$ is a PS sequence iff $f'(u_n) \to 0$ and $f(u_n)$ is bounded; the PS condition hold in a set $Y \subset \Lambda$ (respectively, at a level $\lambda \in \mathbf{R}$) iff every PS sequence $\{u_n\} \subset Y$ (respectively, with $f(u_k) \to \lambda$) has a limit point $u \in \Lambda$.

2.4. THEOREM. — *Let* X *be a Banach space with norm* $\|.\|$, Λ *an open subset of* X, *and suppose a functional* $f : \Lambda \to \mathbf{R}$ *is given such that the following conditions hold:*

(i) $\mathrm{Cat}_\Lambda(\Lambda) = +\infty$;

(ii) $f \in \mathscr{C}^1(\Lambda)$ *and* $\forall u_n \to u \in \partial\Lambda, f(u_n) \to +\infty$;

(iii) $\forall \lambda \in \mathbf{R}, \mathrm{Cat}_\Lambda(\{f \leqq \lambda\}) < +\infty$;

suppose in addition that there exist $g \in \mathscr{C}^1(\Lambda)$, $\beta \in]0, 1[$ *and* $\lambda_0 \in \mathbf{R}$ *such that*

(iv) $\mathrm{Cat}_\Lambda(\{f \leqq g\}) < +\infty$;

(v) *the* PS *condition holds in the set* $\{f \geqq g\}$;

(vi) $\beta \|f'(u)\| \geqq \|g'(u)\|, \forall u \in \{f = g \geqq \lambda_0\}$.

Then f has a sequence $\{u_n\} \subset \Lambda$ *of critical points such that* $f(u_n) \to +\infty$ *and* $f(u_n) \geqq g(u_n) - 1$.

Proof. – Suppose by contradiction that $Z_f \subset \{f < \max(g, \lambda_*) - 1\}$ for some $\lambda_* \geqq \lambda_0$. Let $h = \max(g. \lambda_*)$ and take $\alpha \in]\beta, 1[$: then Lemma 2.3 applies yielding a map η verifiying (η i-iv). The set $A = \{f \leqq h\}$ is positively invariant for the flow η: indeed, if $u \in \partial A$, either $f(u) = \lambda_*$, or $g(u) = f(u) \geqq \lambda^*$. In the former case we have from (η iii) $\eta([0, \infty[, u) \subset \{f \leqq \lambda_*\} \subset A$; in the latter one we get from (η iv) and (η ii)

$$\frac{d}{dt}(f-g)(\eta(t, u))\Big|_{t=0} = \frac{d}{dt} f(\eta(t, u))\Big|_{t=0} - \langle g'(u), \dot{\eta}(0, u)\rangle$$
$$\leqq -\alpha \| f'(u) \| + \| g'(u) \|;$$

and from condition (vi) (since $u \in \{f = g \geqq \lambda_0\}$)

$$-\alpha \| f'(u) \| + \| g'(u) \| \leqq -\alpha \| f'(u) \| + \beta \| f'(u) \| = -(\alpha - \beta) \| f'(u) \|.$$

Note that $f(u) = h(u)$ implies $u \notin Z_f$, since we have assumed $Z_f \subset \{f < h - 1\}$. Therefore

$$\forall u \in \partial A \quad \frac{d}{dt}(f-g)(\eta(t, u))\Big|_{t=0} < 0.$$

Hence $\forall u \in \partial A \, \exists \varepsilon > 0$ such that $\eta([0, \varepsilon[, u) \subset A$, which proves that A is positively invariant for η.

Since Λ can be written as

$$\Lambda = \left(\bigcup_{k \in \mathbf{Z}} \{2k - 1 \leqq f \leqq 2k\} \right) \cup \left(\bigcup_{k \in \mathbf{Z}} \{2k \leqq f \leqq 2k + 1\} \right),$$

and since both $\{\{2k - 1 \leqq f \leqq 2k\}\}_{k \in \mathbf{Z}}$ and $\{\{2k \leqq f \leqq 2k + 1\}\}_{k \in \mathbf{Z}}$ are locally finite families of pairwise disjoints sets, we get, using Proposition 2.1,

$$\begin{aligned} \infty &= \operatorname{Cat}_\Lambda(\Lambda) \\ &= \operatorname{Cat}_\Lambda\left(\bigcup_{k \in \mathbf{Z}} \{2k - 1 \leqq f \leqq 2k\} \right) + \operatorname{Cat}_\Lambda\left(\bigcup_{k \in \mathbf{Z}} \{2k \leqq f \leqq 2k + 1\} \right) \\ &= 2 \sup_{\lambda \in \mathbf{R}} \operatorname{Cat}_\Lambda(\{f \leqq \lambda\}). \end{aligned}$$

On the other hand, by (iii) and (iv)

$$\operatorname{Cat}_\Lambda(A) \leqq \operatorname{Cat}_\Lambda(\{f \leqq g\}) + \operatorname{Cat}_\Lambda(\{f \leqq \lambda_*\}) < \infty$$

Thus there exists a $\lambda^* > \lambda_*$ such that

$$\operatorname{Cat}_\Lambda(\{f \leqq \lambda^*\}) > \operatorname{Cat}_\Lambda(A). \tag{2.9}$$

Consider the deformations

$$\eta_{|[0, n]} : [0, n] \times \{f \leqq \lambda^*\} \to \Lambda, \qquad n \in \mathbf{N}.$$

From (2.2) and (2.9) we infer that $\forall n \in \mathbf{N} \, \eta(n, \{f \leqq \lambda^*\}) \not\subset A$, that is, $\forall n \, \exists u_n \in \{f \leqq \lambda^*\}$ such that $\eta(n, u_n) \in \Lambda \setminus A$; moreover, since A is positively invariant, we have in fact

$$(2.10) \qquad \eta(t, u_n) \in \Lambda \setminus A = \{ f > h \} \subset \{ f \geqq \lambda_* \}, \qquad \forall t \in [0, n].$$

By the mean value theorem there exists $t_n \in [0, n]$ such that

$$(2.11) \qquad \frac{d}{dt} f(\eta(t_n, u_n)) = \frac{1}{n}(f(\eta(u, u_n)) - f(\eta(0, u_n))).$$

Since from (η iii) and (2.10)

$$\lambda^* \geqq f(\eta(0, u_n)) \geqq f(\eta(n, u_n)) \geqq \lambda_*,$$

(2.11) implies that $\frac{d}{dt} f(\eta(t_n, u_n)) \to 0$, therefore, again from (2.10) and (η iv), we have

$$f'(\eta(t_n, u_n)) \to 0.$$

Hence $u_n = \eta(t_n, u_n)$ is a PS sequence in $\{ f \geqq g \} \cap \{ f \geqq \lambda_* \}$. By condition (v) we get a critical point $u \in \Lambda$ with $f(u) \geqq h(u)$, a contradiction.

Q.E.D.

2.5. *Remark.* – In the case $g = \lambda_0$, a constant, condition (iv) and (vi) are contained in the other ones, while condition (v) reduces to the more standard PS condition

(v') *There exists a* $\lambda_0 \in \mathbf{R}$ *such that the PS condition holds on* $\{ f \geqq \lambda_0 \}$.

Namely one has

2.6. Theorem. – *Let* (i), (ii), (iii), (v') *hold. Then there exists a sequence* $\{ u_n \}$ *of critical points of* f *such that* $f(u) \to \infty$.

The idea of using this principle in Singular Potentials is due to [1] (Rem. 2.15). We introduce conditions (iv)-(vi) because in the applications they allow us to handle a larger and more stable class of potentials than (v').

3. APPLICATION TO *T*-PERIODIC SOLUTIONS OF SINGULAR TIME-DEPENDENT HAMILTONIAN SYSTEMS

We recall that a potential $W \in \mathscr{C}^1(S^1_T \times (\mathbf{R}^N \setminus \{ 0 \}))$ satisfies the Strong Force condition [6], if the following holds:

(SF) *There exists a* $U \in \mathscr{C}^1(\mathbf{R}^N \setminus \{ 0 \})$ *and a* $\rho > 0$ *such that*

$$\begin{cases} \lim_{x \to 0} U(x) = \infty \\ W(t, x) \leqq -|U'(x)|^2, \qquad \forall (t, x) \in S^1_T \times (\mathbf{R}^N \setminus \{ x \}) \text{ with } |x| < \rho. \end{cases}$$

Throughout this section we shall deal with a (singular) potential V of the form

$$(V) \qquad V(t, x) = \frac{1}{2} a |x|^2 + W(t, x),$$

where

(V1) $a<\left(\frac{\pi}{T}\right)^2$;

(V2) $W\in\mathscr{C}^1(S^1_T\times(\mathbf{R}^N\setminus\{0\}))$ *satisfies* (SF);

(V3) $\exists\, c,\ \theta<2,\ r>0$ *such that* $\forall\,|x|\geqq r,\ \forall\, t\in S^1_T$

$$W(t,x)\leqq c\,|x|^\theta,\qquad W'(t,x).x-2W(t,x)\leqq c\,|x|^\theta$$

If these hypotheses hold we can also assume without loss of generality that

(V4) $W(t,x)\leqq b,\ \forall\, x\in\mathbf{R}^N\setminus\{0\}$.

Indeed, if we take $\tilde{a}\in\left]a,\left(\frac{\pi}{T}\right)^2\right[$ and pose

$$\tilde{W}(t,x)=-\frac{1}{2}(\tilde{a}-a)\,|x|^2+W(t,x),$$

(V) can be written as

$$V(t,x)=\frac{1}{2}\tilde{a}\,|x|^2+\tilde{W}(t,x),$$

satisfying (V1)-(V4).

A non-collision T-periodic solution of

$$\ddot{u}+V'(t,u)=0 \tag{3.1}$$

is a $u\in\mathscr{C}^2(S^1_T,\mathbf{R}^N\setminus\{0\})$ which solves (3.1). According to the usual notation, we denote by

$$\Lambda=\{u\in H^1(S^1_T,\mathbf{R}^N): u(t)\neq 0\ \forall\, t\in S^1_T\}$$

the space of H^1 non-collision orbits. It is well known that the non-collision solutions of system (3.1) are the singular points of the action functional $f\in\mathscr{C}^1(\Lambda)$ defined by

$$f(u)=\int_0^T\left\{\frac{1}{2}|\dot{u}|^2-V(t,u)\right\}dt, \tag{3.2}$$

whose differential at $u\in\Lambda$ is the linear form

$$\langle f'(u),h\rangle=\int_0^T\{\dot{u}.\dot{h}-V'(t,u).h\}\,dt. \tag{3.3}$$

If $u\in\Lambda$, we denote the pericentrum of the orbit u by

$$p(u)=\min_{t\in S^1_T}|u(t)|. \tag{3.4}$$

Let us draw some consequences of conditions (V1)-(V4).

First of all we have a well known property that motivates the (SF) condition.

3.1. LEMMA. – *Let* $\{u_n\} \subset \Lambda$ *and* $u_n \rightharpoonup u \in \delta\Lambda$. *Then* $f(u_n) \to +\infty$.

Proof. – *See* [6].

Q.E.D.

3.2. LEMMA. – *For every* $\lambda \in \mathbf{R}$ *there exists a constant* $k = k(\lambda)$ *such that*

$$\|\dot{u}\|_2 \leqq k(\lambda)\, p(u), \qquad \forall u \in \{f \leqq \lambda\} \tag{3.5}$$

Proof. – By the Poincaré inequality we know that

$$\|v\|_2 \leqq \frac{T}{\pi} \|\dot{v}\|_2, \qquad \forall v \in H_0^1(0, T; \mathbf{R}^N).$$

Thus if $u \in \Lambda$ and $t_0 \in S_T^1$ is a point where $|u(t)|$ attains its minimum value $p(u)$, since the curve $v(t) = u(t + t_0) - u(t_0)$ is in $H_0^1(0, T; \mathbf{R}^N)$ we obtain

$$\|u\|_2 \leqq \frac{T}{\pi} \|\dot{u}\|_2 + \sqrt{T}\, p(u). \tag{3.6}$$

Condition (V) implies

$$f(u) \geqq \frac{1}{2} \|\dot{u}\|_2^2 - \frac{a}{2} \|u\|_2^2 - bT, \qquad \forall u \in \Lambda, \tag{3.7}$$

which yields, together with (3.6), to

$$f(u) \geqq \frac{1}{2} \|\dot{u}\|_2^2 - \frac{a}{2} \left(\frac{T}{\pi} \|\dot{u}\|_2 + \sqrt{T}\, p(u) \right)^2 - bT. \tag{3.8}$$

Now if the claim of the lemma is false, then there exists a sequence $\{u_k\} \subset \Lambda$ such that $f(u_k)$ is bounded and

$$\|\dot{u}_k\|_2 \geqq k\, p(u_k). \tag{3.9}$$

Putting (3.9) into (3.8), we get

$$f(u_k) \geqq \frac{1}{2} \|\dot{u}_k\|_2^2 \left[1 - a \left(\frac{T}{\pi} + \frac{\sqrt{T}}{k} \right)^2 \right] - bT.$$

Since $a < \left(\frac{\pi}{T} \right)^2$, the term into square brackets is bounded away from zero for large k; since $f(u_k)$ is bounded we conclude that $\|\dot{u}_k\|_2$ is bounded too. Then from (3.9) $p(u_k)$ tends to zero and, extracting a subsequence as needed, we may suppose that the u_k converge weakly to some $u \in \partial\Lambda$. Due to Lemma 3.1 we have $f(u_k) \to \infty$, a contradiction which proves the assertion.

Q.E.D.

3.3. LEMMA. — *For every* $c \in \mathbf{R}$ *the set* $\Lambda_c = \left\{ u \in \Lambda : \frac{\|\dot{u}\|_2}{p(u)} \leqq c \right\}$ *is of finite category in* Λ.

Proof. — Due to Proposition (2.1) it suffices to give a deformation $h:[0, 1] \times \Lambda_c \to \Lambda$ such that $h(1, \Lambda_c) \subset\subset \Lambda$. Take $\delta \in]0, T[$ such that $c\sqrt{\delta} \leqq \frac{1}{2}$, and define

$$\begin{cases} \varphi(t) = \frac{1}{\delta} & \text{if } t \in [0, \delta]; \\ \varphi(t) = 0, & \text{otherwise.} \end{cases}$$

For any $u \in \Lambda$ let $(u * \varphi)(t)$ be the convolution $\int_0^T u(t-s)\varphi(s)\,ds$: then we have for any t, by standard inequalities

$$|u(t) - (u * \varphi)(t)| \leqq \frac{1}{\delta}\int_0^\delta |u(t) - u(t-s)|\,ds$$
$$\leqq \sup_{|s| \leqq \delta} |u(t) - u(t-s)| \leqq \sqrt{\delta}\,\|\dot{u}\|_2 = p(u)\left(\frac{\|\dot{u}\|_2}{p(u)}\right)\sqrt{\delta}.$$

Hence if u is in Λ_c,

$$(3.10) \qquad |u(t) - (u * \varphi)(t)| \leqq p(u)\,c\sqrt{\delta} \leqq \frac{1}{2}p(u) \leqq \frac{1}{2}|u(t)|,$$

so that $\forall (s, t) \in [0, 1] \times [0, T]$

$$(3.11) \qquad (1-s)\,u(t) + s\frac{(u * \varphi)(t)}{p(u)} \neq 0.$$

Thus the left-hand side of (3.11) defines a homotopy $h:[0, 1] \times \Lambda_c \to \Lambda$; furthermore $\overline{h(1, \Lambda_c)} \subset \Lambda$. Finally $h(1, \Lambda_c)$ is relatively compact since it is the image of the bounded set $\{u/p(u) : u \in \Lambda_c\}$ through the convolution operator $T_\varphi : H^1 \ni u \mapsto u * \varphi \in H^1$, which is compact.

Q.E.D.

3.4. LEMMA. — *Let* $V \in \mathscr{C}^1(S^1_T \times (\mathbf{R}^N \setminus \{0\}))$ *and let SF hold. The functional f verify the PS condition on the bounded sets.*

Proof. — Let $\{u_k\}$ be a H^1-bounded PS sequence. Then, up to a subsequence, it converges weakly in H^1 and strongly in L^∞ to an element u of $H^1(S^1_T, \mathbf{R}^N)$ which belongs to Λ by Lemma (3.1). Hence $V'(t, u_k).(u - u_k)$ converges uniformly to zero. Since $f'(u_k) \to 0$ in H^{-1} and

$u-u_k$ is H^1-bounded we have, from (3.3)

$$\|\dot u\|_2^2-\lim_{k\to\infty}\|\dot u_k\|_2^2=\lim_{k\to\infty}\int_0^T \dot u_k.(\dot u-\dot u_k)$$
$$=\lim_{k\to\infty}\left\{\langle f'(u),\, u-u_k\rangle+\int_0^T V'(t,\, u_k).(u-u_k)\right\}=0.$$

Therefore u_k converges to u strongly in H^1.

Q.E.D.

3.5. THEOREM. – *Let* V *be a* T*-periodic time-dependent potential satisfying* (V). *Then the dynamical system*

$$\ddot u+V'(t,\, u)=0$$

has infinitely many T*-periodic non-collision solutions.*

Proof. – We have to check the hypotheses of Theorem 2.4

(i) *See* [3].

(ii) Lemma 3.1.

(iii) Lemma 3.2 and Lemma 3.3.

Now we shall define $g\in\mathscr{C}^1(\Lambda)$, $\beta\in]0,1[$ and $\lambda_0\in\mathbf{R}$ verifying (iv, v, vi). Let k_∞ be a constant such that

$$\|u\|_\infty\leqq k_\infty\|u\|_{1.2},\qquad \forall u\in H^1(S_T^1,\mathbf{R}^N), \tag{3.12}$$

[e.g., $k_\infty=:(T+T^{-1})^{1/2}$], and choose $\beta\in\left]\frac{\theta}{2},1\right[$. We define

$$g(u)=\gamma\|u\|_{1.2}^\theta,\ \forall u\in\Lambda,$$

where

$$\gamma\geqq\frac{\beta c T k_\infty^\theta}{2\beta-\theta}. \tag{3.13}$$

(iv) We have to show that $\{f\leqq g\}$ is a set of finite category in Λ. Let us take $\varepsilon>0$ such that

$$a_\varepsilon=:\frac{a+2\varepsilon}{1-2\varepsilon}<\left(\frac{\pi}{T}\right)^2,$$

$M\in\mathbf{R}$ such that $\forall s\in\mathbf{R}\ \gamma|s|^\theta\leqq\varepsilon s^2+(1-2\varepsilon)M$, and define

$$f_\varepsilon(u)=\int_0^T\left\{\frac{1}{2}|\dot u|^2-\frac{a_\varepsilon}{2}|u|^2-\frac{W(t,u)}{1-2\varepsilon}\right\}dt.$$

Then

$$\{f\leqq g\}\subset\{f\leqq\varepsilon\|u\|_{1.2}^2+(1-2\varepsilon)M\}=\{f_\varepsilon\leqq M\}.$$

Again we have from Lemma 3.2 that there exists $k\in\mathbf{R}$ such that

$$\|\dot u\|_2\leqq kp(u),\qquad \forall u\in\{f_\varepsilon\leqq M\} \tag{3.14}$$

and by Lemma 3.3,

$$\mathrm{Cat}_\Lambda(\{f \leqq g\}) \leqq \mathrm{Cat}_\Lambda(\{f_\varepsilon \leqq M\}) < \infty.$$

(v) For any $\lambda \in \mathbf{R}$ $\{f \geqq g\} \cap \{f \leqq \lambda\} \subset \{g \leqq \lambda\}$ is a bounded set because g is coercive. Therefore by Lemma 3.5 the PS condition holds in $\{f \geqq g\}$.

(vi) From (3.6) and (4.14) we find, for some $k_1 > 0$,

$$(3.15) \qquad \|u\|_{1.2} \leqq k_1 p(u), \qquad \forall u \in \{f \leqq g\}.$$

We take $\lambda_0 =: \gamma (k_1 r)^\theta$. Then if $u \in \{f = g \geqq \lambda_0\}$, there results

$$\|u\|_{1.2} \geqq \left(\frac{\lambda_0}{\gamma}\right)^{1/\theta} = k_1 r$$

so we have from (3.12) and (3.15)

$$r \leqq p(u) \leqq |u(t)| \leqq \|u\|_\infty \leqq k_\infty \|u\|_{1.2}, \qquad \forall t \in S^1_T.$$

Now, taking account of (V3) we get

$$(3.16) \quad \int_0^T \{W'(t, u).u - 2W(t, u\}\, dt$$
$$\leqq T \sup \{W'(t, x).x - 2W(t, x) : t \in S^1_T,\ r \leqq |x| \leqq k_\infty \|u\|_{1.2}\}$$
$$\leqq c\,T (k_\infty \|u\|_{1.2})^\theta.$$

From (3.2) and (3.3) we get

$$(3.17) \quad \|f'(u)\|\ \|u\|_{1.2} \geqq \langle f'(u), u\rangle = 2f(u) - \int_0^T \{W'(t, u).u - 2W(t, u)\}\, dt.$$

From (3.16) and (3.17)

$$\|f'(u)\| \geqq (2\gamma - c\,T k^\theta_\infty)\, \|u\|^{\theta-1}_{1.2}:$$

since $\|g'(u)\| = \gamma\theta \|u\|^{\theta-1}_{1.2}$, we have, from our choice of γ (3.13)

$$\beta \|f'(u)\| - \|g'(u)\| \geqq 0, \qquad \forall u \in \{f = g \geqq \lambda_0\}.$$

Q.E.D.

3.6. *Remark.* — Theorem 3.5 can be improved stating that there exists a sequence $\{u_n\} \subset \mathbf{Z}_f$ such that $f(u_n) \geqq n \|u\|^\theta + n$. This follows at once from Theorem 2.4, for in the definition of the function g we can choose the constant γ arbitrarily large [eq. (3.13)]. We shall use this fact in the following corollary, as a trick to avoid the constant solutions (*see* also [1], § 7).

3.7. Corollary (Autonomous case). — *Let* $W \in \mathscr{C}^1(\mathbf{R}^N \setminus \{0\})$ *be a potential such that* (SF) *holds,* $W'(x).\dfrac{x}{|x|^2} \to +\infty$ *as* $x \to 0$, *and* $W'(x).x - 2W(x) \leqq c|x|^\theta$ *for* $|x| \geqq r$, *with* $\theta < 2$. *Then for any* $T > 0$ *and* *for any* $a \in \mathbf{R}$, *the system*

$$\ddot{u} + au + W'(u) = 0 \tag{3.18}$$

has infinitely many T*-periodic non-constant non-collision solution.*

Proof. – The inequality $W'(x).x - 2W(x) \leqq c|x|^\theta$, $\forall |x| \geqq r$ yields by integration $W(x) \leqq c_1 |x|^2$, $\forall |x| \geqq r$. Hence, replacing if needed W with $W - c_1 |x|^2$ and a with $a + c_1$, we can suppose without loss of generality that W is bounded from above. We take $k \in \mathbf{N}$ so large that $a < k^2 \left(\frac{\pi}{T}\right)^2$, and we pose $\tilde{T} = \frac{T}{k}$. Now we look for $\tilde{T}$-periodic non collision solutions of system (3.18): Theorem 3.6 applies and we get a sequence $\{u_n\} \subset \Lambda$ of solutions such that $f(u_n) \geqq n \| u_n \|_\infty^\theta + n$ (Rem. 3.6). Only finitely many of these can be constant: for otherwise (taking the subsequence of the constant solutions) we would get from (3.18), by scalar product with u_n

$$a|u_n|^2 + W'(u_n).u_n = 0, \tag{3.19}$$

and

$$f(u_n) = \int_0^{\tilde{T}} \left\{ -\frac{a}{2} |u_n|^2 - W(u_n) \right\} dt = \frac{\tilde{T}}{2} \{ W'(u_n).u_n - 2W(u_n) \} \tag{3.20}$$

Since $f(u_n) \to \infty$ either $|u_n| \to 0$ or $|u_n| \to \infty$. In the former case it follows from our hypothesis on W that $W'(u_n).\frac{u_n}{|u_n|^2} \to \infty$, which is in contradiction with (3.19). In the latter one we have from (3.20) that $f(u_n) \leqq \frac{\tilde{T}}{2} c |u_n|^\theta$ for large n, whereas $f(u_n) \geqq n |u_n|^\theta$: a contradiction again.

Q.E.D.

4. FURTHER REMARKS

We emphasize that condition (V) does not imply the usual PS condition (iii)' of Theorem 2.6, even if we assume $\lim_{x \to \infty} V(t, x) = 0$: we shall show this in Example 4.1. However, if additional hypotheses on V are assumed, such as

$$\limsup_{x \to \infty} |W(t, x)| + |W'(t, x)| < \infty,$$

then (iii)' holds and Theorem 2.6 applies.

4.1. *Example.* – *A potential* $V \in \mathscr{C}1(\mathbf{R}^N)$ *satisfying*

$$V \leqq 0, \qquad \lim_{x \to \infty} V(x) = 0, \qquad |V'(x)| \leqq |x|^{1/2}$$

(*hence also the hypotheses of Theorem* 3.6) *and such that the corresponding action functional f does not verify the usual PS condition at any positive level.*

Let $\{q_n\}_{n \in \mathbf{N}}$ be an enumeration of $\mathbf{Q}_+^*$ and $\{x_n\}_{n \in \mathbf{N}}$ a sequence in $\mathbf{R}^N$ such that $x_n \to \infty$, $|x_n| \geqq (q_n+1)^2 + q_n + 1$ and $|x_n - x_m| > q_n + q_m + 2$ if $n \neq m$. For any $n \in \mathbf{N}$ let $\varphi_n \in \mathscr{C}_c^\infty(\mathbf{R}^+)$ be such that

$$\begin{cases} 0 \geqq \varphi_n(t) \geqq -\dfrac{1}{n}, & \forall t \geqq 0; \\ \varphi_n(t) = 0, & \text{if } \ t \geqq q_n + 1; \\ \varphi_n'(q_n) = q_n; \\ \|\varphi_n'\|_\infty \leqq q_n + 1. \end{cases}$$

Define $V_n(x) = \varphi_n(|x - x_n|)$ for every $x \in \mathbf{R}^N$, and let $w \in \mathscr{C}^\infty(S_{2\pi}^1, \mathbf{R}^N)$ satisfy

$$\begin{cases} \ddot{w} + w = 0, \\ |w(t)| = 1. \end{cases} \tag{4.1}$$

Then $u_n = x_n + q_n w$ is a 2π-periodic solution of the system

$$\ddot{u} + V_n'(u) = 0.$$

Since the V_n have disjoint supports it is defined a potential $V = \sum_n V_n$ of class $\mathscr{C}^\infty$ such that $V \leqq 0$ and $V(x) \to 0$ as $x \to \infty$. Moreover $V(x) \leqq |x|^{1/2} \, \forall x$: if $V(x) \neq 0$, then there exists $n \in \mathbf{N}$ such that $x \in B(x_n, q_n + 1)$, so one has, by the choice of x_n,

$$|x| \geqq |x_n| - (q_n + 1) \geqq (q_n + 1)^2$$

and

$$\|V'(x)\| \leqq \|V_n'(x)\| \leqq \|\varphi_n'\|_\infty \leqq q_n + 1 \leqq |x|^{1/2}.$$

Each u_n solves

$$\begin{cases} \ddot{u} + V'(u) = 0 \\ u(t) = u(t + 2\pi), \end{cases}$$

Thus for any $n \in \mathbf{N}$

$$f'(u_n) = 0,$$
$$f(u_n) = \pi q_n^2 - 2\pi\varphi_n(q_n)$$
$$\|u_n\|_{1,2} \to \infty.$$

Since $\varphi_n(q_n) \to 0$ as $n \to \infty$, one has that for any $\lambda \in \mathbf{R}^+$ there exists a subsequence of $\{u_n\}$ which is a non-compact PS sequence at the level λ

for f. Of course the same example can be done for a singular potential, simply adding to V a singular perturbation with compact support.

4.2. *Remark.* – Notice that if V is autonomous no assumptions on the coefficient a are needed in order to get infinitely many T-periodical solution of (3.1). In the following example we show that if we drop condition $a<\left(\frac{\pi}{T}\right)^2$, (iv) and (v) in general fail to hold.

4.3. *Example.* – *A potential* $V\in\mathscr{C}^1(\mathbf{R}^N\setminus\{0\})$ *such that the corresponding action functional f does not verify conditions* (iv) *and* (v).

Let $a>\left(\frac{\pi}{T}\right)^2$, and let $V\in\mathscr{C}^1(\mathbf{R}^N\setminus\{0\})$ be such that $V(x)\geqq\frac{1}{2}a|x|^2$, $\forall x$ with $|x|\geqq 1$. We show that for any λ_0 and ρ_0,

$$\mathrm{Cat}_\Lambda(\{f\leqq\lambda_0\}\setminus B(0,\rho_0))=\infty;$$

in order to do this it is sufficient to exhibit a deformation of a set of infinite category, e.g., $A=\{u\in\Lambda:|u(t)|=1\ \forall t\}$, in $\{f\leqq\lambda_0\}\setminus B(0,\rho_0)$. Choose T^* in $\left]\frac{\pi}{\sqrt{a}},T\right[$, and define the functions $[0,1]\times[0,T]\to\mathbf{R}$

$$g(s,t)=\begin{cases}0, & \text{if } 0\leqq t\leqq sT^*\\ T\dfrac{t-sT^*}{T-sT^*}, & \text{if } sT^*<t\leqq T,\end{cases}$$

$$l(s,t)=\begin{cases}s\sin\left(\dfrac{\pi t}{sT^*}\right), & \text{if } 0\leqq t\leqq sT^*\\ 0, & \text{if } sT^*<t\leqq T.\end{cases}$$

Consider the homotopy $h:[0,1]\times A\to\Lambda$:

$$h(s,u)=u\circ g(s,\,.),$$

and set $B=h(1,A)$: clearly every $u\in B$ is constant on $[0,T^*]$. For $r\in\mathscr{C}(B)$, $r\geqq 0$ consider the homotopy $k:[0,1]\times B\to\Lambda$:

$$k(s,u)=u+r(u)\,u(0)\,l(s,\,.),$$

We shall choose r in such a way that $k(1,B)\subset\{f\leqq\lambda_0\}\setminus B(0,\rho_0)$. In order to do this, we note that

$$(4.2)\quad \|k(1,u)\|_{1,2}\geqq C\|k(1,u)\|_\infty\geqq C\left|k(1,u)\left(\frac{T^*}{2}\right)\right|$$
$$=C\left|u\left(\frac{T^*}{2}\right)+r(u)\,u(0)\right|=C(r(u)+1)|u(0)|\geqq C\,r(u)|u(0)|.$$

Thus $\|k(1, u)\|_{1,2} \geqq \rho_0$ whenever $r(u) \geqq \frac{\rho_0}{C}$. Furthermore, making the positions

$$r_1(u) = \int_{T^*}^{T} \left\{ \frac{1}{2} |\dot{u}|^2 - V(u) \right\} dt$$

and

$$\mu = -\frac{1}{2} \int_0^{T^*} \left\{ \left(\frac{\pi}{T^*} \right)^2 \cos^2\left(\frac{\pi}{T^*} t \right) - a \sin^2\left(\frac{\pi}{T^*} t \right) \right\} dt = \frac{T^*}{4} \left[a - \left(\frac{\pi}{T^*} \right)^2 \right],$$

there results

(4.3) $f(k(1, u))$

$$\leqq \frac{1}{2} \int_0^{T^*} \left\{ \left(\frac{\pi}{T^*} \right)^2 r(u)^2 \cos^2\left(\frac{\pi}{T^*} t \right) - a \left[r(u) \sin\left(\frac{\pi}{T^*} t \right) + 1 \right]^2 \right\} dt + r_1(u)$$

$$\leqq \frac{1}{2} r(u)^2 \int_0^{T^*} \left\{ \left(\frac{\pi}{T^*} \right)^2 \cos^2\left(\frac{\pi}{T^*} t \right) - a \sin^2\left(\frac{\pi}{T^*} t \right) \right\} dt + r_1(u)$$

$$= -\mu\, r(u)^2 + r_1(u).$$

Since $\mu > 0$ and $r_1 \in \mathscr{C}(B)$, if we take

$$r(u) = \max\left(\frac{\rho_0}{C}, \sqrt{\frac{|r_1(u) - \lambda_0|}{\mu}} \right)$$

we have from (4.2) and (4.3) that $\|k(1, u)\|_{1,2} \geqq \rho_0$ and $f(k(1, u)) \leqq \lambda_0$. Then we have

$$\mathrm{Cat}_\Lambda(\{f \leqq \lambda_0\} \setminus B(0, \rho_0)) \geqq \mathrm{Cat}_\Lambda(B) = \mathrm{Cat}_\Lambda(h(1, A)) \geqq \mathrm{Cat}_\Lambda(A) = \infty.$$

Q.E.D.

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